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Saturation and the "Universal" Spectrum for Vertical Profiles of Horizontal Scalar Winds in the Atmosphere

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A theory is presented which explains the universal nature of one-dimensional vertical wave number, k, power spectral densities (PSDs) of horizontal winds as measured in the atmosphere and predicted by VanZandt. The theory is that the PSD amplitude at any given wave number (greater than a certain minimum, k_*) is determined by its saturation value due either to shear instability (i.e., critical Richardson Number) or, more likely, to convective instability. This explains why the PSD amplitudes observed do not grow exponentially with increasing altitude. This saturation theory assumption plus other considerations leads to a PSD of the form N^2/k^n, where n is in the range of about 2.5 to 3 and N is the Brunt frequency. A simplified model involving superimposed narrow bands of gravity waves as well as a model based merely on dimensional arguments both lead to $n = 3$. The full model not only explains the observed spectral slopes but also predicts the PSD amplitude in the troposphere to be 3.5 times smaller than in the stratosphere. The derivation of the model is based on the saturation condition that $\int k^2 \text{PSD}(k)\, dk = N^2$. The model may also apply to the ocean and explain the Garrett-Munk vertical wave number spectrum. Keywords: reprints; fluid mechanics; atmospheric physics. (JD/KT)

1. INTRODUCTION

In previous communications [*Dewan et al.*, 1984] it was shown that the vertical wave number power spectral densities (PSDs) of horizontal winds in the stratosphere were nearly identical over the wavelength range of 40 m to 1 km. This occurred in spite of the fact that the data were obtained at different times of day (dawn versus dusk), during different seasons (fall versus spring), and from different geographic locations (White Sands Missile Range, Wallops Island, and Fort Churchill). As can be seen in Figure 1, both the slopes and amplitudes of the spectra are strikingly similar. *VanZandt* [1982] and Cira (1984) were the first to make the suggestion that the internal buoyancy waves of the atmosphere may have universal PSDs and to give the empirical basis behind it. The purpose of the present report is to present a physical explanation of both the slope ($n = -3$ approximately) and universal amplitude of the vertical wave number PSDs. The theory has an important, testable prediction, namely, that there should be small departures from a constant amplitude of the PSD, the largest being that the amplitudes of the tropospheric PSDs should be smaller by a factor of 3 or 4 than those from the stratosphere. The latter prediction is included in the spectral model of *Garrett and Munk* [1972]; however, the present theory gives a physical mechanism to explain it as regards atmospheric spectra.

2. SATURATION MODEL DUE TO KELVIN-HELMHOLTZ INSTABILITY

The remarkably constant PSD amplitudes found in the work by *Dewan et al.* [1984] strongly suggest that the explanation involves "saturation." In a work by *Phillips* [1977, section 5.4] there is a description of a saturation theory for the wave PSD. It involves internal waves saturating by means of shear or Kelvin-Helmholtz instability induced turbulence. The first model to be given here is closely related to Phillips's theory; however, it differs from it in several ways. For example, we assume a constant buoyancy frequency over the altitude of interest, whereas Phillips assumes it to be appreciable only within a narrow thermocline and negligible outside of it. In addition, we consider a continuum of modes, whereas he considers only the lowest mode. Finally, we consider the one-dimensional vertical wave number spectrum of the horizontal velocities, whereas he obtained the two-dimensional horizontal wave number spectrum for the vertical displacement of a surface in the thermocline. Despite these differences, the key element (namely, saturation by means of shear-induced turbulence) is exactly the same. In addition, our mathematical approach will be identical to that of Phillips. Our second, but related, model involves saturation through convective instability induced turbulence. Both models give the same spectral shape and dependence on buoyancy frequency. Convective instability-induced saturation and saturation in general have been extensively reviewed by *Fritts* [1984]. The first treatment of saturation by convective instability was by *Hodges* [1967]. In section 5 a more exact but less physically obvious approach will be given.

In the following we shall very briefly review some basic information that we need for this problem. The reader will find extensive discussions by *Rayleigh* [1883], *Gill* [1982], *Phillips* [1977, section 5.2], *Turner* [1973], and *Lighthill* [1978]. We commence by listing the assumptions that we shall need. These too are discussed at length in the cited references. (1) The earth's rotation has negligible effects [see *Gill*, 1982, p. 207]. (2) The fluid is statically stable. (3) The wave frequency, ω, is less than the buoyancy frequency, N. Finally, (4) we shall assume the validity of the Boussinesq approximation.

