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LOCAL CORNER CUTTING AND THE SMOOTHNESS OF THE LIMITING CURVE

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ABSTRACT

Stimulated by recent work by Gregory and Qu, it is shown that the limit of local corner cutting is a continuously differentiable curve in case the corners of the iterates become increasingly flat.

AMS (MOS) Subject Classifications: 65D10

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(in earlier work)

Local corner cutting and the smoothness of the limiting curve

Carl de Boor

It was proved in ~~[B]~~ that corner cutting of any kind converges to a Lipschitz-continuous curve, but the question of how one might guarantee that the limiting curve be smoother than that was not considered there. Recently, Gregory and Qu [GQ] took up this question and established sufficient conditions for a certain systematic and local corner cutting scheme to give a limiting curve in C^1. Since [GQ] use the same parametrization of the successive broken lines that made the argument in [B] so simple, I became intrigued and took a look at what one might say in greater generality. Specifically, I looked for conditions under which continuous differentiability of the limiting curve could be inferred from the fact that the corners of the broken lines flatten out eventually.

It is the purpose of this note to prove that the limit of any 'local' corner cutting scheme is in C^1 provided the corners of the broken lines become increasingly flatter. A simple example is given to show that this condition is not necessary, while another simple example shows that, without 'localness', the condition is not sufficient, in general. Finally, as an application, Gregory and Qu's nice argument in [GQ] is redone.

1. Cutting corners

In this section, we recall the setup of [B].

We deal with a sequence $(b_n)_{n=0}^{\infty}$ of broken lines in which, for $n > 0$, b_n is obtained from b_{n-1} by a 'cut', i.e., by replacing a curve segment by the subtended secant to the curve. This means that all the vertices of b_n lie on b_{n-1}, i.e., b_n can be thought of having been obtained from b_{n-1} by interpolation. This observation is used in [B] to prove that, no matter just how the cutting was done to generate the sequence (b_n) from an initial broken line b_0 with finitely many vertices, $b_\infty := \lim_{n\to\infty} b_n$ exists as a Lipschitz-continuous curve which is approached uniformly by b_n, i.e., $\lim_{n\to\infty} \operatorname{dist}(b_n, b_\infty) = 0$.

The argument in [B] was based on parametrizing the curves appropriately. If (v_i) is the sequence of vertices of b_n and (t_i) is a corresponding arbitrary increasing sequence of numbers, then b_n can be parametrized by

$$(1.1) \qquad b_n(t) := v_{i-1}\frac{t_i - t}{\nabla t_i} + v_i \frac{t - t_{i-1}}{\nabla t_i}, \qquad t_{i-1} \le t \le t_i, \quad \text{all } i.$$

Since b_n is obtained from b_{n-1} by interpolation, it is natural to choose the sequence (t_i) in dependence on the parametrization of b_{n-1}, i.e., so that $b_n(t_i) = v_i = b_{n-1}(t_i)$ for all i. With this,

$$b_n = P_n b_{n-1},$$

where P_n is broken line interpolation at the points (t_i). Therefore, ultimately, $b_n = P_n \cdots P_1 b_0$, with $P_n \cdots P_1$ a *linear* map. Hence, although the process of generating the sequence (b_n) is nonlinear (in that it is quite arbitrary), once we have decided on how to cut, we can think of each b_n as a linear function of b_0. In particular, writing b_0 in any one of many reasonable ways as a sum

$$b_0 = \sum_i w_i \varphi_i$$

of scalar-valued functions φ_i with vector coefficients $w_i \in \mathbb{R}^d$, we have

