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Large Deviation Local Limit Theorems for Ratio Statistics†

By

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Abstract

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of non-lattice random variables and let $\{S_n, n \geq 1\}$ be another sequence of positive random variables. Assume that the sequences are independent. In this paper we obtain asymptotic expression for the density function of the ratio statistic $R_n = T_n/S_n$ based on simple conditions on the moment generating functions of T_n and S_n . When $S_n = n$, our main result reduces to that of Chaganty and Sethuraman[*Ann. Probab.* 13(1985):97-114]. We also obtain analogous results when T_n and S_n are both lattice random variables. We call our theorems large deviation local limit theorems for R_n , since the conditions of our theorems imply that $R_n \rightarrow c$ in probability for some constant c . We present some examples to illustrate our theorems.



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1. Introduction. Let $\{R_n, n \geq 1\}$ be a sequence of random variables which converge in distribution to a non-degenerate random variable R . It is well known that convergence in distribution does not guarantee convergence of the corresponding density functions pointwise. Let g_n be the probability density function (p.d.f.) of R_n and let g be the p.d.f. of R . A theorem which asserts that g_n converges to g pointwise is known as a local limit theorem. Now suppose R_n converges to a constant c as $n \rightarrow \infty$. Let $\{r_n, n \geq 1\}$ be a sequence of real numbers bounded away from c . A theorem which obtains the limit of $g_n(r_n)$ or an asymptotic expression for $g_n(r_n)$ is known as a large deviation local limit theorem. The event $\{R_n \geq r_n\}$ is known as a large deviation event. The study of the probabilities of large deviation events and its many uses are well described in the books by Ellis (1985) and Varadhan (1984).

Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of random variables and let $\{S_n, n \geq 1\}$ be another arbitrary sequence of positive random variables. Assume that the two sequences are independent. In this paper we obtain large deviation local limit theorems for the ratio statistic $R_n = T_n/S_n$, based on some mild and easily verifiable conditions on the cumulant generating functions of T_n and S_n . In statistical applications T_n can be viewed as an estimate of a location parameter and S_n can be viewed as an estimate of a scale parameter and a function of the ratio statistic $R_n = T_n/S_n$ can be used to test a hypothesis about the location parameter. In the case where T_n is the sum of i.i.d. random variables and S_n is also the sum of i.i.d. positive random variables the conditions of our theorems are easily verified and the conclusion of our theorems agrees with the heuristic result of Daniels (1954). In the case where S_n is taken to be degenerate at n , our results reduce to the theorems of Chaganty and Sethuraman (1985). However, one should note that Condition (C) of our main result, Theorem 2.1, is weaker than Condition (C) that appears in the paper of Chaganty and Sethuraman (1985).

The organization of this paper is as follows: In Section 2 we consider the case where T_n is a nonlattice random variable and S_n is a positive random variable independent of T_n , and obtain an asymptotic expression for the p.d.f. of $R_n = T_n/S_n$. In Section 3 we

consider lattice valued random variables T_n and S_n and obtain asymptotic expressions for the probability $P(T_n = r_n S_n)$. We illustrate the usefulness of our theorems with three examples in Section 4.

2. Main Results. Let $\{T_n, n \geq 1\}$ be an arbitrary sequence of nonlattice random variables and $\{S_n, n \geq 1\}$ be a sequence of positive random variables. Let ϕ_{1n} and ϕ_{2n} denote the moment generating functions of T_n and S_n respectively. Assume that $\phi_{in}(z)$ is nonzero and analytic in $\Omega_i = \{z \in \mathcal{C} : |z| < c_i\}$ for $i = 1, 2$, where $\mathcal{C}$ denotes the set of all complex numbers and $c_i, i = 1, 2$, are some positive constants. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$. Let

$$(2-1) \quad \psi_{in}(z) = \frac{1}{a_n} \log \phi_{in}(z), \quad z \in \Omega_i, \quad i = 1, 2.$$

Let $J_i = (-b_i, b_i)$, where $0 < b_i < c_i$, for $i = 1, 2$. We are now in a position to state the main theorem of this section. Theorem 2.1 below obtains a large deviation local limit theorem for the ratio statistic $R_n = T_n/S_n$.

