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An hierarchy of uniformly high order accurate essentially non-oscillatory shock capturing algorithms was developed. Some theory and numerical experimentation was done.

A correction to the unsteady full potential equation for flows with strong shocks was obtained. This modification inputs the correct entropy jumps at shocks. Numerical experiments on airfoils were sucessfully performed.

A new family of paraxial wave approximations was derived and was applied to computational problems in seismology, underwater acoustics and artificial boundaries. Theoretical and experimental results were obtained. The family also included

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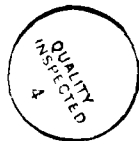| 20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
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1. A. Harten and S. Osher, "Uniformly High-Order Accurate Non-Oscillatory schemes, I", SINUM, V. 24 #2, (1987) pp. 279-309.

2. Y. Brenier and S. Osher, "The Discrete One-Sided Lipschitz Condition for Convex Scalar Conservation Laws", SINUM, V. 25 #1, (1988) pp. 8-23.

3. A. Harten, B. Engquist, S. Osher, and S.R. Chakravarthy, "Uniformly High Order Accurate Essentially Non-Oscillatory Schemes, III", J. of Comp. Physics, V. 71, #2, August 1987.

4. C.-W. Shu and S. Osher, "Efficient Implementation of Essentially Non-Oscillatory Shock-Capturing Schemes, submitted to J. Comp. Physics.

5. S. Osher and J.A. Sethian, "Fronts Propagating with Curvature Dependent Speed; Algorithms Based on Hamilton-Jacobi Formulations", submitted to J. Comp. Physics.

6. Woodrow Whitlow, Jr., Mohamed M. Hafez, and S. Osher, "An Entropy Correction Method for Unsteady Full Potential Flows with Strong Shocks", submitted to AIAA Journal.

7. A. Bamberger, B. Engquist, L. Halpern, and P. Joly, "Higher Order Paraxial Wave Equation Approximation in Heterogeneous Media", Submitted to SIAM J. of Appl. Math..

8. A. Bamberger, B. Engquist, L. Halpern, and P. Joly, "Parabolic Wave Equation Approximations in Heterogeneous Media", submitted to SIAM J. of Appl. Math.

9. Bjorn Engquist, "Computation of Oscillatory Solutions to Hyperbolic Differential Equations", to be submitted.

10. A. Harten, S. Osher, B. Engquist, and S.R. Chakravarthy, "Some Results on Uniformly High-Order Accurate Essentially Non-Oscillatory Schemes", Appl. Num. Math., V. 2, (1986), pp. 347-377.

11. Thomas Y. Hou, "Convergence of Vortex Methods for Highly Oscillatory Vor-

ticity Distribution", to be submitted.

12. B. Engquist and T.Y. Hou, "Particle Method Approximation of Oscillatory Solutions to Hyperbolic Differential Equations", to be submitted.

13. Thomas Y. Hou, "Some New Homogenization Results for Discrete Boltzmann Equations with Oscillatory Data", to be submitted.

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Outline of research findings:

UNIFORMLY HIGH-ORDER ACCURATE NONOSCILLATORY SCHEMES. I*

AMI HARTEN† AND STANLEY OSHER‡

Abstract. We begin the construction and the analysis of nonoscillatory shock capturing methods for the approximation of hyperbolic conservation laws. These schemes share many desirable properties with total variation diminishing schemes, but TVD schemes have at most first-order accuracy, in the sense of truncation error, at extrema of the solution. In this paper we construct a uniformly second-order approximation, which is nonoscillatory in the sense that the number of extrema of the discrete solution is not increasing in time. This is achieved via a nonoscillatory piecewise-linear reconstruction of the solution from its cell averages, time evolution through an approximate solution of the resulting initial value problem and an average of this approximate solution over each cell.