$$b_n = \sum_i w_i P_n \cdots P_1 \varphi_i,$$

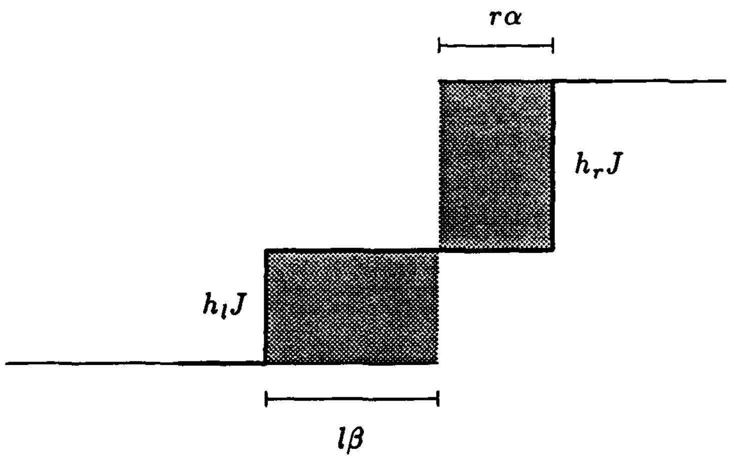
and questions of convergence or of smoothness of the limit can be settled by settling them for the (presumably simpler) sequences $(P_n \cdots P_1 \varphi)$, with φ any one of the φ_i. A particularly simple choice for the φ_i are the truncated powers $(\cdot - \tau_i)_+$ (in addition to the constant function), and this leads to the conclusion that the nature of the limiting curve can be understood if one understands what the particular corner cutting process does to the **standard corner**, i.e., the broken line with vertices $(0,0), (1,0), (2,1)$.

2. Examples

We are now ready to consider the smoothness of the limiting curve, having understood that it is sufficient to consider the case that b_0 is a piecewise linear **(real-valued) function** on some interval. Then each b_n is of the same nature, and its derivative, $d_n := b'_n$, is a step function. In the notation adopted in the preceding section,

$$d_n = \sum_i (\cdot - t_i)_+^0 \, \mathrm{jump}_{t_i} d_n, \tag{2.1}$$

with $\mathrm{jump}_t d := d(t+) - d(t-)$ the difference between the limit from the right and the limit from the left at t. We can take the absolutely largest jump, i.e., the number $\|\mathrm{jump}_{()} d_n\|_\infty$, as a measure of the extent to which b_n fails to be in C^1.



(2.2) Figure The change in the derivative as the result of a corner cut. The two areas are of equal size.

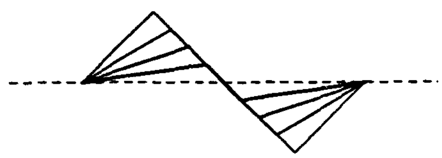
It will be useful to visualize the process by which d_n is obtained from d_{n-1}. Suppose that b_n is obtained from b_{n-1} by replacing b_{n-1} on $[s, t]$ by the linear interpolant to b_{n-1} at s and t. Then

$$\int_s^t (d_n - d_{n-1})(x)dx = (b_n - b_{n-1})(t) - (b_n - b_{n-1})(s) = 0.$$

Also, d_n is a constant (viz., the difference quotient $[s,t]b_{n-1}$) on $[s,t]$. If now b_{n-1} has just one vertex in $[s,t]$, then d_{n-1} has just two steps there, hence $d_n - d_{n-1}$ has just two steps there and, as $\int_s^t(d_n - d_{n-1}) = 0$, the two rectangles which make up this integral must balance; see (2.2)Figure.

More precisely, let J be the jump in d_{n-1} at that sole vertex in $[s,t]$, let l and r be the parametric distances of the corner from its left and right neighbor, and assume that the two new vertices (which replace the vertex being cut off) occur at parametric distances $l\beta$ and $r\alpha$, respectively (with $\alpha, \beta \in [0,1]$). Then the two new jumps are of size $h_l J$ and $h_r J$, with $h_l + h_r = 1$ and $l\beta h_l = r\alpha h_r$, or,

(2.3)
$$h_l = \frac{r\alpha}{l\beta + r\alpha} = 1 - h_r.$$



(2.4) Figure These uniformly nonsmooth broken lines converge to a smooth limit.

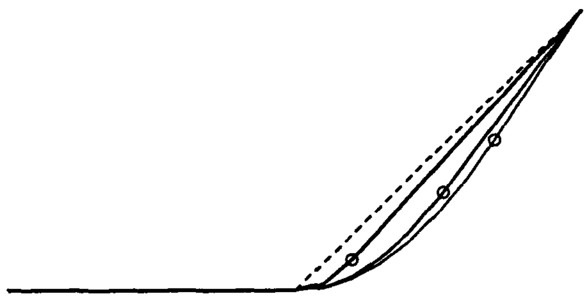
As a first example, we show that the maximum jump need not go to zero for the limit function to be C^1. As shown in (2.4)Figure, the limiting function is smooth (it is the zero function), while the maximum jump in the first derivative stays above 1 (in absolute value). Note that, in this example, we have done two cuts simultaneously, i.e., in terms of the setup adopted earlier, we are showing only every other iterate. Note also that, strictly speaking, each of our cuts involves two corners. It will be important later on to know that such an example (of a limiting curve being smooth even though the absolutely largest jump in the derivative of the iterates is bounded away from zero) can also be given when each cut involves only exactly one corner. Such an example can be supplied by applying, e.g., Chaikin's algorithm to the initial broken line in the above example, except that the points on the two segments flanking the middle segment are chosen closer and closer to the farther endpoint.

