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HOW GOOD ARE GLOBAL NEWTON METHODS?

Part 2

Allen Goldstein

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19 ABSTRACT (Continue on reverse if necessary and identify by block number) Newton's method applied to certain problems with a discontinuous derivative operator is shown to be effective. A global Newton method in this setting is exhibited and its computational complexity is estimated. As an application a method is proposed to solve problems of linear inequalities (linear programming, phase I). Using an example of the Klee-Minty type due to Blair, it was found that the simplex method (used in super-lindo) required over 2,000 iterations, while the method above required an average of 8 iterations (Newton steps) over 15 random starting values. <i>Keywords:</i>					
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How good are global Newton methods. Part 2

A. A. Goldstein*

In the first part of this ms. we made some observations about global Newton methods on a simple class of mappings from a real separable Hilbert space H into H . The map F was assumed to be everywhere differentiable with a derivative D that was onto and satisfied $\mu\|h\| \leq \|D(x)h\| \leq \lambda\|h\|$, for all $h \in H$ and some $\mu > 0$. Relying on theorems of Nemerovsky and Yudin(1977), we showed that no formulation of a global Newton method could achieve better than linear convergence at a certain specified rate for every member of the class. Assuming that D was uniformly Lipschitz continuous with constant L it was easy to utilize the Kantorovich inequalities (1948). Points satisfying the Kantorovich inequalities are guaranteed to be a satisfactory initial point for Newton's method. By satisfactory, we mean that the sequence converges to a root at a quadratic rate, as will be seen below. Following a similar terminology of Smale(1986), such points will be called approximate roots.

We used the Kantorovich inequalities to formulate a coarse version of Smale's global Newton method to obtain the complexity of the algorithm in this setting. The Kantorovich condition is : if x_0 satisfies $\|D^{-1}(x_0)\| \|D^{-1}(x_0)F(x_0)\| \bar{L} = \beta \eta \bar{L} < .5$ then x_0 is an approximate root. Here $\bar{L}$ is the Lipschitz constant for D on the ball $B = \{x \in H : \|x - x_0\| \leq 2\|D^{-1}(x_0)F(x_0)\|\}$. We found that starting at x_0 we could achieve a point satisfying the Kantorovich inequalities in less than:

$$11.46 [\|F(x_0)\| L / \mu^2] \ln(1.443 \ln 8Q) \text{ steps}$$

where: L is the global Lipschitz constant, and $Q = \lambda/\mu$ the condition number. It is noteworthy that this algorithm is insensitive to the condition number. When the word 'steps' appears in this paper, as above, we mean the preceding formula to be rounded up to the nearest integer.

Some numerical experiments however were disappointing. The poor performance stemmed from our reliance on the Kantorovich inequality. This inequality is overconservative when

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the dimension of H exceeds 1. In a test example even though we estimated $\bar{L}$ along only one ray (in the direction of the Newton step) and thus determined a $\bar{L}$ that was a lower bound, we found approximate roots where the Kantorovich constant $\beta \eta \bar{L}$ was several thousand instead of $< 1/2$. (See definition 1.5 below). For this reason we undertook to revise the Kantorovich inequalities, hoping to obtain better sufficient conditions. We shall do this by avoiding the Schwarz inequality at a key spot. In general we were guided by Kantorovich's original proof. A parameter K replaces $\bar{L}$ on the ball B . Estimates are given for the decrease in the norm of the vector valued function. This leads to the definition of an ϵ approximate root. We believe this concept will be useful in the further development of global Newton algorithms. The estimates we give for the convergence rates can probably be improved by a one dimensional analysis, as was done by Gragg and Tapia(1974) for the Kantorovich Theorem..

One consequence of the revised proof is that the derivative operator need not be continuous, although Lipschitz continuity is convenient in a neighborhood of each root. As an example we consider the problem of solving linear inequalities. (Phase 1 linear programming) By the use of Lagrange multipliers the method presented here can be extended to linear programming and other types of programs. We shall however only consider linear inequalities in this ms. The idea of using Newton's method for linear programming may be found in Smale(1986). Our technique will be to use a penalty function approach. In this setting we have a twice differential convex function to minimize, with jump discontinuities in the second derivative. For the code we refurbished an old global Newton method of our own. This was easy because it was already up and running. We tried the method on an example due to C. Blair(in Kortanek and Shi 1987) that is a version of the famous Klee-Minty example. This problem was suggested to us by Ken Kortanek, and we are thankful for his kind help. While Ken was visiting, he ran his scaling (modified Karmarkar) algorithm on my computer for a benchmark in the 10 variable case. The robust 'Superlindo' by Linus Schrage was also used as a benchmark for these problems. For 8, 10, and 12 variables Superlindo took 35, 311, and 2236 iterations respectively. Our algorithm hardly changed in the number of steps versus dimension. Ken Kortanek remarked he noticed the same phenomena with their algorithm. In the appendix appears a list of 15 consecutive runs for the case $n=12$ with the components of the initial point being fed by a random number generator with values between -1000 and 1000. The number of steps until termination ran between 6 Newton steps plus 2 gradient steps, and 12 Newton steps plus 3 gradient steps.

The average was 8 Newton steps and 19/15 gradient steps. Full double precision accuracy was achieved. The Kortanek-Shi algorithm in the 10 variable case was almost as fast as ours for this case, but it achieved only 3 significant figure accuracy.

These spectacular results were unexpected. Unlike the Smale global Newton method, which we as yet have not implemented, the method we used is sensitive to the condition number of the system. See 3.0 below.

There remains to test the algorithm on a general mix of problems to see if it is worth adding to the armamentarium of linear programming. Thus the question of whether we have a viable method is not settled at this time.

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In what follows, the Kantorovich inequality will be sharpened. Let f be Frechet differentiable mapping between real Banach spaces E and F . Let $f'(x)$ denote the derivative at x and $f'_{-1}(x)$ its inverse. The Kantorovich inequality states that if x_0 is given such that f' is Lipschitz continuous with constant K on the ball $\{x \in E : \|x - x_0\| \leq 2 \|f'_{-1}(x_0)f(x_0)\| = 2\eta_0\}$, if $\|f'_{-1}(x_0)\| = \beta_0$ and if $\eta_0\beta_0K < 1/2$, then x_0 is an approximate root. This means that the sequence generated by Newton's method will converge quadratically to a root at least as fast as the rate estimated by the theorem below.

THEOREM 1.0 Let f be a map between real Banach spaces E and F . Assume f is Frechet differentiable on an open subset E' of E . Let $x_0 \in E'$ be given such that $(f'(x_0))^{-1} = f'_{-1}(x_0)$ exists. Set

$$\eta_0 = \|f'_{-1}(x_0)f(x_0)\|$$

$$S = \{x \in E : \|x - x_0\| \leq 1.68\eta_0\}. \text{ Assume that } S \subset E'$$

Set

$$\beta_0 = \|f'_{-1}(x_0)\|, \text{ and let}$$

$$S' = \{x \in S : \|f'_{-1}(x)\| \leq 2.3\beta_0\}.$$

Finally set:

$$K = \left\{ \sup \frac{\|f'_{-1}(x)(f'(x) - f'(\xi))\|}{\|f'_{-1}(x)\| \|f'_{-1}(x)f(x)\|} : x \in S', \xi = x + tf'_{-1}(x)f(x) \text{ and } t \in (0, 1) \right\}.$$

If $\eta_0\beta_0K = h_0 \leq 1/3$, then x_0 is an approximate root.

PROOF By hypothesis x_1 is well defined. Let $H_1(x_0, x_1) = H_1 = f'_{-1}(x_0)f'(x_1) = I - f'_{-1}(x_0)(f'(x_0) - f'(x_1))$. H_1 maps E into itself. Our hypotheses imply that $\|f'_{-1}(x_0)(f'(x_0) - f'(x_1))\| \leq h_0 \leq 1/3$, whence H_1 has an inverse. We have the estimate $\|(H_1)^{-1}\| = (1 - h_0)^{-1} \leq 3/2$. Also $\|H_1\| \leq 4/3$. Observe that $f'(x_0)H_1 = f'(x_1)$ and $(H_1)^{-1}f'_{-1}(x_0) = f'_{-1}(x_1)$. Thus $f'_{-1}(x_1)$ exists and x_2 is well defined.

Let $\beta_1 = \|f'_{-1}(x_1)\|$ and $\eta_1 = \|x_1 - x_2\|$. Thus $\beta_1 = \|f'_{-1}(x_1)\| \leq 1.5 \|f'_{-1}(x_0)\|$. Let F_1 be defined by the formula $F_1(x) = x - f'_{-1}(x_0)f(x)$. Since $F_1(x_0) = x_1$ and $F_1(x_1) = x_1 - f'_{-1}(x_0)f(x_1)$,

$$f'_{-1}(x_0)f(x_1) = F_1(x_0) - F_1(x_1).$$

By the generalized mean value theorem of Graves(1927),

$$\|F_1(x_1) - F_1(x_0)\| \leq \sup\{ \|F'_1(\xi)\| : t \in (0, 1) \text{ and } \xi = tx_0 + (1-t)x_1 \} \|x_1 - x_0\|.$$

But $\|F'_1(\xi)\| = \|I - f'_{-1}(x_0)f'(\xi)\| = \|f'_{-1}(x_0)(f'(x_0) - f'(\xi))\| \leq K\eta_0\beta_0$. Thus:

$$\|f'_{-1}(x_1)f(x_1)\| = h_0\eta_0,$$

$$\eta_1 = \|f'_{-1}(x_1)f(x_1)\| = \|H_1^{-1}f'_{-1}(x_0)f(x_1)\| \leq h_0\eta_0(1-h_0)^{-1}$$

and

$$h_1 = \beta_1\eta_1K \leq (1-h_0)^{-2}h_0^2 \leq 9h_0^2/4 \leq 1/4$$

At this juncture our estimate will differ from that of Kantorovich by a factor of 1/2 that will appear in the recursion for h_n . Kantorovich gains a factor of 1/2 because of the Lipschitz condition (or 2nd differentiability condition) on H . Clearly x_1 and x_2 belong to S , and also to S' . Now define in turn H_2 , β_2 , η_2 , F_2 , and h_2 , mutatis mutandis, to obtain β_2 , η_2 , and h_2 in terms of β_1 , η_1 , and h_1 .