Regarding assumption (4), it implies the following conditions. (1) Effects of variations in density are negligible with regard to acceleration or inertial effects. (2) The velocity fluctuations are small enough to make nonlinear effects negligible. (3) Density fluctuations consist of small deviations from equilibrium values. (4) The vertical scale of motion or vertical wavelength is much less than a scale height, H. We shall also ignore viscous effects and assume that the wavelengths are small enough to allow one to ignore acoustic effects. This last condition is assured by the fact that our wavenumbers, k, are significantly larger than N/c, where c is the speed of sound and N is the buoyancy frequency. Also, we assume that variations of density, ρ, with respect to altitude, z, are much smaller than variations of vertical velocity with respect to height (a more explicit form of condition (4) above). Finally,

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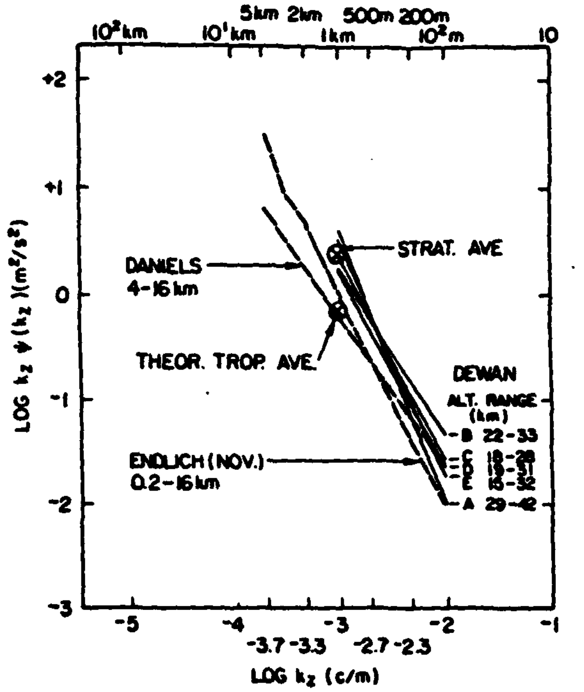

Fig. 1. PSDs from tropospheric and stratospheric measurements of vertical profiles of horizontal winds: Dashed curves (tropospheric) are from *Daniels* [1982] and *Endlich and Singleton* [1969] (see Figure 2, curve marked November); solid curves (stratospheric) are from *Dewan et al.* [1984]. With the exception of Endlich's PSD, these PSDs are the fitted lines. The stratospheric curves are based on Table 1 but have been extrapolated to $\lambda = 100$ m.

we leave out of account mean winds and consider only the perturbation velocities.

Newton's second law, in terms of fluctuations, is then given by

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial P'}{\partial x} \quad (1)$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial P'}{\partial y} \quad (2)$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial P'}{\partial z} - \rho' g \quad (3)$$

where ρ_0 is the equilibrium density, ρ' is the departure from ρ_0, g is the acceleration of gravity, P' is the departure of pressure from equilibrium, t is the time, and u, v, and w are components of velocity fluctuations in the x, y, and z directions, respectively. The equations of continuity for the incompressible fluid are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

and

$$\frac{\partial \rho'}{\partial t} + w \frac{d\rho_0}{dz} = 0 \quad (5)$$

We next employ the standard procedure to eliminate the pressure and density terms. Details are to be found in, for example, the work by *Gill* [1982, p. 129]. One thus finds the well known equation:

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + N^2 \nabla_H{}^2 w = 0 \quad (6)$$

where

$$N^2 = -\left[\frac{g}{\rho_0}\right]\frac{d\rho_0}{dz} \quad (7)$$

is the buoyancy frequency squared [*Rayleigh*, 1883]. With compressible fluids, a term $(-g/c^2)$ is added to the right side of (7). The approximation is used that

$$\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{\partial}{\partial z} w\right) \approx \frac{\partial^2 w}{\partial z^2} \quad (8)$$

(this corresponds to $|k_z| \gg |1/H|$) and where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
$$\nabla_H{}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (9)$$

Equation (6) is very closely related to the Taylor-Goldstein equation.

Under our assumptions (N^2 is constant with respect to z) it is known that (6) possesses wavelike solutions of the form [*Phillips*, 1977]

$$w = W_0 \exp\{i(k_x x + k_y y + k_z z - \omega t)\} \quad (10)$$

provided that the following condition is met:

$$\frac{\omega^2}{N^2} = \frac{k_x{}^2 + k_y{}^2}{k_x{}^2 + k_y{}^2 + k_z{}^2} \quad (11)$$

where the right side is the square of the cosine of the angle between k and the horizontal.

