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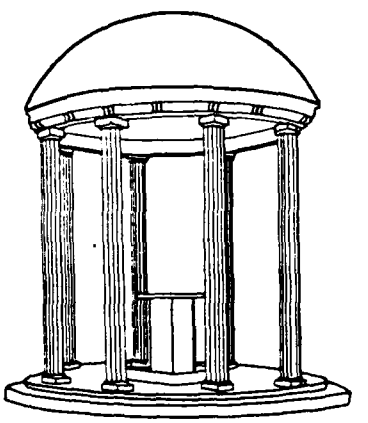
Sensitivity Analysis Using the Monte Carlo Acceptance-Rejection Method

George S. Fishman

Technical Report No. UNC/OR/TR–88/3

September 1988

UNIVERSITY OF NORTH CAROLINA
AT CHAPEL HILL



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Abstract

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The paper derives the variances of the proposed estimators and shows how to use worst case bounds on these or on corresponding coefficients of variation to choose the arguments, at which to sample, that minimize the worst case bounds. Individual and simultaneous confidence intervals are derived and an example based on s-t reliability illustrates the method. The paper also compares the proposed method and an alternative Monte Carlo approach that uses an importance function.

Abstract

This paper describes a Monte Carlo sampling plan for estimating how a function varies in response to changes in its arguments. Most notably, the plan effects this sensitivity analysis by applying the acceptance-rejection technique to data sampled at only one specified setting for the arguments, thus saving considerable computing time when compared to alternative methods. The plan which applies for a 0-1 response on each replication has immediate application when estimating variation in system performance measures in reliability analysis.

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Key Words: acceptance-rejection sampling, Monte Carlo method sensitivity analysis, importance function, reliability

Many Monte Carlo sampling experiments aim at estimating quantities of the form

$$g(\boldsymbol{q}) = \sum_{\boldsymbol{x} \in \mathcal{X}} \phi(\boldsymbol{x}) P(\boldsymbol{x}, \boldsymbol{q}) \tag{1}$$

where $\{\phi(\boldsymbol{x})\}$ is a 0-1 binary function, $\{P(\boldsymbol{x}, \boldsymbol{q}),\ \boldsymbol{x} \in \mathcal{X}\}$ is a probability mass function (p.m.f.) with given parameter vector $\boldsymbol{q}$, and domain of support $\mathcal{X}$ so large as to make exact evaluation via (1) intractable. Occasionally, the objective is to estimate the function $\{g(\boldsymbol{q}),\ \boldsymbol{q} \in \mathcal{Q}\}$ where $\mathcal{Q} = \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_w\}$. Problems of this type arise in reliability theory where $g(\boldsymbol{q})$ represents system reliability and $\boldsymbol{q}_j = (q_{j1}, \ldots, q_{jr})$ denotes the reliabilities of components of types 1 through r which compose the system in the jth of w component reliability vectors of interest. Analysis of $g(\boldsymbol{q}_1), \ldots, g(\boldsymbol{q}_w)$ enables one to assess the benefits of the alternative reliabilities vectors $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_w$ on system reliability.

Although one can simply run w experiments, sampling from $\{P(\boldsymbol{x}, \boldsymbol{q}_1)\}, \ldots, \{P(\boldsymbol{x}, \boldsymbol{q}_w)\}$ to produce estimates of $g(\boldsymbol{q}_1), \ldots, g(\boldsymbol{q}_w)$, respectively, a more efficient method samples from $\{P(\boldsymbol{x}, \boldsymbol{p})\}$ on a single experiment and uses these data together with the Monte Carlo importance function technique or the acceptance-rejection technique to produce the desired estimates. These approaches are not new, the importance function technique being implicit in Kahn (1950) and Kahn and Harris (1949) and the acceptance-rejection technique being implicit in von Neumann (1949). Beckman and McKay (1987) have more recently discussed both methods. However, until recently little was known about how the binary property of $\{\phi(\boldsymbol{x})\}$ affected the sampling properties of these techniques for estimating (1). Fishman (1987) provides a comprehensive account of these properties for the importance function approach. The present paper focuses on the acceptance-rejection method and provides a comprehensive description of the sampling properties of the resulting estimators that exploit the binary property of $\{\phi(\boldsymbol{x})\}$ and the use of a modified

p.m.f. $\{Q(\boldsymbol{x},\boldsymbol{p})\}$, based on $\{P(\boldsymbol{x},\boldsymbol{p})\}$ and information on bounds for $\{\phi(\boldsymbol{x})\}$ and $\{g(\boldsymbol{q}),\ \boldsymbol{q}\in\mathcal{Q}\}$, to sample the data. This last modification allows the acceptance-rejection method to work with considerably improved efficiency. Although the paper focuses on applying the proposed technique to reliability estimation, we emphasize that the methodology applies to the considerably wider class of problems with binary $\{\phi(\boldsymbol{x})\}$.

Section 1 gives basic definitions and Section 2 describes estimation at a single point. Section 3 then describes how to perform function estimation using the acceptance-rejection method. Section 4 shows how to choose the design parameter $\boldsymbol{p}$ to minimize either the worst-case variance or the coefficient of variation of the resulting function estimator, thereby dramatically increasing the efficiency of the proposed Monte Carlo procedure. Sections 5 and 6 show that even in the worst case, the proposed technique is at least as good as crude Monte Carlo sampling. Sections 7 and 8 derive individual and simultaneous confidence intervals. Section 9 illustrates the proposed technique with an example and Section 10 compares the characteristics of the acceptance-rejection method with those of the importance function method.

1. Problem Setting

Consider a network $G = (\mathcal{V},\mathcal{E})$ with node set $\mathcal{V}$ and edge set $\mathcal{E}$. Assume that nodes function perfectly and that edges fail randomly and independently. Let

r = number of distinct types of edges

q_i = probability that an edge of type i functions $i = 1,\ldots,r$

$\boldsymbol{q} = (q_1,\ldots,q_r)$

k_i = number of edges of type i

$\boldsymbol{k} = (k_1,\ldots,k_r)$

e_{ij} = jth edge of type i $\quad j = 1,\ldots,k_i;\ \ i = 1,\ldots,r$

$x_{ij} = 1$ if edge e_{ij} functions

$\quad = 0$ otherwise

$$x_i = \sum_{j=1}^{k_i} x_{ij} = \text{number of functioning edges of type } i$$

$$\boldsymbol{x} = (x_{11},\ldots,x_{1k_1};\ldots;x_{r1},\ldots,x_{rk_r})$$

$\mathcal{X}$ = set of all edge states $\boldsymbol{x}$

$$P(\boldsymbol{x},\boldsymbol{q}) = P(\boldsymbol{x},\boldsymbol{k},\boldsymbol{q}) = \prod_{i=1}^{r}\prod_{j=1}^{k_i}[x_{ij}q_i+(1-x_{ij})(1-q_i)] = \prod_{i=1}^{r} q_i^{x_i}(1-q_i)^{k_i-x_i} \tag{2}$$

= p.m.f. of state $\boldsymbol{x}\in\mathcal{X}$

$\phi(\boldsymbol{x}) = 1$ if the system functions when in state $\boldsymbol{x}$

$= 0$ otherwise

$$g(\boldsymbol{q}) = \sum_{\boldsymbol{x}\in\mathcal{X}} \phi(\boldsymbol{x})\, P(\boldsymbol{x},\boldsymbol{q}) \tag{3}$$

= probability that the system functions.

We also assume that G describes a *coherent* system. A system of components is coherent if its structure function $\{\phi(\boldsymbol{x})\}$ is nondecreasing in each argument and each component is relevant (Barlow and Proschan 1981, p. 6).

Let $\mathcal{Q}$ denote a set of w component reliability vectors of interest. Then the purpose of analysis is to estimate the s-t reliability function $\{g(\boldsymbol{q}),\ \boldsymbol{q}\in\mathcal{Q}\}$.

2. Estimation at a Point

Crude Monte Carlo sampling offers a baseline against which potentially more efficient sampling plans can be compared. Let $\boldsymbol{X}^{(1)},\ldots,\boldsymbol{X}^{(K)}$ denote K independent samples drawn from $\{P(\boldsymbol{x},\boldsymbol{q}),\ \boldsymbol{x}\in\mathcal{X}\}$. Then

$$\bar{g}_K(\boldsymbol{q}) = \frac{1}{K}\sum_{i=1}^{K}\phi(\boldsymbol{X}^{(i)}) \tag{4}$$

is an unbiased estimator of $g(\boldsymbol{q})$ with

$$\text{var } \bar{g}_K(q) = g(q)[1-g(q)]/K. \tag{5}$$

To compute $\bar{g}_K(q)$, one performs K trials sampling X from $\{P(x,k,q)\}$ and evaluates $\phi(X)$ on each trial. The corresponding mean total computation time has the form

$$T(\bar{g}_K(q)) = \alpha_0 + K[\alpha_1+\alpha_2|\mathcal{S}|+\alpha_3(\mathcal{X}, q)]$$

where

$$\alpha_3(\mathcal{X}, q) = \sum_{x\in\mathcal{S}} P(x,q)\ C(x)$$

and

$$C(x) = \text{expected time to evaluate } \phi(x).$$

The quantities α_0, α_1, α_2 and $\alpha_3(\mathcal{X}, q)$ are machine dependent.