For $k=1,2,3,\dots,n$, assume that $\beta_k\eta_kK \leq 1/3$ and $x_k \in S'$. Assume moreover that the recursion formulae:

$$\beta_n = \beta_{n-1}(1-h_{n-1})^{-1} \quad \eta_n = h_{n-1}\eta_{n-1}(1-h_{n-1})^{-1} \quad h_n = h_{n-1}^2(1-h_{n-1})^{-1}$$

hold for $k=1,2,3,\dots,n$. Since these recursion formulae hold also for $k = n+1$, we have the following sequence for $\{h_k\}$

$$\frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{64}, \quad \frac{1}{3968}, \quad \dots, \quad \frac{1}{(\frac{1}{h_n} - 1)^2}$$

Since:

$$\beta_{n+1} \leq \beta_0 \left(\frac{1}{(1-h_0)} \frac{1}{(1-h_1)}, \dots, \frac{1}{(1-h_n)} \right)$$

$$\eta_{n+1} \leq \eta_0 \left(\frac{h_0}{1-h_0} \frac{h_1}{1-h_1} \frac{h_2}{1-h_2}, \dots, \frac{h_n}{1-h_n} \right)$$

and

$$\|f'(x_{n+1})\| \leq \|f'(x_0)\| (1+h_0)(1+h_1), \dots, (1+h_n)$$

We get from above the formulae:

$$\beta_{n+1} \leq \beta_0 \left(\frac{3 \cdot 4 \cdot 9 \cdot 64 \cdot 3969 \cdot 1575024 \cdot \dots \cdot 1}{2 \cdot 3 \cdot 8 \cdot 63 \cdot 3968 \cdot 1575023 \cdot \dots \cdot (1 - h_n)} \right) \leq 2.3 \beta_0, \text{ for } n = 1, 2, 3, \dots$$

$$\eta_{n+1} \leq \eta_0 \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{8} \cdot \frac{1}{63} \cdot \frac{1}{3967} \cdot \dots \cdot \frac{h_n}{(1 - h_n)} \right)$$

$$\|f'(x_n)\| \leq \|f'(x_0)\| \left(\frac{4 \cdot 5 \cdot 10 \cdot 65 \cdot 3970 \cdot \dots \cdot (1 + (h_n)^{-1})}{3 \cdot 4 \cdot 9 \cdot 64 \cdot 3969 \cdot \dots \cdot (h_n)^{-1}} \right)$$

$$\|f'(x_{n+1})\| \leq 1.9 \|f'(x_0)\| \text{ for } n = 1, 2, 3, \dots$$

To complete the induction we show that $x_{n+1} \in S'$. The triangle inequality gives:

$$\begin{aligned} \|x_{n+1} - x_0\| &\leq \eta_0 + \eta_1 + \dots + \eta_n \\ &\leq \eta_0 \left(1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{48} + \frac{1}{3024} + \dots + \prod_{k=1}^n \frac{h_k}{1 - h_k} \right) \leq 1.68 \eta_0 \text{ for } n = 1, 2, 3, \dots \end{aligned}$$

Thus $x_n \in S$. Since $\beta_{n+1} \leq 2.3 \beta_0$ by the above inequality, the induction is completed.

Similarly we have:

$$\|x_{n+p+1} - x_n\| \leq \eta_n + \eta_{n+1} + \eta_{n+p} < 1.68 \eta_n.$$

Since $\{\eta_n\}$ converges to 0, the sequence $\{x_n\}$ is Cauchy with limit, say x^* . Since $f(x_n) = f'(x_n)(x_{n+1} - x_n)$, $\{f(x_n)\}$ converges to 0, and $f(x^*) = 0$.

We now estimate the rate of convergence. We have, at worst, that $h_n \leq \frac{4}{9} \left(\frac{9}{4} h_0 \right)^{2^n} \leq \left(\frac{3}{4} \right)^{2^n}$. The recursion formula for η_n implies that

$$\eta_n \leq h_{n-1} h_{n-2} \dots h_0 \eta_0 (1 - h_{n-1})^{-1} (1 - h_{n-2})^{-1} \dots (1 - h_0)^{-1}.$$

Also $\eta_n \leq \left(\frac{4}{9} \right)^n \left(\frac{9}{4} h_0 \right)^{2^{n-1}} \left(\frac{9}{4} h_0 \right)^{2^{n-2}} \dots \left(\frac{9}{4} h_0 \right) \eta_0$. Since $\sum_{k=1}^n 2^{n-k} = 2^n - 1$,

$$\frac{\eta_n}{\eta_0} \leq \left(\frac{4}{9} \right)^n \left(\frac{9}{4} h_0 \right)^{2^n - 1}.$$

Since $\|f(x_n)\| = \|f'(x_n)\| \eta_n$, while $\|f'(x_n)\| \leq 1.9 \|f'(x_0)\|$, we have the estimates

$$(I) \quad \frac{\|f(x_n)\|}{\|f(x_0)\|} \leq 1.9 \frac{\eta_n}{\eta_0} \quad \text{and} \quad (II) \quad \frac{\|x_n - x^*\|}{\|x_1 - x_0\|} \leq 1.68 \frac{\eta_n}{\eta_0}.$$

REMARK 1.1 Assume the hypotheses of the theorem with suitable changes in S and S' . Take $h_0 < 1/2$. Then x_0 is an approximate root.

REMARK 1.2 The theorem does not require the continuity of f' . Let $f(x) = 4x$, if $\infty < x \leq 1$ and $f(x) = 6x - 2$, if $x > 1$. Then every point is an approximate root for f . For example let $x_0 = 100$. Then $h_0 = 1/3$, $x_1 = 1/3$, $h_1 = 0$, and $x_2 = 0$.

REMARK 1.3 Let S, S' and K be defined as above. Assume that f' is Lipschitz continuous with constant L on S . Let $h_k = \sup\{\|f'_{-1}(x_k)(f'(x_0) - f'(\xi))\| : x \in S' \text{ and } \xi = x_k + tf'_{-1}(x_k)f(x_k), t \in (0, 1)\}$. Assume the Kantorovich inequality holds. Then $h_k \leq \beta_k \eta_k L \leq 1/3$. Whence $h_k/\beta_k \eta_k = K \leq L$. We have replaced L by K . Another hypothesis to ensure the boundedness of K (which we've assumed outright) is to require that f' be bounded on S' and, for any root x^* of f there is a neighborhood $N(x^*)$ such that f' is Lipschitz continuous on $N(x^*) \cap S'$.

REMARK 1.4 The Kantorovich type theorems are important for the insight they furnish, but in general are not helpful in concrete instances. For example, the above theorem and the original Kantorovich theorem are dependent upon the knowledge of the constant K or L , but the determination of upper bounds for these quantities is not trivial. Moreover the conditions given by these inequalities is sufficient but not in general necessary for a point to be an approximate root. Thus the determination of approximate roots is usually not practical with these theorems. Using the rough estimates furnished by local application of the modified Kantorovich inequality we propose proceeding by trial.

DEFINITION 1.5 Given $\epsilon > 0$, let n be the smallest integer such that

$$1.9 \left(\frac{4}{9}\right)^n \left(\frac{3}{4}\right)^{2^n - 1} < \epsilon.$$

A point x_0 is an ϵ -approximate root for the mapping f if the Newton sequence $\{x_i\}$ starting at x_0 is well defined for $i=1, 2, \dots, n$, and

$$\frac{\|f(x_n)\|}{\|f(x_0)\|} < \epsilon,$$

For example if $\epsilon = 10^{-18}$ then $n = 7$. This test follows from the estimates I and II of the above theorem.

LEMMA 2.0 Let A be a positive definite matrix. Let μ be its least eigenvalue and λ its greatest. Let $Q = \lambda/\mu$. Then:

$$\frac{[Ax, x]}{\|Ax\| \|x\|} \geq \frac{2}{Q^{\frac{1}{2}}} \frac{Q}{(Q+1)}.$$

The bound is the best possible.

PROOF. Assume A has been diagonalized, and let

$$[Ax, x]/\mu = x_1^2 + c_2 x_2^2 + \dots, c_n x_n^2.$$

We seek to minimize $f(x) = (x_1^2 + c_2 x_2^2 + \dots, c_4 x_4^2)/(x_1^2 + c_2^2 x_2^2 + \dots, c_4^2 x_4^2)^{\frac{1}{2}}$, subject to $\|x\| = 1$. The reason for restricting to polynomials of degree 4 in x_i^2 will become apparent in what follows.

Set $x_1^2 = 1 - x_2^2 - x_3^2 - x_4^2$. Using this equation to eliminate x_1^2 we get:

$$f(x) = \frac{1 + (c_2 - 1)x_2^2 + (c_3 - 1)x_3^2 + (c_4 - 1)x_4^2}{(1 + (c_2^2 - 1)x_2^2 + (c_3^2 - 1)x_3^2 + (c_4^2 - 1)x_4^2)^{\frac{1}{2}}}$$

Let P denote the numerator of the above fraction and $Q^{\frac{1}{2}}$ the denominator. Let $x = (x_2^2, x_3^2, x_4^2)$ and

$$A = (c_2 - 1, c_3 - 1, c_4 - 1)$$

and

$$B = (c_2^2 - 1, c_3^2 - 1, c_4^2 - 1)$$

The components of the equation $\nabla f(x) = 0$ may be written as :

$$x_2(c_2 - 1)[2B - (c_2 + 1)A, x] = x_2(c_2 - 1)(c_2 - 1) \quad (a)$$

$$x_3(c_3 - 1)[2B - (c_3 + 1)A, x] = x_3(c_3 - 1)(c_3 - 1) \quad (b)$$

$$x_4(c_4 - 1)[2B - (c_4 + 1)A, x] = x_4(c_4 - 1)^2 \quad (c)$$

If we assume distinct eigenvalues and that x_2 , x_3 and x_4 are non-zero we may cancel appropriately and obtain from (a), (b), and (c), a system of 3 linear equations of rank 2.

since the rows of the system are linear combinations of A and B. Thus one unknown say x_2 or x_3 (but not x_4) may be set to 0. Similarly if we had n variables only one variable among x_2 and x_{n-1} would be non-zero. If we set $x_2 = 0$, then the following system obtains:

$$(c_3^2 - 1)x_3^2 + (c_4 - 1)(c_4 - c_3)x_4^2 = c_3 - 1$$

$$(c_3^2 - 1)x_3^2 + (c_4^2 - 1)x_4^2 = c_4 - 1$$

One finds that

$$x_4^2 = \frac{(c_3^2 - 1)(c_4 - c_3)}{-(c_3 - 1)(c_3 + 1)(c_4 - 1)(c_4 - c_3)} < 0.$$

Since x_4^2 is negative, it is inadmissible. Thus x_3^2 must also vanish. Suppose now that $x_3 = 0$ but $x_2 \neq 0$. We again get $x_4^2 < 0$. Thus for these cases we must set x_2 and x_3 to 0.

