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Triangularity of the Basis in Linear Programs for
Material Requirements Planning

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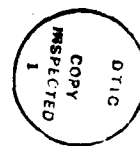
Abstract

It is shown that the basis in a class of linear programs arising from material requirements planning can be triangularized. This allows for efficient adaptation of the Simplex Method similar to those for network problems. It also suggests that for finite-loading (i.e. capacitated) MRP, a decomposition approach exploiting both subproblem structure and parallel processing can be effective for handling complex problems in multiproduct, multistage, multiperiod production systems.

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1. The Single-Product Infinite-Loading MRP Model

The manufacturing of a product usually consists of its assembly from parts which are themselves the products of other parts. A schematic representation of a product structure (also known as a Bill of Materials) is exemplified in Figure 1. Each oblong represents an item indexed by the number in bold type. A basic assumption of the model studied in this paper is that each item except the first contributes directly to the production of only one other item, known as its parent. We shall call this the tree property of the product structure. Item 1 is the finished product. The number of units of an item required per unit production of its parent is given in parenthesis in the figure.

In Material Requirements Planning (MRP), the net demands for each item in every time period over a finite planning horizon are given. The purpose is to determine the levels of production and inventory for the items so as to meet the demands at minimum costs. For the present purpose, we assume the cost function to be linear in the production and inventory variables. Since the assembly of an item takes time, a lead time in units of time periods is specified for each item. We shall show that with many products competing for limited production capacities, the planning problem can be modeled as large, structured linear programs. The dimensions and complexities of such problems make their routine application either infeasible or very expensive with conventional LP software. In this paper, we demonstrate certain properties of the problems that may lead to more efficient solution techniques.

We first consider the case with a single finished product (Item 1) and no production capacity or inventory storage limits on any item. The absence of capacity constraints is commonly known as infinite-loading in the MRP literature. We call this the Single-Product Infinite-Loading MRP model (SPILMRP). The more important case of multiproduct, capacitated (finite-loading) MRP will be discussed later.

2. The Linear Programming Formulation

To formulate the above MRP problem as an LP, the following terminology is used.

Given the parameters:

N = number of items in the product structure;

T = number of time periods in the planning horizon;

d_{it} = (exogenous) demand of item i in period t ;

h_{it} = unit holding cost for item i inventory in period t ;

c_{it} = unit production cost for item i in period t ;

$j(i)$ = index of parent of item i ($i \neq 1$);

m_i = number of units of item i required per unit of its parent item $j(i)$;

L_i = production lead time for item i ;

define the variables:

P_{it} = number of units of item i to be completed at the beginning of period t ;

I_{it} = number of units of item i in inventory at the end of period t ;

where $i = 1, \dots, N$ and $t = 1, \dots, T$ throughout.

Note that the definition of P_{it} does not imply that production is instantaneous since lead times are allowed. The convention used here to account for material balancing is that demand be fulfilled at the beginning of the period using previous inventory and actual completed products. The balance equation is

$$I_{it} = I_{i,t-1} + P_{it} - d_{it}.$$

The ending inventory I_{it} is carried throughout period t and incurs holding cost of h_{it} per unit.

Because of production lead times, certain variables defined above may be eliminated from the model (or equivalently set to zero values as parameters) . For example,

$$P_{it} = 0; \quad i=1,\dots,N, \quad t=1,\dots,L_i.$$

Also, let $R_1 = T$ and $R_i = R_{j(i)} - L_{j(i)}$ for $i=2,\dots,N$ be the production horizon for item i . Then production of item i in periods $t > R_i$ will be too late to be useful in production of its parent item. Therefore,

$$P_{it} = 0; \quad i=2,\dots,N, \quad t=R_i+1,\dots,T.$$

Finally, $I_{iR_i} = 0$ since allowing ending inventory will incur unnecessary holding cost. Also, initial inventories are subtracted from the first period demands so that I_{i0} may be set to zero. Note that such net demands may be negative in value.

Then the LP for the SPILMRP model can be written as (LP1):

$$\begin{aligned} & \sum_{i=1}^N \sum_{t=1}^{R_i} \{ h_{it} I_{it} + c_{it} P_{it} \} \\ \text{minimize} \quad & \\ \text{subject to} \quad & I_{i,t-1} - I_{it} + P_{it} - m_i P_{j(i),t+L_{j(i)}} = d_{it}; \quad i=1,\dots,N \\ & \quad \quad \quad t=1,\dots,R_i \\ & P_{it} = 0; \quad i=1,\dots,N, \quad t=1,\dots,L_i; \\ & I_{i0} = 0; \quad I_{iR_i} = 0; \quad i=1,\dots,N; \\ & P_{it} \geq 0; \\ & I_{it} \geq 0; \quad i=1,\dots,N; \quad t=1,\dots,R_i. \end{aligned}$$

Denote the constraint matrix in (LP1) by M and let its dimensions be m rows by n columns.