THEOREM 2.1. Assume that the two sequences $\{T_n, n \geq 1\}$ and $\{S_n, n \geq 1\}$ are independent. Let $\{r_n\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ contained in J_1 satisfying

$$(2-2) \quad \psi'_{1n}(\tau_n) - r_n \psi'_{2n}(-r_n \tau_n) = 0$$

and $r_n \tau_n \in J_2$ for all $n \geq 1$. Assume that the following conditions are satisfied:

- (A) There exists β_i such that $|\psi_{in}(z)| < \beta_i$ for $n \geq 1$ and $z \in \Omega_i, i = 1, 2$.
- (B) There exist $\alpha_i > 0, i = 1, 2$, such that $\psi''_{1n}(\tau_n) > \alpha_1$ and $\psi'_{2n}(-r_n \tau_n) > \alpha_2$ for all $n \geq 1$.
- (C) For any given $\delta > 0$, there exist $0 < \eta < 1$ and $q > 0$ such that

$$(2-3) \quad \limsup_n \sup_{|t| > \delta} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} < \eta$$

and

$$(2-4) \quad \sup_{|t| > \delta} |\psi'_{2n}(-r_n(\tau_n + it))| = O(a_n^q).$$

(D) There exist $p > 0$, $\ell > 0$ such that

$$(2-5) \quad \int_{-\infty}^{\infty} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{\ell/a_n} dt = O(a_n^p).$$

Then an asymptotic expansion for the density function g_n of T_n/S_n at the point r_n is given by

$$(2-6) \quad g_n(r_n) = \frac{\sqrt{a_n} \psi'_{1n}(-r_n \tau_n)}{[2\pi(\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n))]^{1/2}} \exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n))] \left[1 + O\left(\frac{1}{a_n}\right)\right].$$

We shall postpone the proof of the theorem until the end of Lemma 2.9. At this point some remarks about the conditions (A) thru (D) are in order.

Remark 2.2. Condition (A) of Theorem 2.1 requires that ψ_{1n} and ψ_{2n} be bounded uniformly in n in a circle around the origin in the complex plane. Therefore the derivatives of ψ_{in} , $i = 1, 2$ are also uniformly bounded in a neighborhood of the origin and hence $E(T_n)/a_n$, $Var(T_n)/a_n$, $E(S_n)/a_n$ and $Var(S_n)$ are all uniformly bounded in n . Thus, we can find a subsequence $\{m\}$ such that T_m/a_m and S_m/a_m approach constants in probability as $m \rightarrow \infty$. Therefore the ratio statistic $R_m = T_m/S_m$ converges to a constant in probability as $m \rightarrow \infty$.

Remark 2.3. Condition (D) of Theorem 2.1 implies that the characteristic function (c.f.) $\phi_{1n}(\tau_n + it)/\phi_{1n}(\tau_n)$ is absolutely integrable for sufficiently large n and hence the random variable corresponding to this c.f. is absolutely continuous. Therefore, given $\delta > 0$, for each $n \geq n_0$ we can find $0 < \eta_n < 1$ such that

$$(2-7) \quad \sup_{|t| > \delta} |\phi_{1n}(\tau_n + it)/\phi_{1n}(\tau_n)|^{1/a_n} < \eta_n.$$

Condition (C) requires that the $\limsup_n (\eta_n)$ should be less than 1. We use this condition mainly in Lemma 2.7 to show that the term I_{n1} defined in (2-15) goes to zero exponentially fast.

Remark 2.4. Condition (D) guarantees the existence of the density function of T_n and permits the use of the inversion formula to get an expression for the p.d.f. of T_n . This condition is also used to show that the term I_{n1} defined in (2-15) goes to zero exponentially fast.

Remark 2.5. It is interesting to note that if S_n is a non-lattice random variable the conclusion of Theorem 2.1 holds if ϕ_{1n} is replaced by ϕ_{2n} in (2-3).

We will need the following Lemmas 2.6 thru 2.9 in the proof of Theorem 2.1.

LEMMA 2.6. Let ψ_{in} be as defined in (2-1), for $i = 1, 2$. Assume that Condition (A) of Theorem 2.1 holds. For $i = 1, 2$, let

$$(2-8) \quad R_{in}(\tau + it) = \psi_{in}(\tau + it) - \psi_{in}(\tau) - (it)\psi'_{in}(\tau) - \frac{(it)^2}{2}\psi''_{in}(\tau) - \frac{(it)^3}{6}\psi'''_{in}(\tau)$$

and

$$(2-9) \quad R_n(\tau + it) = \psi'_{2n}(\tau + it) - \psi'_{2n}(\tau) - (it)\psi''_{2n}(\tau) - \frac{(it)^2}{2}\psi'''_{2n}(\tau).$$

Then the following holds:

$$(2-10) \quad \sup_{z \in \Omega'_i} |\psi_{in}^{(k)}(z)| \leq \frac{k! \beta_i}{(c_i - b_i)^k} \quad \text{for all } k \geq 1$$

where $\Omega'_i = \{z \in \mathcal{C} : |z| < b_i\}$, $i = 1, 2$. Also there exists $\delta_0 > 0$ such that whenever $|t| < \delta_0$,

$$(2-11) \quad \sup_{\tau \in J_i} |R_{in}(\tau + it)| \leq \frac{2\beta_i t^4}{(c_i - b_i)^4} \quad \text{for } i = 1, 2$$

and

$$(2-12) \quad \sup_{\tau \in J_2} |R_n(\tau + it)| \leq \frac{2\beta_2 |t|^3}{(c_2 - b_2)^4}.$$

Proof. The proof of this lemma follows from Cauchy's theorem and is similar to the proof of Lemma 2.10 of Chaganty and Sethuraman (1985) and hence is omitted.