If $c_2 > 1$ and $c_4 > c_2$ then the rank of $\{A, B\} = 2$. This case is treated just as the case above of distinct roots. If all the eigenvalues are equal, the ratio in the statement of the lemma is 1, verifying the claim for that case. For all other cases the rank of $\{A, B\} = 1$, and we may set x_2 and x_3 to 0.

There remains to carry out the minimization. The equation

$$(c_4^2 - 1)x_4^2 = c_4 - 1 \quad \text{implies} \quad x_4^2 = \frac{1}{Q + 1}$$

and

$$f(x) = \frac{2 \frac{Q}{Q+1}}{Q^{\frac{1}{2}}}$$

Algorithm 3.0

Let f be a twice differentiable function defined on real euclidean space E_n . Assume that f is convex and that it has bounded level sets. Given an arbitrary point x_0 let S denote the level set $\{x \in R_n : f(x) \leq f(x_0)\}$. Assume that the hessian of f at x , which we call $H(x)$, has an inverse at x_0 , denoted by $H^{-1}(x_0)$. At the k th step of the algorithm assume that $H^{-1}(x_k)$ exists. Let

$$g(x, \gamma) = \frac{f(x) - f(x - \gamma H^{-1}(x) \nabla f(x))}{\gamma [H^{-1}(x) \nabla f(x), \nabla f(x)]}$$

Choose $\delta \in (0, 1/2]$ and choose $\gamma_k > 0$ so that $\delta \leq g(x_k, \gamma_k) \leq 1 - \delta$, taking $\gamma_k = 1$, if possible. Set $x_{k+1} = x_k - \gamma_k H^{-1}(x_k) \nabla f(x_k)$. Let $S' = \{x_k \in S : k = 1, 2, 3, \dots\}$. Assume that the spectrum of $H(x)$ is bounded above on S' by ΛQ , and below by Λ . Let

$$h = \sup \left\{ \|H^{-1}(x)(H(x) - H(\xi))\| : x \in S' \text{ and } \xi = x + tH^{-1}(x)\nabla f(x), t \in [0, 1] \right\}.$$

(a) Assume that $K =$

$$\sup \left\{ \frac{\|H^{-1}(x)(H(x) - H(\xi))\|}{\|H^{-1}(x)\| \|H^{-1}(x)\nabla f(x)\|} : x \in S' \text{ and } \xi = x + tH^{-1}(x)\nabla f(x), t \in [0, 1] \right\} < \infty.$$

Let N be the least integer exceeding:

$$\frac{f(x_0) - \min f}{\Delta} \quad \text{where} \quad \Delta = \frac{\delta^2 \Lambda^3}{(1 + 2hQ^{1/2})K^2 Q^{1/2}}$$

Then an approximate root of $\nabla f(x)$ will be achieved in at most N steps.

(b) Let

$$\Delta_1 = \frac{\delta^2(1 - \delta)^2 \Lambda^3}{(1 + 2hQ^{1/2})K^2 Q^{1/2}}$$

. Let M be the least integer exceeding:

$$\frac{f(x_0) - \min f}{\min\{\Delta, \Delta_1\}}$$

. Then in at most M steps an approximate root will be achieved, and all subsequent steps will be Newton steps.

PROOF (a) Assume in what follows that $x \in S$. The change in f due to the step $\gamma s(x) = \gamma H^{-1}(x) \nabla f(x)$ may be estimated by Taylor's formula as:

$$(*) \quad \Delta(x) = f(x) - f(x - \gamma s(x)) = \gamma \left[1 - \frac{\gamma}{2} - \frac{\gamma}{2} \frac{[s(x), (H(\xi) - H(x))s(x)]}{[\nabla f(x), s(x)]} \right] [\nabla f(x), s(x)]$$

or

$$(**) \quad \Delta(x) = \gamma g(x, \gamma) [\nabla f(x), s(x)]$$

Here ξ lies between x and $x - \gamma s(x)$. By the above formula we see that the right hand limit of $g(x, \gamma)$ at $\gamma = 0$ is 1. Since S is bounded, for some $\hat{\gamma}$, $g(x, \hat{\gamma}) = 0$. This shows that

γ may be chosen as claimed. We see also that $\{f(x_k)\}$ is a decreasing sequence so that $x_k \in S$ for $k=1,2,3,\dots$

We aim now to find a lower bound for $\Delta(x)$. In the equation (*) above, the third term in the large brackets is

$$\geq -\frac{\gamma}{2} \frac{\|\nabla f(x)\| \|H^{-1}(x)(H(x) - H(\xi))\| \|s(x)\|}{[\nabla f(x), H^{-1}(x)\nabla f(x)]}.$$

Using lemma 2.0 we get the inequalities

$$1 - \frac{\gamma}{2} + \frac{\gamma}{2} \alpha h(x) Q^{\frac{1}{2}} \geq g(x, \gamma) \geq 1 - \frac{\gamma}{2} - \frac{\gamma}{2} \alpha h(x) Q^{\frac{1}{2}}.$$

Where $h(x) = \sup \{\|H^{-1}(x)(H(x) - H(\xi))\| : \xi = x + ts(x) \text{ with } t \in [0, 1]\}$. and:

$$\alpha = \frac{Q+1}{2Q}.$$

so that $.5 < \alpha \leq 1$. Since $g(x, \gamma) \leq 1 - \delta$ we thus obtain the lower bound

$$\gamma \geq \frac{2\delta}{1 + \alpha h(x) Q^{\frac{1}{2}}}$$

If $\{\|\nabla f(x_k)\|\}$ does not converge to 0, an infinite subsequence of it is bounded away from 0. Since $[\nabla f(x), s(x)] \geq \|\nabla f(x)\|^2 / Q\Lambda$, and $\gamma_k g(x_k, \gamma_k)$ is also bounded away from 0, $f(x_k)$ tends to $-\infty$. This contradiction establishes that the sequence $\{\nabla f(x_k)\}$ converges to 0. Moreover every cluster point of the set S' minimizes f , and $\{f(x_k)\}$ converges to $\min f$.

If x does not satisfy our condition for an approximate root then $\|H^{-1}(x)\| \|s(x)\| K \geq 1/2$. By the lemma $[\nabla f(x), s(x)] \geq (Q^{1/2} \alpha)^{-1} \|s(x)\| \|\nabla f(x)\|$. Consequently, $\|s(x)\| \geq 1/2K \|H^{-1}(x)\|$, and $\|\nabla f(x)\| \|s(x)\| \geq 1/4K^2 \Lambda^3$. Thus a lower estimate of (**) is

$$\Delta(x) \geq \frac{\delta^2}{(1 + 2h Q^{\frac{1}{2}} 4\Lambda^3 K^2 Q^{\frac{1}{2}})} = \Delta$$

We may similarly find a number of steps that will guarantee that an approximate root has been reached and that the above algorithm will always produce subsequent Newton steps. We do this by observing that $.5K \|s(x)\| \Lambda \alpha Q^{1/2} \leq .5 - \delta$ implies that $.5K h(x) \alpha Q^{1/2} \leq$

$.5 - \delta$. And this implies that $\delta \leq g(x, 1) \leq 1 - \delta$. Thus we shall count the steps for which $\|s(x_k)\|$ exceeds $(1 - 2\delta)/\alpha K \Lambda Q^{1/2}$. As above we find that

$$\Delta(x) \geq \frac{\delta^2(.5 - \delta)^2 \Lambda^3}{(1 + 2h Q^{\frac{1}{2}} K^2 Q^{\frac{3}{2}})} = \Delta_1$$

We now "count" the steps. Since $\Delta(x_k)$ is bounded away from zero, say by Δ , we have $f(x_0) - f(x_k) \geq k \left(\frac{1}{k} \sum_{i=0}^k \Delta(x_i) \right) \geq k \Delta$. Set

$$N = \frac{f(x_0) - \min f}{\Delta}$$

Set

$$M = \frac{f(x_0) - \min f}{\min\{\Delta, \Delta_1\}}.$$

If $k = N$, and x_k is not an approximate root, we have a contradiction.

REMARK 3.1 The results above are disappointing in that the cost of the algorithm is sensitive to the condition number Q , in spite of the fact that Newton directions are taken. (Recall that Smale's global Newton method was not sensitive to the condition number). Indeed the gradient method is less sensitive to Q than this method. We may classify this algorithm as "greedy" because it is trying to decrease f at each iteration. On the plus side, the algorithm is easily implemented.

The most robust result over the class of strongly convex functions is due to Nesterov. A simple, easily coded algorithm using combinations of gradient steps will drive $f(x_n)/f(x_0)$ to less than ϵ in less than

$$\frac{4\sqrt{Q}}{\ln 2} \ln \left(\frac{1}{\epsilon} \right) \text{ steps}$$

. By the theory of Nemerovsky and Yuden, for some positive number c no algorithm can do better than the following number of steps for every strongly convex function of condition number Q :

$$c \frac{\min(n, \sqrt{Q})}{\ln \min(n, \sqrt{Q})} \ln \frac{1}{\epsilon}.$$

For the case when $n > \sqrt{Q}$ we can assert that Nesterov's method is to within a slowly changing multiplicative factor essentially optimal over the class of strongly convex functions. This method and its generalizations are being studied by Osman Guler at U. of Chicago, School of Business.

Consider the Smale global Newton method applied to minimizing strongly convex functions of condition number Q . This method which is sensitive to a Lipschitz constant for the Hessian would not be a candidate for optimality on the set of strongly convex functions. The reason is that there exist strongly convex functions with condition number Q that have Hessians with arbitrarily large Lipschitz constants. However for the multitude of natural problems with bounded Lipschitz 2nd derivatives, or bounded values of the constant K , we should expect better results for large condition numbers from the Smale global Newton method, than with an algorithm of the Nesterov type.

