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Efficient Nearly Orthogonal Deletion Designs

by

Subir Ghosh and Joan Mahoney

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April, 1988

# Efficient Nearly Orthogonal Deletion Designs

by

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## 0. Summary

This article considers single replicate factorial experiments in incomplete blocks. A single replicate $2^{m_1} \times 3^{m_2}$ deletion design in incomplete blocks is obtained from a single replicate $3^m$ $(m = m_1 + m_2)$ preliminary design by deleting all runs (or treatment combinations) with the first $m_1$ factors at the level two. A systematic method for determining the unbiasedly estimable (u.e.) and not unbiasedly estimable (n.u.e) factorial effects is provided. Although the method is discussed for single replicate ~~$2^{m_1} \times 3^{m_2}$~~ deletion designs in three incomplete blocks, the method can easily be extended to more than three blocks. It is shown that for $m_2 > 0$ all factorial effects of the type $F_1^{\alpha_1} \ldots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \ldots F_m^{\alpha_m}$, $\alpha_i = 0, 1$ for $i = 1, \ldots, m_1$, $\alpha_i = 0,1,2$ for $i = m_1+1, \ldots, m$, $(\alpha_1, \ldots, \alpha_m) \neq (0, \ldots, 0)$, $(\alpha_{m_1+1}, \ldots, \alpha_m) \neq \alpha(1, \ldots, 1)$ where $\alpha = 1$ and 2, are u.e. and the remaining factorial effects are n.u.e. It is noted that $(2^{m_1} - 1)$

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factorial effects of $2^{m_1}$ factorial experiments and $(3^{m_2}-3)$ factorial effects of $3^{m_2}$ factorial experiments, which are embedded in $2^{m_1} \times 3^{m_2}$ factorial experiments, are u.e. The $2 \times 3^{m-1}$ deletion designs were considered in the work of Voss (1986). Defining factorial effects of a $2^{m_1} \times 3^{m_2}$ factorial experiment in a form different than in Voss (1986), a simple representation of u.e. and n.u.e. factorial effects is obtained. In this representation, there are $(2^{m_1+1}+1)$ n.u.e. factorial effects of the type $F_1^{\alpha_1} \ldots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}} \ldots F_m^{\alpha_m}$. This number is smaller than the corresponding number of n.u.e. factorial effects in the representation of Voss (1986). The relative efficiencies in the estimation of factorial effects of $2^{m_1} \times 3^{m_2}$ deletion designs are also given.

KEY WORDS: Confounding, Factorial experiment, Single replicate, Unbiasedly estimable.

## 1. Introduction

There is a vast literature on the construction of orthogonal single replicate factorial designs in incomplete blocks. The reader is referred to Voss (1986) for the list of references. The concept of deletion designs was introduced in Kishen and Srivastava (1959). The deletion technique in deletion designs was then used by many authors (see Addleman (1962, 1972), Margolin (1969), Sardana and Das (1965), Voss (1986)). This article considers $2^{m_1} \times 3^{m_2}$ deletion designs in three incomplete blocks and then presents a systematic method for finding the u.e. and n.u.e. factorial effects. The smaller values of $m_1$ and $m_2$ are the most practically important cases.

For n.u.e. factorial effects, the biased estimators (biased w.r.t block effects) are called the unadjusted estimators. Under the assumption that certain higher order interactions are negligible, the unbiased estimation of block effects contrasts and n.u.e. factorial effects, excluding the general mean, are possible. This makes the deletion design an orthogonal design. The unbiased estimators of n.u.e. factorial effects under the assumption are called the adjusted estimators.

The relative efficiency in the estimation of a factorial effect is the ratio of the variance of the unadjusted estimator divided by the variance of the adjusted estimator. Observe that for u.e. factorial effects there is no need for adjustment and hence the relative efficiency is unity. For n.u.e. factorial effects the relative efficiency is less than unity. The closer the value of the relative efficiency to unity implies the lesser effect of adjustment to the variance of the estimator.

Definition and notation are given in section 2. Section 3 presents the systematic method of determining u.e. and n.u.e. factorial effects. Section 4 discusses the relative efficiency with an illustrative example. Section 5 presents some miscellaneous results.