We now show how to modify the sampling plan to improve statistical efficiency using information on bounds as described in Fishman (1986). Suppose that there exist 0-1 binary functions $\{\phi_L(x), x\in\mathcal{X}\}$ and $\{\phi_U(x), x\in\mathcal{X}\}$ such that

$$\phi_L(x) \le \phi(x) \le \phi_U(x) \qquad \forall\ x\in\mathcal{X}$$

Then $g(q)$ has lower and upper bounds $g_L(q)$ and $g_U(q)$, respectively, where

$$g_i(q) = \sum_{x\in\mathcal{S}} \phi_i(x)\ P(x,q) \qquad i\in\{L,U\}.$$

Suppose that one now samples $X^{(1)},\ldots,X^{(K)}$ independently from the modified p.m.f.

$$Q(x,q) = \left[\frac{\phi_U(x)-\phi_L(x)}{\Delta(q)}\right] P(x,q) \qquad x \in \mathcal{X} \tag{6}$$

where

$$\Delta(q) = g_U(q) - g_L(q).$$

Then

$$\hat{g}_K(q) = g_L(q) + \Delta(q)\frac{1}{K}\sum_{i=1}^{K}\phi(X^{(i)}) \tag{7}$$

is also an unbiased estimator of $g(q)$, but with variance

$$\text{var } \hat{g}_K(q) = [g_U(q)-g(q)][g(q)-g_L(q)]/K \le \Delta^2(q)/4K. \tag{8}$$

Compared to crude Monte Carlo sampling, one has

$$\frac{\text{var } \bar{g}_K(q)}{\text{var } \hat{g}_K(q)} \ge D(q) = 1/\left[\{g_L(q)[1-g_U(q)]\}^{\frac{1}{2}}-\{g_U(q)[1-g_L(q)]\}^{\frac{1}{2}}\right]^2 \tag{9}$$

$$\ge 1,$$

indicating that $\hat{g}_K(q)$ always has a variance no larger that var $\bar{g}_K(q)$.

To compute $\hat{g}_K(q)$ using precomputed bounds, one performs K trials sampling X from $\{Q(x,q)\}$ and evaluates $\phi(X)$ on each trial Here mean total time assumes the form

$$T(\hat{g}_K(\boldsymbol{q})) = \beta_0 + K[\beta_1+\beta_2|\mathcal{E}|+\alpha_3(\mathcal{X}_{01},\boldsymbol{p})/\Delta(\boldsymbol{q})]$$

where

$$\mathcal{X}_{01} = \{\boldsymbol{x}\in\mathcal{X}\colon \phi_L(\boldsymbol{x})=0 \text{ and } \phi_U(\boldsymbol{x})=1\}$$

and β_0,β_1 and β_2 denote machine dependent constants.

Observe that

$$K(\boldsymbol{q}) = K \text{ var } \bar{g}_K(\boldsymbol{q})/\text{var } \hat{g}_K(\boldsymbol{q})$$

denotes the number of observations one would have to take with crude Monte Carlo to achieve the same variance that arises in K observations using $\{Q(\boldsymbol{x},\boldsymbol{p})\}$. Then $\Lambda_1(\boldsymbol{q}) = T(\bar{g}_{K(\boldsymbol{q})}(\boldsymbol{q}))/T(\hat{g}_K(\boldsymbol{q}))$ measures the efficiency of $\hat{g}_K(\boldsymbol{q})$ relative to $\bar{g}_K(\boldsymbol{q})$ and for large K and $|\mathcal{E}|$ has the approximate form

$$\Lambda_1(\boldsymbol{q}) \approx \left[\frac{\alpha_2+\alpha_3(\mathcal{X},\boldsymbol{q})/|\mathcal{E}|}{\beta_2+\alpha_3(\mathcal{X}_{01},\boldsymbol{q})/\Delta(\boldsymbol{q})|\mathcal{E}|}\right]\frac{g(\boldsymbol{q})[1-g(\boldsymbol{q})]}{[g_U(\boldsymbol{q})-g(\boldsymbol{q})][g(\boldsymbol{q})-g_L(\boldsymbol{q})]}. \tag{10}$$

$$\geq \left[\frac{\alpha_2+\alpha_3(\mathcal{X},\boldsymbol{q})/|\mathcal{E}|}{\beta_2+\alpha_3(\mathcal{X}_{01},\boldsymbol{q})/\Delta(\boldsymbol{q})|\mathcal{E}|}\right] D(\boldsymbol{q})$$

where (9) defines $D(\boldsymbol{q}) \geq 1$. A ratio greater than unity favors the alternative sampling plan. Experience (Fishman 1986) has shown this to be the case for moderate and high component reliabilities for s-t reliability.

3. Function Estimation Based on the Acceptance–Rejection Method

To estimate $g(\boldsymbol{q})$ for each component reliability vector $\boldsymbol{q} \in \mathcal{Q} = \{\boldsymbol{q}_1,...,\boldsymbol{q}_w\}$, one can perform w separate experiments, sampling from $\{Q(\boldsymbol{x},\boldsymbol{q}_i)\}$ in (6) on the ith experiment for $i = 1,...,w$. This procedure incurs the cost of running w individual sampling experiments. However, one can actually avoid this cost by performing a single experiment, sampling data from $\{Q(\boldsymbol{x},\boldsymbol{p})\}$ and then using these same data to estimate $g(\boldsymbol{q}_1),...,g(\boldsymbol{q}_w)$. We later show that if the component reliability vector $\boldsymbol{p}$ at which sampling occurs belongs to $\mathcal{Q}$, the proposed approach leads to estimates of specified accuracy at a cost no larger than that incurred by performing all w individual experiments to achieve the identical accuracies.

Consider the p.m.f.

$$f(\boldsymbol{x}) = ab(\boldsymbol{x})\ c(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathcal{X} \tag{11}$$

where

$$c(\boldsymbol{x}) \geq 0,\ \sum_{\boldsymbol{x}\in\mathcal{X}} c(\boldsymbol{x}) = 1,\ 0 \leq b(\boldsymbol{x}) \leq 1 \text{ and } a = 1/\sum_{\boldsymbol{x}\in\mathcal{X}} b(\boldsymbol{x})\ c(\boldsymbol{x}).$$

Suppose one samples $\boldsymbol{X}$ from the p.m.f. $\{c(\boldsymbol{x})\}$ and Z from $\mathcal{U}(0,1)$. If $Z \leq b(\boldsymbol{X})$, then $\boldsymbol{X}$ has the p.m.f. $\{f(\boldsymbol{x})\}$ in (11). This acceptance-rejection method of sampling is due to von Neumann (1949). For the current problem,

$$c(\boldsymbol{x}) = Q(\boldsymbol{x},\boldsymbol{p})$$

$$b(\boldsymbol{x}) = R(\boldsymbol{x},\boldsymbol{q},\boldsymbol{p})/R^*(\boldsymbol{q},\boldsymbol{p}), \tag{12}$$

where

$$R(\boldsymbol{x},\boldsymbol{q},\boldsymbol{p}) = P(\boldsymbol{x},\boldsymbol{q})/P(\boldsymbol{x},\boldsymbol{p})$$

$$= \prod_{i=1}^{r} (q_i/p_i)^{x_i} \left[(1-q_i)/(1-p_i)\right]^{k_i-x_i} \tag{13}$$

and

$$R^*(\boldsymbol{q},\boldsymbol{p}) = \max_{\boldsymbol{x}\in\mathcal{S}} R(\boldsymbol{x},\boldsymbol{q},\boldsymbol{p}) \quad \text{s.t. } \phi_L(\boldsymbol{x})=0 \text{ and } \phi_U(\boldsymbol{x})=1.$$

The quantity

$$a = \Delta(\boldsymbol{p})\, R^*(\boldsymbol{k},\boldsymbol{q},\boldsymbol{p})/\Delta(\boldsymbol{q}) \tag{14}$$

denotes the mean number of trials required until one successfully obtains an $\boldsymbol{X}$ from $\{Q(\boldsymbol{x},\boldsymbol{q})\}$.