We now digress to the problem of linear inequalities. Let A be an m by n matrix of rank n , and b an m by 1 matrix. Denote the i th row of A by A^i . Set

$$R(x) = Ax - b.$$

We seek a solution of the system of inequalities :

$$R_i(x) \leq 0 \quad i = 1, 2, \dots, m.$$

Or, if this system has no solution, to establish its inconsistency. Let $G(x) = \max\{R_i(x) : 1 \leq i \leq m\}$. We assume that G is bounded below, and consequently, that G has bounded level sets. To ensure that this is the case, a phony half-space may be added to our system of inequalities. Let $A_0 = -\sum_{i=1}^m A_i$. Let $R_0(x) = [A_0, x] - b$. Let $G^*(x) = \max\{R_i(x) : 0 \leq i \leq m\}$. Let $z(b) = \operatorname{argmin} G^*$. If the original system is consistent then for b sufficiently large $G(z(b)) \leq 0$.

We shall employ the penalty function:

$$\sum_{i=1}^m [\max(0, R_i(x))]^p \quad \text{with } p \geq 2$$

We note that for $p \geq 2$, F is convex. If $p=2$, F' is continuous, and F'' has jump discontinuities. If $p > 3$ then F''' is continuous. Our numerical experiments using F to solve inequalities gave most satisfactory results with $p=2$, even though more smoothness is obtained with larger values of p . Thus in what follows we shall take $p=2$. We assume that F has bounded level sets.

Let $I^+(x)$ denote the set of all indices from $\{1, 2, 3, \dots, m\}$ for which $R_i(x) > 0$ when $i \in I^+(x)$. If $\dim\{A_i : i \in I^+(x)\} < n$, then $F''(x)$ does not have an inverse. We seek the following construction.

CONSTRUCTION B 4.0

Assume that at x_k the dimension of the set of gradients of the active residuals is less than n . That is, $\dim\{A_i : 1 \leq i \leq q\} \leq n$. Take $h \neq 0$ such that $[A_i, h] = 0$, $1 \leq i \leq q$. Let $\epsilon_k = .5(F(x_{k-1}) - F(x_k))$. Find the smallest t such that $\sum\{R_i^2(x + th) : q+1 \leq i \leq m \text{ and } R_i > 0\} = \epsilon_k^2/2(m-q)$. Set $x_{k+1} = x_k + th$. Repeat this process if necessary. After at most $m-q$ steps we obtain x_{k+1} .

ALGORITHM 5.0 Take x_0 arbitrarily in E_n . If $\dim\{A_i : i \in I^+(x_0)\} < n$ use (4.0) to find x_1 with $\dim\{A_i \in I^+(x_1)\} = n$. At the k th iteration we are given x_k and a non-singular $H(x_k)$. Choose γ_k such that $.75 \geq g(x_k, \gamma_k) \geq .25$, taking $\gamma_k = 1$, if possible. Set $\bar{x}_{k+1} = x_k - \gamma_k H^{-1}(x_k) \nabla f(x_k)$. If H has an inverse at $\bar{x}_{k+1}$ then set $x_{k+1} = \bar{x}_{k+1}$. Otherwise update $\bar{x}_{k+1}$ with the construction (4.0) or (4.1) to obtain x_{k+1} .

CLAIM 5.1 Assume the hypotheses of (4.0). The algorithm is well defined and will terminate in a finite number of steps, with a positive value of F if the system is inconsistent and with $F = 0$ if the system is consistent. For the latter case the argument of F is a solution of the inequalities.

PROOF Let $X = \{x \in E_n : [A^i, x] \leq b_i, 1 \leq i \leq m\}$. Assume that X is not empty. Let w denote the number of vertices of the solution set X , and let $v_s, s=1,2,\dots,w$ denote the vertices. Each v_s lies in the intersection of at least n hyperplanes supporting X . Any point in X is on the right side ($R_i(x) \leq 0$) of these hyperplanes. Stated otherwise: $[A^i, v_s] = b_i, i \in I_s, \text{card } I_s \geq n$. Let $C(v_s) = \{x \in E_n : [A^i, x] > b_i, i \in I_s\} = \{x \in E_n : [A^i, x - v_s] > 0, i \in I_s\}$, and let $\bar{C}(v_s)$ denote its closure. Let B_s be a ball of radius r_s centered at v_s . If some hyperplane $[A^i, x] - b_i$ such that $i \in I \sim I_s$ meets $C(v_s)$ then there exists a minimal value of r_s such that $B_s \cap \bar{C}(v_s)$ meets the closest hyperplane $[A^i, x] - b_i$ such that $i \in I \sim I_s$. Points in $B_s \cap C(s) \sim \{v_s\}$ are on the wrong side of the same hyperplanes, n or more in number. In all the above sets s is understood to range over $1,2,\dots,w$.

It follows from the proof of (3.0) that the above algorithm generates a sequence $\{x_k\} \in S$ (the level set of F at x_0) such that $F(x_k)$ converges downward to $\min F$. We claim that the only cluster points of $\{x_k\}$ are the vertices $v_s, s = 1,2,\dots,w$. The sets $I^+(x_k)$ are collections of n or more out of m indices that are repeated infinitely often. Thus at least one of the collections must be frequently repeated. Let $\{x_{k_i}\}$ denote a subsequence converging to a limit v . Take a thinner subsequence if necessary such that only one index set is represented. Since $F(v) = 0$, it follows that $v \in X$. Since $\{x_{k_i}\}$ is on the wrong side of at least n hyperplanes at points arbitrarily close to X , the index set I must be one of the sets $I_s, s=1,2,\dots,w$. Let $\bar{k}$ denote the least value of k such that $x_{\bar{k}} \in B_s \cap C(s)$ for some $s=1,2,\dots,w$. At most one more step at $x_{\bar{k}}$ terminates the process, because a Newton step minimizes the restriction of F to $B_s \cap C(v_s)$ —a quadratic function,—in one step.

Assume now that X is empty. A necessary condition that a system of inequalities be inconsistent is that 0 belongs to the convex hull of the rows of the matrix A . Let x^* minimize F . We claim that x^* is unique, $\text{card } I^+(x^*) \geq n+1$, and 0 belongs to the convex hull of the rows of A . If $\text{card } I^+(x^*) \leq n$ choose h so that $[A^i, h] = -1, i \in I^+(x^*)$. Then for some h sufficiently small $F(x^*)$ can be decreased. Since x^* is a strict local minimum it is unique because of the convexity of F .

Let B be a ball centered at x^* with the property that $I^+(x) = I^+(x^*)$ for all $x \in B$. This existence of such a ball follows from the continuity of $R_i(x)$ for each $i, 1 \leq i \leq m$, and because $R_i(x^*) \neq 0$ for all $i \in I^+(x^*)$. Because x^* is unique the sequence $\{x_k\}$ converges to it. Thus for some least $k = \bar{k}$ $x_{\bar{k}} \in B$. Thus the solution occurs here or at the next step.

Notice that the parameter K of (3.0) is actually finite. Consider the denominator of K for $k=0,1,2,\dots,\bar{k}$. Since $\|s(x)\| \geq \|\nabla f(x)\|/Q\mathcal{L}$ the denominator is always positive. (If $\nabla f(x) = 0$ we have a solution) One more step at $x_{\bar{k}}$ terminates the process. The Newton step minimizes the quadratic function (restriction of F to $B_{\bar{k}} \cap C(v_{\bar{k}})$) in one step. Note that the numerator of K for this last step is 0 , while the denominator is positive.

NUMERICAL COMPUTATIONS 6.0

The Blair example may be found in Kortanek and Shi. The coefficients for the case $n=8$ are written out (Example 2a, p55) and a program is given to generate the coefficients for any n . For $n = 12$, the case for which we offer extensive calculations, the Blair problem is a linear programming problem (LP) of the form: minimize $L(x)$ subject to

$$R_i(x) \leq 0 \quad 1 \leq i \leq 24$$

of which 12 inequalities constrain x to lie in the first orthant. The coefficients of the system are integers between 1 and 305,175,780. Thus the system is very poorly conditioned. The problem has a unique solution $\hat{x}_i = 0, i=1,2,\dots,11, \hat{x}_{12} = 1$, with $L(\hat{x}) = 305175780$. The system we solve contains following inequalities :

$$r_i(x) \leq 0, \quad i = 1, 2, \dots, 25 \quad \text{where}$$

$$r_{14}(x) = L(x) - 305175780,$$

$$r_i(x) = R_i(x), \quad i = 1, 13 \quad \text{and}$$

$$r_{i+1}(x) = R_i(x) \quad i = 14, \dots, 24$$

Superlindo required 2438 iterations to solve the above LP problem with full accuracy. Because one cannot enter linear inequalities directly into Superlindo we entered the following problem: $\min L(x)$ subject to

$$r_i(x) \leq 0 \quad 1 \leq i \leq 25.$$

Thus when phase I was finished the problem would terminate. This took 2236 iterations. Let

$$r(x) = Ax - b.$$

The components of A and b are listed at the end of this section.

A simple scheme of row scaling was used to convert to a better scaled problem. Let bm be the maximal component of the right hand side. $bm=305175780$. The scaled system is:

$$(bm/|b_i|) r_i(x) \leq 0 \quad i = 1, 2, 3, \dots, 14$$

$$bm r_i(x) \leq 0 \quad i = 15, \dots, 25$$

Thus 4.0 should be used. Some computations were made before 4.0 was coded using an ad hoc procedure of an overrelaxed gradient step to move off a degenerate point x . The step-length chosen was $100/\|H(x)\|$. This has worked for this problem in at most 3 steps for every example (over 100) tried. However the penalty function usually increases during these perturbations. Subsequently, the method of 4.0 was coded. The results were similar to those given below, with the moves in the null spaces (which we think of as gear shifts) replacing the gradient steps.

There follows 15 successive runs with random initial vectors with components lying between -1000 and 1000. The 12 numbers after x_0 are the components of the starting vector.

The last column contains a list of integers in order 1,2,3 interspersed with the symbol *. Each integer numbers a normal step of the algorithm while * indicates a gradient step. In what follows disregard this last column. We explain first the lines containing tt, info,

rmax, number, number, number. Here tt means the number of active residuals; its value is the first number. Info is a test for degeneracy; its value is the second number. If info = 0 the point is nondegenerate, info = 1 , degenerate. Rmax is the value of the maximum residual. It is the 3rd number.