As for the horizontal velocities, call them v_H, they also have a wave solution of form

$$v_H = V_0 \exp\{i(k_x x + k_y y + k_z z - \omega t)\} \quad (12)$$

provided (from (4))

$$\mathbf{k} \cdot \mathbf{v} = 0 \quad (13)$$

where $\mathbf{v}$ is the three-dimensional fluctuation velocity. In other words, particle motion is limited to directions which are perpendicular to the wavenumber vector. Assuming conditions (13) and (11) and hence the validity of (12) and (10), we can write down the following relations which are crucial for the remainder of our argument:

$$\frac{\partial v}{\partial z} = ik_z v$$
$$\frac{\partial u}{\partial z} = ik_z u \quad (14)$$

We now turn to wave saturation and commence with the one due to shear instability. At the outset it should be mentioned that we treat the shear instability here mainly for reasons of making our explanation clear. It is easier to consider the convective instability (i.e., the one which is clearly the more probable one, since it is 4 times lower in threshold) as a second step in the discussion. As has been pointed out originally by *Miles* [1961, 1963] and *Howard* [1961], the Richardson number, Ri (defined as N^2/S^2, where S is the vertical shear of the horizontal velocity), obeys the following relation as the necessary condition for dynamic instability (leading to turbulence):

$$R_i < \tfrac{1}{4} \quad (15)$$

We shall follow *Phillips* [1977] and take this as being also the sufficient condition for instability. This is valid for practical purposes even though it is not rigorously true mathematically. Thus the condition on shear at the border line of turbulence is

$$S_{cr}^2 = \left(\left[\frac{\partial v}{\partial z}\right]^2 + \left[\frac{\partial u}{\partial z}\right]^2\right)_{max} = 4N^2 \tag{16}$$

Note that the subscript "max" implies that (16) holds for the maximum shears along a wave train. Equation (16) can be simplified, without loss of generality, by choosing a coordinate system so that one horizontal axis, say the y axis, is in the direction of the horizontal velocity. It can be written then as

$$S_{cr}^2 = \left(\frac{\partial u}{\partial z}\right)_{max}^2 = 4N^2 \tag{17}$$

In order to closely examine the physical significance of (17) we consider a numerical example. Let

$$u = u_0 \sin (k_z z) \tag{18}$$

At critical shear, (18) gives

$$\left|\left(\frac{\partial u}{\partial z}\right)_{max}\right| = \left|k_z u_0 \cos k_z z\right| = 2N \tag{19}$$

where the argument of the cosine must be 0° or 180°. For specificity, let $N = 2 \times 10^{-2}$ s^{-1} and $\lambda_z = 500$ m. From (19) the critical velocity amplitude is defined as

$$(u_0)_{cr} = \frac{\lambda_z}{2\pi}(2N) = 3.18 \text{ m/s} \tag{20}$$

If u_0 had exactly the value 3.18 m/s, the wave would be at, but not over, the threshold of instability. In order that the instability grow in a finite time, the value of u_0 must exceed 3.18 m/s. Consider the case of $u_0 = 3.19$ m/s. It is easy to show from (19) that the shear will exceed the critical value over two small regions, of depth 10 m, in a wavelength of 500 m. In other words, each turbulent region in a wavelength will occupy 2% of the wavelength, and there will be only two such regions in one wavelength. From this illustration it is clear that (1) the thickness of a turbulent layer can be very much smaller than the wavelength of the wave that caused it (this is true despite the fact that subsequent spreading or other thickening mechanisms are possible) and (2) there is no fixed relation between λ and turbulent layer thickness. Finally, it is clear that waves which have velocity amplitudes above threshold will lose energy to turbulent dissipation until they reach the critical limit. When the latter occurs, the dissipation mechanism will no longer operate. This is a classic case of "saturation."

Our goal is to relate the one-dimensional PSD, $\psi(k_z)$ to the saturation condition. First, we wish to start (from (20)) with

$$(u_0)_{cr}^2 = \frac{4N^2}{k_z^2} \tag{21}$$

As mentioned above, we duplicate the procedure given by Phillips. The first step is to review the fact that (as is well known) the total variance of the fluctuations is equal to the PSD integrated over all wave numbers [*Phillips*, 1977, section 5.4, p. 220].

$$\int_{\text{all } k_z} \psi(k_z)\, dk_z = \sum_{\text{all } k_z} \langle u^2(k_z)\rangle_{k_z\text{-band}} = \langle u^2\rangle \tag{22}$$