The next Lemma 2.7 shows that the term I_{n1} appearing in the proof of Theorem 2.1 goes to zero exponentially fast.

LEMMA 2.7. Let ψ_{in} be as defined in (2-1), for $i = 1, 2$. Let $\{r_n\}$ be a sequence of real numbers. Assume that (2-2) and conditions (A) thru (D) of Theorem 2.1 are satisfied. Let

$$(2-13) \quad f_n(z) = \psi_{1n}(z) + \psi_{2n}(-r_n z)$$

and

$$(2-14) \quad D_n(t) = \psi'_{2n}(-r_n(\tau_n + it)) / \psi'_{2n}(-r_n \tau_n).$$

Then

$$(2-15) \quad I_{n1} = \left[\frac{a_n f''_n(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta_1} \exp[a_n(f_n(\tau_n + it) - f_n(\tau_n))] D_n(t) dt$$

goes to zero exponentially fast for all δ_1 , $0 < \delta_1 < \delta_0$, where δ_0 is as in Lemma 2.6.

Proof. Note that

$$(2-16) \quad |I_{n1}| \leq \left[\frac{a_n f''_n(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta_1} \left| \exp[a_n(f_n(\tau_n + it) - f_n(\tau_n))] \right| |D_n(t)| dt$$

Substituting $\psi_{1n}(z) + \psi_{2n}(-r_n z)$ for $f_n(z)$ in the integrand we get

$$(2-17) \quad \begin{aligned} |I_{n1}| &\leq \left[\frac{a_n f''_n(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| \geq \delta_1} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right| \left| \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} \right| dt \\ &\leq \left[\frac{a_n f''_n(\tau_n)}{2\pi} \right]^{1/2} \sup_{|t| \geq \delta_1} \left[\left| \frac{\psi'_{2n}(-r_n(\tau_n + it))}{\psi'_{2n}(-r_n \tau_n)} \right| \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1-\ell/a_n} \right] \\ &\quad \times \int_{-\infty}^{\infty} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{\ell/a_n} dt \end{aligned}$$

where ℓ is as in Condition (D). Using (2-10) and Conditions (B) thru (D) we get for sufficiently large n ,

$$\begin{aligned} |I_{n1}| &\leq O(a_n^{(q+p+\frac{1}{2})}) \eta^{a_n(1-\ell/a_n)} \\ (2-18) \quad &= O(a_n^{(q+p+\frac{1}{2})}) e^{-\eta_1(a_n-\ell)} \end{aligned}$$

where $\eta_1 = -\log(\eta) > 0$. Hence I_{n1} goes to zero exponentially fast since $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

We need the following Lemma 2.8 in the proof of the next Lemma 2.9.

LEMMA 2.8. Let ψ_{in} , be as defined in (2-1), for $i = 1, 2$. Let $\{r_n\}$ be a sequence contained in J_1 satisfying (2-2) and $r_n \tau_n \in J_2$ for all $n \geq 1$. Assume that Conditions (A), (B) of Theorem 2.1 hold. Let $D_n(t)$ be as defined in (2-14) and let

$$(2-19) \quad L_n(s) = [\exp(z_n(s)) D_n\left(\frac{s}{\sqrt{a_n}}\right) - 1 - z_n(s)]$$

where

$$\begin{aligned} (2-20) \quad z_n(s) = &\left[-\frac{is^3}{6\sqrt{a_n}} \psi_{1n}'''(\tau_n) + \frac{ir_n^3 s^3}{6\sqrt{a_n}} \psi_{2n}'''(-r_n \tau_n) \right. \\ &\left. + a_n R_{1n}\left(\tau_n + i\frac{s}{\sqrt{a_n}}\right) + a_n R_{2n}\left(-r_n\left(\tau_n + i\frac{s}{\sqrt{a_n}}\right)\right) \right] \end{aligned}$$

Then there exists δ_1 , $0 < \delta_1 < \delta_0$, such that

$$(2-21) \quad Q_n = \left[\frac{f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta_1} \exp\left(-\frac{s^2 f_n''(\tau_n)}{2}\right) L_n(s) ds = O\left(\frac{1}{a_n}\right).$$