THE DISCRETE ONE-SIDED LIPSCHITZ CONDITION FOR CONVEX SCALAR CONSERVATION LAWS*

YANN BRENIER† AND STANLEY OSHER‡

Abstract. Physical solutions to convex scalar conservation laws satisfy a one-sided Lipschitz condition (OSLC) that enforces both the entropy condition and their total variation boundedness. Consistency with this condition is therefore desirable for a numerical scheme and was proved for both the Godunov and the Lax-Friedrichs scheme—also, in a weakened version, for the Roe scheme, all of them being only first order accurate. A new, fully second order scheme is introduced here, which is consistent with the OSLC. The modified equation is considered and shows interesting features. Another second order scheme is then considered and numerical results are discussed.

Some New Homogenization Results for Discrete Boltzmann Equations with Oscillatory Data

Thomas Y. Hou

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Abstract. The behaviors of multi-dimensional discrete Boltzmann systems with highly oscillatory data are studied. Homogenized equations for the mean solutions have been obtained. Uniform convergence of the solutions of the oscillatory equations to the solutions of the homogenized equations has been established. Moreover, we find that the weak limits of the oscillatory solutions for the Broadwell type of models are not continuous functions of the discrete velocities. Generalization of the above results to problems with multiple scale initial data is established.

Particle Method Approximation of Oscillatory
Solutions to Hyperbolic Differential Equations

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ABSTRACT

Particle methods approximating hyperbolic partial differential equations with oscillatory solutions are studied. Convergence is analyzed for approximations for which the continuous solution is not well resolved on the computational grid. Highly oscillatory solution to the Broadwell and variable coefficient Carleman model is considered. Numerical approximation of more general systems are discussed. Numerical examples are given.

SOME RESULTS ON UNIFORMLY HIGH-ORDER ACCURATE ESSENTIALLY NONOSCILLATORY SCHEMES

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We continue the construction and the analysis of essentially nonoscillatory shock capturing methods for the approximation of hyperbolic conservation laws. These schemes share many desirable properties with total variation diminishing schemes, but TVD schemes have at most first-order accuracy in the sense of truncation error, at extrema of the solution. In this paper we construct an hierarchy of uniformly high-order accurate approximations of any desired order of accuracy which are tailored to be essentially nonoscillatory. This means that, for piecewise smooth solutions, the variation of the numerical approximation is bounded by that of the true solution up to $O(h^{R-1})$, for $0 < R$ and h sufficiently small. The design involves an essentially nonoscillatory piecewise polynomial reconstruction of the solution from its cell averages, time evolution through an approximate solution of the resulting initial value problem, and averaging of this approximate solution over each cell. To solve this reconstruction problem we use a new interpolation technique that when applied to piecewise smooth data gives high-order accuracy whenever the function is smooth but avoids a Gibbs phenomenon at discontinuities.

1. Introduction

In this paper we consider numerical approximations to weak solutions of the hyperbolic initial value problem (IVP)

$$u_t + f(u)_x = 0 = u_t + a(u)u_x, \tag{1.1a}$$

$$u(x, 0) = u_0(x). \tag{1.1b}$$

Here u and f are m vectors, and $a(u) = \partial f/\partial u$ is the Jacobian matrix, which is assumed to have only real eigenvalues and a complete set of linearly independent eigenvectors.

The initial data $u_0(x)$ are assumed to be piecewise-smooth functions that are either periodic or of compact support.

Let $v_j^n = v_h(x_j, t_n)$, $x_j = jh$, $t_n = n\tau$, denote a numerical approximation in conservation form.

$$v_j^{n+1} = v_j^n - \lambda \left(\hat{f}_{j+1/2} - \hat{f}_{j-1/2} \right) = (E_h \cdot v^n)_j. \tag{1.2a}$$