Finally, we illustrate the fact that having the maximum jump in the first derivative go to zero is, in general, no guarantee that the limiting curve is C^1. For that, we consider the 1-parameter family of functions f_α (see (2.6)Figure) given by the rule that

(2.5)
$$f_\alpha(t) := \begin{cases} 0, & t \le 0; \\ \alpha t^2, & 0 \le t \le t_\alpha; \\ 1 + (t-1)2\alpha t_\alpha, & t_\alpha \le t, \end{cases}$$

with $t_\alpha := 1 - \sqrt{1 - 1/\alpha}$. One verifies that, for $\alpha > 1$, f_α is in C^1, and that f_α converges, on [-1,1] say, monotonely and uniformly to the 'standard corner' which is not

C^1. The actual example is provided by replacing selected f_α by interpolating broken lines of sufficiently fine spacing to make the jumps in the first derivatives as small as one pleases.



(2.6) Figure These smooth broken lines converge monotonely to a nonsmooth one. (The circles are the points $(t_\alpha, f_\alpha(t_\alpha))$ of (2.5).)

Nevertheless, if the corner cutting is local, then having the absolutely largest jump in the first derivative go to zero does imply that the limiting function is in C^1. This is the content of the next section.

3. Local corner cutting

We say that the corner cutting is **local** in case any cut involves exactly one corner. This means that the cut endpoints must lie in the interior of the two segments which form the corner being cut. Schemes that cut all corners simultaneously fall into this category as long as the cuts of neighboring corners do not share an endpoint. For we can then think of them as having been carried out one cut at a time. In particular, the corner cutting scheme considered in [GQ] is local in this sense, as are the schemes considered in [R] so many years ago. It follows that every segment, of the original broken line as well as of any subsequently generated broken line, is tangent to the limiting curve, hence the situation depicted in (2.6)Figure could not have been generated by *local* corner cutting.

(3.1)Theorem. *The limiting curve produced by a* **local** *corner cutting scheme is C^1 in case the maximum jump in the first derivative of the iterates b_n goes to zero as $n \to \infty$. The converse holds in case b_0 is convex and for arbitrary corner cutting.*

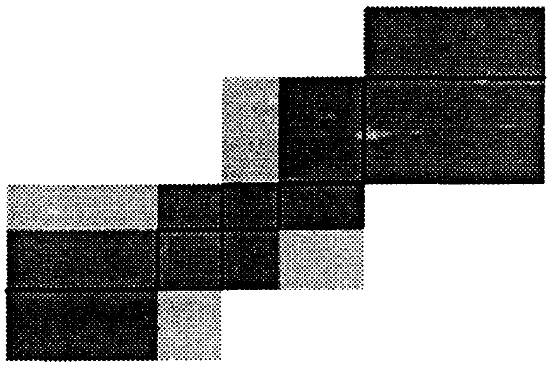
Proof. It is sufficient to consider the special case that b_0 is the 'standard corner'. Then b_0 is convex, hence so are all the iterates, with b_n growing uniformly, and pointwise monotonely, toward the limit function b_∞, which is also convex. Let b be one of the iterates. Then $d := b'$ is a monotone increasing step function. Recall from (2.1) that

$$d = \sum_j (\cdot - t_j)_+^0 \operatorname{jump}_{t_j} d,$$

with $0 < t_1 < \ldots < t_m < 2$ its breakpoints. We consider also the two step functions d^+ and d^-, given by the rule

$$d^{\pm} = \sum_{j=1}^{m} (\cdot - t_{j\mp 1})_+^0 \,\mathrm{jump}_{t_j} d,$$

with $t_0 := 0$ and (for the sake of neatness) $t_{m+1} := 2$. Then $\|d^+ - d^-\|_\infty \le 2\|\mathrm{jump}_{()} d\|_\infty$, while $d^- \le d \le d^+$ pointwise, since d is monotone increasing.