3. Triangularity of the Basis

Observe that although (LP1) consists of essentially flow-balance type constraints, it is not a network LP. While the coefficients m_i can be considered as gain factors in a generalized network, the proportionality requirement on the production of items supplying a common parent still needs to be expressed separately. For formulations of this type of problems as networks with side constraints, see e.g. Chen and Engquist [2], Steinberg and Napier [8], and Zahorik et al [10]. The allowance of nonzero initial inventories and lead times also distinguishes (LP1) from Leontief substitution systems studied by Dantzig [3] and Veinott [9]. However, the following result shows that (LP1) has a very important network-like property, namely, that any basis can be triangularized. This was first derived in [7] using a generalization of the concepts of network models. In this paper we present a direct, algebraic proof.

Lemma Given any m by m nonsingular submatrix B of the coefficient matrix M in (LP1), there exists either a row with a single nonzero coefficient (row singleton), or a column with a single nonzero coefficient (column singleton).

Proof. We assume the assertion is false, and show that a nonzero vector π exists for which $\pi \cdot B = 0$, contradicting the non-singularity of B .

Consider a column of B corresponding to the variable I_{it} . I_{it} appears in the inventory balance constraint of item i in period t with a coefficient of $+1$; if the assertion is false, then it also appears in the inventory balance constraint of item i in period $t + 1$ with a coefficient of -1 . Now consider a column of B corresponding to the variable P_{it} . P_{it} appears in the inventory balance constraint of item i in period t with a coefficient of $+1$. To produce P_{it} item i must withdraw the quantity $m_k P_{it}$ at times $t - L_i$ from the inventory balance constraints of item k in period $t - L_i$ with a coefficient of $-m_k$.

As a consequence of the above remarks $\pi \cdot B = 0$ if and only if

$$\pi_{it}^I - \pi_{i,t+1}^I = 0 \quad (1)$$

$$\pi_{it}^P - \sum_{k \in P(i)} \pi_{k,t-L_i}^I m_k = 0 \quad (2)$$

where π_{it}^I (respectively π_{it}^P) is the component of π corresponding to the column of B associated with the variable I_{it} (respectively P_{it}), and $P(i)$ denotes the set of items required in the production of item i (i.e. those having item i as parent). (When the assertion is false, $P(i)$ can never be empty; in words, columns corresponding to production of those items which require no other items as input can never be in the basis.) Let E denote the set of items i which have a column corresponding to the variable P_{it} or I_{it} in the basis. A non-zero π satisfies (1) and (2) if a non-zero π^* exists which satisfies

$$\pi_{i,t}^* - \sum_{k \in P(i)} \pi_{k,t-L_i}^* m_k = 0; \quad i \in E. \quad (3)$$

(Equate π_{it}^P and π_{it}^I to $\pi_{i,t}^*$ for each $i \in E$ and t .) Recursively, assign items to "levels" as follows: assign item i to level 0 if $P(i) = \emptyset$; for those items not yet assigned, assign item i to level r if the level of each $k \in P(i)$ is no greater than $r - 1$. For each item i assigned to level 0 set $\pi_{i,t}^* = 1$. For each item i assigned to level 1 set $\pi_{i,t}^* = \sum_{k \in P(i)} \pi_{k,t-L_i}^* m_k$, etc. Continuing in this fashion a π^* is

determined which satisfies (3).

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Theorem A basis in (LP1) can be transformed to a lower triangular matrix by row and column permutations.

Proof. By the lemma, we can find either a row or a column singleton in the basis. In the case of a row singleton, permute the nonzero coefficient to the upper diagonal. In the case of a column singleton, permute the nonzero to the lower diagonal. Deleting the row and column corresponding to the singleton, the remaining submatrix must also be nonsingular. Therefore, the same procedure can be repeated until the basis is lower triangularized.

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4. An Example

To illustrate, we use an example based on the product structure in Figure 1. The production lead times in the following Table apply.

<u>Item</u>	<u>Production Lead Time (in number of time periods)</u>
1	1
2	1
3	2
4	1
5	1

Table 1. Production Lead Times in the Example.

Figure 2 shows the constraint matrix for the single product, infinite loading MRP model. A basis is exemplified by the shaded columns of the matrix in Figure 3.

