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SMALL INTERNAL WAVES IN TWO-FLUID SYSTEMS

C. J. Amick and R. E. L. Turner

**UNIVERSITY  
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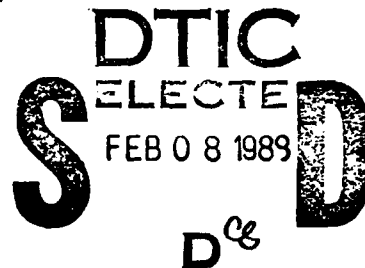


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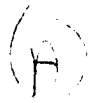
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ABSTRACT

This paper treats travelling waves in a heterogeneous, inviscid, non-diffusive fluid bounded between two horizontal boundaries. The fluid has two incompressible components of different, but constant density and is acted on by gravity. The flow is steady when viewed in a moving reference frame and gives rise to a quasilinear elliptic problem with an eigenvalue parameter related to the wave speed. The small amplitude solutions are analyzed using a dynamical systems approach. A center manifold reduction in combination with a conserved quantity for the flow is used to parametrise all 'small' solutions of the full elliptic system in terms of solutions of an autonomous first order ordinary differential equation for a principal component of the wave amplitude. The result is a characterization of all small waves, irrotational in each fluid, near the critical speed for the system. They are: solitary waves; surges, connecting distinct conjugate flows at extreme ends of the channel; conjugate flows; and periodic waves.

AMS (MOS) Subject Classifications: 34C35, 35J60, 76B15, 76C10

Key Words: Internal wave, solitary wave, surge, dynamical system

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# SMALL INTERNAL WAVES IN TWO-FLUID SYSTEMS

C. J. Amick and R. E. L. Turner

## 1. INTRODUCTION

### 1.1. The physical and mathematical problem

This paper treats travelling waves in a heterogeneous, inviscid, non-diffusive fluid bounded between two horizontal boundaries. The fluid has two incompressible components of different, but constant density, and the flow is steady when viewed in a suitable reference frame, to which we shall restrict attention. Further details about the physical problem may be found in [4],[7],[9], and [20].

The vertical height of the channel is normalized to be unity, and the flow domain is  $S = \mathbf{R} \times I$  where  $I = (-h, 1 - h)$ ,  $h \in (0, 1)$  being the specified depth of the lower fluid in an undisturbed state. For a nontrivial flow the two components are separated by an interface  $\Gamma = \{(x, Y(x) : x \in \mathbf{R}\} \subset S$ , where  $Y$  is unknown. The density function  $\tilde{\rho}(x, y)$  is normalized to equal unity in the region  $S^-$  where  $y < Y(x)$  and to equal a prescribed  $\rho \in (0, 1)$  in the region  $S^+$  where  $y > Y(x)$ . A pseudo-stream-function  $\psi$  is defined in  $S^\pm$  by the formula  $\tilde{\rho}^{1/2} \mathbf{q} = (\psi_y, -\psi_x)$ , where  $\mathbf{q} = (q_1, q_2)$  denotes the fluid velocity. Although  $\mathbf{q}$  is, in general, discontinuous across  $\Gamma$ , the assumption of non-diffusivity ensures that it is tangent to  $\Gamma$  as approached from within  $S^+$  and  $S^-$ ; in particular,  $\Gamma$  is a streamline. We assume there are no stagnation points; that is,  $|\mathbf{q}| \neq 0$ , whence all streamlines go from minus to plus infinity. The top and bottom walls,  $\{y = 1 - h\}$  and  $\{y = -h\}$ , are streamlines.

The total-head pressure  $H$  is defined by

$$(1.1) \quad H = p + \frac{1}{2} |\nabla \psi|^2 + \tilde{\rho} g y,$$

where  $p$  is the pressure and  $g > 0$  is the gravitational constant. A calculation from the Euler equations ensures that  $\nabla H$  and  $\nabla \psi$  are proportional. In fact  $\nabla H = \Delta \psi \nabla \psi$  in  $S^\pm$  so that  $\mathbf{q} \cdot \nabla H = 0$  and  $H$  is constant on each streamline. If we assume that  $q$  approaches a constant vector  $(c, 0)$  as  $x \rightarrow -\infty$  (or  $+\infty$ ), then the Euler equations show that  $H$  is constant in each fluid at infinity. Since streamlines go from  $-\infty$  to  $+\infty$ , it follows that  $H$  takes constant values  $H^\pm$  in  $S^\pm$ . Since  $|\mathbf{q}| \neq 0$ , it follows that  $\Delta \psi = 0$  and so the flow is irrotational in  $S^\pm$ . This will occur, then, if