A small modification increases the efficiency of this procedure. Let

$$R_0(\boldsymbol{q},\boldsymbol{p}) = \max_{\boldsymbol{x}\in\mathcal{S}} R(\boldsymbol{x},\boldsymbol{q},\boldsymbol{p}) \quad \text{s.t. } \phi(\boldsymbol{x})=0 \text{ and } \phi_U(\boldsymbol{x})=1 \tag{15a}$$

$$R_1(\boldsymbol{q},\boldsymbol{p}) = \max_{\boldsymbol{x}\in\mathcal{S}} R(\boldsymbol{x},\boldsymbol{q},\boldsymbol{p}) \quad \text{s.t. } \phi_L(\boldsymbol{x})=0 \text{ and } \phi(\boldsymbol{x})=1 \tag{15b}$$

and

$$F(\boldsymbol{x},i,\boldsymbol{q},\boldsymbol{p}) = R(\boldsymbol{x},\boldsymbol{q},\boldsymbol{p})/R_i(\boldsymbol{q},\boldsymbol{p}) \qquad i = 0,1. \tag{16}$$

Suppose one samples $\boldsymbol{X}$ from $\{Q(\boldsymbol{x},\boldsymbol{p})\}$, samples Z from $\mathcal{U}(0,1)$ and determines $\phi(\boldsymbol{X})$. If $Z \le F(\boldsymbol{X},\phi(\boldsymbol{X}),\boldsymbol{q},\boldsymbol{p})$, then $\boldsymbol{X}$ has the p.m.f. $\{Q(\boldsymbol{x},\boldsymbol{q})\}$ with mean number of trials until

success

$$a = \frac{g_U(p) - g(p)}{\Delta(q)} R_0(q,p) + \frac{g(p) - g_L(p)}{\Delta(q)} R_1(q,p)$$

$$\leq \Delta(p)\, R^*(q,p)/\Delta(q),$$

since $\max [g_U(p) - g(p),\ g(p) - g_L(p)] \leq \Delta(p)$ and $\max [R_0(q,p),\ R_1(q,p)] \leq R^*(q,p)$. The computations of $R_0(q,p)$ and $R_1(q,p)$ depend on the choice of bounding functions $\{\phi_L(x)\}$ and $\{\phi_U(x)\}$ and are discussed in the example in Section 9.

We next describe the statistical properties of data generated by the acceptance-rejection method.

Theorem 1. Let X and Z denote samples drawn from $\{Q(x,p)\}$ in (6) and $\mathcal{U}(0,1)$ respectively. Define $R_0 \equiv R_0(q,p)$, $R_1 \equiv R_1(q,p)$,

$$\varphi_i(x,u,q,p) = 1 \quad \text{if } 0 \leq u \leq R(x,q,p)/R_i \qquad i = 0,1 \qquad (17)$$

$$\mu_0(x,u,q,p) = g_U(q) - \Delta(p) R_0 [1 - \phi(x)] \varphi_{\phi(x)}(x,u,q,p) \qquad (18a)$$

and

$$\mu_1(x,u,q,p) = g_L(q) + \Delta(p) R_1 \phi(x) \varphi_{\phi(x)}(x,u,q,p). \qquad (18b)$$

Then

i. $E\{[1-\phi(X)]\varphi_{\phi(X)}(X,Z,q,p)\} = [g_U(q) - g(q)]/\Delta(p)R_0$

ii. $E[\phi(X)\varphi_{\phi(X)}(X,Z,q,p)] = [g(q) - g_L(q)]/\Delta(p)R_1$

iii. $E\mu_i(X,Z,q,p) = g(q) \qquad i = 0,1$

iv. $\text{var } \mu_0(X,Z,q,p) = [g_U(q) - g(q)][g(q) - g_U(q) + \Delta(p)R_0]$

$$= v(q) + [g_U(q) - g(q)][\Delta(p)R_0 - \Delta(q)]$$

v. $\text{var } \mu_1(X,Z,q,p) = [g(q) - g_L(q)][\Delta(p)R_1 + g_L(q) - g(q)]$

$$= v(q) + [g(q) - g_L(q)][\Delta(p)R_1 - \Delta(q)]$$

vi. $\text{cov}[\mu_0(X,Z,q,p), \mu_1(X,Z,q,p)] = v(q) = [g_U(q) - g(q)][g(q) - g_L(q)].$

Proof. Straightforwardly,

$$E\{[1-\phi(X)]\varphi_{\phi(X)}(X,Z,q,p)\} = \text{pr}[\phi(X) = 0, \varphi_0(X,Z,q,p) = 1]$$

$$= \sum_{x \in \mathcal{X}} \left\{[1-\phi(x)] \frac{R(x,q,p)}{R_{\phi(x)}}\right\} \frac{[\phi_U(x) - \phi_L(x)]}{\Delta(p)} P(x,p)$$

$$= [g_U(q) - g(q)]/\Delta(p)R_0,$$

establishing i. Part ii follows in analogous fashion and the proofs of parts iii through vi are then immediately obvious.

Suppose one performs K independent replications generating $X^{(1)},...,X^{(K)}$ from (6) and $Z^{(1)},...,Z^{(K)}$ from $\mathcal{U}(0,1)$. Then

$$\bar{g}_{iK}(q,p) = \frac{1}{K}\sum_{j=1}^{K} \mu_i(X^{(j)},Z^{(j)},q,p) \qquad i = 0,1 \tag{19}$$

have expectations $g(q)$ with var $\bar{g}_{iK}(q,p)$ = var $\mu_i(X,Z,q,p)/K$. Observe that the inequalities var $\mu_0(X,Z,q,p) > v(q)$ and var $\mu_1(X,Z,q,p)] > v(q)$ for $q \neq p$, when they occur, signal an inflation of variances over what obtains if one were to sample from $\{Q(x,q)\}$ directly. Therefore, it is of interest to assess how much these variances and corresponding coefficients of variation grow when using the proposed acceptance-rejection method. Theorems 2 and 3 provide worst case upper bounds.

Theorem 2. Let X and Z denote samples from $\{Q(x,p)\}$ in (6) and $\mathcal{U}(0,1)$ respectively. Then

$$\text{var } \mu_i(X,Z,q,p) \leq M_i(q,p) = \begin{cases} [\Delta(p)R_i]^2/4 & \text{if } \Delta(q) > \Delta(p)R_i/2 \\ \Delta(q)[\Delta(p)R_i - \Delta(q)] & \text{if } \Delta(q) \leq \Delta(p)R_i/2 \end{cases} \tag{20}$$

$$i = 0,1.$$

Proof. Since $g_L(q) \leq g(q) \leq g_U(q)$, $A = [g_U(q) - g(q)][g(q) - g_U(q) + \Delta(p)R_0]$ has its maximum at $g^*(q) = g_L(q) + \max[0,\Delta(q) - \Delta(p)R_0/2]$, from which (20) follows for $i = 0$. Similarly, $B = [g(q) - g_L(q)][\Delta(p)R_1 + g_L(q) - g(q)]$ has its maximum at $g^*(q) = g_U(q) - \max[0,\Delta(q) - \Delta(p)R_1/2]$, from which (20) follows for $i = 1$.

Observe that evaluation of (20) for $i = 0,1$, prior to sampling, enables one to determine which estimator has the smallest worst case variance.

Theorem 3. Let

$$\gamma_i(q,p) = [\text{var } \mu_i(X,Z,q,p)]^{\frac{1}{2}}/[1-g(q)] \qquad i = 0,1.$$

Then

$$\max_{g_L(q)\le g(q)\le g_U(q)} \gamma_0^2(q,p) = N_0(q,p) = [\Delta(p)R_0]^2/4[1-g_U(q)+\Delta(p)R_0][1-g_U(q)] \tag{21a}$$

$$\text{if } \Delta(p)R_0][1-g_U(q)-\Delta(q)] \le 2\Delta(q)[1-g_U(q)]$$

$$= \Delta(q)[\Delta(p)R_0-\Delta(q)]/[1-g_L(q)]^2 \quad \text{otherwise} \tag{21b}$$

and

$$\max_{g_L(q)\le g(q)\le g_U(q)} \gamma_1^2(q,p) = N_1(q,p) = [\Delta(p)R_1]^2/4[1-g_L(q)-\Delta(p)R_1][1-g_L(q)] \tag{22a}$$

$$\text{if } \Delta(p)R_1][1-g_L(q)+\Delta(q)] \le 2\Delta(q)[1-g_L(q)]$$

$$= \Delta(q)[\Delta(p)R_1-\Delta(q)]/[1-g_U(q)]^2 \quad \text{otherwise.} \tag{22b}$$

Proof. We give the proof for $\max \gamma_0^2(q,p)$. Let

$$A = \text{var } \mu_0(X,Z,q,p)]/[1-g(q)]^2. \tag{23}$$

Then

$$\frac{\partial A}{\partial g} = \frac{-g(q)\{2[1-g_U(q)]+\Delta(p)R_0\}+2g_U(q)[1-g_U(q)]-\Delta(p)R_0[1-2g_U(q)]}{[1-g(q)]^3}.$$

Since $\partial A/\partial g(q)\Big|_{g(q)=g_U(q)} < 0$ and $\partial A/\partial g(q) = 0$ at

$$g^*(q) = \{2g_U(q)[1-g_U(q)] - \Delta(p)R_0[1-2g_U(q)]\}/\{2[1-g_U(q)]+\Delta(p)R_0]\},$$

A has its maximum at $g^*(q)$ if $g^*(q) \geq g_L(q)$, which upon substitution of $g^*(q)$ for $g(q)$ in (23) gives (21a). If $g^*(q) < g_L(q)$ then the maximum occurs at $g_L(q)$, giving (21b). A completely analogous result holds for $\max \gamma_1^2(q,p)$.