Now, the line that contains step-length, gamma, number, number. Here step-length = $\|H^{-1}(x)\nabla f(x)\|$. Its value is the first number. The meaning of gamma is that of algorithm 3.0, the fraction of a Newton step. The value of gamma is the second number. At the bottom of the page we have x,r number,number. This the row-wise print out of the solution vector followed by the residuals at solution.

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APPENDIX THE COMPONENTS OF THE BLAIR MATRICES A & B
FIFTEEN RUNS WITH RANDOM STARTING POINTS

a(1,1)=-5.
a(2,1)=-1.
a(3,1)=0.
a(4,1)=0.
a(5,1)=0.
a(6,1)=0.
a(7,1)=0.
a(8,1)=0.
a(9,1)=0.
a(10,1)=0.
a(11,1)=0.0
a(12,1)=0.
a(13,1)=0.0
a(14,1)=5.
a(15,1)=-1.
a(16,1)=0.
a(17,1)=0.
a(18,1)=0.
a(19,1)=0.
a(20,1)=0.
a(21,1)=0.
a(22,1)=0.
a(23,1)=0.
a(24,1)=0.
a(25,1)=0.
a(1,2)=-10.
a(2,2)=-5.
a(3,2)=-1.
a(4,2)=0.
a(5,2)=0.
a(6,2)=0.
a(7,2)=0.
a(8,2)=0.
a(9,2)=0.
a(10,2)=0.
a(11,2)=0.
a(12,2)=0.
a(13,2)=0.
a(14,2)=30.
a(15,2)=0.
a(16,2)=-1.
a(17,2)=0.
a(18,2)=0.
a(19,2)=0.
a(20,2)=0.
a(21,2)=0.
a(22,2)=0.
a(23,2)=0.
a(24,2)=0.
a(25,2)=0.
a(1,3)=-20.
a(2,3)=-10.
a(3,3)=-5.
a(4,3)=-1.
a(5,3)=0.
a(6,3)=0.

a(7,3)=0.
a(8,3)=0.
a(9,3)=0.
a(10,3)=0.
a(11,3)=0.
a(12,3)=0.
a(13,3)=0.
a(14,3)=155.
a(15,3)=0.
a(16,3)=0.
a(17,3)=-1.
a(18,3)=0.
a(19,3)=0.
a(20,3)=0.
a(21,3)=0.
a(22,3)=0.
a(23,3)=0.
a(24,3)=0.
a(25,3)=0.
a(1,4)=-40.
a(2,4)=-20.
a(3,4)=-10.
a(4,4)=-5.
a(5,4)=-1.
a(6,4)=0.
a(7,4)=0.
a(8,4)=0.
a(9,4)=0.
a(10,4)=0.
a(11,4)=0.
a(12,4)=0.
a(13,4)=0.
a(14,4)=780.
a(15,4)=0.
a(16,4)=0.
a(17,4)=0.
a(18,4)=-1.
a(19,4)=0.
a(20,4)=0.
a(21,4)=0.
a(22,4)=0.
a(23,4)=0.
a(24,4)=0.
a(25,4)=0.
a(1,5)=-80.
a(2,5)=-40.
a(3,5)=-20.
a(4,5)=-10.
a(5,5)=-5.
a(6,5)=-1.
a(7,5)=0.
a(8,5)=0.
a(9,5)=0.
a(10,5)=0.
a(11,5)=0.
a(12,5)=0.

a(13,5)=0.
a(14,5)=3905.
a(15,5)=0.
a(16,5)=0.
a(17,5)=0.
a(18,5)=0.
a(19,5)=-1.
a(20,5)=0.
a(21,5)=0.
a(22,5)=0.
a(23,5)=0.
a(24,5)=0.
a(25,5)=0.
a(1,6)=-160.
a(2,6)=-80.
a(3,6)=-40.
a(4,6)=-20.
a(5,6)=-10.
a(6,6)=-5.
a(7,6)=-1.
a(8,6)=0.
a(9,6)=0.
a(10,6)=0.
a(11,6)=0.
a(12,6)=0.
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a(14,6)=19530.
a(15,6)=0.
a(16,6)=0.
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a(23,6)=0.
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a(2,7)=-160.
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a(6,7)=-10.
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a(19,7)=0.
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 a(21,7)=-1.
 a(22,7)=0.
 a(23,7)=0.
 a(24,7)=0.
 a(25,7)=0.
 a(1,8)=-640.
 a(2,8)=-320.
 a(3,8)=-160.
 a(4,8)=-80.
 a(5,8)=-40.
 a(6,8)=-20.
 a(7,8)=-10.
 a(8,8)=-5.
 a(9,8)=-1.
 a(10,8)=0.
 a(11,8)=0.
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 a(14,8)=488280.
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 a(20,8)=0.
 a(21,8)=0.
 a(22,8)=-1.
 a(23,8)=0.
 a(24,8)=0.
 a(25,8)=0.
 a(1,9)=-1280.
 a(2,9)=-640.
 a(3,9)=-320.
 a(4,9)=-160.
 a(5,9)=-80.
 a(6,9)=-40.
 a(7,9)=-20.
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 a(19,9)=0.
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 a(23,9)=-1.0
 a(24,9)=0.

a(25,9)=0.
 a(1,10)=-2560.
 a(2,10)=-1280.
 a(3,10)=-640.
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 a(5,10)=-160.
 a(6,10)=-80.
 a(7,10)=-40.
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 a(19,11)=0.
 a(20,11)=0.
 a(21,11)=0.
 a(22,11)=0.
 a(23,11)=0.
 a(24,11)=0.
 a(25,11)=-1.
 a(1,12)=-10240.
 a(2,12)=-5120.
 a(3,12)=-2560.
 a(4,12)=-1280.
 a(5,12)=-640.

a(6,12)=-320.
 a(7,12)=-160.
 a(8,12)=-80.
 a(9,12)=-40.
 a(10,12)=-20.
 a(11,12)=-10.
 a(12,12)=-5.
 a(13,12)=-1.
 a(14,12)=305175780.
 a(15,12)=0.
 a(16,12)=0.
 a(17,12)=0.
 a(18,12)=0.
 a(19,12)=0.
 a(20,12)=0.
 a(21,12)=0.
 a(22,12)=0.
 a(23,12)=0.
 a(24,12)=0.
 a(25,12)=0.
 b(1)=-4096.
 b(2)=-2048.
 b(3)=-1024.
 b(4)=-512.
 b(5)=-256.
 b(6)=-128.
 b(7)=-64.
 b(8)=-32.
 b(9)=-16.
 b(10)=-8.
 b(11)=-4.
 b(12)=-2.
 b(13)=-1.
 b(14)=305175780.
 b(15)=0.
 b(16)=0.
 b(17)=0.
 b(18)=0.
 b(19)=0.
 b(20)=0.
 b(21)=0.
 b(22)=0.
 b(23)=0.
 b(24)=0.
 b(25)=0.

RUN #1 11 Newton steps and 4 gradient steps.

x0	-157.239626709990	-743.322103338538	-971.373266891785
	507.770637071353	581.201493434374	-593.593433850401
	-877.746807236177	-361.133546071757	425.520516159178
	894.635221028124	563.412949712823	268.597848237653
tt.info.rmax,	7.00000000000000	1	236624527029.611 *
tt.info.rmax,	14.00000000000000	1	9653257644971.96 *
steplength.gamma	47765.5239230591	1.00000000000000	1
tt.info.rmax,	14.00000000000000	0	2357700197.06739
steplength.gamma	19.2575641130305	1.00000000000000	2
tt.info.rmax,	12.00000000000000	0	2428295204.71995
steplength.gamma	5454.13223736253	0.305175781250000D-00	3
tt.info.rmax,	14.00000000000000	0	2428221099.03133
steplength.gamma	17.3705077615239	1.00000000000000	4
tt.info.rmax,	12.00000000000000	0	1185139268.73383
steplength.gamma	5472.94705380250	0.762939453125000D-00	5
tt.info.rmax,	13.00000000000000	0	1185130226.88878
tt.info.rmax,	1.00000000000000	1	620675091.14548 *
tt.info.rmax,	19.00000000000000	1	163062290989.794 *
steplength.gamma	198.732501363480	1.00000000000000	6
tt.info.rmax,	13.00000000000000	0	153611593.072991
steplength.gamma	196.752514409908	0.152587890625000D-00	7
tt.info.rmax,	14.00000000000000	0	153610594.293101
steplength.gamma	66.3907250047904	0.107421875000000D-001	8
tt.info.rmax,	13.00000000000000	0	152948384.537403
steplength.gamma	83.3248672145210	0.131835937500000D-001	9
tt.info.rmax,	16.00000000000000	0	152548504.421150
steplength.gamma	0.584973381718277	1.00000000000000	10
tt.info.rmax,	12.00000000000000	0	133329412.581193
steplength.gamma	0.504481166507457	1.00000000000000	11
tt.info.rmax,	12.00000000000000	0	0.250339508056641D-005
steplength.gamma	0.936594908882387D-014	1.00000000000000	
tt.info.rmax,	12.00000000000000	0	0.00000000000000
x . r	0.165629975217302D-031	0.608593862426366D-031	
	0.175644810928116D-030	0.412919380076623D-030	0.850490663441403D-030
	0.173795918181504D-029	0.345126646034193D-029	0.690253292068385D-029
	0.136078506150625D-028	0.268212707775144D-028	0.544314024602498D-028
	1.00000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.00000000000000	0.00000000000000
	-0.505462562158010D-023	-0.185728104234804D-022	-0.536025414753610D-022
	-0.126012992240322D-021	-0.259549148196485D-021	-0.530383041966730D-021
	-0.105324292021762D-020	-0.210648584043524D-020	-0.415278637114376D-020
	-0.818520212283408D-020	-0.166111454845750D-019	