Proof. Let δ_1 be less than δ_0 , where δ_0 is as in Lemma 2.6. Using (2-19) we can write Q_n as the sum of two integrals as follows:

$$\begin{aligned} (2-22) \quad Q_n &= \left[\frac{f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta_1} \exp\left(-\frac{s^2 f_n''(\tau_n)}{2}\right) \left[(\exp(z_n(s)) - 1 - z_n(s)) D_n\left(\frac{s}{\sqrt{a_n}}\right) \right] ds \\ &\quad + \left[\frac{f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta_1} \exp\left(-\frac{s^2 f_n''(\tau_n)}{2}\right) (1 + z_n(s)) \left[D_n\left(\frac{s}{\sqrt{a_n}}\right) - 1 \right] ds \\ &= Q_{n1} + Q_{n2} \quad (\text{say}). \end{aligned}$$

We complete the proof of the lemma by showing that $Q_{ni} = O(1/a_n)$ for $i = 1, 2$. In order to show that $Q_{n1} = O(1/a_n)$, we get an upper bound for $|\exp(z_n(s)) - 1 - z_n(s)|$ first by obtaining an upper bound for $z_n(s)$. For $|s| < \sqrt{a_n}\delta_1$, using Condition (A), (2-10) and (2-11) we get that

$$\begin{aligned}
 |z_n(s)| &\leq \frac{|s|^3}{\sqrt{a_n}} \left[\frac{\beta_1}{(c_1 - b_1)^3} + \frac{|r_n|^3 \beta_3}{(c_2 - b_2)^3} \right] + \frac{s^4}{a_n} \left[\frac{2\beta_1}{(c_1 - b_1)^4} + \frac{2r_n^4 \beta_2}{(c_2 - b_2)^4} \right] \\
 (2-23) \quad &\leq s^2 \delta_1 \left[\frac{\beta_1}{(c_1 - b_1)^3} + \frac{r^3 \beta_2}{(c_2 - b_2)^3} \right] + s^2 \delta_1^2 \left[\frac{2\beta_1}{(c_1 - b_1)^4} + \frac{2r^4 \beta_2}{(c_2 - b_2)^4} \right] \\
 &= s^2 M(\delta_1) \quad (\text{say})
 \end{aligned}$$

where $r = \sup_n |r_n|$. Let δ_1 be such that $M(\delta_1) < \alpha_1/2$. We are now in a position to show that $Q_{n1} = O(1/a_n)$. Using Condition (B) and (2-10) it is easy to check that $f_n''(\tau_n) \geq \alpha_1$ and $f_n''(\tau_n) = O(1)$ and $D_n(\frac{s}{\sqrt{a_n}}) = O(1)$ for $|s| < \sqrt{a_n}\delta_1$. Therefore

$$(2-24) \quad |Q_{n1}| \leq O(1) \int_{|s| < \sqrt{a_n}\delta_1} \exp\left(-\frac{s^2 \alpha_1}{2}\right) |\exp(z_n(s)) - 1 - z_n(s)| ds.$$

Using the simple inequality $|\exp(z) - 1 - z| \leq |z|^2 \exp(|z|)$ and the upper bounds in (2-23) for $z_n(s)$ we get that

$$\begin{aligned}
 |Q_{n1}| &\leq O\left(\frac{1}{a_n}\right) \int_{|s| < \sqrt{a_n}\delta_1} \exp\left(-\frac{s^2}{2}(\alpha_1 - 2M(\delta_1))\right) \\
 (2-25) \quad &\times \left[\frac{|s|^3 \beta_1}{(c_1 - b_1)^3} + \frac{|s|^3 r_n^3 \beta_2}{(c_2 - b_2)^3} + \frac{2\beta_1 s^4}{\sqrt{a_n}(c_1 - b_1)^4} + \frac{2\beta_2 r_n^4 s^4}{\sqrt{a_n}(c_2 - b_2)^4} \right]^2 ds \\
 &= O\left(\frac{1}{a_n}\right)
 \end{aligned}$$

since $M(\delta_1) < \alpha_1/2$. The second integral Q_{n2} can be handled similarly after noting that for $|s| < \sqrt{a_n}\delta_1$,

$$(2-26) \quad \left[D_n\left(\frac{s}{\sqrt{a_n}}\right) - 1 \right] = \frac{-ir_n s}{\sqrt{a_n}} \frac{\psi_{2n}''(-r_n \tau_n)}{\psi_{2n}'(-r_n \tau_n)} - \frac{r_n^2 s^2}{a_n} \frac{\psi_{2n}'''(-r_n \tau_n)}{\psi_{2n}'(-r_n \tau_n)} + \frac{R_n(-r_n(\tau_n + i\frac{s}{\sqrt{a_n}}))}{\psi_{2n}'(-r_n \tau_n)}.$$

Using Condition (B), (2-26) and Lemma 2.6 we can easily verify that $Q_{n2} = O(1/a_n)$. This completes the proof of Lemma 2.8.