(3.2) Figure Corner cutting contracts the 'envelope' formed around the derivative d by the step functions d^- and d^+.

Further, if b^* is obtained from b by cutting off exactly one corner, and d^* is, correspondingly, the derivative of b^*, then (see (3.2)Figure)

$$d^- \le (d^*)^- \le (d^*)^+ \le d^+. \tag{3.3}$$

This implies that the first derivative of all subsequent iterates lies between d^- and d^+. Hence, if $\|\mathrm{jump}_{()} d_n\|_\infty \longrightarrow 0$ as $n \to \infty$, then (d_n) is a Cauchy sequence in the complete normed linear space of all bounded functions on $[0,2]$ (with the max-norm). Consequently, $d_n = b_n'$ converges uniformly to some bounded function d_∞. Now consider the modulus of continuity ω_∞ of this limiting function. For any $h > 0$ and any n,

$$\omega_\infty(h) = \sup_{0<t-s<h} (d_\infty(t) - d_\infty(s)) \le \sup_{0<t-s<h} (d_n^+(t) - d_n^-(s)) =: \overline{\omega}_n(h).$$

Note that $\overline{\omega}_n$ is a nondecreasing step function, with $0 \le \overline{\omega}_n(0+) = \sup_j(d_n(t_{j+2}+) - d_n(t_{j-1}-)) \le 3\|\mathrm{jump}_{()} d_n\|_\infty \longrightarrow 0$ as $n \to \infty$. Consequently, for every $\epsilon > 0$, we can find n and $\delta > 0$ so that, for all $h < \delta$, $\overline{\omega}_n(h) < \epsilon$. This proves that $\omega_\infty(0+) = 0$, and so establishes that d_∞ is continuous.

We now know that b_n converges uniformly to some Lipschitz-continuous function b_∞, while $d_n := b_n'$ converges uniformly to some continuous function d_∞. It is a standard result that, therefore, $b_\infty' = d_\infty$, i.e., the limiting function b_∞ is in C^1.

For the converse, assume that the sequence (b_n) of convex broken lines, all defined on the interval [0,2] say, converges uniformly to some function b. Assume further that b is continuously differentiable at the interior point p. This means that, for some modulus of continuity ω (i.e., some positive function ω on $(0,\infty)$ with $\omega(0+)=0$),

$$[t,p]b := (b(t)-b(p))/(t-p) = b'(p) + O(\omega(|t-p|)).$$

Now consider $J := \text{jump}_p b_n$ for some n, and let $\epsilon := \|b - b_n\|_\infty$. Then, for any small positive h,

$$0 \le J \le [p+h,p]b_n - [p,p-h]b_n \le [p+h,p]b - [p,p-h]b + 4\epsilon/h = O(\omega(h)) + 4\epsilon/h.$$

Since $\omega(0+)=0$ and $\epsilon = \|b-b_n\| \to 0$ as $n \to \infty$, this implies that J must be small when n is large. ♠

By going to a larger envelope, the argument can be applied to somewhat more general corner cutting, viz. one in which no cut reduces the number of corners. Since each cut generates two corners, this means that any cut is restricted to cut away no more than two corners. To handle this more general situation, one would change the definition of $d^\pm$ to

$$d^\pm = \sum_{j=1}^{m} (\cdot - t_{j\mp 2})_+^0 \,\text{jump}_{t_j} d,$$

with further adjustments at the endpoints.