RUN #2

7 Newton steps and 3 gradient steps

x0	-930.292890085242	-863.844227490599	-322.317913607396	
	244.628299272343	719.519989562319	543.254998043481	
	-508.677056980336	-162.246891100147	618.731966442102	
	416.323740083475	-708.495647012945	253.088220718418	
tt.info.rmax.	8.00000000000000	1	283902854639.046	*
tt.info.rmax.	1.00000000000000	1	1371116280676.87	*
tt.info.rmax.	16.00000000000000	1	359774939814244.	*
steplength.gamma	435728.172539526	1.00000000000000		1
tt.info.rmax.	12.00000000000000	0	161170658.157925	
steplength.gamma	10514.7126566587	0.274658203125000D-003		2
tt.info.rmax.	14.00000000000000	0	161126391.314553	
steplength.gamma	60.4510944982947	0.312500000000000D-001		3
tt.info.rmax.	14.00000000000000	0	156091441.193965	
steplength.gamma	52.5504577006773	0.125000000000000		4
tt.info.rmax.	15.00000000000000	0	148358483.404517	
steplength.gamma	126.089758342685	0.644531250000000D-001		5
tt.info.rmax.	14.00000000000000	0	147307230.642514	
steplength.gamma	0.649075995843773	1.00000000000000		6
tt.info.rmax.	12.00000000000000	0	129555772.393379	
steplength.gamma	0.490203091522860	1.00000000000000		7
tt.info.rmax.	12.00000000000000	0	0.244379043579102D-005	
steplength.gamma	0.913113886853910D-014	1.00000000000000		
tt.info.rmax.	12.00000000000000	0	0.000000000000000	
x . r	0.331259950434605D-031	0.100148357108136D-030		
	0.200296714216273D-030	0.409837892165604D-030	0.819675784331207D-030	
	0.167632942359465D-029	0.337731075047746D-029	0.680392530753123D-029	
	0.134106353887572D-028	0.264268403249039D-028	0.528536806498078D-028	
	1.000000000000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	0.000000000000000	0.000000000000000	
	-0.101092512431602D-022	-0.305628525956006D-022	-0.611257051912012D-022	
	-0.125072596775842D-021	-0.250145193551685D-021	-0.511575132677130D-021	
	-0.103067342307010D-020	-0.207639318557188D-020	-0.409260106141704D-020	
	-0.306483150338064D-020	-0.161290620067613D-019		

RUN #3

7 Newton steps and 1 gradient step

x0	384.636186040327	521.532991478957	-700.058083640171
	890.556864074000	22.8991136970037	-639.569422439330
	823.899904502448	823.568592993691	996.340118297574
	-94.6125941635709	-466.466448786907	-450.878854236014
tt.info.rmax.	17.00000000000000	0	529467259301.064 *
steplength.gamma	2212.27366081213	1.00000000000000	1
tt.info.rmax.	14.00000000000000	0	194786857.815711
steplength.gamma	11.5597435264857	0.937500000000000D-001	2
tt.info.rmax.	15.00000000000000	0	179212381.151034
steplength.gamma	17.4100728479677	0.312500000000000D-001	3
tt.info.rmax.	15.00000000000000	0	173957718.602083
steplength.gamma	13.1910259009825	0.109375000000000	4
tt.info.rmax.	16.00000000000000	0	156186399.016667
steplength.gamma	4.83788792413538	0.375000000000000	5
tt.info.rmax.	17.00000000000000	0	126699425.005225
steplength.gamma	0.654804890885739	1.00000000000000	6
tt.info.rmax.	12.00000000000000	0	134111089.546827
steplength.gamma	0.507437943699474	1.00000000000000	7
tt.info.rmax.	12.00000000000000	0	0.250339508056641D-005
steplength.gamma	0.936796982418018D-014	1.00000000000000	
tt.info.rmax.	12.00000000000000	0	0.000000000000000
x , r	0.130963236218332D-031	0.485334345985583D-031	
	0.140207699951391D-030	0.372860037233369D-030	0.850490663441403D-030
	0.173795918181504D-029	0.345126646034193D-029	0.690253292068385D-029
	0.138050658413677D-028	0.268212707775144D-028	0.544314024602493D-028
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.399668072404008D-023	-0.148112285655603D-022	-0.427879936338408D-022
	-0.113787851202082D-021	-0.259549148196485D-021	-0.530383041966730D-021
	-0.105324292021762D-020	-0.210648584043524D-020	-0.421297168037048D-020
	-0.818520212283408D-020	-0.166111454845750D-019	

RUN #4

7 Newton steps

x0	-732.342233747725	741.538604529571	244.383834286300
	341.271154817054	324.279857033038	112.113379925855
	262.873709250308	604.307441254551	-321.373886549000
	-966.821403447874	-693.073136470057	-623.141231756370
tt.info.rmax,	17.00000000000000	0	1096227398623.85 1
steplength, gamma	2002.80968276938	1.00000000000000	2
tt.info.rmax,	13.00000000000000	0	153540383.712848
steplength, gamma	130.062162441505	0.634765625000000D-002	3
tt.info.rmax,	14.00000000000000	0	153117350.272783
steplength, gamma	29.8103139169651	0.253906250000000D-001	4
tt.info.rmax,	15.00000000000000	0	151769052.907618
steplength, gamma	7.61286793105905	0.937500000000000D-001	5
tt.info.rmax,	16.00000000000000	0	149354307.884918
steplength, gamma	0.480797575295140	1.00000000000000	6
tt.info.rmax,	12.00000000000000	0	133329412.581193
steplength, gamma	0.504481166507457	1.00000000000000	7
tt.info.rmax,	12.00000000000000	0	0.244379043579102D-005
steplength, gamma	0.918104743739594D-014	1.00000000000000	
tt.info.rmax,	1.00000000000000	1	0.596046447753906D-007
x . r	0.770371977754894D-033	0.433334237487128D-032	
	0.208000433993821D-031	0.962964972193618D-031	0.440652771275800D-030
	0.202145606962884D-029	0.917050802319426D-029	0.410207670714926D-028
	0.183015730011275D-027	0.795171792462780D-027	0.338263556158770D-026
	1.000000000000002	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.596046447753906D-007	-0.596046447753906D-007
	-0.235093666120005D-024	-0.132243112192503D-023	-0.634766938524012D-023
	-0.293873582650006D-022	-0.134476551420643D-021	-0.616899424698892D-021
	-0.273861690229253D-020	-0.125135444231580D-019	-0.558519674263973D-019
	-0.242667168818140D-018	-0.103229843243272D-017	

RUN #5

7 Newton steps

x0	-114.147717658216	558.470022662264	-490.039243519173
	624.461662419657	-654.960097720481	-156.588812028271
	98.7877842091593	675.860880011940	297.942678453505
	-427.420645454674	-607.602961981098	-603.144542822476
tt.info.rmax,	19.0000000000000	0	762408473253.473
steplength, gamma	1696.80530559346	1.00000000000000	1
tt.info.rmax,	13.0000000000000	0	194664482.932851
steplength, gamma	78.7277480487342	0.122070312500000D-003	2
tt.info.rmax,	15.0000000000000	0	194642039.668371
steplength, gamma	10.5301245257422	0.125000000000000	3
tt.info.rmax,	16.0000000000000	0	171746426.699385
steplength, gamma	8.22887713006761	0.156250000000000D-001	4
tt.info.rmax,	16.0000000000000	0	169258335.804595
steplength, gamma	4.40534461134211	0.312500000000000	5
tt.info.rmax,	18.0000000000000	0	125867764.378646
steplength, gamma	0.637037280856529	1.00000000000000	6
tt.info.rmax,	12.0000000000000	0	134638584.993735
steplength, gamma	0.509430809427019	1.00000000000000	7
tt.info.rmax,	12.0000000000000	0	0.250339508056641D-005
steplength, gamma	0.936745800551893D-014	1.00000000000000	
tt.info.rmax,	12.0000000000000	0	0.000000000000000
x . r	0.108815041857879D-031	0.396741568543771D-031	
	0.117866912596499D-030	0.311230279012977D-030	0.764209001932655D-030
	0.173795918181504D-029	0.350057026691824D-029	0.700114053383648D-029
	0.138050658413677D-028	0.268212707775144D-028	0.544314024602498D-028
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.332077148394506D-023	-0.121075916051802D-022	-0.359701265163607D-022
	-0.949799419124819D-022	-0.233218075191045D-021	-0.530383041966730D-021
	-0.106828924764930D-020	-0.213657849529860D-020	-0.421297166067048D-020
	-0.818520212283408D-020	-0.166111454845750D-019	

RUN #6

7 Newton steps and 2 gradient steps

x0	472.361079681048	-831.003678005437	-740.935969423671
	899.660706261435	814.735410238548	-652.488692051425
	64.0326036041202	-59.8063268680018	72.9986403563025
	955.477280014122	640.867128180124	712.585229961250
tt.info.rmax,	5.00000000000000	1	268083811943.665 *
tt.info.rmax,	16.00000000000000	1	23715049130988.6 *
steplength.gamma	50637.9972940535	1.00000000000000	1
tt.info.rmax,	12.00000000000000	0	161169141.153651
steplength.gamma	10514.7126636119	0.274658203125000D-003	2
tt.info.rmax,	14.00000000000000	0	161124874.726943
steplength.gamma	60.4510874755542	0.312500000000000D-001	3
tt.info.rmax,	14.00000000000000	0	156089971.999712
steplength.gamma	52.5504645937000	0.125000000000000	4
tt.info.rmax,	15.00000000000000	0	148358479.968610
steplength.gamma	126.089752134237	0.644531250000000D-001	5
tt.info.rmax,	14.00000000000000	0	147307227.428061
steplength.gamma	0.649075033909147	1.00000000000000	6
tt.info.rmax,	12.00000000000000	0	129555772.393379
steplength.gamma	0.490203091522860	1.00000000000000	7
tt.info.rmax,	12.00000000000000	0	0.244379043579102D-005
steplength.gamma	0.911924546507179D-014	1.00000000000000	
tt.info.rmax,	12.00000000000000	0	0.000000000000000
x , r	0.331259950434605D-031	0.986076131526265D-031	
	0.200296714216273D-030	0.406756404254584D-030	0.819675784331207D-030
	0.167632942359465D-029	0.332800694390114D-029	0.670531769437860D-029
	0.134106353887572D-028	0.264268403249039D-028	0.520648197445868D-028
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.101092512431602D-022	-0.300926548633606D-022	-0.611257051912012D-022
	-0.124132201311362D-021	-0.250145193551685D-021	-0.511575132677130D-021
	-0.101562710163842D-020	-0.204630053070852D-020	-0.409260106141704D-020
	-0.806463150338064D-020	-0.158889217678544D-019	