The next lemma shows that the term I_{n2} appearing in the proof of the main Theorem 2.1 is $1 + O(1/a_n)$.

LEMMA 2.9. Let $f_n(z)$ and $D_n(t)$ be as defined in (2-13) and (2-14) respectively. Let $\delta_1 > 0$ be as in Lemma 2.8. Assume that Conditions (A) and (B) of Theorem 2.1 hold. Then

$$(2-27) \quad I_{n2} = \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|t| < \delta_1} \exp[a_n(f_n(\tau_n + it) - f_n(\tau_n))] D_n(t) dt$$

$$= 1 + O\left(\frac{1}{a_n}\right).$$

Proof. Making a change of variable $t = s/\sqrt{a_n}$, we get that

$$(2-28) \quad I_{n2} = \left[\frac{f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta_1} \exp[a_n(f_n(\tau_n + is/\sqrt{a_n}) - f_n(\tau_n))] D_n\left(\frac{s}{\sqrt{a_n}}\right) ds.$$

Note that for $|s| < \sqrt{a_n} \delta_1$, we can write

$$(2-29) \quad a_n(f_n(\tau_n + i\frac{s}{\sqrt{a_n}}) - f_n(\tau_n)) = -\frac{s^2}{2} f_n''(\tau_n) + z_n(s)$$

where $z_n(s)$ is as defined by (2-20). Hence

$$(2-30) \quad I_{n2} = \left[\frac{f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta_1} \exp\left[-\frac{s^2}{2} f_n''(\tau_n) + z_n(s)\right] D_n\left(\frac{s}{\sqrt{a_n}}\right) ds$$

$$= \left[\frac{f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{|s| < \sqrt{a_n} \delta_1} \exp\left[-\frac{s^2}{2} f_n''(\tau_n)\right] [1 + z_n(s) + L_n(s)] ds$$

where $L_n(s)$ is as defined by (2-19). The r.h.s. of (2.30) can be written as the sum of three integrals. The first integral is $1 + O(1/a_n)$ follows from Mill's ratio. Using (2-23) we can easily verify that the second integral is $O(1/a_n)$. The third integral is $O(1/a_n)$ as shown in Lemma 2.8. Thus $I_{n2} = 1 + O(1/a_n)$. This completes the proof of Lemma 2.9.

We now proceed with the proof of the main Theorem 2.1.

Proof of Theorem 2.1. Let F_{1n}, F_{2n} and G_n be the distribution functions of T_n, S_n and $R_n = T_n/S_n$ respectively. Since T_n and S_n are independent we have $G_n(r) =$

$\int_0^\infty F_{1n}(ry) dF_{2n}(y)$, for any r . Hence the probability density function, g_n , of R_n is given by

$$(2-31) \quad g_n(r) = \int_0^\infty y f_{1n}(ry) dF_{2n}(y)$$

where f_{1n} is the p.d.f. of T_n . Proceeding as in the proof of Theorem 2.1 of Chaganty and Sethuraman (1985) we get that

$$(2-32) \quad f_{1n}(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \exp(-x(\tau + it)) dt$$

for any $\tau \in J_1$. Therefore

$$(2-33) \quad \begin{aligned} g_n(r_n) &= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty y \phi_{1n}(\tau + it) \exp(-r_n y(\tau + it)) dt dF_{2n}(y) \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \left[\int_0^\infty y \exp(-r_n y(\tau + it)) dF_{2n}(y) \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \phi_{1n}(\tau + it) \phi'_{2n}(-r_n(\tau + it)) dt \\ &= \frac{a_n}{2\pi} \int_{-\infty}^\infty \exp[a_n(\psi_{1n}(\tau + it) + \psi_{2n}(-r_n(\tau + it)))] \psi'_{2n}(-r_n(\tau + it)) dt. \end{aligned}$$