In the special case of a sine wave (e.g., (18)) which has a single wave number it is well known that

$$\tfrac{1}{2}u_0^2 = \langle u^2\rangle \tag{23}$$

Of course, if one considers a band of Fourier components around a given k, the variance $\langle u^2\rangle$ would not be exactly equal to $\frac{1}{2}u_0^2$ of the main component. However, the variance of such a band would be proportional to u_0^2,

$$\langle u^2\rangle_{k_z\text{-band}} \propto u_0^2(k_z) \tag{24}$$

For such a band the integral of the PSD reduces to

$$\int_{k_z\text{-band}} \psi(k_z)\, dk_z \doteq \psi(k_z)\Delta k_z \propto u_0^2(k_z) \tag{25}$$

Next, we again follow Phillips and assume that the band width, Δk_z, is proportional to the central wave number, i.e.,

$$\Delta k_z \propto k_z \tag{26}$$

and we turn our attention to the case where there are many waves containing such bandwidths. As Phillips put it, one can visualize the field of gravity waves as "a random succession of wave groups at different wavenumbers." In this case, the contribution to the mean square velocity fluctuations from each wave number region is

$$\langle u^2(k_z)\rangle_{k_z\text{-band}} \propto \psi(k_z)k_z \tag{27}$$

where

$$\langle u^2\rangle = \sum_{\text{all } k_z} \langle u^2(k_z)\rangle_{k_z\text{-band}} = \int_{\text{all } k_z} \psi(k_z)\, dk_z \tag{28}$$

Next, as in the work by *Phillips* [1977], we make the assumption that all the wave number components are limited individually by the saturation condition, (17). We then have, with the help of (24) and (27),

$$\psi(k_z)k_z \propto u_0^2(k_z) \tag{29}$$

and, from (21),

$$\psi(k_z) \propto \frac{4N^2}{k_z^3} \tag{30}$$

This is the result we were seeking.

Note that in (30) one has a proportionality, not an equality (as was also true in Phillips's treatment). Note also the basic assumptions: (1) each wave number band saturates by the shear instability, (2) the wave bands Δk_z scale as k_z, and (3) the wave bands saturate independently. These assumptions of Phillips have met with success for the case he studied. However, these assumptions are not needed in the more exact treatment shown in section 5 below.

There is a far shorter and more direct method to obtain (30). It is by the "principle of similitude" or "dimensional analysis." This approach's validity rests completely on the initial assumption made. From (15) we can assume that if saturation due to the shear instability determines ψ, then ψ must depend upon the critical shear $S_c^2 = 4N^2$ (let N be constant). The only other variable that ψ would depend upon in this case is k_z. Using the fact that the dimensions of ψ are $[L^3T^{-2}]$ and those of S_c and k_z are $[T^{-1}]$ and $[L^{-1}]$, respectively, one immediately obtains

$$\psi \propto \frac{S_c^2}{k_z^3} = \frac{4N^2}{k_z^3} \tag{31}$$

where a universal constant of proportionality of order unity is left to be determined by means of experiment. This does raise the question of how many of the assumptions made above are actually necessary to arrive at (31). On the other hand, it appears that our use of $\Delta k_z \propto k_z$ is a crucial one, and the dimensional argument leaves this rather well hidden. It is for this reason that we have given both derivations in this paper.

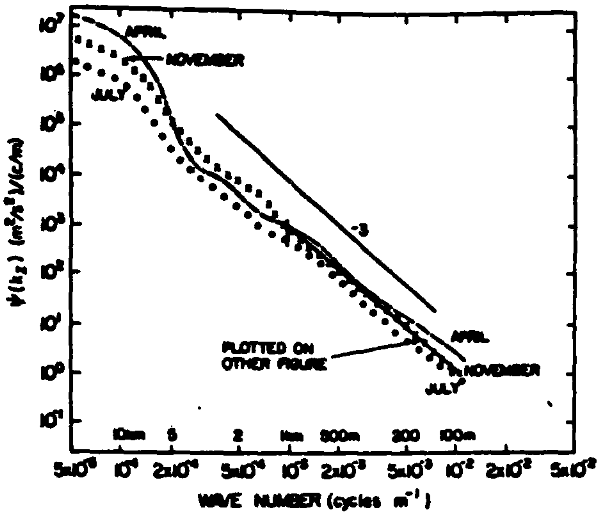

Fig. 2. After *Endlich and Singleton* [1969, Figure 10]. Smoothed spectra for sequences of wind profiles on April 8, 1966, July 5, 1966, and November 10, 1965. The "November" curve is plotted to $\lambda = 5$ km in Figure 1 for comparison.

Note that *Lumley* [1964] arrived at a similar dependence to that given in (31), for ψ, but he did so in an entirely different context, namely, that of buoyancy subrange turbulence. All dimensional arguments are subject to possible omission of underlying physics, as this case illustrates.