RUN #7

8 Newton steps

x0	-980.614385562064	754.663860379469	-849.966118423708
	872.092335185815	903.372305432807	903.822716644623
	-787.423383854608	-889.882406466205	-656.189370753424
	-801.412660064975	-257.581608353113	-733.888854017488
tt, info, rmax,	20.00000000000000	0	930810709180.125
steplength, gamma	2781.62252337554	1.000000000000000	1
tt, info, rmax,	12.00000000000000	0	153611829.496706
steplength, gamma	3648.13836926195	0.391006469726563D-004	2
tt, info, rmax,	15.00000000000000	0	153605823.174790
steplength, gamma	64.0782140937533	0.830078125000000D-002	3
tt, info, rmax,	16.00000000000000	0	153096180.228943
steplength, gamma	27.6115425905722	0.195312500000000D-001	4
tt, info, rmax,	15.00000000000000	0	152568067.708317
steplength, gamma	73.2258353230729	0.305175781250000D-004	5
tt, info, rmax,	15.00000000000000	0	152567395.728954
steplength, gamma	0.340599172069112	1.000000000000000	6
tt, info, rmax,	12.00000000000000	0	132047268.790020
steplength, gamma	0.499630146021219	1.000000000000000	7
tt, info, rmax,	12.00000000000000	0	0.250339508056641D-005
steplength, gamma	0.934835255493608D-014	1.000000000000000	8
tt, info, rmax,	12.00000000000000	0	0.000000000000000
x , r	0.215704153771370D-031	0.801186856865090D-031	
	0.206459690038312D-030	0.412919380076623D-030	0.844327687619364D-030
	0.168865537523873D-029	0.342661455705377D-029	0.680392530753123D-029
	0.136078506150625D-028	0.268212707775144D-028	0.536425415550288D-028
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.658276825136013D-023	-0.244502820764805D-022	-0.630064961201612D-022
	-0.126012992240322D-021	-0.257668357267525D-021	-0.515336714535050D-021
	-0.104571975650178D-020	-0.207639318557188D-020	-0.415278637114376D-020
	-0.818520212283408D-020	-0.163704042456682D-019	

RUN #8

9 Newton steps

x0	489.859138456204	20.9052331088909	-633.600054840564
	174.179628824158	-653.211700218227	675.884989041090
	425.973625697280	-171.141846780673	-754.508656207635
	955.040074874622	-896.536088135900	206.621797933154
tt, info, rmax,	18.0000000000000	0	273601096408.878
steplength, gamma	2023.42010222668	1.00000000000000	1
tt, info, rmax,	12.0000000000000	0	159079979.545477
steplength, gamma	15612.1105994219	0.190734863281250D-005	2
tt, info, rmax,	14.0000000000000	0	159079676.124495
steplength, gamma	23.5758796432499	0.781250000000000D-002	3
tt, info, rmax,	16.0000000000000	0	158553666.463820
steplength, gamma	18.9507539629753	0.125000000000000	4
tt, info, rmax,	15.0000000000000	0	154534990.248806
steplength, gamma	34.1524046774300	0.244140625000000D-003	5
tt, info, rmax,	16.0000000000000	0	154529135.198641
steplength, gamma	92.0011728271753	0.280761718750000D-001	6
tt, info, rmax,	14.0000000000000	0	153934008.178371
steplength, gamma	0.213780894549538	1.00000000000000	7
tt, info, rmax,	12.0000000000000	0	129555772.393379
steplength, gamma	0.490203091522860	1.00000000000000	8
tt, info, rmax,	12.0000000000000	0	0.238418579101563D-005
steplength, gamma	0.895106789666395D-014	1.00000000000000	9
tt, info, rmax,	1.00000000000000	1	0.596046447753906D-007
x , r	0.776390508831104D-033	0.433334237487128D-032	
	0.206074504049434D-031	0.962964972193618D-031	0.446815747097839D-030
	0.204610797291700D-029	0.917050802319426D-029	0.410207670714926D-028
	0.183015730011275D-027	0.795171792462780D-027	0.343312265952184D-026
	1.00000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.596046447753906D-007	-0.596046447753906D-007
	-0.236935576011567D-024	-0.132243112192503D-023	-0.628889466871012D-023
	-0.293873582650006D-022	-0.136357342349603D-021	-0.624422588414732D-021
	-0.279861690229253D-020	-0.125185444231580D-019	-0.558519674263973D-019
	-0.242667168818140D-018	-0.104770587172276D-017	

RUN #9 10 Newton steps and 2 gradient step

x0	-834.972506153810	-983.769175140755	447.075901089682
	-875.078848947569	924.303418741492	38.9442774086471
	-795.176371801571	-0.500703910808475	487.150834676203
	-346.782467450234	322.616508754233	721.511346799489
tt.info.rmax,	7.00000000000000	1	300222521428.460 *
tt.info.rmax,	15.00000000000000	1	17359103350870.1 *
steplength.gamma	54027.8046190082	1.00000000000000	1
tt.info.rmax,	12.00000000000000	0	1786211071.36421
steplength.gamma	15776.5628664350	0.316619873046875D-003	2
tt.info.rmax,	14.00000000000000	0	1785645521.44156
steplength.gamma	27.7719244099574	0.25000000000000	3
tt.info.rmax,	14.00000000000000	0	1339238345.83148
steplength.gamma	8.49827918067834	1.00000000000000	4
tt.info.rmax,	12.00000000000000	0	590430919.804070
steplength.gamma	2736.52420367105	0.762939453125000D-005	5
tt.info.rmax,	13.00000000000000	0	590426415.173640
steplength.gamma	47.6924304507833	0.781250000000000D-002	6
tt.info.rmax,	14.00000000000000	0	585835987.155582
steplength.gamma	8.13685607249363	0.625000000000000D-001	7
tt.info.rmax,	15.00000000000000	0	549466999.076904
steplength.gamma	42.5352574148447	0.328125000000000	8
tt.info.rmax,	16.00000000000000	0	370527291.858912
steplength.gamma	1.55797420666023	1.00000000000000	9
tt.info.rmax,	12.00000000000000	0	133329412.581193
steplength.gamma	0.504481166507457	1.00000000000000	10
tt.info.rmax,	12.00000000000000	0	0.250339508056641D-005
steplength.gamma	0.936377296560471D-014	1.00000000000000	
tt.info.rmax,	12.00000000000000	0	0.00000000000000
x , r	0.165629975217302D-031	0.597038282760043D-031	
	0.175644810928116D-030	0.422163843809682D-030	0.844327687619364D-030
	0.173795918181504D-029	0.342661455705377D-029	0.690253292068385D-029
	0.138050658413677D-028	0.268212707775144D-028	0.544314024602498D-028
	1.00000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.00000000000000	0.00000000000000
	-0.505462562158010D-023	-0.182201621243004D-022	-0.536025414753610D-022
	-0.128834178633763D-021	-0.257668357267525D-021	-0.530383041966730D-021
	-0.104571975650178D-020	-0.210648584043524D-020	-0.421297168087048D-020
	-0.818520212283408D-020	-0.166111454845750D-019	

RUN #10

7 Newton steps

x0	168.562088607904	879.936512051979	282.756764008947
	-215.631795284871	386.541205751119	-272.694289835621
	-312.386754660958	-847.804830474609	-918.030969523965
	-783.756813578536	-462.177283109723	318.376253985250
tt,info,rmax,	18.00000000000000	0	280160813516.508
steplength,gamma	1935.29079830120	1.000000000000000	1
tt,info,rmax,	12.00000000000000	0	851810879.278383
steplength,gamma	7019.68685306682	0.122070312500000D-003	2
tt,info,rmax,	14.00000000000000	0	851706898.458158
steplength,gamma	78.7484599706272	0.781250000000000D-001	3
tt,info,rmax,	13.00000000000000	0	785166074.844281
steplength,gamma	335.697096172363	0.209960937500000D-001	4
tt,info,rmax,	14.00000000000000	0	768681659.969552
steplength,gamma	3.57321680138793	1.000000000000000	5
tt,info,rmax,	12.00000000000000	0	129555772.393379
steplength,gamma	0.490203091522860	1.000000000000000	6
tt,info,rmax,	12.00000000000000	0	0.244379043579102D-005
steplength,gamma	0.913177239355039D-014	1.000000000000000	7
tt,info,rmax,	12.00000000000000	0	0.000000000000000
x , r	0.325482160601443D-031	0.986076131526265D-031	
	0.200296714216273D-030	0.409837892165604D-030	0.819675784331207D-030
	0.167632942359465D-029	0.332800694390114D-029	0.680392530753123D-029
	0.134106353887572D-028	0.264268403249039D-028	0.528536806498078D-028
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.993292709357019D-023	-0.300926548633606D-022	-0.611257051912012D-022
	-0.125072596775842D-021	-0.250145193551685D-021	-0.511575132677130D-021
	-0.101562710163842D-020	-0.207639318557188D-020	-0.409260106141704D-020
	-0.806483150338064D-020	-0.161296630067613D-019	