We note that the integral on the r.h.s. of (2-33) remains the same for all τ in J_1 . The saddle point method suggests that the appropriate choice of τ is τ_n which satisfies the equation (2-2), that is, $\psi'_{1n}(\tau_n) = r_n \psi'_{2n}(-r_n \tau_n)$. Replacing τ by τ_n in the r.h.s. of (2-33) we can rewrite

$$(2-34) \quad \begin{aligned} g_n(r_n) &= \frac{a_n}{2\pi} \int_{-\infty}^\infty \exp[a_n(\psi_{1n}(\tau_n + it) + \psi_{2n}(-r_n(\tau_n + it)))] \psi'_{2n}(-r_n(\tau_n + it)) dt \\ &= \frac{\sqrt{a_n} \psi'_{2n}(-r_n \tau_n) \exp[a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n))]}{[2\pi(\psi''_{1n}(\tau_n) + r_n^2 \psi''_{2n}(-r_n \tau_n))]^{1/2}} I_n \end{aligned}$$

where

$$(2-35) \quad I_n = \left[\frac{a_n f''_n(\tau_n)}{2\pi} \right]^{1/2} \int_{-\infty}^\infty \exp[a_n(f_n(\tau_n + it) - f_n(\tau_n))] D_n(t) dt$$

where $f_n(z)$ and $D_n(t)$ are as defined in (2-13) and (2-14) respectively. We can write the integral on the r.h.s. of (2-35) as the sum of two integrals, the first integral over the region $\{t \geq \delta_1\}$ and the second integral over the region $\{|t| < \delta_1\}$. Thus

$$I_n = I_{n1} + I_{n2}$$

where I_{n1} and I_{n2} are as defined in (2-15) and (2-27) respectively. Lemmas 2.7 and 2.9 show that $I_{n1} = O(1/a_n)$ and $I_{n2} = 1 + O(1/a_n)$. Thus

$$I_n = 1 + O(1/a_n)$$

and this completes the proof of the Theorem 2.1.

In the case where T_n and S_n are chosen to be the sums of n i.i.d. random variables, the Conditions (A) thru (D) of Theorem 2.1 are very much simplified and they are easy to verify. We state this case as a separate theorem because of its importance in mathematical statistics. Later, in Section 4 we shall apply Theorem 2.10 to some examples.

THEOREM 2.10. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. non-lattice random variables with moment generating function ϕ_1 . Let $\{Y_n, n \geq 1\}$ be a sequence of i.i.d. positive valued random variables with moment generating function ϕ_2 . Assume that the two sequences are independent. Let $\phi_i(z)$ be non-vanishing and analytic in $\Omega_i = \{z \in \mathbb{C} : |z| < c_i\}$ for $i = 1, 2$. Let $J_i = (-b_i, b_i)$ where $0 < b_i < c_i, i = 1, 2$. Let $\{r_n\}$ be a sequence of real numbers such that there exists $\{\tau_n\}$ contained in J_1 satisfying*

$$(2-36) \quad \psi'_1(\tau_n) - r_n \psi'_2(-r_n \tau_n) = 0$$

and $r_n \tau_n \in J_2$ for all $n \geq 1$. Assume that the following conditions hold:

(A1) There exist $\beta_i < \infty$ such that $|\psi_i(z)| < \beta_i$ for all $z \in \Omega_i, i = 1, 2$.

(B1) There exist $\alpha_i > 0, i = 1, 2$, such that $\psi''_1(\tau_n) > \alpha_1$ and $\psi'_2(-r_n \tau_n) > \alpha_2$ for all $n \geq 1$.

(C1) For any given $\delta > 0$, there exists $q > 0$ such that

$$(2-37) \quad \sup_{|t| > \delta} |\psi'_2(-r_n(\tau_n + it))| = O(n^q).$$

(D1) There exists $\ell > 0$ such that

$$(2-38) \quad \limsup_n \int_{-\infty}^{\infty} \left| \frac{\phi_1(\tau_n + it)}{\phi_1(\tau_n)} \right|^\ell dt = M < \infty.$$

Let $T_n = X_1 + \dots + X_n$ and $S_n = Y_1 + \dots + Y_n$. If g_n denotes the p.d.f. of $R_n = T_n/S_n$ then

$$(2-39) \quad g_n(r_n) = \frac{\sqrt{n}\psi'_2(-r_n\tau_n)}{[2\pi(\psi''_1(\tau_n) + r_n^2\psi''_2(-r_n\tau_n))]^{1/2}} \exp[n(\psi_1(\tau_n) + \psi_2(-r_n\tau_n))] [1 + O(\frac{1}{n})].$$

Proof. The conclusion of this theorem follows easily from Theorem 2.1 where we choose $a_n = n$. Note that in this case (2-3) is automatically satisfied (see Remark 2.3).

3. The Lattice Case. In this section we obtain large deviation local limit theorems for the ratio statistic $R_n = T_n/S_n$ analogous to the results of Section 2 in the case where T_n and S_n are independent lattice valued random variables. The main result of this section is stated as Theorem 3.1. We shall not deal with the case where T_n is lattice valued and S_n is non-lattice valued, since this problem can be reduced to the case covered by Theorem 2.1 if we consider the ratios S_n/T_n^+ and S_n/T_n^- where T_n^+ and T_n^- are the positive and negative parts of T_n respectively. We shall continue to use the notation introduced in Section 2.