$$(1.2) \quad \lim_{x \rightarrow -\infty} Y(x) = 0,$$

$$(1.3) \quad \lim_{x \rightarrow -\infty} q(x, y) = (c, 0)$$

where  $c$  is a constant. In the original 'laboratory' coordinates one would see a wave moving from right to left with speed  $c$ , into an undisturbed fluid with an interface having height zero infinitely far to the left. In the moving frame the pressure is hydrostatic at  $x = -\infty$ , and may be normalized by  $p(-\infty, 0) = 0$ . Then (1.1)-(1.3) yield  $H = H^+ = \rho c^2/2$  in  $S^+$  and  $H^- = c^2/2$  in  $S^-$ . The pressure is to be continuous across  $\Gamma$ , whence (1.1) implies

$$(1.4) \quad 1/2\{|\nabla\psi^+|^2 - |\nabla\psi^-|^2\} = (\rho - 1)\{c^2/2 - gy\} \quad \text{on } \Gamma$$

where  $\nabla\psi^\pm$  denote the limiting values of  $\nabla\psi$  on  $\Gamma$  from within  $S^\pm$ .

This paper considers a slightly wider class of flows than those satisfying (1.2)-(1.3) in that we shall also allow periodic flows. Since small solutions are of interest, the function  $\psi(x, y)$  should be close, in some sense, to the trivial solution

$$(1.5) \quad \Psi(y) = c \int_0^y \sqrt{\rho_\infty(s)} ds,$$

where

$$(1.6) \quad \rho_\infty(y) = \begin{cases} 1, & y < 0, \\ \rho, & y > 0, \end{cases}$$

for a constant  $\rho \in (0, 1)$ . In particular, we shall restrict attention to flows for which

$$(1.7) \quad \psi(x, -h) = \Psi(-h) = -ch; \quad \psi(x, 1-h) = \Psi(1-h) = -c\sqrt{\rho}(1-h), \quad x \in \mathbf{R}.$$

We also want the total-head pressure to be constant in the two domains whence

$$(1.8) \quad \nabla \times q = \Delta\psi = 0 \quad \text{in } S^+ \cup S^-.$$

The mathematical problem is the following: given  $h$  and  $\rho$  in  $(0, 1)$  find unknowns  $\Gamma$ ,  $c$ , and  $\psi$  satisfying (1.7) and (1.8) with  $\psi = 0$  and (1.4) holding on the unknown interface  $\Gamma$ .

We shall consider only flows for which no reversal occurs; that is,  $q_1 > 0$ , or, equivalently,  $\psi_y > 0$ . For such flows one can solve for  $y$  as a function of the spatial variable  $x$  and the material coordinate  $\psi$ . The advantage of this semi-Lagrangian formulation is that the unknown interface function  $Y$  is now  $y(x, \psi)$  evaluated at  $\psi = 0$ . The disadvantage is

that Laplace's equation (1.8) is replaced by a singular, quasi-linear equation for  $y(x, \psi)$ . Let  $Y(\psi)$  denote the function inverse to that in (1.5) so that  $Y(\Psi(y)) = y$ ,  $y \in I$ . As in [4] a final change of variables is performed: let  $x_1 = x$ ,  $x_2 = Y(\psi)$ , and

$$(1.9) \quad w(x_1, x_2) = y(x_1, \Psi(x_2)) - x_2$$

so that  $w$  represents the deviation of the streamline height  $y(x, \psi)$  from its value  $Y(\psi)$  in a trivial flow. For ease of notation, we subsequently write  $x$  for  $x_1$  and  $y$  for  $x_2$ .