## 2. Definition and Notation

Consider a single replicate $2^{m_1} \times 3^{m_2}$ factorial experiment in incomplete blocks. There are m, $m = m_1 + m_2$, factors in the experiment. The runs are denoted by $(x_1,\ldots,x_{m_1}, x_{m_1+1},\ldots,x_m)$, where $x_i = 0,1$, for $i = m_1 + 1,\ldots,m_1$ and $x_i = 0,1,2$, for $i = m_1+1,\ldots,m$. The runs and their effects are denoted by the same notation. The factorial effects are denoted by $F_1^{\alpha_1}\ldots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\ldots F_m^{\alpha_m}$, where $\alpha_i = 0,1$ for $i = 1,\ldots,m_1$ and $\alpha_i = 0,1,2$ for $i = m_1 + 1,\ldots,m$. The observation on the run $(x_1,\ldots x_m)$ is denoted by $y(x_1,\ldots,x_m)$. The fixed effect model assumed is

$$
\begin{aligned}
E(y(x_1,\ldots,x_m)) &= (x_1,\ldots,x_m) + \beta_j,\\
V(y(x_1,\ldots,x_m)) &= \sigma^2,\\
\mathrm{Cov}(y(x_1,\ldots,x_m),\ y(x_1',\ldots,x_m')) &= 0, \qquad (1)
\end{aligned}
$$

where $\beta_j$ is the fixed effect of the jth block containing the run $(x_1,\ldots,x_m)$, $\sigma^2$ and $\beta_j$ $(j = 0,1,2)$ are unknown constants. Recall that the effect of the run $(x_1,\ldots,x_m)$ is denoted by the same notation $(x_1,\ldots,x_m)$. The notation $\{\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = u_1\}$ represents the sum of all $2^{m_1-1}$ points $(x_1,\ldots,x_{m_1})$ which are solutions of $\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = u_1$ over the Galois Field GF(2), $u_1 = 0,1$. Again the notation $\{\alpha_{m_1+1} x_{m_1+1}+\ldots+\alpha_m x_m = u_2\}$ represents the sum of all $3^{m_2-1}$ points

$(x_{m_1+1},\ldots,x_m)$ which are solutions of $\alpha_{m_1+1}x_{m_1+1}+\ldots+\alpha_m x_m = u_2$ over the Galois Field GF(3), $u_2 = 0,1$. The Kronecker product of $\{\alpha_1 x_1+\ldots+\alpha_{m_1}x_{m_1} = u_1\}$ and $\{\alpha_{m_1+1}x_{m_1+1}+\ldots+\alpha_m x_m = u_2\}$ is denoted by $\{\alpha_1 x_1+\ldots+\alpha_{m_1}x_{m_1} = u_1\} \otimes \{\alpha_{m_1+1}x_{m_1+1}+\ldots+\alpha_m x_m = u_2\}$ and it represents the sum of all $2^{m_1-1}\ 3^{m_2-1}$ run effects $(x_1,\ldots,x_{m_1},x_{m_1+1},\ldots,x_m)$ where $(x_1,\ldots,x_{m_1})$ is a solution of $\alpha_1 x_1+\ldots+\alpha_{m_1}x_{m_1} = u_1$ over GF(2) and $(x_{m_1+1},\ldots,x_m)$ is a solution of $\alpha_{m_1+1}x_{m_1+1}+\ldots+\alpha_m x_m = u_2$ over GF(3).

Example 1. Consider a $2^2 \times 3^2$ factorial experiment. We have $m_1 = 2$, $m_2 = 2$ and $m = m_1 + m_2 = 4$. The notation $\{x_1 + x_2 = 0\}$ represents the sum $(0,0) + (1,1)$. The notation $\{x_3 + 2x_4 = 1\}$ represents the sum $(1,0) + (0,2) + (2,1)$. The Kronecker product $\{x_1 + x_2 = 0\} \otimes \{x_3 + 2x_4 = 1\}$ represents the sum of run effects, $(0,0,1,0) + (0,0,0,2) + (0,0,2,1) + (1,1,1,0) + (1,1,0,2) + (1,1,2,1)$.