Note that except for illustration, there is no need to physically permute the basis in triangularization. It suffices to identify a pivot sequence specifying which row and column to use at each step of eliminating a variable from the system of equations. The pivot sequence for the triangularization of the basis in Figure 3 is displayed in the left-most column in the figure. The pivots are enclosed in circles. The first pivot uses row one and column one, the second row two and column six, the third row seven and column eleven, and so on.

With a triangular basis, the major operations in the Simplex Method are greatly simplified. Both the computation of the simplex prices and the updating of a column reduce to back-substitutions. It should be remarked that basis triangularity in this case does not imply integer solutions as the m_i 's may appear on the diagonal.

5. The Multi-Product Finite-Loading MRP Model

Most real production systems involve a multitude of finished products. The assembly of these products and their parts requires production capacity at every stage. When production capacities are limited, we have the finite-loading model. For a survey, see Billington et al [1]. Suppose there are K types of capacities with s_{kt} units of type k available in time period t . Let a_{ik} be the unit requirement of type k capacity in the production of item i . Then the LP for the Multi-Product Finite-Loading MRP model (MPFLMRP) is (LP2) below.

$$\begin{aligned}
 & \text{minimize } \sum_{i=1}^N \sum_{t=1}^{R_i} \{ h_{it} I_{it} + c_{it} P_{it} \} \\
 & \text{subject to } \sum_{i=1}^N a_{ik} P_{it} \leq s_{kt} ; \quad k=1, \dots, K; \quad t=1, \dots, T \quad (\text{LP2.1}) \\
 & I_{i,t-1} - I_{it} + P_{it} - m_i P_{j(i),t+L_j(i)} = d_{it} ; \quad i=1, \dots, N \quad (\text{LP2.2}) \\
 & \quad \quad \quad t=1, \dots, R_i \\
 & P_{it} = 0; \quad i=1, \dots, N, \quad t=1, \dots, L_i \text{ or } t > R_i ; \\
 & I_{i0} = 0; \quad I_{iR_i} = 0; \quad i=1, \dots, N; \\
 & P_{it} \geq 0; \\
 & I_{it} \geq 0; \quad i=1, \dots, N; \quad t=1, \dots, R_i.
 \end{aligned}$$

Here, the N items can be partitioned into mutually exclusive subsets, each corresponding to a distinct finished product. Therefore (LP2) has the block-angular structure with (LP2.1) as the coupling constraints and (LP2.2) decomposing into as many independent blocks as there are finished products. In [7] McKenney proposed to solve (LP2) using Dantzig-Wolfe decomposition [4]. Generalizing the concepts

of networks, trees and paths, he devised a network-simplex type procedure to take advantage of the triangular basis property of the subproblems.

Ongoing work in LP decomposition with parallel computers (Ho [5], Ho et al [6]) will be specialized to solve (LP2). For a 10-period MRP system with 100 products, each with 100 parts, (LP2.2) alone will have on the order of 100,000 constraints and 200,000 variables. For this reason, most previous attempts to MPFLMRP are deemed impracticable due to "prohibitive" computational requirements (see, e.g. [8]). However, multicomputers having 2^n parallel processors are becoming increasingly cost-effective. Currently, practical values of n are already between 6 and 8 (i.e. 64 to 256 processors). The power of individual processors is also well suited to handle the subproblem for one product (say, with about 1,000 constraints) if one exploits the special property discussed in this paper. Therefore, the implementation of Multi-Product, Finite-Loading Material Requirements Planning systems on parallel computers should be an important advance in production and operations management in the near future.

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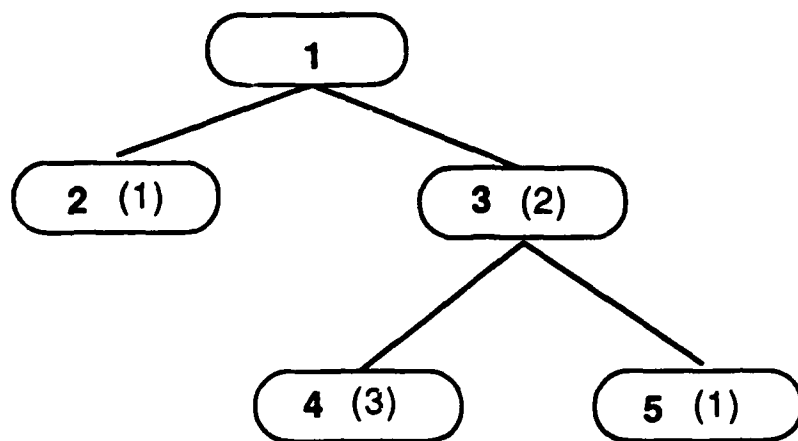