To describe the problem in the new coordinates we define

$$(1.10) \quad f_1(\nabla w) = \frac{w_x}{1 + w_y} \quad \text{and} \quad f_2(\nabla w) = \frac{w_y}{1 + w_y} - \frac{|\nabla w|^2}{2(1 + w_y)^2}.$$

The problem becomes the following: find an eigenvalue  $\lambda = \frac{g}{c^2}$  and a function  $w(x, y)$ , continuous in  $T = \mathbf{R} \times I$ , satisfying

$$(1.11) \quad \frac{\partial}{\partial x} (f_1(\nabla w)) + \frac{\partial}{\partial y} (f_2(\nabla w)) = 0 \quad \text{in} \quad T^- \cup T^+,$$

$$(1.12) \quad f_2(\nabla w^-) - \lambda w^- - \rho(f_2(\nabla w^+) - \lambda w^+) = 0 \quad \text{on} \quad y = 0,$$

$$(1.13) \quad w(x, -h) = w(x, 1 - h) = 0, \quad x \in \mathbf{R}.$$

Here  $T^- = \mathbf{R} \times (-h, 0)$ ,  $T^+ = \mathbf{R} \times (0, 1 - h)$ , and  $\pm$  are used to denote values or limits taken within  $T^\pm$ . We ultimately use a standard weak formulation for a continuous  $w$  which is  $C^1$  in  $\overline{T^\pm}$ . The theory in [4] shows that such solutions have real analytic extensions to  $\overline{T^\pm}$  so regularity is not an issue. The condition (1.12) is merely the statement that the pressure is continuous across the fluid interface. Clearly, for any  $\lambda$ , the trivial solution  $w \equiv 0$  satisfies (1.11)-(1.13); we shall be interested in nontrivial solutions.

One might append to (1.11)-(1.13) the condition suggested by (1.2)-(1.3) that  $w \rightarrow 0$  as  $x \rightarrow -\infty$  and, in addition, (i) that  $w \rightarrow 0$  as  $x \rightarrow +\infty$ , giving a solitary wave; or (ii) that  $w \rightarrow 0$  as  $x \rightarrow -\infty$  and approaches a nonzero conjugate flow as  $x \rightarrow +\infty$ , giving what we shall call a surge; or (iii) that  $w$  be periodic in  $x$ . Rather than any of these conditions we shall demand that  $w$  (and suitable derivatives) be small in  $T$  and that  $\lambda$  be near the critical value

$$(1.14) \quad \lambda_d = \frac{1}{1 - \rho} \left( \frac{1}{h} + \frac{\rho}{1 - h} \right).$$

It will be shown that such solutions satisfy one of (i)-(iii). After  $(\lambda, w)$  has been found, it is straightforward to reverse the changes of variables to arrive at  $\psi$  and  $Y(x)$ . Therefore attention will be restricted to (1.11)-(1.13) for the rest of this paper.

## 1.2. Methods and results

In [4] there was shown to exist an unbounded branch  $\mathcal{C}$  of nontrivial solutions  $(\lambda, w)$  of (1.11)-(1.13) bifurcating from the line of trivial solutions at the point  $(\lambda_d, 0)$ . The set  $\mathcal{C}$  was considered in the space  $\mathbf{R} \times (C^{0,1}(\bar{T}) \cap \overset{\circ}{W}^{1,2}(T))$  and satisfied the following: (i)  $\lambda \in (0, \lambda_d)$ , (ii)  $w$  is even in  $x$ , (iii) the sign of  $w$  on  $T$  and of  $w_x$  on  $(-\infty \times I)$  is the same as that of the important parameter

$$(1.15) \quad e = \frac{1}{h^2} - \frac{\rho}{(1-h)^2}.$$

In [4] it was assumed that either  $e > 0$ , giving rise to a global branch of waves of elevation, or  $e < 0$ , giving waves of depression.

Since the branch  $\mathcal{C}$  is unbounded while  $\lambda$  is bounded (cf. (i)),  $\mathcal{C}$  contains a sequence of solutions, unbounded in  $C^{0,1}(\bar{T}) \cap \overset{\circ}{W}^{1,2}(T)$ . Under the assumption that they remain bounded in  $C^{0,1}(\bar{T})$  (which says, roughly speaking, that the corresponding interfaces tend not to develop vertical tangents) suitable horizontal translates of the sequence were shown to approach a surge (heteroclinic orbit)  $\bar{w}$  satisfying (1.11)-(1.13). The amplitude of  $\bar{w}$  is proportional to  $|e|$  and the corresponding (explicit) value  $\bar{\lambda}$  differs from  $\lambda_d$  by  $O(e^2)$ . However, the existence of a surge was dependent on the assumption of boundedness in  $C^{0,1}(\bar{T})$ . Evidence for its existence can be found in the analysis of model equations done in [10],[12],[18] and in the numerical work of [19].