The factorial effects of a $2^{m_1} \times 3^{m_2}$ factorial experiment are defined in terms of run effects by

$$\begin{aligned} & F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\cdots F_m^{\alpha_m} \\ = & \left[c_0\{\alpha_1 x_1+\ldots+\alpha_{m_1}x_{m_1} = 0\} + c_1\{\alpha_1 x_1+\ldots+\alpha_{m_1}x_{m_1} = 1\}\right] \\ \otimes & \left[d_0\{\alpha_{m_1+1}x_{m_1+1}+\ldots+\alpha_m x_m = 0\} + d_1\{\alpha_{m_1+1}x_{m_1+1}+\ldots+a_m x_m = 1\}\right. \\ & \left. + d_2\{\alpha_{m_1+1}x_{m_1+1}+\ldots+\alpha_m x_m = 2\}\right], \end{aligned} \qquad (2)$$

where the coefficients $c_0$, $c_1$, $d_0$, $d_1$ and $d_2$ are given in Table 1.

Table 1

The coefficients $c_0$, $c_1$, $d_0$, $d_1$ and $d_2$ in the equation (2)

| | $c_0$ | $c_1$ | $d_0$ | $d_1$ | $d_2$ |
|---|---|---|---|---|---|
| $(\alpha_1,\dots,\alpha_{m_1})' = \underline{0}$, $(\alpha_{m_1+1},\dots,\alpha_m)' = \underline{0}$ | 1 | 1 | 1 | 1 | 1 |
| $\neq \underline{0}$ $\quad = \underline{0}$ | -1 | 1 | 1 | 1 | 1 |
| $= \underline{0}$ $\quad \neq \underline{0}$ | | | | | |
| (i) the first nonzero element in $(\alpha_{m_1+1},\dots,\alpha_m)$ is 1. | 1 | 1 | -1 | 0 | 1 |
| (ii) the first nonzero element in $(\alpha_{m_1+1},\dots,\alpha_m)$ is 2. | 1 | 1 | 1 | -2 | 1 |
| $\neq \underline{0}$ $\quad \neq \underline{0}$ | | | | | |
| (i) the first nonzero element in $(\alpha_{m_1+1},\dots,\alpha_m)$ is 1. | -1 | 1 | -1 | 0 | 1 |
| (ii) the first nonzero element in $(\alpha_{m_1+1},\dots,\alpha_m)$ is 2. | -1 | 1 | 1 | -2 | 1 |

Example 2. In Example 1, the factorial effect $F_2F_3^2$ is defined by

$$F_2F_3^2 = [-\{x_2 = 0\} + \{x_2 = 1\}] \otimes [\{x_3 = 0\} -2\ \{x_3 = 1\} + \{x_3 = 2\}]$$

$$= [-(0,0) - (1,0) + (0,1) + (1,1)]$$

$$\otimes\ [(0,0) + (0,1) + (0,2) -2\ (1,0) -2\ (1,1) -2\ (1,2)$$

$$+(2,0) + (2,1) + (2,2)]$$

$$= -\ (0,0,0,0) -\dots+2\ (0,0,1,0) +\dots-\ (0,0,2,0) -\dots+\ (1,1,2,2,).$$

A $2^{m_1} \times 3^{m_2}$ deletion design D in three incomplete blocks is described below. The deletion design D is used throughout the discussion. Consider a $3^m$ factorial experiment in 3 blocks by confounding the two degrees of freedom in $F_1F_2\ldots F_m$ and $F_1^2F_2^2\ldots F_m^2$. The block u consists of runs which are solutions of the equation $x_1+\ldots+x_m = u$, $u = 0,1,2$. From every block, the runs with the level 2 for the first $m_1$ factors are deleted. The resulting design is D with $2^{m_1} \times 3^{m_2-1}$ runs in every block. It is assumed that $m_2 \geq 1$. The design D for $m_2 = 0$ is discussed in Section 5.

Example 3. The runs in the three blocks of a $2^2 \times 3^2$ deletion design D are given below.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Block 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 0 | 1 | 2 | 2 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 2 |
| | 0 | 2 | 1 | 0 | 2 | 1 | 0 | 2 | 1 | 0 | 1 | 2 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Block 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 0 | 2 | 0 | 1 | 2 | 0 | 1 | 2 | 2 | 0 | 1 |
| | 0 | 1 | 2 | 0 | 2 | 1 | 0 | 2 | 1 | 0 | 2 | 1 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Block 2 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 2 | 0 | 1 | 1 | 0 | 2 | 1 | 0 | 2 | 0 | 1 | 2 |
| | 0 | 2 | 1 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 2 | 1 |

It is to be noted that in every block there are 12 runs and the columns represent the runs.