4. Choosing the Sampling Probabilities p

The results in Theorems 2 and 3 play a critical role in deciding at which component reliability vector p one should conduct the Monte Carlo sampling experiment. For each $i = 0,1$, one procedure finds the $p_i \in \mathcal{Q}$ that minimizes $\max_{q\in\mathcal{Q}} M_i(q,p)$ where (20) defines $M_i(q,p)$ as the worst case var $\mu_i(X,Z,q,p)$. Then one uses

$$\begin{aligned} p &= p_0 && \text{if } \max_{q\in\mathcal{Q}} M_0(q,p_0) \leq \max_{q\in\mathcal{Q}} M_1(q,p_1) \\ &= p_1 && \text{otherwise,} \end{aligned} \tag{24}$$

so that sampling from $\{Q(x,p)\}$ with p as in (24) minimizes the worst case variance that can arise. Finding p_i takes w^2 evaluations of $M_i(q,p)$. Also, note that

$$K_* = \lceil \min[\max_{q\in\mathcal{Q}} M_0(q,p_0), \max_{q\in\mathcal{Q}} M_1(q,p_1)]/v_* \rceil$$

gives the worst case sample size required to obtain estimates of $g(q_1),\ldots,g(q_w)$ with variances no greater than a specified v_*. This valuable information can assist a user of the Monte Carlo method before any sampling begins.

The proposed technique can also accommodate a relative accuracy specification. For $i = 0,1$, an alternative procedure finds the $p_i \in \mathcal{Q}$ that minimizes $\max_{q \in \mathcal{Q}} N_i(q,p)$ where (21) and (22) define $N_0(q,p)$ and $N_1(q,p)$, and then uses

$$p = p_0 \qquad \text{if } \max_{q \in \mathcal{Q}} N_0(q,p_0) \le \max_{q \in \mathcal{Q}} N_1(q,p_1)$$

$$= p_1 \qquad \text{otherwise.}$$

Sampling from $\{Q(x,p)\}$ with this p minimizes the worst case coefficient of variance. Also,

$$K_{**} = \lceil \min[\max_{q \in \mathcal{Q}} N_0(q,p_0), \max_{q \in \mathcal{Q}} N_1(q,p_1)]/u_*^2 \rceil \tag{25}$$

provides the worst case sample size needed to estimate $g(q_1),\ldots,g(q_w)$ with coefficients of variation no greater than a specified u_*.

5. Efficiency

Naturally, the appeal of any proposed sampling plan depends on the cost saving it offers, when achieving a specified accuracy as compared to other more conventional methods. These cost considerations have two components, one based on variances and the other based on computer times expended per replication. Theorem 4 derives an expression for the smallest variance ratio that one can expect to achieve when comparing a crude Monte Carlo estimate $\bar{g}_K(q)$ to an estimate $\tilde{g}_{iK}(q,p)$ based on the proposed method. This smallest ratio is analogous to $D(q)$ in (9) and reveals the least favorable circumstance that one can expect to encounter. The ratio can be computed prior to sampling, thereby

providing a lower bound on what to expect.

Theorem 4. Let Y denote a sample from $\{P(x,p)\}$, X a sample drawn from $\{Q(x,p)\}$ and Z a sample drawn from $\mathcal{U}(0,1)$. Let

$$B_i(g(q),p) = \text{var } \phi(Y)/\text{var } \mu_i(X,Z,q,p) \qquad i = 0,1.$$

Then

$$\min_{g_L(q)\le g(q)\le g_U(q)} B_0(g(q),p) = 1/\{\{g_U(q)[1-g_U(q)+\Delta(p)R_0]\}^{\frac{1}{2}}-\{[1-g_U(q)][g_U(q)-\Delta(p)R_0]\}^{\frac{1}{2}}\}^2 \tag{26}$$

$$\text{if } \Delta(p)R_0 \le \frac{\Delta(q)\{g_L(q)[1-g_U(q)]+g_U(q)[1-g_L(q)]\}}{\Delta^2(q)+g_U(q)[1-g_U(q)]}$$

$$= g_L(q)[1-g_L(q)]/\Delta(q)[\Delta(p)R_0-\Delta(q)] \qquad \text{otherwise}$$

and

$$\min_{g_L(q)\le g(q)\le g_U(q)} B_1(g(q),p) = 1/\{\{g_L(q)[1-g_L(q)-\Delta(p)R_1]\}^{\frac{1}{2}}-\{[1-g_L(q)][g_L(q)+\Delta(p)R_1]\}^{\frac{1}{2}}\}^2 \tag{27}$$

$$\text{if } \Delta(p)R_1 \le \frac{\Delta(q)\{g_L(q)[1-g_U(q)]+g_U(q)[1-g_L(q)]\}}{\Delta^2(q)+g_L(q)[1-g_L(q)]}$$

$$= g_U(q)[1-g_U(q)]/\Delta(q)[\Delta(p)R_1-\Delta(q)] \qquad \text{otherwise.}$$

Proof. We prove the result for $B_1(g(q),p)$. Observe that $\partial B_1/\partial g(q) = 0$ has roots

$$r_i = 1/\left\{1+(-1)^i\left[\frac{1-g_L(q)}{g_L(q)} \bullet \left(\frac{1-g_L(q)-\Delta(p)R_1}{g_L(q)+\Delta(p)R_1}\right)\right]^{\frac{1}{2}}\right\} \quad i=1,2. \qquad (28)$$

If $\Delta(p)R_1 \leq 1 - g_L(q)$, the roots are real with either $r_2 \leq 0$ or $r_2 \geq 1$ and $g_L(q) \leq r_2 \leq 1$. Since $\partial B_1/\partial g(q)\Big|_{g(q)=g_U(q)} < 0$, then

$$\min_{g_L(q)\leq g(q)\leq g_U(q)} B_1(g(q),p) = B_1(r_2,p) \qquad \text{if } r_2 \leq g_U(q)$$

$$= B_1(g_U(q),p) \qquad \text{if } r_2 \geq g_U(q).$$

Expression (27) follows from substituting (28) for r_2 in the inequality. If $\Delta(p)R_1 > 1 - g_L(q)$, then the roots are complex and $\partial B_1/\partial g(q) < 0$ for all $g(q) \in [g_L(q), g_U(q)]$ so that the minimum occurs at $g(q) = g_U(q)$. Moreover, complex roots imply that the condition in the upper branch of (27) is always true, thus completing the proof. An analogous result holds for $B_0(g(q),p)$. ■

The availability of (26) and (27) for each q in $\mathcal{Q}$ again provides valuable information to the Monte Carlo user prior to experimentation. In particular, it identifies at which q adverse variance ratios may occur. However, measuring the statistical efficiency of $\{\tilde{g}_{0K}(q,p),\ q \in \mathcal{Q}\}$ and $\{\tilde{g}_{1K}(q,p),\ q \in \mathcal{Q}\}$ as estimators of $\{g(q),\ q \in \mathcal{Q}\}$ calls for a more elaborate analysis than that for estimation at a single point. In particular, the sobering observation that R_0 and R_1 in (26) and (27) increases exponentially with $|\mathcal{E}|$ makes one circumspect about the benefit of the proposed method as the size of G grows. We now show that this benefit is assured for finite $w = |\mathcal{Q}|$ and number of edge types r,

provided that $\boldsymbol{p} \in \mathcal{Q}$.

Recall that $\mathcal{Q} = \{\boldsymbol{q}_1,...,\boldsymbol{q}_w\}$ where $\boldsymbol{q}_j = (q_{1j},...,q_{rj})$ and q_{ij} is the reliability assigned to components of type i in the jth component reliability vector for $j = 1,...,w$. Let $\mathcal{H} = \{1,...,r\}$ and

$$\mathcal{H}^* = \{i \in \mathcal{H}:\ p_i \neq q_{ij} \text{ for at least one } j;\quad j = 1,...,w\},$$

so that $|\mathcal{H}^*|$ component reliability types vary in $\mathcal{Q}$.

Algorithm A-R describes the steps for computing the estimates and provides the basis for measuring efficiency. In addition to computing $\{\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p}),\ \tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p});\ \boldsymbol{q} \in \mathcal{Q}\}$, it computes $\{V[\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p})],\ V[\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p})];\ \boldsymbol{q} \in \mathcal{Q}\}$ as unbiased estimators of $\{\text{var}\ \tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p}),\ \text{var}\ \tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p});\ \boldsymbol{q} \in \mathcal{Q}\}$. Observe that preprocessing in step 1 takes $O(|\mathcal{H}^*|w)$ time, postprocessing in step 3 takes $O(w)$ time and, on each replication, sampling in step 2a takes $O(|\mathcal{S}|)$ time using Procedure Q in Fishman (1986), summation in step 2c takes $O(\sum_{i \in \mathcal{H}^*} k_i) \leq O(|\mathcal{S}|)$ time and step 2d takes $O(|\mathcal{H}^*|w)$ time. One can also show that the mean total time for *K replications* using Algorithm A-R has the form

$$T(\{\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p}),\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p})\}) = \omega_0 + \omega_1|\mathcal{H}^*|w + \omega_2 w + K[\omega_3 + \omega_4|\mathcal{S}| + \alpha_3(\mathcal{X}_{01},\boldsymbol{p})/\Delta(\boldsymbol{p})$$

$$+ \omega_5|\mathcal{H}^*|w + \omega_6 \sum_{i \in \mathcal{H}^*} k_i]$$

time where $\omega_0,...,\omega_6$ denote machine dependent constants. To reduce numerical error, all computation in step 3 should be performed in extended precision arithmetic.