(3.4)Corollary. *The limiting curve produced by a corner cutting scheme which never cuts across more than two corners is C^1 in case the maximum jump in the first derivative of the iterates b_n goes to zero as $n \to \infty$.*

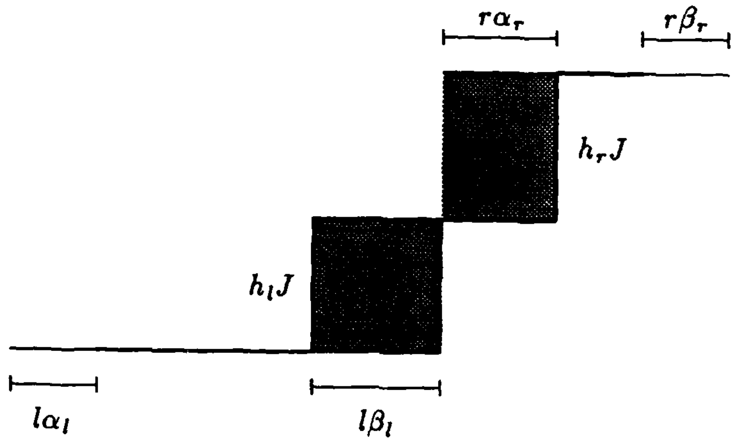
4. The Gregory-Qu result

As an application of (3.1)Theorem, we now consider the Gregory-Qu scheme, in which b_n is obtained from b_{n-1} by a simultaneous, non-interfering, cutting of all corners; hence the scheme is local in the sense defined earlier. In fact, [GQ] assumes that the new vertices generated are in the interior of old segments (i.e., that all α and β are positive, in the notation used there and introduced below). But if we allow also trivial cuts, i.e., cuts that begin and end at the same vertex ($\alpha = 0$ or $\beta = 0$), then this scheme models any local corner cutting.

To prove that the limit is in C^1, it is therefore sufficient to prove that the jumps in the first derivative go to zero. For this, we discuss the scheme in terms of the step function which is the first derivative of the broken line in question. A look at (4.1)Figure might be helpful.

The single jump of height J, with left and right segments of length l, r, spawns two jumps, a left one of height Jh_l, with $h_l := r\alpha_r/(l\beta_l + r\alpha_r)$ and with segments $l_l := l(1-\alpha_l-\beta_l)$ and $r_l := l\beta_l + r\alpha_r$, and a right one of height Jh_r, with $h_r := 1 - h_l$ and with segments $l_r := r_l$ and $r_r := r(1-\alpha_r-\beta_r)$. Since

$$h_l = \frac{1}{1+(\beta_l/\alpha_r)l/r} = 1 - h_r = 1 - \frac{1}{1+(\alpha_r/\beta_l)r/l},$$



(4.1) Figure The change in the derivative due to one step of the Gregory-Qu process.

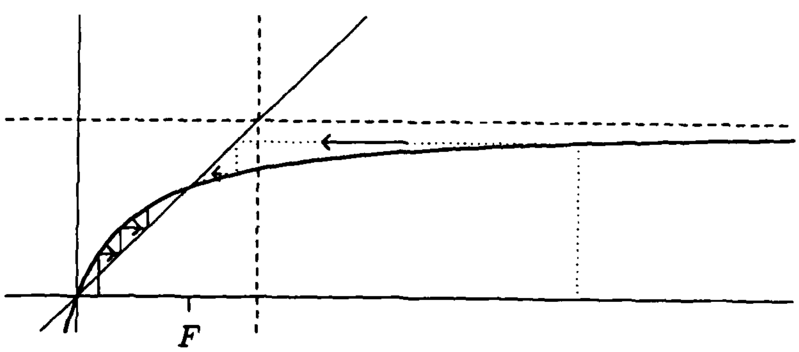
we can be assured that h_l and h_r are uniformly smaller than 1 (hence the limiting curve is in C^1) provided we can show that the local mesh ratio l/r is bounded away from 0 and ∞. For this, consider the local mesh ratios l_l/r_l and l_r/r_r spawned by the cutting of this corner. We find

$$l_l/r_l = \frac{l(1-\alpha_l-\beta_l)}{l\beta_l + r\alpha_r} = L(l/r),$$

with

$$L(t) := \frac{t(1-\alpha_l-\beta_l)}{t\beta_l + \alpha_r}.$$

This function is increasing on $[0,\infty)$, starting at 0 with a value of 0 and a slope of $L'(0) = (1-\alpha_l-\beta_l)/\alpha_r$ at 0 and taking the limiting value $L(\infty) = (1-\alpha_l-\beta_l)/\beta_l$. Consequently, L maps the interval $[0,\infty)$ into the interval $[0, L(\infty)]$. Further, if $L'(0) > 1$, then L has an attracting fixed point in that interval, viz. the point $F := (1-\alpha_l-\beta_l-\alpha_r)/\beta_l = (L'(0)-1)\alpha_r/\beta_l$. This means that L maps any interval $[1/M, M]$ containing F (and contained in $[0,\infty]$) into itself.