The least squares estimators of u.e. factorial effects $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\cdots F_m^{\alpha_m}$ is $\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\cdots F_m^{\alpha_m}}$ which is obtained by replacing the run effect $(x_1,\ldots,x_m)$ with the observation $y(x_1,\ldots,x_m)$ in (2). For n.u.e. factorial effects, the same method yields biased (non-adjusted) estimators.

Let $B_u (u = 0,1,2)$ be the sum of all run effects in the uth block. Let $X = - B_1 + B_2$ and $Y = 2B_0 - B_1 - B_2$. Clearly X and Y are confounded with the blocks in D. Let $B_u(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = i)$, $i = 0, 1$, $u = 0,1,2,$ denote the sum of all run effects satisfying $\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = i$ in the uth block. Notice that $B_u = B_u(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 0) +$
$B_u(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 1)$.

<u>Example 4</u>. Consider the block 0 in Example 3. Observe that

$$\begin{aligned}B_0(x_1 + x_2 = 0) = {} & (0,0,0,0) + (0,0,1,2) + (0,0,2,1)\\ & + (1,1,0,1) + (1,1,1,0) + (1,1,2,2),\\ B_1(x_1 + x_2 = 1) + {} & (1,0,2,0) + (1,0,0,2) + (1,0,1,1)\\ & + (0,1,2,0) + (0,1,0,2) + (0,1,1,1).\end{aligned}$$

Denote

$$\begin{aligned}F_1^{\alpha}\cdots F_{m_1}^{\alpha_m} X = {} & - [B_1(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 1) - B_1(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 0)]\\ & +[B_2(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 1) - B_2(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 0)],\end{aligned}$$

$$\begin{aligned}F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} Y = {} & 2\ [B_0(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 1) - B_0(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 0)]\\ & -[B_1(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 1) - B_1(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 0)],\\ & -[B_2(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 1) - B_2(\alpha_1 x_1+\ldots+\alpha_{m_1} x_{m_1} = 0)].\end{aligned} \quad (3)$$

## 3. Properties.

In this section the u.e. and n.u.e. factorial effects under D are given. It is assume that $m_2 \geq 1$.

Theorem 1. The factorial effects $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\cdots F_m^{\alpha_m}$ for $(\alpha_{m_1+1},\ldots,\alpha_m) \neq \alpha\,(1,\ldots,1)$, $\alpha = 1,2$ and $(\alpha_1,\ldots,\alpha_m) \neq (0,\ldots,0)$, are u.e. under D.

Proof. When $(\alpha_{m_1+1},\ldots,\alpha_m) \neq \alpha\,(1,\ldots,1)$ and $(\alpha_1,\ldots,\alpha_m) \neq (0,\ldots,0)$, it can be seen that $2^{m_1}\,3^{m_2-1}$ runs in a block can be divided into six sets of $2^{m_1-1}\,3^{m_2-2}$ runs satisfying $\alpha_1 x_1+\cdots+\alpha_{m_1} x_{m_1} = u_1$ and $\alpha_{m_1+1}x_{m_1+1}+\cdots+\alpha_m x_m = u_2$, $u_1 = 0,1$ and $u_2 = 0,1,2$. It now follows from (1) and (2) that in $E(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\cdots F_m^{\alpha_m}})$, the block effects cancel and it becomes equal to $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^{\alpha_{m_1+1}}\cdots F_m^{\alpha_m}$. This completes the proof.

Example 5. In Example 3, the factorial effects $F_1$, $F_2$, $F_1F_2$, $F_3$, $F_3^2$, $F_4$, $F_4^2$, $F_3F_4^2$, $F_3^2F_4$, $F_1F_3$, $F_1F_3^2$, $F_1F_4$, $F_1F_4^2$, $F_1F_3F_4^2$, $F_1F_3^2F_4$, $F_2F_3$, $F_2F_3^2$, $F_2F_4$, $F_2F_4^2$, $F_2F_3F_4^2$, $F_2F_3^2F_4$, $F_1F_2F_3$, $F_1F_2F_3^2$, $F_1F_2F_4$, $F_1F_2F_4^2$, $F_1F_2F_3F_4^2$, $F_1F_2F_3^2F_4$ are u.e. under D by Theorem 1.