In this paper we not only show the existence of small surges, but characterize *all* small solutions when  $|\lambda - \lambda_d|$  is small. A 'dynamical systems' approach is employed, whereby  $x$  is treated as a new 'time' variable and  $w(x, \cdot)$  as the 'state' of a system at time  $x$ . One can then bring to bear a center manifold reduction, parametrising all 'small' solutions of the full elliptic system in terms of solutions of an ordinary differential equation for a function  $Q : \mathbf{R} \rightarrow \mathbf{R}$ . This point of view was developed by Kirchgässner starting in the paper [13] and has been considerably expanded in recent years (cf. [3],[14]-[17] and their references). Here it is shown (cf. (1.19)) that the dynamics of the small solutions considered are governed by an autonomous first-order ordinary differential equation for  $Q$  which depends on  $\lambda$ . With this equation in hand it is a simple matter to determine all solutions and their dependence on parameters.

The plan of the paper is as follows. In section 2 the necessary mathematical notation is introduced. The problem (1.11)-(1.13) is considered in the form of a linear operator (the formal Frechet derivative of (1.11)-(1.13) at  $(\lambda_d, 0)$ ) acting on  $w$ , equal to a nonlinear function of  $(\lambda, w)$ . The linear operator is studied in section 3 and the nonlinear problem solved in section 4 with the aid of the contraction mapping principle. The main idea is to seek  $w(x, y)$  in the form  $Q(x)t(y) + R(x, y)$  where  $t$  is an eigenfunction of a suitable boundary value problem in the variable  $y$  and the 'remainder'  $R$  is orthogonal to  $t(y)$  at each  $x$  with respect to the weight  $\rho_\infty$ . Ultimately  $R$  will be of higher order than linear in  $Q$  and  $\lambda_d - \lambda$ . An important reduction theorem (Theorem 4.6) states that  $R$  and its partial derivatives are pointwise functions of  $Q$  and  $Q'$ . Thus  $R(x, \cdot) = F_1(Q(x), Q'(x), \lambda)$ . To obtain this striking reduction we are led to pose the problem in such a way that the natural space consists of functions which may *grow* exponentially at infinity. While this may appear rather far from our goal of obtaining small, bounded solutions, the success of the contraction mapping hinges on working in a suitable closed subset of these growing functions. Section 3 is necessarily long due to the technicalities of working with possibly unbounded functions.

In the nonlinear equation satisfied by  $w$  one then expresses  $R$  and its derivatives in terms of  $Q$  and  $Q'$  to arrive at an ordinary differential equation for  $Q$ :

$$(1.16) \quad Q''(x) = F(Q(x), Q'(x), \lambda), \quad x \in \mathbf{R}$$

where  $F$  is even in its second variable. Analogous equations arise for water waves in the presence or absence of surface tension [3]. Although (1.16) may be analyzed by scaling and using the implicit function theorem (cf. [5]), there is a first integral of (1.16) which simplifies the analysis, carried out in section 5. In an important paper [7] on internal waves, Benjamin considered the flow force defined by

$$(1.17) \quad \int_I \{H(x, y) + \frac{1}{2}(\psi_y^2(x, y) - \psi_x^2(x, y)) - g\tilde{\rho}(x, y)y\} dy$$

for each  $x \in \mathbf{R}$ , and showed it was independent of  $x$ . In terms of  $w$  the result becomes

$$(1.18) \quad \int_I \rho_\infty \frac{w_y^2 - w_x^2}{(1 + w_y)} dy + \lambda(\rho - 1)w^2(x, 0) = \text{constant}$$

and when all quantities are again expressed in terms of  $Q$  and  $Q'$ , (1.18) yields a relation

$$(1.19) \quad Q'(x)^2 = h(Q(x), \lambda), \quad x \in \mathbf{R}.$$

Since in (1.16) the dependence of  $F$  on  $Q'$  is even, (1.19) shows that  $Q''$  is a function of  $Q$  and  $\lambda$  alone, and (1.19) is just the first integral. Its analysis is straightforward and yields the totality of nontrivial small waves: solitary waves, surges, conjugate flows, and periodic waves. For the problem of solitary water waves considered in [3] equations analogous to (1.16) and (1.19) hold when the Bond number vanishes or exceeds  $1/3$ . In the range  $(0, 1/3)$  (1.16) is replaced by a fourth-order equation and (1.19), by a relation among third and lower order derivatives.