(4.2) Figure The function L contracts around the point F.

By symmetry, $r_r/l_r = R(r/l)$, with

$$R(t) := \frac{t(1-\alpha_r-\beta_r)}{\beta_l + t\alpha_r}$$

a function which maps $[0,\infty)$ into the interval $[0, R(\infty)]$, and which has the fixed point $G := (1-\alpha_r-\beta_r-\beta_l)/\alpha_r = (R'(0)-1)\beta_l/\alpha_r$ in case $R'(0) = (1-\alpha_r-\beta_r)/\beta_l$ is greater than one. This means that R maps any interval $[1/M, M]$ containing G (and contained in $[0,\infty)$) into itself.

We conclude that the local meshratios are bounded away from 0 and ∞ provided the fixed points F and G are eventually bounded away from 0 and ∞. As [GQ] point out, this can be guaranteed by having α and β eventually bounded away from zero, i.e. having both $\underline{\alpha} := \liminf \alpha$ and $\underline{\beta} := \liminf \beta$ be positive, and having $L'(0), R'(0) > 1$ when formed with $\alpha_l, \alpha_r = \overline{\alpha} := \limsup \alpha$ and $\beta_l, \beta_r = \overline{\beta} := \limsup \beta$. This amounts to the conditions

$$0 < \underline{\alpha}, \underline{\beta} \quad \text{and} \quad \overline{\alpha}, \overline{\beta} < 1 - \overline{\alpha} - \overline{\beta}.$$

When these conditions are violated, we cannot be certain that the local meshratios stay away from 0 or ∞, hence the reduction factors h_l or h_r may come close to 1. This does not, of itself, imply that the limiting curve has corners. But the above discussion is sufficient to show that the limiting curve has corners if $L'(0)$ or $R'(0)$ are uniformly below 1.

We discuss this only for the case of *constant α and constant β*. Assume, for example, that α and β are such that

$$R'(0) = \frac{1-\alpha-\beta}{\beta} < 1.$$

Then, starting with the 'standard corner', the cutting process generates a sequence of vertices proceeding to the right with associated local mesh ratios r/l equal to

$$R(1), R^2(1) = R(R(1)), R^3(1), \ldots$$

which decay geometrically to zero. In fact, $R^n(1) \sim (R'(0))^n$ as $n \to \infty$. The corresponding reduction factors therefore satisfy

$$\text{(4.3)} \quad h_r^{(n)} = \frac{1}{1+(\alpha/\beta)R^n(1)} = 1 - (\alpha/\beta)R^n(1) + O((R^n(1))^2) \sim 1 - (\alpha/\beta)(R'(0))^n.$$

We want to show that the corresponding sequence of jumps is bounded away from zero. Since this is a decreasing sequence, it is sufficient to show that its limit, the infinite product

$$\prod_{n=1}^{\infty} h_r^{(n)},$$

is positive. This is the same as proving that the infinite series

$$\sum_{n=1}^{\infty} \ln h_r^{(n)}$$

is finite. From (4.3), $\ln h_r^{(n)} \sim -(\alpha/\beta)(R'(0))^n$, i.e., the terms of the sum behave like that of a convergent geometric series, hence the series converges.

The foregoing analysis also explains the **fractal** nature of the resulting curves (when using constant α and β). For it shows that the height of a particular jump or the meshratio at a particular breakpoint of the nth iterate is the result of two contending fixed point iterations, L and R, with the influence of each entirely determined by the particular sequence of right and left turns taken to reach the breakpoint in question from the original breakpoint. In particular, we expect any collection of jumps sharing the first few of these turns to look like any other collection of jumps sharing the first few of these turns.

It would be interesting to explore further special situations when the fixed points coincide. For example, both fixed points (for L and R) are 1 exactly when $\alpha + \beta = 1/2$. In this case, all the mesh ratios are the same, hence the left factors h_l are all the same as are all the right factors h_r. The requirement that $h_l = h_r$ is satisfied exactly when $\alpha = \beta$. Thus the imposition of both requirements leads to $\alpha = 1/4 = \beta$ which is Chaikin's algorithm.

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