Theorem 2. The factorial effects $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}\cdots F_m$ and $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}} F_{m_1+1}^2\cdots F_m^2$ are n.u.e. under D (i.e., they are confounded with blocks in D).

Proof. Consider the uth $(u = 0,1,2)$ block in D. Out of $2^{m_1}3^{m_2-1}$ runs in the uth block, $2^{m_1-1}3^{m_2-1}$ runs satisfy $x_1+\cdots+x_{m_1} = 0$ over GF(2) and

$x_{m_1+1}+\dots+x_m = u$ over GF(3). The remaining $2^{m_1-1}3^{m_2-1}$ runs satisfy $x_1+\dots+x_{m_1} = 1$ over GF(2) and $x_{m_1+1}+\dots+x_m = u-1$ over GF(3). Out of $2^{m_1-1}3^{m_2-1}$ runs satisfying $x_1+\dots+x_{m_1} = i$, $i = 0,1$, $2^{m_1-2}3^{m_2-1}$ runs satisfy $\alpha_1x_1+\dots+\alpha_{m_1}x_{m_1} = j$, $j = 0,1$, and $(\alpha_1,\dots,\alpha_{m_1}) \neq (1,\dots,1)$, $(0,\dots,0)$. It is now clear from the definition (2) of $F_1^{\alpha_1}\dots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha_{m_1+1}}\dots F_m^{\alpha_m}$ with $\alpha_{m_1+1}=\dots=\alpha_m = \alpha$, $\alpha = 1,2$, that in

$E(F_1^{\alpha_1}\dots \widehat{F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha_{m_1+1}}}\dots F_m^{\alpha_m})$ the block effects do not cancel. This completes the proof.

<u>Example 6</u>. In Example 3, the factorial effects $F_3F_4$, $F_3^2F_4^2$, $F_1F_3F_4$, $F_2F_3F_4$, $F_1F_2F_3F_4$, $F_1F_3^2F_4^2$, $F_2F_3^2F_4^2$ and $F_1F_2F_3^2F_4^2$ are not u.e. in addition to the general mean $\mu$.

<u>Theorem 3</u>. Under D, $F_1^{\alpha_1}\dots F_{m_1}^{\alpha_{m_1}}X$ and $F_1^{\alpha_1}\dots F_{m_1}^{\alpha_{m_1}}Y$ with $(\alpha_1,\dots,\alpha_{m_1}) \neq (0,\dots,0)$, defined in (3) are u.e.

Proof. In the uth $(u = 0,1,2)$ block of D, $2^{m_1}3^{m_2-1}$ runs can be divided into 2 sets of $2^{m_1-1}3^{m_2-1}$ runs each satisfying $\alpha_1x_1+\dots+\alpha_m x_m = i$, $i = 0,1$. It now follows from (1) that in $E(\widehat{F_1^{\alpha_1}\dots F_{m_1}^{\alpha_{m_1}}}X)$ and $E(\widehat{F_1^{\alpha_1}\dots F_{m_1}^{\alpha_{m_1}}}Y)$ the block effects cancel. The rest is clear. This completes the proof.

Observe that $\mu$, X, Y are confounded with blocks in D. The $(2^{m_1}(3^{m_1}-2)-1)$ factorial effects $F_1^{\alpha_1}\dots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha_{m_1+1}}\dots F_m^{\alpha_m}$ with $(\alpha_{m_1+1},\dots,\alpha_m) \neq \alpha(1,\dots,1)$, $\alpha = 1,2$ and $(\alpha_1,\dots,\alpha_m) \neq (0,\dots,0)$, are u.e. under D. The $(2^{m_1}-1)2$ linear functions of factorial effects $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}X$ and $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}Y$ with $(\alpha_1,\ldots,\alpha_{m_1}) \neq (0,\ldots,0)$, are u.e. under D. The above $[3 + (2^{m_1}(3^{m_2}-2)-1) + (2^{m_1}-1)2] = 2^{m_1}3^{m_2}$ linear functions of factorial effects are othogonal to each other.