Convergence of Vortex Methods for Highly Oscillatory Vorticity Distribution

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Abstract. Vortex methods for two dimensional inviscid, incompressible flow with highly oscillatory vorticity distribution are studied. A homogenization result for Euler equations is obtained up to a finite time. Convergence of vortex methods in the case when continuous vorticity is not well resolved on the computational grid is analylized. Numerical results are given. Comparisons are made with the corresponding finite difference approximation.
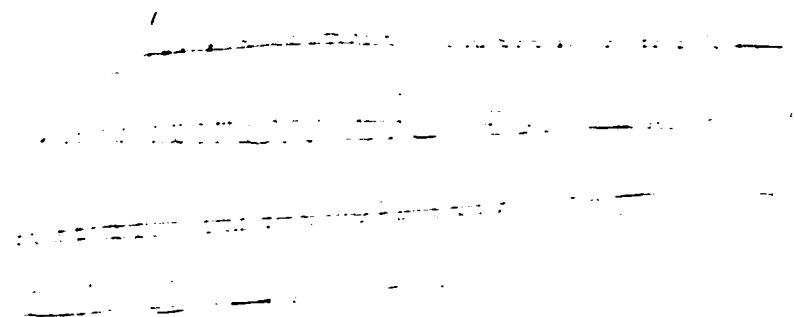
An hierarchy of uniformly high order accurate essentially non-oscillatory shock capturing algorithms was developed. Some theory and numerical experimentation was done.

A correction to the unsteady full potential equation for flows with strong shocks was obtained. This modification inputs the correct entropy jumps at shocks. Numerical experiments on airfoils were sucessfully performed.

A new family of paraxial wave approximations was derived and was applied to computational problems in seismology, underwater acoustics and artificial boundaries. Theoretical and experimental results were obtained. The family also included variants of parabolic approximations of scalar wave equations.

A method for the computation of highly oscillatory solutions to hyperbolic equations was obtained. A convergence concept which makes analysis possible in the practical situation in which not all frequencies are well resolved is developed. Convergence of an average approximation is established for a general class of methods. Applications to particle methods were also obtained.

AN ENTROPY CORRECTION METHOD FOR UNSTEADY FULL POTENTIAL FLOWS WITH STRONG SHOCKS

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Abstract

An entropy correction method for the unsteady full potential equation is presented. The unsteady potential equation is modified to account for entropy jumps across shock waves. The conservative form of the modified equation is solved in generalized coordinates using an implicit, approximate factorization method. A flux-biasing differencing method, which generates the proper amounts of artificial viscosity in supersonic regions, is used to discretize the flow equations in space. Comparisons between the present method and solutions of the Euler equations and between the present method and experimental data are presented. The comparisons show that the present method more accurately models solutions of the Euler equations and experiment than does the isentropic potential formulation.

Nomenclature

a	speed of sound
A_1, A_2, A_3	metrics of coordinate transformation, Equation (7)
c	airfoil chord
h	time step in computational space
i,j	indices of grid points
I	identity matrix
J	Jacobian of coordinate transformation
k	reduced frequency based on semichord
M	Mach number
q	total speed
r	distance from point of rotation to a point in the flow field
R	gas constant
s	entropy
t	time
U,W	contravariant velocities, Equation (6)
x,z	cartesian coordinates
α	angle of attack
β	$\rho^{2-\gamma}/J$
γ	ratio of specific heats
Γ	jump in potential across the wake
δ	difference operator
ξ, ζ	computational coordinate directions
θ	polar angle
θ_s	airfoil pitch angle
ρ	density
$\bar{\rho}$	ρ/J
$\tilde{\rho}$	biased density
τ	computational time
ϕ	velocity potential
$\langle \rangle$	average of quantity across the wake

Subscripts

i	isentropic
n	normal to shock
o	steady airfoil location
p	pitch axis location
s	points on airfoil
1	minimum airfoil pitch angle
2	maximum airfoil pitch angle
∞	free stream conditions

Superscripts

ℓ	lower side of wake
n	iteration number
u	upper side of wake
*	sonic conditions

Introduction

Modern aircraft typically operate at high speeds where aeroelastic instabilities are more likely to occur. To successfully predict and analyze such aeroelastic phenomena, the aircraft designer needs methods that accurately predict the aerodynamic loads--steady and unsteady--that the vehicle experiences. Since many critical aeroelastic phenomena occur at transonic speeds, methods based on linear aerodynamic theory cannot accurately predict many aeroelastic responses. Thus, it is necessary to use an aerodynamic method that can predict time-accurate solutions of nonlinear flows and that can accurately model shock waves and their unsteady motions.