THEOREM 3.1. Let $\{T_n, n \geq 1\}$ be a sequence of lattice valued random variables with spans $\{h_n > 0, n \geq 1\}$. Let $\{S_n, n \geq 1\}$ be an independent sequence of positive lattice valued random variables. Let $\{\tau_n\}$ be a sequence of real numbers as in Theorem 2.1 satisfying (2-2). Assume that T_n and S_n satisfy Condition (A) of Theorem 2.1. Further replace Conditions (B), (C) and (D) by the following:

(B') There exists $\alpha_1 > 0$ such that $\psi''_{1n}(\tau_n) > \alpha_1$ for $n \geq 1$.

(C') Given $\delta > 0$, there exists $\eta, 0 < \eta < 1$, such that

$$(3-1) \quad \limsup_n \sup_{\delta < |t| \leq \pi/h_n} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{1/a_n} < \eta.$$

(D') There exist positive constants p and ℓ such that

$$(3-2) \quad \int_{-\infty}^{\infty} \left| \frac{\phi_{1n}(\tau_n + it)}{\phi_{1n}(\tau_n)} \right|^{\ell/a_n} dt = O(a_n^p).$$

Let $P_n(r_n) = P(T_n = r_n S_n)$. Then

$$(3-3) \quad \frac{\sqrt{a_n}}{h_n} P_n(r_n) = \frac{\exp [a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n))]}{[2\pi(\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n \tau_n))]^{1/2}} [1 + O(\frac{1}{a_n})].$$

Proof. Consider

$$(3-4) \quad \begin{aligned} P_n(r_n) &= P(T_n = r_n S_n) \\ &= \sum_y P(T_n = r_n y) P(S_n = y) \end{aligned}$$

since T_n and S_n are independent. Proceeding as in the proof of Theorem 2.2 of Chaganty and Sethuraman (1985) and using Condition (D') we can show that

$$(3-5) \quad P(T_n = r_n y) = \frac{h_n}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \phi_{1n}(\tau_n + it) \exp(-r_n y(\tau_n + it)) dt$$

Combining (3-4) and (3-5) and interchanging the order of summation and integration we get that

$$(3-6) \quad \begin{aligned} P_n(r_n) &= \frac{h_n}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \phi_{1n}(\tau_n + it) \phi_{2n}(-r_n(\tau_n + it)) dt \\ &= \frac{h_n}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \exp [a_n(\psi_{1n}(\tau_n + it) + \psi_{2n}(-r_n(\tau_n + it)))] dt. \end{aligned}$$

Therefore

$$(3-7) \quad \begin{aligned} \frac{\sqrt{a_n}}{h_n} P_n(r_n) &= \frac{\sqrt{a_n}}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \exp [a_n(\psi_{1n}(\tau_n + it) + \psi_{2n}(-r_n(\tau_n + it)))] dt \\ &= \frac{\exp [a_n(\psi_{1n}(\tau_n) + \psi_{2n}(-r_n \tau_n))]}{[2\pi(\psi_{1n}''(\tau_n) + r_n^2 \psi_{2n}''(-r_n \tau_n))]^{1/2}} I_n \end{aligned}$$

where

$$(3-8) \quad I_n = \left[\frac{a_n f_n''(\tau_n)}{2\pi} \right]^{1/2} \int_{-\pi/h_n}^{\pi/h_n} \exp[a_n(f_n(\tau_n + it) - f_n(\tau_n))] dt$$

and $f_n(z)$ is as defined in (2-13). Using Conditions (A), (B'), (C') and (D') and imitating Lemmas 2.6 thru 2.9 we can show that

$$(3-9) \quad I_n = 1 + O\left(\frac{1}{a_n}\right).$$

The two identities (3-7) and (3-9) complete the proof of Theorem 3.1.

Remark 3.2. When $a_n = n$ and S_n is taken to be degenerate at n , our Theorems 2.1 and 3.1 reduce to Theorems 2.1 and 2.2 respectively of Chaganty and Sethuraman (1985). Thus the main results of this paper generalize the results of Chaganty and Sethuraman (1985).

4. Applications. In this section we present four examples to illustrate the theorems of Sections 2 and 3. These example cover all the combinations of non-lattice and lattice cases for T_n and S_n . The examples clearly demonstrate the wide applicability of our theorems. The conditions of our theorems are easily verified in these examples because both T_n and S_n are sums of n i.i.d. random variables. One should note that in all these examples the exact density does not have a closed form, however our theorems provide a simple asymptotic expressions for the density functions.