## 4. Relative Efficiency

In this section the relative efficiences of n.u.e. factorial effects are calculated. First note that

$$E(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}}) = F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha} + (d_0\beta_0+d_1\beta_1+d_2\beta_2), \quad (4)$$

where $d_0$, $d_1$ and $d_2$ depends on the values of $\alpha_i$, $i = 1,\ldots,m_1$ and $\alpha$, $\alpha = 1,2$. The estimator $\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}}$ *is called the unadjusted* estimator of $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}$ and it is denoted by $(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}})_{unadj}$. It can be checked that

$$\mathrm{Var}(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}})_{unadj} = \begin{cases} \sigma^2 2^{m_1+1}3^{m_2-1} & \text{for } \alpha = 1, \\ \sigma^2 2^{m_1+1}3^{m_2} & \text{for } \alpha = 2. \end{cases} \quad (5)$$

It can be seen that out of $2^{m_1-1}$ points $(x_1,\ldots,x_{m_1})$ satisfying $x_1+\cdots+x_{m_1} = 0$ over GF(2), $n_{ou}$ points satisfy $x_1+\cdots+x_{m_1} = u$, $u = 0,1,2,$ over GF(3). Again, out of $2^{m_1-1}$ points $(x_1,\ldots,x_{m_1})$ satisfying $x_1+\cdots+x_{m_1} = 1$ over GF(2), $n_{1u}$ points satisfy $x_1+\cdots+x_{m_1} = u$, $u = 0,1,2,$ over GF(3). Clearly, $n_{00} + n_{01} + n_{02} = n_{10} + n_{11} + n_{12} = 2^{m_1-1}$. It can be check that

$$n_{00} = \sum_{\substack{w \geq 0 \\ w \text{ even integer}}} \binom{m_1}{3w}, \quad n_{01} = \sum_{\substack{w \geq 0 \\ w \text{ odd integer}}} \binom{m_1}{3w+1},$$

$$n_{02} = \sum_{\substack{w \geq 0 \\ w \text{ even integer}}} \binom{m_1}{3w+2}, \quad n_{10} = \sum_{\substack{w \geq 0 \\ w \text{ odd integer}}} \binom{m_1}{3w},$$

$$n_{11} = \sum_{\substack{w \geq 0 \\ w \text{ even integer}}} \binom{m_1}{3w+1}, \quad n_{12} = \sum_{\substack{w \geq 0 \\ w \text{ odd integer}}} \binom{m_1}{3w+2}. \tag{6}$$

Under the assumption that the factorial effects $F_1 \cdots F_{m_1} F^{\alpha}_{m_1+1} \cdots F^{\alpha}_m$, $\alpha = 1,2$, are negligible, it follows that

$$E(\widehat{F_1 \cdots F_{m_1} F_{m_1+1} \cdots F_m})_{unadj} = 3^{m_2-1} [(n_{10}-n_{12}-n_{00}+n_{02})\beta_2$$

$$+ (n_{12}-n_{11}-n_{02}+n_{01})\beta_1 + (n_{11}-n_{10}-n_{01}+n_{00})\beta_0],$$

$$E(\widehat{F_1 \cdots F_{m_1} F^2_{m_1+1} \cdots F^2_m})_{unadj} = 3^{m_2-1} [(n_{10}-2n_{11}+n_{12}-n_{00}+2n_{01}-n_{02})\beta_2$$

$$+ (n_{12}-2n_{10}+n_{11}-n_{02}+2n_{00}-n_{01})\beta_1 + (n_{11}-2n_{12}+n_{10}-n_{01}+2n_{02}-n_{00})\beta\ ]. \tag{7}$$

For $(\alpha_1, \ldots, \alpha_{m_1}) \neq (1, \ldots, 1)$, the adjusted estimators of factorial effects $F_1^{\alpha_1} \cdots F_{m_1}^{\alpha_{m_1}} F^{\alpha}_{m_1+1} \cdots F^{\alpha}_m$ are

$$(\widehat{F_1^{\alpha_1} \cdots F_{m_1}^{\alpha_{m_1}} F^{\alpha}_{m_1+1} \cdots F^{\alpha}_m})_{adj} = (\widehat{F_1^{\alpha_1} \cdots F_{m_1}^{\alpha_{m_1}} F^{\alpha}_{m_1+1} \cdots F^{\alpha}_m})_{unadj}$$