3. Saturation Model Due to Convective Instability

Hodges [1967] was the first person to suggest that the saturation due to convective instability could limit the amplitudes of upward propagating gravity waves. *Fritts* [1984] has given an excellent review of this work and related subjects. We now consider the effect of this particular mechanism upon $\psi(k_z)$, the vertical wave number PSD.

The convective instability condition [see *Fritts*, 1984] is given by

$$\frac{d\bar{\theta}}{dz} + \frac{d\theta'}{dz} \le 0 \qquad (32)$$

where $\bar{\theta}$ and θ' are the mean and perturbation values of the potential temperature. Using a truncated Taylor expansion, it is easy to show

$$\theta' = -h\frac{d\bar{\theta}}{dz} \qquad (33)$$

where h is the vertical displacement of the fluid parcel causing the fluctuation θ', but, with the help of (10),

$$h = \int w\,dt = -\frac{w}{i\omega} \qquad (34)$$

hence, from (33)

$$\theta' = \left(\frac{d\bar{\theta}}{dz}\right)\left(\frac{w}{i\omega}\right) \qquad (35)$$

Taking the derivative of (35) with respect to z with $d\bar{\theta}/dz$ constant,

$$\frac{d\theta'}{dz} = \frac{d\bar{\theta}}{dz}\left(\frac{ik_z}{i\omega}\right)w \qquad (36)$$

and using (32) with the equality sign to denote the onset of instability, we obtain the critical velocity as

$$(w)_{cr} = -\frac{\omega}{k_z} \qquad (37)$$

To simplify calculations, let us consider this in the two dimensions of x and z, i.e., let u stand for horizontal velocity. Then, from (13) and (37),

$$-k_x(u)_{cr} = (w)_{cr}k_z \qquad (38)$$

Using (38) in (37) gives

$$(u)_{cr} = \frac{\omega}{k_x} \qquad (39)$$

i.e., u is equal to the horizontal phase velocity, and using (11) with $k_y = 0$,

$$(u)_{cr} = \frac{N}{(k_x^2 + k_z^2)^{1/2}} \sim \left(\frac{N}{k_z}\right) \qquad (40)$$

where we have followed the argument of *Fritts* [1984] that $k_z \gg k_x$.

At this point one needs merely to repeat the arguments following (21) in order to arrive at

$$\psi(k_z) \propto \frac{N^2}{k_z^3} \qquad (41)$$

which shows that the saturation amplitude is 4 times smaller than would be given by the shear instability mechanism. We shall assume that the convective instability controls saturation in the more exact treatment of section 5 below. (See *Gossard and Hooke* [1975] for further discussion of this instability.) The approach for the shear instabiity would be altered by replacing $4N^2$ by N^2. In addition, the same dimensional analysis derivation would be valid, as was given above. There seems to be little question that the convective mechanism is the controlling factor in saturation when compared to the shear mechanism.

4. Comparison With Experimental Results

4.1. *PSD Log-Log Slopes*

Endlich and Singleton [1969] published PSDs from vertical profiles of horizontal winds that were measured by means of radar-tracked jimspheres. These measurements were in the 200 m to 16 km altitude region, and the PSDs covered a wavelength range of 100 m to 20 km (see Figures 1 and 2). Over the wavelength range of 100 m to 5 km they observed slopes of $-\frac{5}{2}$. When the range is reduced to 100 m to 7 km, the slope is close to -3, in agreement with our theory. At very large wavelengths ($\lambda > 5$ km) they saw a pronounced change (see

TABLE 1. Fitted Log PSD Amplitudes and Slopes of PSDs From *Dewan et al.* [1984].

Label	Wind Profile	Log PSD Amplitudes (From Fitted Line) at $\lambda = 1$ km, $(m/s)^2/(c/m)$	Slope
A	May 22, 1978	3.60	-3.59 ± 0.16
B	May 20, 1978	3.28	-2.61 ± 0.21
C	April 26, 1977	3.41	-2.99 ± 0.09
D	May 2, 1978	3.27	-2.91 ± 0.19
E	Sept. 12, 1978	3.55	-3.26 ± 0.07
Average		3.42 ± -0.15	-3.0 ± 0.33

The range is from $\lambda = 1$ km to $\lambda = 200$ m.