Algorithm A-R

Purpose: To estimate the reliability function $\{g(\boldsymbol{q}),\ \boldsymbol{q} \in \mathcal{Q}\}$.

Input: Network $G = (\mathcal{V}, \mathcal{E})$; number of type of components r; k_i = number of components of type i for $i=1,\ldots,r$; sampling distribution $\{Q(\boldsymbol{x},\boldsymbol{p}),\ \boldsymbol{x} \in \mathcal{X}\}$; $\mathcal{H}^*$ = set of component types that vary in $\mathcal{Q}$; lower and upper bounds $\{g_L(\boldsymbol{q}),\ g_U(\boldsymbol{q});\ \boldsymbol{q} \in \mathcal{Q} \cup \{\boldsymbol{p}\}\}$; and number of independent replications K.

Output: $\{\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p}),\ \tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p}),\ V[\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p})],\ V[\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p})];\ \boldsymbol{q} \in \mathcal{Q}\}$ as unbiased estimates of $\{g(\boldsymbol{q}),\ g(\boldsymbol{q}),\ \text{var}\ \tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p}),\ \text{var}\ \tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p});\ \boldsymbol{q} \in \mathcal{Q}\}$.

Method:

1. Initialization

   a. $\Delta(\boldsymbol{p}) \leftarrow g_U(\boldsymbol{p}) - g_L(\boldsymbol{p})$.

   b. For each $\boldsymbol{q} \in \mathcal{Q}$:

      $K(0,\boldsymbol{q}) = K(1,\boldsymbol{q}) \leftarrow 0$.

      For each $i \in \mathcal{H}^*$:

      $\alpha_i(\boldsymbol{q}) \leftarrow \log[q_i(1-p_i)/p_i(1-q_i)]$ and $\beta_i(\boldsymbol{q}) \leftarrow \log(1-q_i)/(1-p_i)]$.

2. On each of K independent trials:

   a. Sample $X_{ij}\ j = 1,\ldots,k_i\ \ i = 1,\ldots,r$ from $\{Q(\boldsymbol{x},\boldsymbol{p})\}$.

   b. Determine $\phi(\boldsymbol{X})$.

   c. For each $i \in \mathcal{H}^*$: $X_i \leftarrow \sum_{j=1}^{k_i} X_{ij}$.

   d. Sample Z from $\mathcal{U}(0,1)$.

   e. For $\boldsymbol{q} \in \mathcal{Q}$:

      $T(\boldsymbol{q}) \leftarrow 0$.

      For each $i \in \mathcal{H}^*$: $T(\boldsymbol{q}) \leftarrow T(\boldsymbol{q}) + k_i\beta_i(\boldsymbol{q}) + X_i\alpha_i(\boldsymbol{q})$.

      $R(\boldsymbol{X},\boldsymbol{q},\boldsymbol{p}) \leftarrow \exp[T(\boldsymbol{q})]$.

      $F(\boldsymbol{X},\phi(\boldsymbol{X}),\boldsymbol{q},\boldsymbol{p}) \leftarrow R(\boldsymbol{X},\boldsymbol{q},\boldsymbol{p})/R_{\phi(\boldsymbol{X})}(\boldsymbol{q},\boldsymbol{p})$.

      $\varphi_{\phi(\boldsymbol{X})}(\boldsymbol{X},Z,\boldsymbol{q},\boldsymbol{p}) \leftarrow \lfloor Z + F(\boldsymbol{X},\phi(\boldsymbol{X}),\boldsymbol{q},\boldsymbol{p}) \rfloor$.

      $K(\phi(\boldsymbol{X}),\boldsymbol{q}) \leftarrow K(\phi(\boldsymbol{X}),\boldsymbol{q}) + \varphi_{\phi(\boldsymbol{X})}(\boldsymbol{X},Z,\boldsymbol{q},\boldsymbol{p})$.

3. Computation of summary statistics

   For each $\boldsymbol{q} \in \mathcal{Q}$:

$$\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p}) \leftarrow g_U(\boldsymbol{q}) - \Delta(\boldsymbol{p})R_0(\boldsymbol{q},\boldsymbol{p})K(0,\boldsymbol{q})/K.$$

$$\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p}) \leftarrow g_L(\boldsymbol{q}) + \Delta(\boldsymbol{p})R_1(\boldsymbol{q},\boldsymbol{p})K(1,\boldsymbol{q})/K.$$

$$V[\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p})] \leftarrow [\Delta(\boldsymbol{p})R_0(\boldsymbol{q},\boldsymbol{p})]^2[K(0,\boldsymbol{q})/K][1-K(0,\boldsymbol{q})/K]/(K-1).$$

$$V[\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p})] \leftarrow [\Delta(\boldsymbol{p})R_1(\boldsymbol{q},\boldsymbol{p})]^2[K(1,\boldsymbol{q})/K][1-K(1,\boldsymbol{q})/K]/(K-1).$$

Let us now compare this approach to estimating $\{g(q),\ q \in \mathcal{Q}\}$ with the alternative approach based on the w point estimates $\{\bar{g}_{K(q,p)}(q),\ q \in \mathcal{Q}\}$ using (4), where one chooses the sample sizes $\{H(q,p),\ q \in \mathcal{Q}\}$ to achieve equal variances under the two methods. That is,

$$\text{var}\ \bar{g}_{H(q,p)}(q) = g(q)[1-g(q)]/H(q,p) \tag{29}$$

where

$$H(q,p) = K\,\lambda(q,p)$$

and

$$\lambda(q,p) = \frac{g(q)[1-g(q)]}{\min\limits_{j\in\{0,1\}} \text{var}\ \mu_j(X,Z,q,p)}.$$

Observe that

$$\lambda(p,p) = g(p)[1-g(p)]/[g_U(p)-g(p)][g(p)-g_L(p)]$$

and, except in special cases, for any edge type $i \in \mathcal{H}^*$

$$\lim_{k_i\to\infty} \lambda(q,p) = 0 \qquad \text{for } q \neq p.$$

Let

$$\lambda(p) = \sum_{q\in\mathcal{Q}} \lambda(q,p). \tag{30}$$

and observe that

$$\lim_{k_i \to \infty} \lambda(p) = \lambda(p,p). \tag{31}$$

Therefore, the time ratio

$$\Lambda_1(\mathcal{Q},p) = \frac{T(\{\bar{g}_{H(q,p)}(q)\})}{T(\{\tilde{g}_{0K}(q,p), \tilde{g}_{1K}(q,p)\})}, \tag{32}$$

where

$$T(\{\bar{g}_{H(q,p)}(q)\}) = \sum_{q \in \mathcal{Q}} T(\bar{g}_{H(q,p)}(q)),$$

measures the efficiency of the proposed method relative to using crude Monte Carlo sampling with (4) w times to obtain estimates with equal variances $\text{var}\ \bar{g}_{H(q,p)}(q) = \min_{j \in \{0,1\}} \text{var}\ \tilde{g}_{jK}(q,p)$ for each $q \in \mathcal{Q}$. As k_i increases, (32) assumes the form

$$\Lambda_1(\mathcal{Q},p) \approx \frac{\sum_{q \in \mathcal{Q}} [\alpha_2 + \alpha_3(\mathcal{X},q)/k_i]\lambda(q,p)}{\omega_4 + \alpha_3(\mathcal{X}_{01},p)/\Delta(p)k_i + \omega_6} \tag{33}$$

$$\geq \frac{\alpha_2 + \alpha_3(\mathcal{X},p)/k_i}{\omega_4 + \omega_6 + \alpha_3(\mathcal{X}_{01},p)/\Delta(p)k_i}\lambda(p,p)$$

where the lower bound is analogous to (10). This implies that one should expect efficiency to exceed that which obtains from estimating $g(p)$ only. As the example in Section 9 shows, the realized efficiency can be considerably greater.

6. Improving Computational Efficiency

The special, but common, case $\mathcal{Q} = \{q_1 < ... < q_w\}$ provides an opportunity for improving the computational efficiency of Algorithm A-R. Write $F(x,i,q)$ for $F(x,i,q,p)$ defined in (16) and note that for fixed x and i

$$0 < F(x,i,q_w) < F(x,i,q_{w-1}) < ... < F(x,i,q_1) \leq 1$$

so that $\{F(x,i,q_{w-j+1}),\ j = 1,...,w;\ F(x,i,q_0) \equiv 1\}$ is a distribution function (d.f.). Suppose that one draws X from $\{Q(x,p)\}$ in (6) and Z from $\mathcal{U}(0,1)$, and let

$$W = \min[z:\ F(X,\phi(X),q_{w-z+1}) \geq Z]. \tag{34}$$

Then $\varphi_{\phi(X)}(X,Z,q_j,p) = 1$ for $W \leq w$ and $= 0$ otherwise for $j = 1,...,W$.