$$+ w_1(\widehat{F_1 \cdots F_{m_1} F_{m_1+1} \cdots F_m})_{unadj} + w_2(\widehat{F_1 \cdots F_{m_1} F^2_{m_1+1} \cdots F^2_m})_{unadj}, \tag{8}$$

where $w_1$ and $w_2$ are constants depending on $\alpha$ and $(\alpha_1, \ldots, \alpha_{m_1})$. Notice that under the assumption that $F_1 \cdots F_{m_1} F^{\alpha}_{m_1+1} \cdots F^{\alpha}_m$ are negligible, the factorial effects $F_1^{\alpha_1} \cdots F_{m_1}^{\alpha_{m_1}} F^{\alpha}_{m_1+1} \cdots F^{\alpha}_m$, $(\alpha_1, \ldots, \alpha_{m_1}) \neq (1, \ldots, 1)$,

$\alpha = 1,2$, are u.e. and the adjusted estimators of $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}$, $(\alpha_1,\ldots,\alpha_{m_1}) \neq (1,\ldots,1)$, $\alpha = 1,2$, are in fact unbiased estimators. The unbiased estimators of factorial effects (except the general mean) are orthogonal to each other and hence the deletion design is orthogonal under the assumption that $F_1\cdots F_{m_1}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}$, $\alpha = 1,2$, are negligible. The effect of adjustment is now evaluated in terms of the variance of the estimators. It cn be seen from (8) that for $(\alpha_1,\ldots,\alpha_{m_1}) \neq (1,\ldots,1)$,

$$V(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}})_{adj} = \begin{cases} \sigma^2 2^{m_1+1}3^{m_2-1}(1 + w_1^2 + 3w_2^2) & \text{for } \alpha = 1, \\ \sigma^2 2^{m_1+1}3^{m_2-1}(3 + w_1^2 + 3w_2^2) & \text{for } \alpha = 2. \end{cases} \tag{9}$$

The relative efficiency in the estimation of $F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}$, $(\alpha_1,\ldots,\alpha_{m_1}) \neq (1,\ldots,1)$ is

$$RE = \frac{V(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}})_{unadj}}{V(\widehat{F_1^{\alpha_1}\cdots F_{m_1}^{\alpha_{m_1}}F_{m_1+1}^{\alpha}\cdots F_m^{\alpha}})_{adj.}} = \begin{cases} \dfrac{1}{1 + w_1^2 + 3w_2^2} & \text{for } \alpha = 1, \\[2ex] \dfrac{3}{3 + w_1^2 + 3w_2^2} & \text{for } \alpha = 2. \end{cases} \tag{10}$$

Notice that $0 < RE \leq 1$. For u.e. factorial effects RE = 1 and for n.u.e. factorial effects RE < 1. Further the value of E away from 1 the more is the effect of the adjustment to the variance of the estimator.

Example 7. In Example 3, $m_1$ equals to 2 and moreover, $n_{00} = \binom{2}{0} = 1$, $n_{01} = 0$, $n_{02} = \binom{2}{2} = 1$, $n_{10} = 0$, $n_{11} = \binom{2}{1} = 2$ and $n_{12} = 0$. Under the assumption that $F_1F_2F_3F_4$ and $F_1F_2F_3^2F_4^2$ are negligible, it follows from (7) that

$$E(\widehat{F_1F_2F_3F_4})_{unadj} = 9\,(-\,\beta_1 + \beta_0),$$

$$E(\widehat{F_1F_2F_3^2F_4^2})_{unadj} = 9(-\,2\beta_2 + \beta_1 + \beta_0).$$

It can be seen that

$$E(\widehat{F_1F_3F_4})_{unadj} = F_1F_3F_4 + 3(-\,2\beta_2 + \beta_1 + \beta_0).$$

Thus

$$(\widehat{F_1F_2F_4})_{adj} = (\widehat{F_1F_3F_4})_{unadj} - \frac{1}{3}\,(\widehat{F_1F_2F_3^2F_4^2})_{unadj}.$$

Therefore, from (8), $\alpha = 1$, $w_2 = -\frac{1}{3}$ and $w_1 = 0$. Hence from (10),

$$RE = \frac{1}{1+3(\frac{1}{3})^2} = \frac{3}{4} = .75.$$

Table 2 presents the values of $w_1$, $w_2$ and the relative efficiencies for factorial effects. It is to be noted that the relative efficiences for all 6 factorial effects are more than .75 and therefore the adjustments do not have large effects on the variances of the estimators. The deletion design with such high relative efficiencies can be considered as a near orthogonal design.