Figure 1. Example of a Product Structure

ITEM	1						2					3					4			5							
	Inv			Prod			Inv			Prod		Inv			Prod		Inv		Prod	Inv		Prod					
	1	2	3	4	5	6	1	2	3	4	2	3	4	5	1	2	3	4	5	1	2	2	3	1	2	2	3
TIME	1	2	3	4	5	6	1	2	3	4	2	3	4	5	1	2	3	4	5	1	2	2	3	1	2	2	3
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Figure 2. The LP Matrix for a Single Product MRP Problem

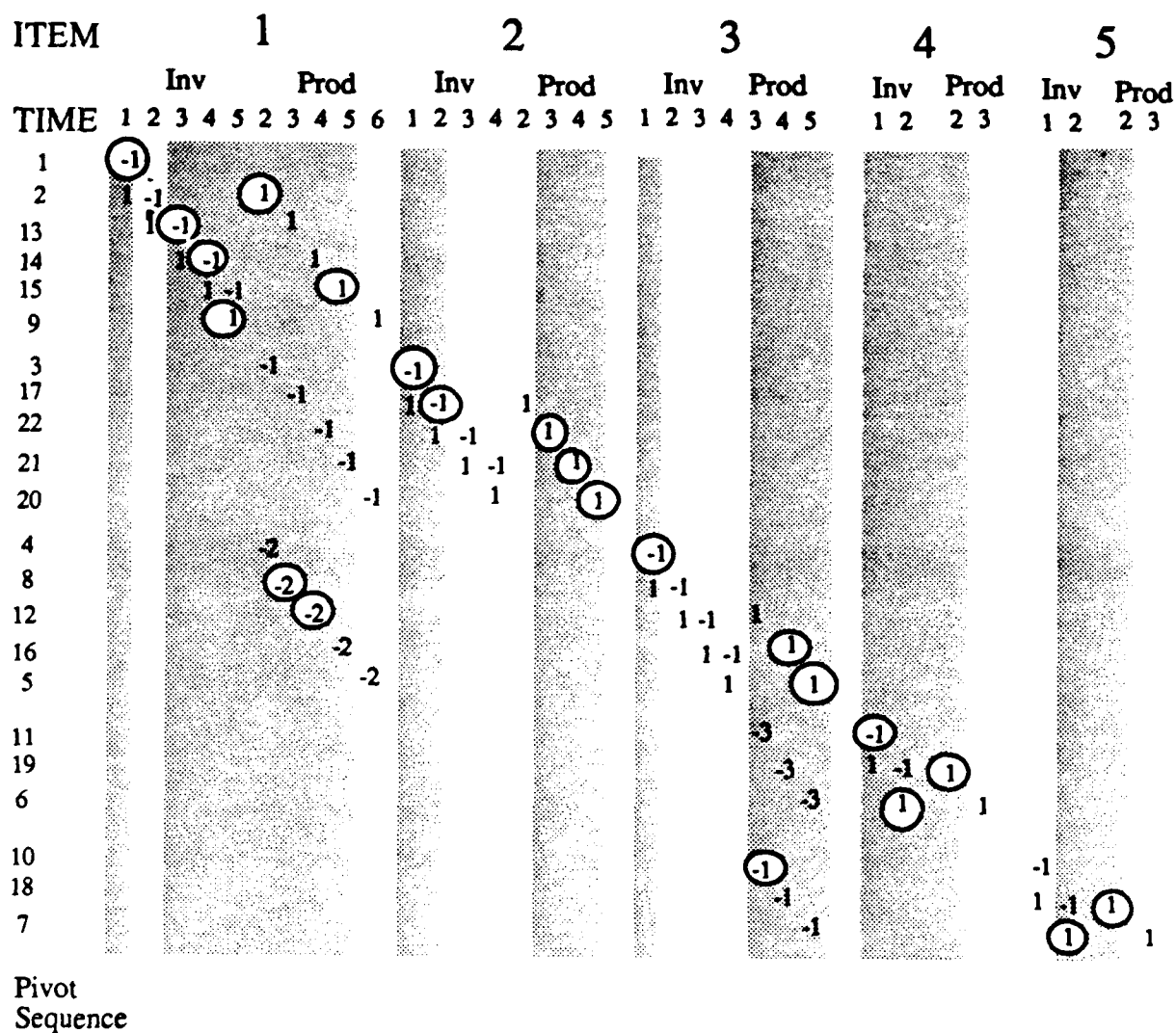


Figure 3. Pivot Sequence to Triangularize the Basis

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