## 2. NOTATION

Let  $I$  denote a bounded open set in  $\mathbf{R}^n$  or a point and let  $T = \mathbf{R} \times I$ . Thus when  $I$  is a point we merely identify  $T$  with  $\mathbf{R}$ . Let  $C^j(T)$  denote the space of continuous real-valued functions on  $T$  with continuous derivatives through order  $j$ . For  $h$  in  $C^0(T)$  and  $0 < \alpha < 1$  we measure the Hölder constant of  $h$  'at  $x$ ' by

$$(2.1) \quad (H_\alpha h)(x) = \sup_{\substack{|(x,y)-(x',y')| < 1 \\ y \in I}} \frac{|h(x,y) - h(x',y')|}{|(x,y) - (x',y')|^\alpha}$$

where  $|\cdot|$  denotes the Euclidean distance in  $T$ . For each real number  $\mu$  we define a norm

$$(2.2) \quad |h|_{j,\alpha,\mu} = \sum_{|\beta| \leq j} \sup_{x \in \mathbf{R}, y \in I} e^{-\mu|x|} |\partial^\beta h(x,y)| + \sum_{|\beta|=j} \sup_{x \in \mathbf{R}} e^{-\mu|x|} (H_\alpha \partial^\beta h)(x)$$

using the standard multi-index notation for derivatives, and let

$$(2.3) \quad C_\mu^{j,\alpha} = \{h \in C^j : |h|_{j,\alpha,\mu} < \infty\}.$$

Let  $C_\mu^{j,\alpha}(T, \mathbf{R}^n)$  denote  $\mathbf{R}^n$ -valued functions, normed by the supremum of the norms of the components; and  $C_\mu^j$ , the space with the  $H_\alpha$  estimate omitted from (2.2). For  $\delta > 0$  let

$$(2.4) \quad C_{\mu,\delta}^{j,\alpha} = \{u \in C_\mu^{j,\alpha}(T) : |\partial^\beta u| \leq \delta, |\beta| = k, 1 \leq k \leq j; H_\alpha \partial^\beta u \leq \delta, |\beta| = j\}$$

and  $C_{\mu,\delta}^{j,\alpha}(T, \mathbf{R}^n)$ , the vector-valued version, or simply  $C_{\mu,\delta}^{j,\alpha}$  when the context is clear. Note that  $C_{\mu,\delta}^{j,\alpha}$  is a closed subset of  $C_\mu^{j,\alpha}$  and that  $u \in C_{\mu,\delta}^{j,\alpha}$  can be unbounded.

For the internal wave problem  $I$  will denote the interval  $(-h, 1-h)$ . We shall repeatedly refer to sets and functions restricted to the regions  $\{y > 0\}$  and  $\{y < 0\}$  and will use the symbols  $+$  and  $-$  for this purpose. Thus  $I^+ = (0, 1-h)$ ,  $I^- = (-h, 0)$ ,  $T^+ = \mathbf{R} \times I^+$ ,

and  $T^- = \mathbf{R} \times I^-$ . For a function  $u$  defined on  $I$  or  $T$ ,  $u^\pm$  denote its respective restrictions and if  $u^\pm$  are initially defined,  $u$  will denote the corresponding function defined on the union of the domains. In general,  $u$  will not extend continuously to  $\{y = 0\}$ .