Table 2

Efficiencies for $2^2 \times 3^2$ deletion designs

| Factorial Effects | $\alpha$ | $w_1$ | $w_2$ | RE |
|---|---|---|---|---|
| $F_3F_4$ | 1 | $-\frac{1}{3}$ | 0 | .90 |
| $F_3^2F_4^2$ | 2 | 0 | $-\frac{1}{3}$ | .90 |
| $F_1F_3F_4$ | 1 | 0 | $-\frac{1}{3}$ | .75 |
| $F_2F_3F_4$ | 1 | 0 | $-\frac{1}{3}$ | .75 |
| $F_1F_3^2F_4^2$ | 2 | 1 | 0 | .75 |
| $F_2F_3^2F_4^2$ | 2 | 1 | 0 | .75 |

## 5. Miscellaneous Results

In this section the case $m_2 = 0$ i.e., $m_1 = m$ is considered for the sake of completeness. The u.e. and n.u.e. factorial effects for a $2^m$ deletion design are displayed. It is a feeling that the deletion design for the case $m_2 = 0$ is of lesser practical importance than the deletion designs for the case $m_2 > 0$.

Theorem 4. Under a $2^m$ deletion design D, the factorial effects $F_1^{\alpha_1} \ldots F_m^{\alpha_m}$ for all $\alpha_1, \ldots, \alpha_m$ are not u.e.

Proof. First observe that three blocks in D can not be of equal sizes and therefore the block sizes can not all be even. The rest is clear from the definition of $F_1^{\alpha_1} \ldots F_m^{\alpha_m}$. This completes the proof.

Denote the number of nonzero elements in a vector $(\alpha_1, \ldots, \alpha_m)$ by $W(\alpha_1, \ldots, \alpha_m)$. For $w = 0, 1, \ldots, m$, denote

$$A_w = \{F_1^{\alpha_1} \ldots F_m^{\alpha_m};\ W(\alpha_1, \ldots, \alpha_m) = w\}. \tag{11}$$

Notice that $A_0$ consists of the general mean, $A_1$ consists of all main effects, $A_2$ consists of all two factor interactions and so on.

Theorem 5. For a $w$ $(\neq 0, m)$ all contrasts of the elements in $A_w$ are u.e.

Proof. Consider two vectors $(\alpha_1, \ldots, \alpha_m)$ and $(\alpha_1^*, \ldots, \alpha_m^*)$ so that $W(\alpha_1, \ldots, \alpha_m) = W(\alpha_1^*, \ldots, \alpha_m^*) = w$ $(\neq 0)$. It can now be seen that in every block, the number of runs satisfying $\alpha_1 x_1 + \ldots + \alpha_m x_m = u$ is exactly identical to the number of runs satisfying $\alpha_1^* x_1 + \ldots + \alpha_m^* x_m = u$ for $u = 0, 1$. The rest is clear from the definition of factorial effects and the model (1).

Example 8. The three blocks in a $2^4$ deletion design are given below.

Block 0

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 |

Block 1

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 |

Block 2

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 |

Notice that the Blocks 0 and 1 are of the same size 5 and the Block 2 is of the size 6. For the set $A_1 = \{F_1, F_2, F_3, F_3^2, F_4, F_4^2\}$, it follows from Theorems 4 and 5 that all the elements in $A_1$ are n.u.e. but every contrast of elements in $A_1$ is u.e.

Theorem 6.

(a) For a w, $\sum_{A_w} F_1^{\alpha_1} \cdots F_m^{\alpha_m}$ is n.u.e. under D.

(b) The linear function of factorial effects $c_0B_0 + c_1B_1 + c_2B_2$ with $c_0 + c_1 + c_2 = 0$ is n.u.e. under D.

(c) For a $w(\neq 0, m)$,

$$\sum_{A_w} F_1^{\alpha_1} \cdots F_m^{\alpha_m} + (c_0B_0 + c_1B_1 + c_2B_2)$$

with $c_0 + c_1 + c_2 = 0$, is u.e. under D.

Proof. The part (a) can be seen from Theorems 4 and 5. The part (b) is obvious. The part (c) follows from the block structures in D. This completes the proof.

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