Example 4.1. Let T_n be distributed as Normal with mean 0 and variance n . Let S_n be distributed as chi-square with n degrees of freedom. Assume that T_n and S_n are independent. The m.g.f.'s of T_n and S_n are given by

$$(4-1) \quad \phi_{1n}(z) = \exp(nz^2/2), \quad |z| < \infty$$

and

$$(4-2) \quad \phi_{2n}(z) = (1 - 2z)^{-n/2}, \quad |z| < 1/2.$$

Let $\{r_n\}$ be a sequence of real numbers such that $\sup_n |r_n| = r < 1$. Let $\tau_n = (-1 + \sqrt{1 + 8r_n^2})/4r_n$. We can choose $0 < c_1 < \infty, 0 < c_2 < 1/2$ and $0 < b_i < c_i$ for $i = 1, 2$ such that Condition (2-2) and Conditions (A) thru (D) of Theorem 2.1 are satisfied with $a_n = n$. Let g_n be the p.d.f. of T_n/S_n . Then by the conclusion of Theorem 2.1 we have

$$(4-3) \quad g_n(r_n) = \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{1}{(1 + 2r_n\tau_n)^{\frac{3}{2}+1}(1 + 2r_n^2)^{1/2}} \exp\left[\frac{n\tau_n^2}{2}\right] \left[1 + O\left(\frac{1}{n}\right)\right].$$

Note that in this example both T_n and S_n are non-lattice valued random variables.

Example 4.2. Let T_n be as in Example 4.1. Let S_n be Poisson with mean n . Assume that T_n and S_n are independent. The m.g.f.'s of T_n and S_n are given by

$$(4-4) \quad \phi_{1n}(z) = \exp(nz^2/2), \quad |z| < \infty$$

and

$$(4-5) \quad \phi_{2n}(z) = \exp(n(\exp(z) - 1)), \quad |z| < \infty$$

Let $\{r_n\}$ be a bounded sequence of real numbers. Let τ_n be such that the following equation is satisfied:

$$(4-6) \quad \tau_n = r_n \exp(-r_n\tau_n).$$

We can choose finite positive constants c_1, c_2 and b_1, b_2 such that $0 < b_i < c_i$ for $i = 1, 2$ and Condition (2-2) and Conditions (A) thru (D) of Theorem 2.1 are satisfied with $a_n = n$. Note that the ratio random variable $|R_n| = |T_n/S_n|$ takes the value ∞ with probability $\exp(-n)$ and possess an improper density function $g_n(r)$ on the interval $(-\infty, \infty)$. By the conclusion of Theorem 2.1 we have

$$(4-7) \quad g_n(r_n) = \frac{\sqrt{n}}{\sqrt{2\pi}} \frac{\exp(-r_n\tau_n)}{(1 + r_n\tau_n)^{1/2}} \exp\left[n\tau_n^2/2 + n(\exp(-r_n\tau_n) - 1)\right] \left[1 + O\left(\frac{1}{n}\right)\right]$$

Note that in this example we have considered non-lattice over lattice random variables.

Example 4.3. Let T_n and S_n be distributed as Poisson with means $n\lambda_1$ and $n\lambda_2$ respectively. Assume that T_n and S_n be independent. The m.g.f.'s of T_n and S_n are given by

$$(4-8) \quad \phi_{1n}(z) = \exp(n\lambda_1(\exp(z) - 1)), \quad |z| < \infty$$

and

$$(4-9) \quad \phi_{2n}(z) = \exp(n\lambda_2(\exp(z) - 1)), \quad |z| < \infty$$

Let $\{r_n\}$ be a bounded sequence of positive rational numbers. Let

$$r_n = [\log(r_n) + \log(\lambda_2/\lambda_1)]/(1 + r_n).$$

We can find constants c_1, c_2 and b_1, b_2 such that $0 < b_i < c_i$, for $i = 1, 2$ and Condition (2-2) and all the Conditions (A), (B'), (C') and (D') of Theorem 3.1 are satisfied with $a_n = n$. Let $P_n(r_n) = P(T_n = r_n S_n)$. Then from the conclusion of Theorem 3.1 we get

$$(4-10) \quad \sqrt{n}P_n(r_n) = \frac{\exp[n(\lambda_1(\exp(r_n) - 1) + \lambda_2(\exp(-r_n r_n) - 1))]}{[2\pi(\lambda_1 \exp(r_n) + \lambda_2 r_n^2 \exp(-r_n r_n))]^{1/2}} \left[1 + O\left(\frac{1}{n}\right)\right].$$

5. References

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