TABLE 2. N^2 Scaling of Spectral Amplitudes

Altitude Range, km	Region	$N^2 \times 10^{-4}$, $(rad/s)^2$	N, rad/s	Ratio of N_t^2/N^2 Upper Stratosphere	Inverse Ratio
0–10	Troposphere (3, 6, 10 km)	1.32	0.0115	0.286	3.50
12–20	Lower stratosphere	4.41	0.0210	0.955	1.05
21–50	Upper stratosphere	4.62	0.0215	1	1.00
50–72	Lower mesosphere	2.74	0.0166	0.593	1.69
72–86	Upper mesosphere	3.63		0.786	1.272
86–91	Thermosphere	4.86		1.05	0.952

Based on US Standard Atmosphere (1976).

Figure 2). It is necessary to choose wavelengths less than 1 km to eliminate effects from the long wavelengths in order to observe the saturation effects.

Dewan et al. [1984] reported PSDs of such wind profiles measured by the smoke trail method in the altitude range of 13–37 km. They reported a slope of -3.07 ± 0.3 in the wavelength range of 200 m to 1 km, in agreement with theory (see Table 1). They also reported a slope, if a line were fitted from 1 km to 40 m, of -2.7 ± 0.2; however, in the upper end of their spectra, which had a Nyquist wavelength of 20 m, showed much curvature. By fitting up to 40 m, the half Nyquist value, it was presumed that aliasing effects would be negligible; however, other artifacts such as measurement error noise could extend to larger wavelengths. In view of this, one may possibly regard the slope over 1 km to 200 m as being the more valid slope. On the other hand, see section 5, which shows that the saturation hypothesis is compatible with both of these slopes.

Rosenberg et al. [1974] also reported PSDs based on NASA smoke trail measurements (vertical) in the altitude range of 5–18 km and wavelength range 100 m to 3.6 km. They cited an average slope of -2.75 ± 0.1. If they had restricted their wavelength range to the interval 1 km–100 m, then one would expect a somewhat steeper slope. This expected decrease in slope at smaller wave number can be seen in Figure 2, which is based on Endlich's data.

Finally, *Daniels* [1982] reported on a total of 1200 vertical profile measurements made with rawinsondes in the 4- to 16-km altitude range and wavelengths between 4 km and 80 m. His spectral slopes were in the neighborhood of -2.5 (see Figure 1). Since only the resulting fit and not the actual PSD is shown by Daniels, we believe the determination of the slope was dominated by the long-wavelength region. If the wavelength range were limited to 1 km to 80 m, then one would expect the slope to approach -3, as was the case with the Endlich data. Either slope, as is shown in section 5, is compatible with saturation.

In conclusion, observations of spectral slope over the 100 m to 1 km range appear to be consistent with the theoretical value of -3 in the sense that when all the observations are carefully considered, this slope is either directly supported or, at least, not ruled out. Contamination from long wavelength power is a problem, and careful PSD analysis is required to arrive at the theoretical slope.

4.2. *PSD Amplitudes*

Dewan [1984] reported the spectral slopes fitted to the PSD in the range of 1 km to 200 m. The average of the log amplitude of the fitted line (to the PSD) at $\lambda = 1$ km is 3.42 ± 0.15 (see Table 1). This small variability is good evidence for saturation indeed. Equation (30) however, implies that $\psi(k_z)$ scales as N^2, which is to say that the amplitude is only approximately constant in that it should vary between altitude regions where N^2 differs significantly. Table 2 lists values of N^2 for different altitude ranges. As can be seen, a significant difference in N^2, and hence ψ amplitude, should exist between stratospheric and tropospheric values. Theory predicts a factor of 3.5 difference in that case, the troposphere having the smaller amplitudes.

As can be seen from Figure 1, the value of *Daniels's* [1982] PSD at wavelength equal to 1 km (partly in the troposphere) is indeed significantly smaller in relative value than the stratospheric values of *Dewan et al.* [1984]. *Endlich and Singleton's* [1969] value falls between them. A cross is located at theoretical value using $N_{TROP}^2 = 1.32 \times 10^{-4}$ rad^2 s^{-2}. The theoretical tropospheric average (which is calculated from the stratosphere average by multiplying the latter by N_{TROP}^2/N_{STRAT}^2) is in perfect agreement with experiment. This observation can be considered as strong evidence that the PSD is determined by saturation.

Finally, it should be mentioned that the strongest evidence for our general saturation hypothesis lies in the following observation. The amplitude of upward propagating buoyancy waves if unsaturated increases at the inverse of the square root of the atmospheric density $(\rho)^{1/2}$ (see *Gossard and Hooke* [1975, p. 76]. This fact implies that the PSD amplitude, for any given wave number, would increase with altitude as the inverse of ρ. As can be seen from Table 3, this would cause PSD amplitudes to increase rapidly as a function of altitude. Since the experimental results of Figure 1 rule out such a dependence, the saturation hypothesis is strongly supported. (Note, however, that the smoke trail method used for velocity PSD's was limited to fair weather conditions.)