Note that every component state with x_i edges of type i for $i = 1,...,r$ has either $\{F(x,0,q_{w-j+1});\ j = 1,...,w\}$ or $\{F(x,1,q_{w-j+1});\ j = 1,...,w\}$ as its d.f. and there are $m = 2 \prod_{i \in \mathcal{K}^*}(k_i+1)$ of these d.fs. If m is sufficiently small, as it will be if there are a small number of component types, then before sampling begins one can compute these d.fs. and use them to create tables needed for the cutpoint sampling method (Fishman and Moore 1984). On each trial, one samples W from these specially prepared tables in O(1) time regardless of how large w is. Algorithm A-R* shows how to incorporate this alternative sampling method. It replaces step 2e of Algorithm A-R, which takes $O(K|\mathcal{K}^*|w)$ time with a new step 2e that takes $O(K)$ time, thus reducing computing time per replication.

Algorithm A-R*

Purpose: To estimate the reliability function $\{g(q),\ q \in \mathcal{Q}\}$ where $\mathcal{Q} = \{q_1 < \ldots < q_w\}$.

Input: Network $G = (\mathcal{V}, \mathcal{E})$; number of type of components r ; k_i = number of components of type i for $i=1,\ldots,r$; sampling distribution $\{Q(x,p),\ x \in \mathcal{X}\}$; $\mathcal{H}^* \subseteq$ set of component types that vary in $\mathcal{Q}$; lower and upper bounds $\{g_L(q),\ g_U(q);\ q \in \mathcal{Q} \cup \{p\}\}$; and number of independent replications K.

Output: $\{\tilde{g}_{0K}(q,p),\ \tilde{g}_{1K}(q,p),\ V[\tilde{g}_{0K}(q,p)],\ V[\tilde{g}_{1K}(q,p)];\ q \in \mathcal{Q}\}$ as unbiased estimates of $\{g(q),\ g(q),$ var $\tilde{g}_{0K}(q,p),$ var $\tilde{g}_{1K}(q,p);\ q \in \mathcal{Q}\}$.

Method:

1. Initialization
   a. $\Delta(p) \leftarrow g_U(p) - g_L(p)$.
   b. For $i=1,\ldots,w$: $\tau(0,i) = \tau(1,i) \leftarrow 0$.
2. On each of K independent trials:
   a. Sample X_{ij}, $j=1,\ldots,k_i$, $i=1,\ldots,r$ from $\{Q(x,p)\}$.
   b. Determine $\phi(X)$.
   c. Sample Z from $\mathcal{U}(0,1)$.
   d. $W \leftarrow \min[z:\ F(X,\phi(X),\ q_{w-z+1}) \geq Z]$. (Fishman and Moore 1984).
   e. $\tau(\phi(X), W) \leftarrow \tau(\phi(X), W) + 1$.
3. Computation of summary statistics

   $K(0,q_w) \leftarrow \tau(0,w)$ and $K(1,q_w) \leftarrow \tau(1,w)$.

   For $i=0,1$:

   For $j=2,\ldots,w$: $K(i,q_{w-j+1}) \leftarrow K(i,q_{w-j}) + \tau(i,w-j+1)$.

   For $j=1,\ldots,w$:

$$\tilde{g}_{0K}(q_j,p) \leftarrow g_U(q_j) - \Delta(p) R_0(q_j,p) K(0,q_j)/K.$$

$$\tilde{g}_{1K}(q_j,p) \leftarrow g_L(q_j) + \Delta(p) R_1(q_j,p) K(1,q_j)/K.$$

$$V[\tilde{g}_{0K}(q_j,p)] \leftarrow [\Delta(p) R_0(q_j,p)]^2 [K(0,q_j)/K][1-K(0,q_j)/K]/(K-1).$$

$$V[\tilde{g}_{1K}(q_j,p)] \leftarrow [\Delta(p) R_1(q_j,p)]^2 [K(1,q_j)/K][1-K(1,q_j)/K]/(K-1).$$

7. Individual Confidence Intervals

Since

$$\lim_{K\to\infty} \text{pr}\left\{ \frac{|\tilde{g}_{iK}(\boldsymbol{q},\boldsymbol{p})-g(\boldsymbol{q})|}{[\text{var}\ \tilde{g}_{iK}(\boldsymbol{q},\boldsymbol{p})]^{\frac{1}{2}}} \le \beta \right\} = 2\Phi(\beta) - 1$$

where $\Phi(\bullet)$ denotes the d.f. of the standard normal distribution, one can immediately compute an *approximating* confidence interval for $g(\boldsymbol{q})$. In particular, based on $\tilde{g} \equiv \tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p})$ and Theorem 1, one has the approximating $100\times(1 - \delta)$ percent confidence interval

$$\frac{\tilde{g}+[2g_U(\boldsymbol{q})-\Delta(\boldsymbol{p})R_0]\beta^2/2K \pm \beta\{\Delta(\boldsymbol{p})R_0\beta^2/K^2+[g_U(\boldsymbol{q})-\tilde{g}][\Delta(\boldsymbol{p})R_0-g_U(\boldsymbol{q})+\tilde{g}]/K\}^{\frac{1}{2}}}{1 + \beta^2/K}.$$

for $g(\boldsymbol{q})$ where

$$\beta \equiv \left(z: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy = 1 - \delta/2\right).$$

An analogous interval can be computed based on $\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p})$.

Because of the nonuniform convergence to normality, this approach inevitably incurs an error of approximation. An alternative approach avoids this error, albeit at the cost of a wider interval.

Theorem 5. Let

$$m(z,\omega) = z\log(\omega/z) + (1-z)\log[(1-\omega)/(1-z)] \qquad 0 < z,\ \omega < 1,$$

let $\omega(z,\delta/2,K)$ denote the solution to $m(z,\omega) = \frac{1}{K}\log(\delta/2)$ for fixed $z \in (0,1]$ and $\delta \in (0,1)$,

and let

$$\omega^*(z,\delta/2,K) = \omega(z,\delta/2,K) \qquad \text{if } 0 < z \leq 1$$
$$= 0 \qquad \text{otherwise.} \tag{35}$$

Then, the interval

i. $$(g_U(q)-\Delta(p)R_0\omega^*(1-K(0,q)/K,\delta/2,K),\ g_U(q)-\Delta(p)R_0\omega^*(K(0,q)/K,\delta/2,K)) \tag{36a}$$

covers $g(q)$ with probability $> 1 - \delta$

and

ii. $$(g_L(q)+\Delta(p)R_1\omega^*(K(1,q)/K,\delta/2,K),\ g_L(q)+\Delta(p)R_1\omega^*(1-K(1,q)/K,\delta/2,K)) \tag{36b}$$

covers $g(q)$ with probability $> 1 - \delta$.

The proof exploits the observation that

$$\text{pr}[g_U(q)-\Delta(p)R_0 \leq \mu_0(X,Z,q,p) \leq g_U(q)] = 1$$

and

$$\text{pr}[g_L(q) \leq \mu_1(X,Z,q,p) \leq g_L(q) + \Delta(p)R_1] = 1.$$

The resulting confidence intervals follow from Theorem 1 in Fishman (1988). ■

Since the slowest convergence to normality for $\tilde{g}_{iK}(q,p)$ occurs for $g(q)$ close to zero and unity and since one is often interested in $g(q)$ near unity the wider confidence intervals that result from this approach seem a reasonable price to pay to be free of the error of approximation inherent in normal intervals. Since $\{m(z,\omega)\}$ is concave in ω, one can compute the required roots by bisection.

8. Simultaneous Confidence Intervals

Although each confidence interval in Section 7 holds with probability $> 1-\delta$, the joint confidence intervals for $\{g(q),\ q \in \mathcal{Q}\}$ hold simultaneously only with probability $> 1 - w\delta$. This result follows from a Bonferroni inequality. See Miller (1981, p. 8). To restore the joint confidence level to $1-\delta$, one replaces $\delta/2$ by $\delta/2w$ in (36a) and (36b) and determines the corresponding solutions. The effect of this substitution is to increase the constant of proportionality in the approximate interval widths from $[2\log(2/\delta)]^{\frac{1}{2}}$ to $[2\log(2w/\delta)]^{\frac{1}{2}}$ (see Fishman 1988). For $\delta = .01$ and $w = 20$ one has $[\log(2w/\delta)/\log(2/\delta)]^{\frac{1}{2}} = 1.25$. For $\delta = .01$ and $w = 100$, it is 1.37 and for $\delta = .01$ and $w = 1000$ it is 1.52. However, if $\mathcal{Q}$ denotes a continuous region in the $|\mathcal{S}|$-dimensional hypercube $(0,1)^{|\mathcal{S}|}$, then the resulting confidence intervals have infinite widths and are therefore useless.