Spaces particular to the internal wave problem are:

$$(2.5) \quad X_\mu^1 = \{u \in C^0(\bar{T}) : u^\pm \in C_\mu^{1,1/2}(T^\pm); u(x, -h) = u(x, 1-h) = 0, x \in \mathbf{R}\},$$

the norm being the sum of those on  $T^\pm$ ,

$$(2.6) \quad U_\mu = C_\mu^{1,1/2}(\mathbf{R}) \times C_\mu^{0,1/2}(\mathbf{R}) \times X_\mu^1,$$

and

$$(2.7) \quad \Lambda_\mu = C_\mu^{0,1/2}(T^+) \times C_\mu^{0,1/2}(T^+) \times C_\mu^{0,1/2}(T^-) \times C_\mu^{0,1/2}(T^-) \times C_\mu^{0,1/2}(\mathbf{R}).$$

Finally,  $U_{\mu,\delta}$  denotes a product analogous to (2.6) but with subscripts  $\mu, \delta$  on each factor.

### 3. A LINEAR PROBLEM

In this section we shall consider the linearization of (1.11)-(1.13) about  $(\lambda, w) = (\lambda_d, 0)$ . Formally, the problem is

$$(3.1) \quad \Delta w^\pm = \text{div}(g_1^\pm, g_2^\pm) \quad \text{in } T^\pm,$$

$$(3.2) \quad \rho(w_y^+ - \lambda_d w^+) - (w_y^- - \lambda_d w^-) = g \quad \text{on } y = 0,$$

$$(3.3) \quad w^-(x, -h) = w^+(x, 1-h) = w^-(x, 0) - w^+(x, 0) = 0, \quad x \in \mathbf{R}.$$

This problem will be analyzed in suitable spaces and the results applied to the nonlinear problem in the next section.

### 3.1. A weak formulation of the linear problem

Define

$$(3.4) \quad \langle \psi, \phi \rangle_I = \int_I \rho_\infty(y) \psi_y \phi_y dy - \lambda_d(1 - \rho) \psi(0) \phi(0), \quad \forall \psi, \phi \in C_0^\infty(I)$$

and

$$(3.5) \quad \langle \psi, \phi \rangle_T = \int \int_T \rho_\infty(y) \nabla \psi \nabla \phi dx dy - \lambda_d(1 - \rho) \int_{-\infty}^{\infty} \psi(x, 0) \phi(x, 0) dx, \quad \forall \psi, \phi \in C_0^\infty(T)$$

where  $\rho_\infty$  is given by (1.6). Fix  $\mu > 0$  and  $G = (g_1^+, g_1^-, g_2^+, g_2^-, g) \in \Lambda_\mu$ . By a weak solution of (3.1)-(3.3) is meant a function  $w \in X_\mu^1$  satisfying

$$(3.6) \quad \langle w, \phi \rangle_T = [G, \phi], \quad \forall \phi \in C_0^\infty(T)$$

where

$$(3.7) \quad [G, \phi] = \int \int_T \rho_\infty(y) (g_1, g_2) \cdot \nabla \phi + \int_{-\infty}^{\infty} \{ \rho g_2^+(x, 0) - g_2^-(x, 0) - g(x) \} \phi(x, 0) dx.$$

### 3.2. The kernel of the linear problem

First we consider  $G = 0$  in (3.6). We shall prove that if  $\mu > 0$  is sufficiently small, then  $w$  belongs to a two-dimensional subspace of  $X_\mu^1$ . The dimension of this subspace determines the order of the differential equation (cf. (1.16)) which governs small solutions.

A simple argument (cf. [4]) shows that

$$(3.8) \quad \max_{\tau \in W} \frac{\lambda_d(1 - \rho) \tau^2(0)}{\int_I \rho_\infty(\tau')^2} = 1$$

where  $W = \overset{\circ}{W}^{1,2}(I)$ , with the maximum attained by

$$(3.9) \quad t(y) = \begin{cases} m(y + h), & y \leq 0, \\ -mh(1 - h)^{-1}(y + h - 1), & y \geq 0, \end{cases}$$

and  $m$  is chosen so that  $\int_I \rho_\infty(y) t^2(y) dy = 1$ . Further

$$(3.10) \quad \max_{\tau \in W, \int_I \rho_\infty t \tau = 0} \frac{\lambda_d(1 - \rho) \tau^2(0)}{\int_I \rho_\infty(\tau')^2} < 1$$

with  $\nu$  so restricted. From (3.8) one sees that neither (3.4) nor (3.5) is coercive, while

$$(3.11) \quad \langle \tau, \phi \rangle \equiv \langle \tau, \phi \rangle_I + \int_I \rho_\infty \tau \phi$$

is so and extends to an inner product on  $\dot{W}^{1,2}(I)$  equivalent to the usual one. Consider the problem of finding all nontrivial solutions  $(\nu, \tau) \in \mathbf{R} \times \dot{W}^{1,2}(I)$  of

$$(3.12) \quad \langle \tau, \phi \rangle = (\nu + 1) \int_I \rho_\infty \tau \phi, \quad \forall \phi \in \dot{W}^{1,2}(I).$$