5. THE BROAD-BAND SATURATION MODEL

As has been previously mentioned, the "narrow band" treatment given above (wherein each band is assumed to saturate independently) is certainly only an approximation. The purpose of this section is to take into account the effects of all the other bands upon the saturation of a given band.

In order to ascertain the mean square shear due to the fluctuations one can, by definition, integrate the shear spec-

TABLE 3. Density, ρ, Falloff With Altitude

Altitude, km	Region	ρ,* (kg/m^3)	$(\rho/\rho_{TROP})^{-1}$
0	Troposphere	1.23	1
10	Lower stratosphere	0.414	2.97
30	Mid stratosphere	1.84×10^{-2}	66.9
50	Stratosphere	1.03×10^{-3}	1.19×10^3
90	Thermosphere	3.42×10^{-6}	3.6×10^5

*US Standard Atmosphere (1976).

TABLE 4. Sensitivity Test for Model

Variable	k_e, c/m	k_{max}, c/m	B(m²/s²)/(c/m)	n	Comments
k_e	...	2.5×10^{-2}	5×10^6	...	n increases a factor
	2×10^{-5}	2.5×10^{-2}	5×10^6	2.46	of 2 for a factor of
	5×10^{-5}	2.5×10^{-2}	5×10^6	3.07	5 increase in k_e
	1×10^{-4}	2.5×10^{-2}	5×10^6	5.97	(quite sensitive)
k_{max}	5×10^{-5}	...	5×10^6	...	n increases a factor
	5×10^{-5}	10^{-3}	5×10^6	2.68	of 1.18 for 3
	5×10^{-5}	5×10^{-3}	5×10^6	2.96	orders of magnitude
	5×10^{-5}	5×10^{-2}	5×10^6	3.10	increase in k_{max}
	5×10^{-5}	10^{-1}	5×10^6	3.12	(not sensitive
	5×10^{-5}	1.0	5×10^6	3.16	for this range of n)
B	5×10^{-5}	2.5×10^{-2}	...	...	n increases a factor
	5×10^{-5}	2.5×10^{-2}	5×10^5	2.45	of 1.3 for a factor
	5×10^{-5}	2.5×10^{-2}	1×10^6	2.61	of 16 increase in B
	5×10^{-5}	2.5×10^{-2}	2×10^6	2.79	(mild sensitivity)
	5×10^{-5}	2.5×10^{-2}	3×10^6	2.90	
	5×10^{-5}	2.5×10^{-2}	8×10^6	3.27	

$N^2 = 3.35 \times 10^{-6}$ Hz² for all cases. These results can be understood from $\int_0^{k_{max}} k^2\psi\, dk = N^2$. A steepened slope (larger n) decreases the integral, while the increase of k_e, k_{max}, and B increases the integral.

trum, $k^2\psi$, over all wave numbers. The borderline of the convective instability occurs (under the assumptions stated previously) when $Ri = 1$. Hence one arrives at (within a constant very close to unity)

$$\int_0^{k_{max}} k^2\psi\, dk = N^2 \tag{42}$$

The upper limit, k_{max}, is the wave number where the waves has been replaced, for the most part, by turbulence.

There is evidence that for wave numbers less than a certain value, k_e, the spectrum bends over from the constant slope regime to a regime which is flat and horizontal for the oceanic case [*Garrett and Munk*, 1972, 1975]. For the atmospheric case, T. E. VanZandt (private communication, 1985) has suggested the analytical expression of the form

$$\psi = B\left(1 + \frac{k}{k_e}\right)^{-n} \tag{43}$$

In the following we shall understand that the region where the PSD goes as k^{-n} is primarily governed by saturation, whereas in the flat, horizontal region there are only indirect contributions to saturation. The shape of the latter region is presumably due to properties of the wave sources as well as to mode interaction, but at present this remains an open question.

It is sufficient for our purposes to represent this situation by means of the simple expression

$$\begin{aligned} \psi &= B \qquad (0 \le k < k_e) \\ \psi &= Ak^{-n} \qquad (k_e \le k \le k_{max}) \end{aligned} \tag{44}$$

where $B \equiv Ak_e^{-n}$. Eliminating A and inserting (44) into (42) and integrating, one obtains

$$N^2 = \frac{Bk_e^3}{3} + \frac{(Bk_e^n)}{(3-n)}\{k_{max}^{3-n} - k_e^{3-n}\} \tag{45}$$

where $n \neq 3$ but where n can be very close to 3 so far as numerical calculations are concerned.