For the case $Q = \{q_1 < \ldots < q_w\}$, an alternative approach derives simultaneous confidence intervals for $\{g(q),\ q \in \mathcal{Q}\}$ by exploiting the fact that $\{K(0,q_j)/K;\ j = 1,\ldots,w\}$ and $\{K(1,q_j)/K;\ j = 1,\ldots,w\}$, in steps 3 of Algorithms A-R and A-R*, satisfy the definition of an empirical distribution function. Since

$$K^{-1}\mathrm{E}K(0,q) = \rho(0,q) = [g_U(q) - g(q)]/\Delta(p)R_0$$

and

$$K^{-1}EK(1,q) = \rho(1,q) = [g(q)-g_L(q)]/\Delta(p)R_1 ,$$

$$\text{pr}\{\bigcap_{j=1}^{w}[|K(0,q_j)/K - \rho(0,q_j)| < d_K(\delta)]\} \geq 1-\delta$$

and

$$\text{pr}\{\bigcap_{j=1}^{w}[|K(1,q_j)/K - \rho(1,q_j)| < d_K(\delta)]\} \geq 1-\delta$$

where $d_K(\delta)$ denotes the critical value of the Kolmogorov-Smirnov distribution for sample size K at significance level δ. Therefore,

$$g_U(q_j) - \Delta(p)R_0(q_j,p)[K(0,q_j)/K \pm d_K(\delta)] \quad \forall\, j = 1,...,w \tag{37a}$$

cover $g(\mathbf{q}_1),...,g(q_w)$ simultaneously with probability $\geq 1 - \delta$ and similarly

$$g_L(q_j) + \Delta(p)R_1(q_j,p)[K(1,q_j)/K \mp d_K(\delta)] \quad \forall\, j = 1,...,w \tag{37b}$$

cover $g(\mathbf{q}_1),...,g(q_w)$ with probability $\geq 1 - \delta$. For $\delta = .05$, $\lim_{K\to\infty} K^{\frac{1}{2}} d_K(.05) = 1.3581$ and for for $\delta = .01$ $\lim_{K\to\infty} K^{\frac{1}{2}} d_K(.01) = 1.6276$. Since $d_\infty(.05)/d_K(.05) \leq 1.013$ for $K \geq 100$ and $d_\infty(.01)/d_K(.01) \leq 1.014$ for $K \geq 80$ (Birnbaum 1952), little error arises when replacing $d_K(.05)$ by $1.3581/K^{\frac{1}{2}}$ and $d_K(.01)$ by $1.6276/K^{\frac{1}{2}}$ above for $K \geq 100$. The appeal of this alternative approach is that the widths of the intervals are all independent of w. The limitation is that all intervals are of the same width. In practice, one can compute the intervals based on (36a) and (36b) with $\delta/2w$ replacing $\delta/2$ and the intervals based on (37a) and (37b), and choose the set with smaller widths.

9. Example

An analysis of the network in Fig. 1 illustrates the proposed method. The network has 30 edges and 20 nodes. The example assumes r=1 so that all edges have identical reliabilities, allowing us to write $\boldsymbol{q} = q$. Note that any other specification with $r > 1$ can

Insert Fig. 1 about here.

be accommodated easily. The objective is to estimate $\{g(\boldsymbol{q}),\ q = .80 + .01(i\text{-}1)\ i = 1,...,20\}$ where $g(\boldsymbol{q})$ = probability that nodes $s = 1$ and $t = 20$ are connected when edge reliabilities are q. For sampling, we use $\boldsymbol{p} = p$, again merely as a convenience. The lower and upper bounding functions $\{g_L(\boldsymbol{q})\}$ and $\{g_U(\boldsymbol{q})\}$ were computed beforehand using edge-disjoint minimal s-t cutsets for $\{g_L(\boldsymbol{q})\}$ and edge-disjoint minimal s-t cutsets for $\{g_U(\boldsymbol{q})\}$, as in Fishman (1986). To determine these paths takes $O(I|\mathcal{E}|)$ time, where I denotes the size of the smallest minimal s-t cutset and to determine the paths takes $O(|\mathcal{E}|)$ time. The determination of R_0 and R_1 is discussed in Fishman (1988). The evaluation of $\phi(\boldsymbol{X})$ using a depth-first search as in Aho, Hopcroft and Ullman (1974) takes $O(\max(|\mathcal{E}|,|\mathcal{V}|))$ time.

An experiment was run with $p = .80$, which minimized the worst case variances as in (24), and with sample size $K = 2^{20} = 1048576$. Since results for $\{\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p})\}$ were considerably more favorable than those for $\{\tilde{g}_{1K}(\boldsymbol{q},\boldsymbol{p})\}$, the analysis focuses on $\{\tilde{g}_{0K}(\boldsymbol{q},\boldsymbol{p})\}$. Table 1 shows individual point estimates and confidence intervals, the latter having been computed as in (36a). Table 2 compares the precomputed worst case and the empirically observed coefficients of variation and variances, and Table 3 shows the worst case and empirically observed variance ratios, where the variance in the numerator corresponds to that for crude Monte Carlo sampling.

Recall that the worst case results can be computed and used prior to sampling. For example, suppose that one wants a coefficient of variation no larger than $u_* = .01$ for all point estimates. Since the largest worst case results in Table 2 is 10.13, one would use (25)

to compute the worst case sample size $n_{**} = 1{,}008{,}016$.

Insert Tables 1,2, and 3 about here.

In contrast to the exact results in col. 3 of Table 1 which took slightly more than one hour *each* to compute, all results in cols. 4 through 8 took 74.9 minutes to compute in total, or 4.28 milliseconds per replication. Whereas the calculated exact results in col. 3 are accurate to sixteen significant digits (reduced to four digits here for comparative purposes), the confidence intervals suggest an accuracy to two significant digits at the .99 level. If two significant digits is acceptable for purposes of analysis, then the Monte Carlo approach clearly prevails.

An experiment with $K = 1048576$ was also run using Algorithm A-R*. It, of course, gave statistical results close to those that Algorithm A-R produced. However, it took 36.6 minutes or 2.09 milliseconds per replication revealing a substantial increase in computing efficiency.

10. A Comparison

At least one alternative method exists for using the data from a single experiment with input vector $\boldsymbol{p}$ to generate estimates of $\{g(\boldsymbol{q}),\ \boldsymbol{q} \in \mathcal{Q}\}$. This method is based on using the *importance function* (13) to form

$$\psi_a(x,\boldsymbol{q},\boldsymbol{p}) = g_L(\boldsymbol{q}) + \Delta(\boldsymbol{p})\phi(x)R(x,\boldsymbol{q},\boldsymbol{p})$$

and

$$\psi_b(x,\boldsymbol{q},\boldsymbol{p}) = g_U(\boldsymbol{q}) + \Delta(\boldsymbol{p})[1-\phi(x)]R(x,\boldsymbol{q},\boldsymbol{p})$$

so that $\psi_a(X,q,p)$ and $\psi_b(X,q,p)$ both have expectation $g(q)$ when X is from $\{Q(x,p)\}$. Fishman (1987) studies these estimates in detail using the same network, and a comparison between these importance function (IF) and the currently proposed acceptance-rejection (A-R) estimators seems appropriate.

For every $q \in \mathcal{Q}$, the IF estimators have smaller variance than the A-R estimators do and both methods have about the same computation time per replication. If variance is the dominant consideration, then the IF method prevails. However, there are other issues that also deserve consideration. The A-R estimators have considerably simpler expressions for variance and coefficient of variation than the IF estimators do. Also, on each trial $\mu_0(X,Z,q,p)$ and $\mu_1(X,Z,q,p)$ for the A-R approach each assume binary values thus allowing standard techniques of analysis for binary data to apply. In contrast $\psi_a(X,q,p)$ and $\psi_b(X,q,p)$ in the IF approach each assume $O(\prod_{i=1}^{r}(k_i+1))$ values precluding the use of the simpler analysis.

With regard to confidence intervals, the A-R approach allows one to compute individual asymptotically normal intervals without nuisance parameters whereas the IF estimators do not. For individual confidence intervals based on Theorem 5, both methods give intervals of about the same length. This is a consequence of ignoring estimated variance information for the IF method. For simultaneous confidence intervals the A-R method allows the development in Section 8 when $q_1,\ldots,q_w$, whereas the IF method does not.