When shock waves appear in transonic flow fields, aerodynamic loads predicted using potential flow theory can be grossly inaccurate and even multivalued. Multiple solutions of the potential equation were first observed in two dimensions by Steinhoff and Jameson.[1] Salas and Gumbert[2] showed that the phenomenon is not

On Numerical Particle Methods

Yizhao Hou

Introduction

Particle Methods have received a great deal of attention in the past few years. In Fluid Dynamics, Vortex Methods have been successfully used to simulate high Reynold number fluid flows. They were first introduced by Rosenhead [33] and were subsequently developed by Chorin , Leonard among other contributors (see Ref[25] for a detailed bibliography). Applications of Vortex Methods include turbulent combustion([15],[34]), aerodynamic calculations([26],[9]), flows through heart valves [25] and flows of variable density [1].

In Physics, on the other hand, Particle Methods have been used for numerical solutions of kinetic equations such as Boltzmann equations, Vlasov equations. They are currently used for studying a number of physical phenomena, for instance the lasermatter interaction in inertial confinement fusion problems. See the book of Hockney and Eastwood [18] for more information along this line.

It has been observed in practice that Particle Methods are often used for the problems in which the number of particles is too small for the traditional asymptotic error estimate to be applicable, and yet very useful results are still possible (see, eg. [30]) . The previous mathematical analysis for Particle Methods (see, eg. [16], [4], [5] and [32]) have not been able to give a satisfactory explanation to this feature of Particle Methods .

PARABOLIC WAVE EQUATION APPROXIMATIONS

IN HETEROGENEOUS MEDIA

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     Work partly supported by an INRIA-IFP-SNEA (P) contract.

ABSTRACT

Different variants of parabolic approximations of scalar wave equations are derived and their properties are analysed. These equations are of general form which includes the ones used in seismology, underwater acoustics and other applications. A new version of the parabolic approximation is derived for heterogeneous media. It has optimal properties with respect to wave reflection at material interfaces. The amplitudes of the reflected and transmitted waves depend continuously on the interface. Existence, uniqueness and energy estimates are proved.

ABSTRACT

A new family of paraxial wave equation approximations is derived.
These approximations are of higher order accuracy than the parabolic approxi-
mation and they can be applied to the same computational problems e.g in
seismology, underwater acoustics, and as artificial boundary counditions. The
equations are written as systems which simplify computations. The support and
singular support are studied, energy estimates are given which prove the well-
posedness. The reflection and transmission are showed to be continuously depen-
ding on material interfaces in heterogeneous media.

COMPUTATION OF OSCILLATORY SOLUTIONS

TO HYPERBOLIC DIFFERENTIAL EQUATIONS

BJORN ENGQUIST[1]

Abstract. Difference approximations of hyperbolic partial differential equations are studied. The error is analyzed in approximations for which the continuous solution is not well resolved on the computational grid. A convergence concept is introduced with the purpose of making analysis possible and describing the practical situation in which not all frequences in the solution are well resolved. Convergence of an averaged approximation for linear problems is established for a general class of methods. For special methods and a weakly nonlinear problem it is showed by numerical experiments and in one case by analysis that the discrete approximation converges to the solution of the homogenized equation.

Efficient Implementation of Essentially Non-Oscillatory Shock Capturing Schemes

Chi-Wang Shu
and
Stanley Osher

In the computation of discontinuous solutions of hyperbolic conservation laws, TVD (total-variation-diminishing), TVB (total-variation-bounded) and the recently developed ENO (essentially non-oscillatory) schemes have proven to be very useful. In this paper two improvements are discussed: a simple TVD Runge-Kutta type time discretization, and an ENO construction procedure based on fluxes rather than on cell averages. These improvements simplify the schemes considerably -- especially for multi-dimensional problems or problems with forcing terms. Preliminary numerical results are also given.