Note that from (3.8) and (3.11),  $\nu \geq 0$  and  $\nu_1 = 0$ ,  $\tau = t = t_1$  is a solution. Since the right-hand side of (3.12) is symmetric and compact with respect to  $\langle \cdot, \cdot \rangle$  standard theory ensures that (3.12) has a sequence of simple characteristic values  $0 = \nu_1 < \nu_2 < \nu_3 \dots$  and corresponding eigenfunctions  $\{t_k\}_{k=1}^\infty$  which form an orthonormal basis for  $L_2(I)$  with respect to the measure  $\rho_\infty dy$ .

Assume  $w \in X_\mu^1$  satisfies (3.6) with  $G = 0$  and set  $\phi(x, y) = \Phi(x)t_k(y)$ , where  $\Phi \in C_0^\infty(\mathbf{R})$ . Although  $\phi \notin C_0^\infty(T)$ , a density argument shows it is a suitable test function. The use of (3.12) yields

$$(3.13) \quad \int_{-\infty}^{\infty} \Phi'(x)A_k'(x)dx = -\nu_k \int_{-\infty}^{\infty} \Phi(x)A_k(x)dx,$$

where  $A_k(x)$  denotes the  $k$ -th Fourier coefficient of  $w(x, \cdot)$ :

$$(3.14) \quad A_k(x) = \int_I \rho_\infty(y)w(x, y)t_k(y)dy, \quad k = 1, 2, \dots$$

It follows that  $A_k''(x) = \nu_k A_k(x)$ . Since  $w \in X_\mu^1$ ,

$$|A_k(x)| \leq \text{const.}e^{\mu|x|}, \quad x \in \mathbf{R}.$$

Hence, if  $\mu$  is restricted to lie in the interval  $(0, \sqrt{\nu_2})$ , then  $A_k \equiv 0$  for  $k \geq 2$ , and  $A_1(x) = \xi_1 + \xi_2 x$  for some pair  $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ . Since the  $t_k$  form a basis, we conclude the following:

**Theorem 3.1.** Suppose  $\mu \in (0, \sqrt{\nu_2})$ . If  $w \in X_\mu^1$  satisfies (3.6) with  $G = (g_1^\pm, g_2^\pm, g) = 0$ , then

$$w(x, y) = (\xi_1 + \xi_2 x)t(y) \quad \text{in } T$$

for some  $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ .

### 3.3. The inhomogeneous linear problem

Next (3.6) will be solved for  $w \in X_\mu^1$  in terms of  $G \in \Lambda_\mu$ . The parameter  $\mu$  will eventually be restricted to lie in  $(0, \bar{\mu})$  where  $\bar{\mu} = \sqrt{\nu_2/2}$ . The function  $w$  is sought in the form  $Q(x)t(y) + R(x, y)$  where  $R(x, \cdot)$  is  $\rho_\infty$  orthogonal to  $t(y)$  for each  $x$  and  $Q(0) = Q'(0) = 0$ . According to Theorem 3.1 the general solution can then be obtained by adding an arbitrary linear function  $\xi_1 + \xi_2 x$  to  $Q$ .

We begin with solutions in the space  $\mathring{W}^{1,2}(T)$ . Let  $E$  denote the closed subspace of  $\mathring{W}^{1,2}(T)$  consisting of functions  $b$  satisfying

$$(3.15) \quad \int_I \rho_\infty(y)t(y)b(x, y)dy = 0, \quad \forall x \in \mathbf{R}$$

Note that  $E \cap C_0^\infty(T)$  is dense in  $E$ . Using the preceding analysis one verifies that for  $a \in W^{1,2}(\mathbf{R})$  and  $b \in E$

$$(3.16) \quad \langle a(x)t(y), \phi \rangle_T = 0, \quad \forall \phi \in E$$

and

$$(3.17) \quad \langle b, \Phi(x)t(y) \rangle_T = 0, \quad \forall \Phi \in W^{1,2}(\mathbf{R})$$

so that each  $w \in \mathring{W}^{1,2}(T)$  has a splitting  $w = a(x)t(y) + b(x, y)$  orthogonal with respect to  $\langle \cdot, \cdot \rangle_T$ .