RUN #11 10 Newton steps and 2 gradient step

x0	-507.462338260756	-745.331526294334	822.626989288406	
	460.053875018011	-77.6430245183148	-406.981993993563	
	-554.345688116258	-506.427349605299	53.2199324551891	
	-905.916211360387	-924.250809940748	901.741966752070	
tt,info,rmax,	9.00000000000000		1	282058958142.296 *
tt,info,rmax,	14.00000000000000		1	9678403149694.32 *
steplength,gamma	45364.5042318625	1.00000000000000		1
tt,info,rmax,	12.00000000000000	0	429107069.641468	
steplength,gamma	15773.5499860498	0.748634338378906D-004		2
tt,info,rmax,	14.00000000000000	0	429074945.212750	
steplength,gamma	29.2671518584511	0.625000000000000D-001		3
tt,info,rmax,	14.00000000000000	0	402258848.273416	
steplength,gamma	10.3875006835552	0.375000000000000		4
tt,info,rmax,	18.00000000000000	0	295236784.148774	
steplength,gamma	2.41364909828324	1.00000000000000		5
tt,info,rmax,	13.00000000000000	0	287724616.031283	
steplength,gamma	106.889816604460	0.244140625000000D-003		6
tt,info,rmax,	16.00000000000000	0	287655027.217895	
steplength,gamma	9.48881403921367	0.156250000000000D-001		7
tt,info,rmax,	18.00000000000000	0	283221899.663543	
steplength,gamma	11.5704315204349	0.156250000000000D-001		8
tt,info,rmax,	19.00000000000000	0	278861295.027645	
steplength,gamma	0.911987664675868	1.00000000000000		9
tt,info,rmax,	12.00000000000000	0	135022355.068402	
steplength,gamma	0.510872132671297	1.00000000000000		10
tt,info,rmax,	12.00000000000000	0	0.250339508056641D-005	
steplength,gamma	0.935628649287364D-014	1.00000000000000		
tt,info,rmax,	12.00000000000000	0	0.00000000000000	
x , r	0.938890847888777D-032	0.337037740267766D-031		
	0.100148357108136D-030	0.263467216392174D-030	0.659438412958189D-030	
	0.156539585879795D-029	0.350057026691824D-029	0.700114053383648D-029	
	0.138050658413677D-028	0.268212707775144D-028	0.536425415550288D-028	
	1.00000000000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	-457763664.000000	-457763664.000000	
	-457763664.000000	0.00000000000000	0.00000000000000	
	-0.286526743083756D-023	-0.102855753927502D-022	-0.305628525956006D-022	
	-0.804038122130416D-022	-0.201244629398724D-021	-0.477720895955849D-021	
	-0.106828924764930D-020	-0.213657849529860D-020	-0.421297168087048D-020	
	-0.818520212283408D-020	-0.163704042456682D-019		

RUN #12

10 Newton steps

x0	-415.465089220958	93.3967863568357	-655.692275827660
	-777.129532256195	503.520149216688	-632.224189726941
	-976.113784378833	686.576335880251	263.850657800655
	-78.1654829784306	169.694047289757	-953.766508610371
tt, info, rmax,	19.00000000000000	0	669201674021.751
steplength, gamma	2075.58548391905	1.00000000000000	1
tt, info, rmax,	12.00000000000000	0	495718564.127140
steplength, gamma	4.19354626531279	0.50000000000000	2
tt, info, rmax,	13.00000000000000	0	356977880.225168
steplength, gamma	581.479889666029	0.610351562500000D-004	3
tt, info, rmax,	14.00000000000000	0	356957986.239844
steplength, gamma	17.0801205817469	0.781250000000000D-001	4
tt, info, rmax,	17.00000000000000	0	331543988.701578
steplength, gamma	6.52560684331984	0.50000000000000	5
tt, info, rmax,	15.00000000000000	0	181610652.924412
steplength, gamma	2.39912596957116	0.12500000000000	6
tt, info, rmax,	15.00000000000000	0	162997667.387076
steplength, gamma	18.3964281018146	0.21875000000000	7
tt, info, rmax,	19.00000000000000	0	134566511.091070
steplength, gamma	0.662869275097632	1.00000000000000	8
tt, info, rmax,	12.00000000000000	0	135022355.068401
steplength, gamma	0.510872132671297	1.00000000000000	9
tt, info, rmax,	12.00000000000000	0	0.250339508056641D-005
steplength, gamma	0.938228545189278D-014	1.00000000000000	10
tt, info, rmax,	12.00000000000000	0	0.00000000000000
x, r	0.938890847888777D-032	0.342815530100928D-031	
	0.100918729085891D-030	0.268089448258703D-030	0.659438412958189D-030
	0.156539585879795D-029	0.350057026691824D-029	0.700114053383648D-029
	0.138050658413677D-028	0.272157012301249D-028	0.544314024602498D-028
	1.00000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.00000000000000	0.00000000000000
	-0.286526743083756D-023	-0.104618995423402D-022	-0.307979514617206D-022
	-0.818144054097616D-022	-0.201244629398724D-021	-0.477720895955849D-021
	-0.106828924764930D-020	-0.213657849529860D-020	-0.421297168087048D-020
	-0.830557274228752D-020	-0.166111454845750D-019	

PAUSE?

RUN #13

6 Newton steps 2 gradient steps

x0	536.477440189505	961.442495391034	449.092889867492
	-565.222855711304	180.182538508556	768.913749734951
	-80.3906713861346	-628.929442909511	273.249274237714
	157.383786486445	42.1332745428034	528.871667806834
tt,info,rmax,	4.000000000000000	1	191934030789.156 *
tt,info,rmax,	15.000000000000000	1	16845691099344.1 *
steplength,gamma	37117.6105912337	1.000000000000000	1
tt,info,rmax,	14.000000000000000	0	939465467.781931
steplength,gamma	11.6912012042131	0.500000000000000	2
tt,info,rmax,	14.000000000000000	0	532153589.895607
steplength,gamma	21.0876406660046	0.500000000000000	3
tt,info,rmax,	15.000000000000000	0	283305905.447369
steplength,gamma	75.6961499181736	0.226562500000000	4
tt,info,rmax,	16.000000000000000	0	219586876.653153
steplength,gamma	1.44623443931224	1.000000000000000	5
tt,info,rmax,	12.000000000000000	0	133329412.531193
steplength,gamma	0.504481166507457	1.000000000000000	6
tt,info,rmax,	12.000000000000000	0	0.250339508056641D-005
steplength,gamma	0.934252011646448D-014	1.000000000000000	
tt,info,rmax,	12.000000000000000	0	0.000000000000000
x . r	0.161778115328528D-031	0.597038282760043D-031	
	0.174104066972606D-030	0.422163843809682D-030	0.844327687619364D-030
	0.173795918181504D-029	0.342661455705377D-029	0.690253292068385D-029
	0.136078506150625D-028	0.268212707775144D-028	0.528536806498073D-028
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.493707618852010D-023	-0.182201621243004D-022	-0.531323437431210D-022
	-0.128834178633763D-021	-0.257668357267525D-021	-0.530383041966730D-021
	-0.104571975650178D-020	-0.210648584043524D-020	-0.415278637114376D-020
	-0.818520212283408D-020	-0.161296630067613D-019	

RUN #14 6 Newton steps 3 gradient steps

x0	-313.188277293696	-74.7127697490351	363.859180360679
	225.308342001151	-449.303067641651	-722.721760127721
	749.187281293977	334.306731674827	566.136306586745
	21.6374771325915	834.136420883962	-471.498311508622
tt.info.rmax,	8.00000000000000	1	220557173979.063 *
tt.info.rmax,	1.00000000000000	1	2819394490440.15 *
tt.info.rmax,	17.00000000000000	1	743317395805483. *
steplength.gamma	895425.145414435	1.00000000000000	1
tt.info.rmax,	13.00000000000000	0	153587264.635352
steplength.gamma	98.7784435622585	0.732421875000000D-002	2
tt.info.rmax,	15.00000000000000	0	153108097.868560
steplength.gamma	13.9259737920949	0.117187500000000D-001	3
tt.info.rmax,	14.00000000000000	0	152485904.282359
steplength.gamma	16.3951704458082	0.625000000000000D-001	4
tt.info.rmax,	15.00000000000000	0	150818862.186507
steplength.gamma	0.717112686602673	1.00000000000000	5
tt.info.rmax,	12.00000000000000	0	132047268.790020
steplength.gamma	0.499630146021219	1.00000000000000	6
tt.info.rmax,	12.00000000000000	0	0.244379043579102D-005
steplength.gamma	0.915426311736677D-014	1.00000000000000	
tt.info.rmax,	1.00000000000000	1	0.596046447753906D-007
x , r	0.770371977754894D-033	0.430926825056644D-032	
	0.208000433993821D-031	0.962964972193618D-031	0.440652771275800D-030
	0.202145606962884D-029	0.917050802319426D-029	0.410207670714926D-028
	0.183015730011275D-027	0.795171792462780D-027	0.338263556158770D-026
	1.000000000000002	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.596046447753906D-007	-0.596046447753906D-007
	-0.235098866120005D-024	-0.131508428235878D-023	-0.634766938524012D-023
	-0.293873582650006D-022	-0.134476551420643D-021	-0.616899424698892D-021
	-0.279861690229253D-020	-0.125185444231580D-019	-0.558519674263973D-019
	-0.242667168818140D-018	-0.103229843243272D-017	

RUN #15

8 Newton steps

x0	296.981125772494	805.619158814028	51.6047080018680
	-40.5170014433711	-670.077290135360	-975.512313377759
	78.1184786292034	-737.687608386972	-6.90543145820155
	95.5765248137808	-613.386207147848	-628.146325780695
tt,info,rmax,	19.00000000000000	0	744985996579.413
steplength,gamma	1863.45202256309	1.00000000000000	1
tt,info,rmax,	14.00000000000000	0	151289322.544588
steplength,gamma	20.6857895261084	0.253906250000000D-001	2
tt,info,rmax,	15.00000000000000	0	150658392.083229
steplength,gamma	2.08038227471001	0.250000000000000	3
tt,info,rmax,	16.00000000000000	0	144578667.929296
steplength,gamma	35.9811215440998	0.488281250000000D-003	4
tt,info,rmax,	18.00000000000000	0	144551342.269987
steplength,gamma	6.59190252887505	0.781250000000000D-002	5
tt,info,rmax,	19.00000000000000	0	144407436.410621
steplength,gamma	0.455879229119055	1.00000000000000	6
tt,info,rmax,	12.00000000000000	0	135022355.068402
steplength,gamma	0.510872132671297	1.00000000000000	7
tt,info,rmax,	12.00000000000000	0	0.250339508056641D-00
steplength,gamma	0.938331531860259D-014	1.00000000000000	8
tt,info,rmax,	12.00000000000000	0	0.00000000000000
x , r	0.938890847888777D-032	0.342815530100928D-031	
	0.100918729085891D-030	0.268089448258703D-030	0.671764364602268D-03
	0.156539585879795D-029	0.350057026691824D-029	0.700114053383648D-02
	0.140022810676730D-028	0.272157012301249D-028	0.544314024602498D-02
	1.000000000000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	-457763664.000000	-457763664.000000
	-457763664.000000	0.000000000000000	0.000000000000000
	-0.286526743083756D-023	-0.104618995423402D-022	-0.307979514617206D-02
	-0.818144054097616D-022	-0.205006211256644D-021	-0.477720895955849D-02
	-0.106828924764930D-020	-0.213657849529860D-020	-0.427315699059720D-02
	-0.830557274228752D-020	-0.166111454845750D-019	

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