Fronts Propagating with Curvature Dependent Speed: Algorithms based on Hamilton-Jacobi Formulations

Stanley Osher
and
James A. Sethian

We devise new numerical algorithms, called PSC algorithms, for following fronts propagating with curvature-dependent speed. The speed may be an arbitrary function of curvature, and the front can also be passively advected by an underlying flow. These algorithms approximate the equations of motion, which resemble Hamilton-Jacobi equations with parabolic right-hand-sides, by using techniques from the hyperbolic conservation laws. Non-oscillatory schemes of various orders of accuracy are used to solve the equations, providing methods that accurately capture the formation of sharp gradients and cusps in the moving fronts. The algorithms handle topological merging and breaking naturally, work in any number of space dimensions, and do not require that the moving surface be written as a function. The methods can be also used for more general Hamilton-Jacobi-type problems. We demonstrate our algorithms by computing the solution to a variety of surface motion problems.

Efficient Implementation of Essentially

Non-Oscillatory Shock Capturing Schemes

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Abstract

In the computation of discontinuous solutions of hyperbolic conservation laws, TVD (total-variation-diminishing), TVB (total-variation-bounded) and the recently developed ENO (essentially non-oscillatory) schemes have proven to be very useful. In this paper two improvements are discussed: a simple TVD Runge-Kutta type time discretization, and an ENO construction procedure based on fluxes rather than on cell averages. These improvements simplify the schemes considerably – especially for multi-dimensional problems or problems with forcing terms. Preliminary numerical results are also given.[23]

Fronts Propagating with Curvature Dependent Speed:

Algorithms based on Hamilton-Jacobi Formulations

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We devise new numerical algorithms, called PSC algorithms, for following fronts propagating with curvature-dependent speed. The speed may be an arbitrary function of curvature, and the front can also be passively advected by an underlying flow. These algorithms approximate the equations of motion, which resemble Hamilton-Jacobi equations with parabolic right-hand-sides, by using techniques from the hyperbolic conservation laws. Non-oscillatory schemes of various orders of accuracy are used to solve the equations, providing methods that accurately capture the formation of sharp gradients and cusps in the moving fronts. The algorithms handle topological merging and breaking naturally, work in any number of space dimensions, and do not require that the moving surface be written as a function. The methods can be also used for more general Hamilton-Jacobi-type problems. We demonstrate our algorithms by computing the solution to a variety of surface motion problems.

Uniformly High Order Accurate Essentially
Non-oscillatory Schemes, III

Ami Harten
Bjorn Engquist
Stanley Osher
Sukumar R. Chakravarthy

We continue the construction and the analysis of essentially non-oscillatory shock capturing methods for the approximation of hyperbolic conservation laws. We present an hierarchy of uniformly high-order accurate schemes which generalizes Godunov's scheme and its second-order accurate MUSCL extension to an arbitrary order of accuracy. The design involves an essentially non-oscillatory piecewise polynomial reconstruction of the solution from its cell averages, time evolution through an approximate solution of the resulting initial value problem, and averaging of this approximate solution over each cell. The reconstruction algorithm is derived from a new interpolation technique that, when applied to piecewise smooth data, gives high-order accuracy whenever the function is smooth but avoids a Gibbs phenomenon at discontinuities. Unlike standard finite difference methods this procedure uses an adaptive stencil of grid points and, consequently, the resulting schemes are highly non-linear.

Uniformly High Order Accurate Essentially Non-oscillatory Schemes, III

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DEDICATED TO PETER LAX ON HIS 60TH BIRTHDAY

We continue the construction and the analysis of essentially non-oscillatory shock capturing methods for the approximation of hyperbolic conservation laws. We present an hierarchy of uniformly high-order accurate schemes which generalizes Godunov's scheme and its second-order accurate MUSCL extension to an arbitrary order of accuracy. The design involves an essentially non-oscillatory piecewise polynomial reconstruction of the solution from its cell averages, time evolution through an approximate solution of the resulting initial value problem, and averaging of this approximate solution over each cell. The reconstruction algorithm is derived from a new interpolation technique that, when applied to piecewise smooth data, gives high-order accuracy whenever the function is smooth but avoids a Gibbs phenomenon at discontinuities. Unlike standard finite difference methods this procedure uses an adaptive stencil of grid points and, consequently, the resulting schemes are highly non-linear.   © 1987 Academic Press, Inc