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Table 1

Reliability Estimation

(p=.80, K=1048576)

q	$1-g_U(q)$	$1-g_L(q)$	$1-g(q)$†	$1-\tilde{g}_{0K}(q,p)$††	$V[\tilde{g}_{0K}(q,p)]$††	Individual 99% Confidence Intervals on $1-g(q)$		
						Lower	Upper	Width
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
.80	.1612D–01	.3039D+00	.3624D–01	.3628D–01	.5144D–08	.3604D–01	.3651D–01	.4669D–03
.81	.1381D–01	.2763D+00	.2976D–01	.2976D–01	.4683D–08	.2954D–01	.2999D–01	.4455D–03
.82	.1173D–01	.2492D+00	.2421D–01	.2420D–01	.4120D–08	.2400D–01	.2441D–01	.4179D–03
.83	.9874D–02	.2227D+00	.1949D–01	.1953D–01	.3512D–08	.1933D–01	.1972D–01	.3858D–03
.84	.8225D–02	.1969D+00	.1552D–01	.1559D–01	.2893D–08	.1542D–01	.1577D–01	.3502D–03
.85	.6773D–02	.1722D+00	.1221D–01	.1227D–01	.2282D–08	.1212D–01	.1243D–01	.3110D–03
.86	.5503D–02	.1485D+00	.9473D–02	.9540D–02	.1728D–08	.9406D–02	.9677D–02	.2707D–03
.87	.4404D–02	.1262D+00	.7241D–02	.7240D–02	.1223D–08	.7127D–02	.7355D–02	.2276D–03
.88	.3462D–02	.1053D+00	.5440D–02	.5442D–02	.8354D–09	.5349D–02	.5537D–02	.1882D–03
.89	.2666D–02	.8611D–01	.4006D–02	.4017D–02	.5407D–09	.3942D–02	.4094D–02	.1514D–03
.90	.2002D–02	.6867D–01	.2880D–02	.2876D–02	.3197D–09	.2819D–02	.2935D–02	.1164D–03
.91	.1459D–02	.5314D–01	.2011D–02	.2002D–02	.1739D–09	.1960D–02	.2046D–02	.8587D–04
.92	.1025D–02	.3962D–01	.1355D–02	.1354D–02	.8754D–10	.1325D–02	.1385D–02	.6093D–04
.93	.6862D–03	.2818D–01	.8720D–03	.8820D–03	.4039D–10	.8620D–03	.9034D–03	.4139D–04
.94	.4321D–03	.1884D–01	.5284D–03	.5334D–03	.1490D–10	.5213D–03	.5465D–03	.2514D–04
.95	.2500D–03	.1158D–01	.2947D–03	.2984D–03	.4515D–11	.2918D–03	.3056D–03	.1384D–04
.96	.1280D–03	.6293D–01	.1456D–03	.1467D–03	.9371D–12	.1437D–03	.1500D–03	.6307D–05
.97	.5400D–04	.2819D–01	.5937D–04	.5981D–04	.1209D–12	.5875D–04	.6102D–04	.2267D–05
.98	.1600D–04	.8869D–03	.1702D–04	.1710D–04	.5882D–14	.1686D–04	.1736D–04	.5000D–06
.99	.2000D–05	.1177D–03	.2062D–05	.2067D–05	.2941D–16	.2051D–05	.2087D–05	.3537D–07

†Provided by J.S. Provan using an algorithm based on cutset enumeration.

††Computed as in Algorithm A-R.

Table 2

Coefficients of Variation and Variances

($p = .80$)

$$\gamma_0(q,p) = \frac{[\text{var } \mu_0(X,Z,q,p)]^{\frac{1}{2}}}{1 - E\mu_0(X,Z,q,p)} \qquad \text{var } \mu_0(X,Z,q,p)$$

q	Worst Case†	Observed††	Worst Case†††	Observed††
.80	2.06	2.02	.207D-01	.539D-02
.81	2.37	2.35	.262D-01	.491D-02
.82	2.72	2.72	.322D-01	.432D-02
.83	3.11	3.11	.383D-01	.368D-02
.84	3.53	3.53	.435D-01	.303D-02
.85	4.00	3.93	.455D-01	.239D-02
.86	4.51	4.46	.443D-01	.181D-02
.87	5.06	4.95	.406D-01	.128D-02
.88	5.64	5.44	.349D-01	.876D-03
.89	6.26	5.93	.282D-01	.567D-03
.90	6.91	6.37	.212D-01	.335D-03
.91	7.57	6.74	.147D-01	.182D-03
.92	8.23	7.08	.928D-02	.918D-04
.93	8.87	7.37	.520D-02	.423D-04
.94	9.44	7.41	.250D-02	.156D-04
.95	9.89	7.29	.982D-03	.473D-05
.96	10.13	6.76	.287D-03	.983D-06
.97	10.04	5.95	.527D-04	.127D-06
.98	9.37	4.59	.415D-05	.617D-08
.99	7.55	2.69	.396D-07	.308D-10

† Computed from (21). †† Estimated from data. ††† Computed from (20).

Table 3

Variance Ratios

$$\frac{\text{var } \bar{g}_K(q)}{\text{var } \tilde{g}_{0K}(q,p)}$$

q	Worst Case†	Observed††
.80	5.15	6.48
.81	4.32	5.88
.82	3.71	5.47
.83	3.27	5.20
.84	2.95	5.06
.85	2.75	5.07
.86	2.60	5.21
.87	2.54	5.61
.88	2.56	6.18
.89	2.68	7.06
.90	2.91	8.55
.91	3.31	10.95
.92	3.97	14.73
.93	5.10	20.81
.94	7.14	34.12
.95	11.23	63.00
.96	20.88	149.23
.97	50.53	471.61
.98	197.50	2771.75
.99	2490.13	67040.00

† Computed from (27). †† Estimated from data.

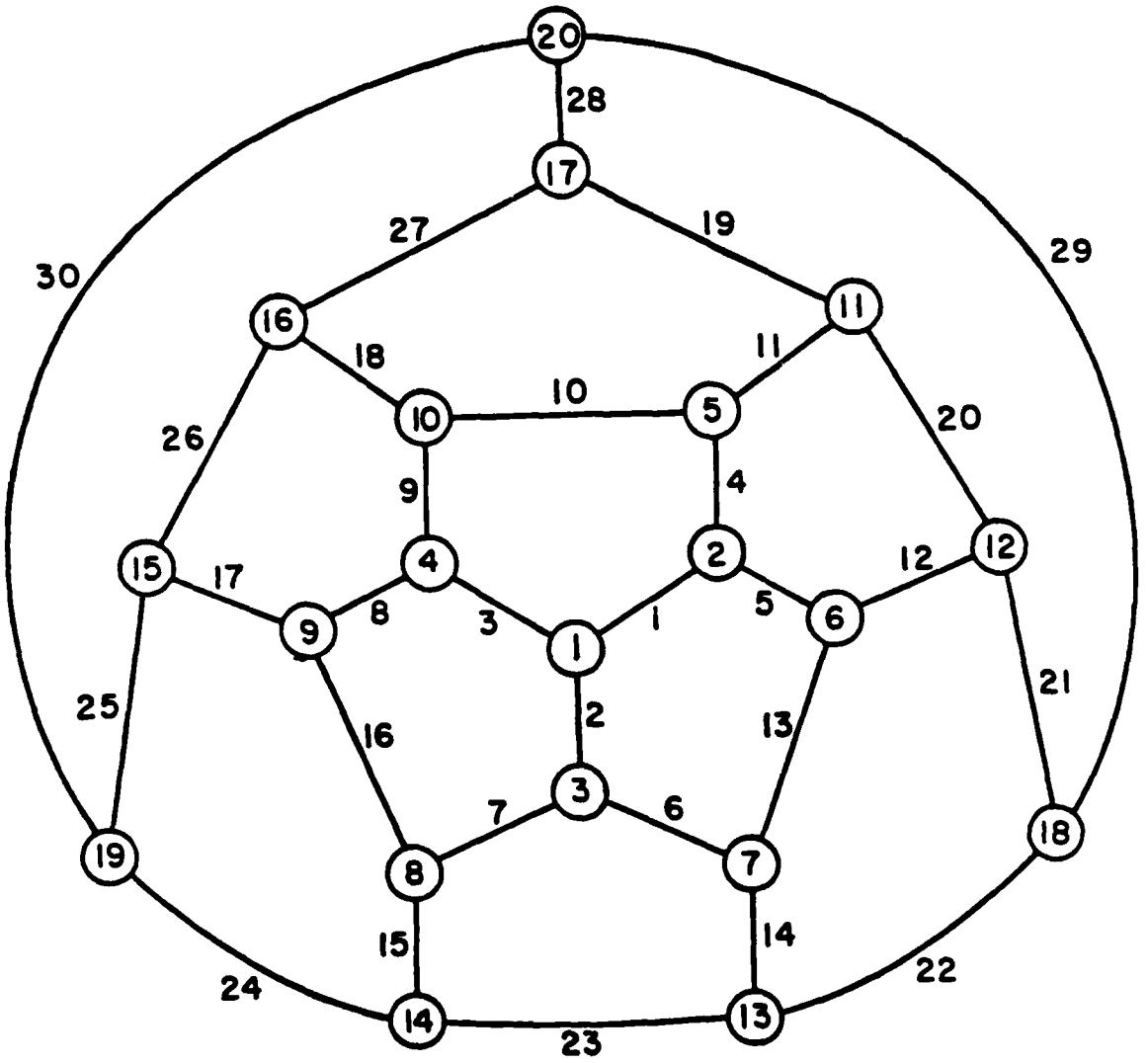

Fig.1 Network