Equation (45) can be easily solved numerically for n for given values of N^2, B, k_e, and k_{max}. In particular, we determine n from parameters based on Figure 2. There we infer that $k_e = 5 \times 10^{-5}$ c/m (cycles/meter) and $B = 5 \times 10^6$ (m²/s²)/(c/m), approximately. One can obtain the average value of N for the troposphere from Table 2; however, this value must be divided by 2π to make the units compatible with those of k (i.e., c/m as opposed to rad/m); hence we take $N^2 = 3.35 \times 10^{-6}$ Hz². For k_{max} we shall use the value (1/40 m) on the basis of *Dewan et al.* [1984]. While it is true that the latter measurements were carried out in the stratosphere, we shall assume that the troposphere does not differ significantly in this regard. As we shall see in Table 4, the results for $n \approx 3$ are very insensitive to the value of k_{max}, hence this approximation is not problematic. Solving for n, we found $n = 3.07$. From this result we find good evidence that in regard to the value of n the narrow-band approximation is a valid one.

Table 4, as has already been mentioned, explores the sensitivity of n in (45) to changes in the input parameters. While there is little sensitivity to k_{max}, there is moderate sensitivity to B (or, equivalently, to N^2) and a large sensitivity to k_e.

In regard to the high sensitivity to k_e values we must respond to a remark made to us by D. Fritts in private conversation (1985). He suggested that at low values of k in Figure 2 there could be significant contributions from nongravitational wave sources such as the jet stream. For this reason, he suggested that one estimate k_e and B from an alternate figure in *Endlich and Singleton's* [1969, Figure 5] paper, in which the PSD is calculated on the basis that mean winds (i.e., the average over a number of profiles taken on different days) are removed by subtraction. On this basis we obtain $B = 10^4$ (m²/s²)/Hz and $k_e = 3 \times 10^{-4}$ (c/m). If we retain $k_{max} = $ (1/40 m) and $N^2 = 3.35 \times 10^{-6}$ Hz², we arrive at $n = 2.70$. This value, while lower than 3.0, is not very much lower, and it is still reasonably consistent with slopes observed by Endlich and others [e.g., *Daniels*, 1982] in the troposphere.

As we see, the present and more accurate model leads to values of n that are not exactly equal to 3 but which are nevertheless consistent with our saturation hypothesis. With this in mind, the values of $n = 2.5$ and $2.75 \pm .1$ reported by *Daniels* [1982] and *Rosenberg et al.* [1974], respectively, etc., may possibly not be due merely to artifact as was suggested previously. They may possibly have actually been closer to the slopes "in nature." In any case, it is important to note that the broadband model permits values of n which are different from 3 but that, to apply this model, one must have information about k_e, k_{max}, B, and N.

In passing, it should be mentioned that the spectrum found by *Garrett and Munk* [1972, 1975] for the upper ocean (with, in their case, $n = 2.5$) may be entirely consistent with our explanation. Their PSD amplitudes certainly scale as N^2. We intend to treat this in detail elsewhere in the future.

Our saturation hypothesis suggests other testable predictions. Some of these have been pointed out to us by Smith, Fritts, and VanZandt, who plan to publish them in the near future in *Geophysical Research Letters*.

6. SUMMARY

A simple physical explanation for the nearly universal vertical wave number PSD for horizontal winds has been presented. It rests on the plausible assumption that such PSD amplitudes and shapes are determined by saturation due to turbulent breakdown. As was pointed out, this general approach has been used in the physics of ocean surface waves and thermocline internal waves. The theory predicts a PSD slope in the neighborhood of −2.5 to −3 and an amplitude dependence that scales as N^2. These predictions are in agreement with experiment. The N^2 dependence as well as the approximate −3 slope gives us two ways to test the theory. In particular, the predicted difference of 3 to 4 between tropospheric and stratospheric PSD amplitudes seems to provide a practical test.

The simplicity of our model is one of its most important assets. It may also provide a physical explanation of the vertical wave number PSD of the universal spectral model proposed by *Garrett and Munk* [1972, 1975] for mesoscale oceanic motions, at least for those situations and wave numbers where saturation can be proved to be a dominant factor.

In the simple initial approach, our main assumptions included (1) $k_z \ll k_g$, (2) $\Delta k_z \propto k_g$, and (3) individual Fourier wave components saturate. Our more exact approach did away with the need of these assumptions, and we found that n can be different from 3. Future publications will address further predictions of our approach.

In this paper our attention has been focused on a range of wavelengths that included 40 m to about 3 km. At the small scale end it is assumed that turbulence effects become important (at k_{max}).

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