Let  $\gamma_0$  be in  $C_0^\infty(\mathbf{R}, [0, 1])$ , supported on  $[-2, 2]$ , equal to 1 on  $[-1, 1]$ , and such that  $\gamma_0(x) - 1/2$  is odd about  $x = 3/2$  on  $[1, 2]$ . Then  $\gamma_k(x) = \gamma_0(x - 3k)$ , for  $k \in \mathbf{Z}$  is a partition of unity on  $\mathbf{R}$ . Consider equation (3.6) with  $G$  replaced by  $\gamma_k G = (\gamma_k g_1^\pm, \gamma_k g_2^\pm, \gamma_k g)$ . It will be shown to have a solution of the form  $a_k(x)t(y) + b_k(x, y)$  where  $b_k \in E$ . Then in Theorem 3.2 it is shown that

$$w \equiv \sum_{k=-\infty}^{\infty} (a_k t + b_k)$$

belongs to  $X_\mu^1$  and is the desired solution of (3.6).

To find the component  $b_k$  one solves

$$(3.18) \quad \langle b_k, \phi \rangle_T = [\gamma_k G, \phi], \quad \forall \phi \in E.$$

From the inequality (3.10) it follows that  $\langle \cdot, \cdot \rangle_I$  is coercive when restricted to the  $\rho_\infty$  orthogonal complement of  $t(y)$  and, by integration,  $\langle \cdot, \cdot \rangle_T$  is coercive on  $E$ . Since  $\gamma_k$  has compact support, the right-hand side of (3.18) is easily seen to define a bounded map on  $E$  and the Lax-Milgram theorem provides a unique solution  $b_k \in E$  for (3.18).

To determine  $a_k$  one may justify setting  $\phi = \Phi(x)t(y)$ ,  $\Phi \in C_0^\infty(\mathbf{R})$  in (3.6) where  $G$  is replaced by  $\gamma_k G$ . The use of (3.17) and the identity  $\langle t, t \rangle_I = 0$  yield

$$\int_{-\infty}^{\infty} a' \Phi' = \int_{-\infty}^{\infty} \Phi t(0) \gamma_k \{ \rho g_2^+(\cdot, 0) - g_2^-(\cdot, 0) - g \} - \int_{-\infty}^{\infty} \int_I \rho_\infty \gamma_k (g_1, g_2) \cdot (\Phi' t, \Phi t').$$

The unique solution  $a_k$  with  $a_k(0) = a'_k(0) = 0$  is

$$(3.19) \quad a_k(x) = - \int_{-\infty}^{\infty} (x-s) \gamma_k(s) \{ t(0) (\rho g_2^+(s, 0) - g_2^-(s, 0) - g(s)) + \int_I \rho_\infty g_2(s, y) t'(y) dy \} ds \\ + \int_0^x \gamma_k(s) \int_I \rho_\infty g_1(s, y) t(y) dy ds - x \gamma_k(0) \int_I \rho_\infty g_1(0, y) t(y) dy.$$

**Theorem 3.2.** Suppose  $\mu \in (0, \bar{\mu})$  and let  $w_k = a_k(x)t(y) + b_k(x, y)$  denote the solution of (3.6) with  $G$  replaced by  $\gamma_k G$ . Then

$$w \equiv \sum_{k=-\infty}^{\infty} w_k$$

is an element of  $X_\mu^1$  satisfying (3.6). Moreover,

$$(3.20) \quad |w|_{X_\mu^1} \leq \text{const.} |G|_{\Lambda_\mu}.$$

*Proof.* Fix  $\zeta \in \mathbf{R}$ , set  $V = (\zeta + 1/4, \zeta + 3/4) \times I$ , and  $U = (\zeta, \zeta + 1) \times I$ . Let  $X$  denote the natural restriction of  $X_\mu^1$  to  $V$  and for  $u \in X$  take as a norm:

$$|u|_\zeta = |u^+|_{C_\mu^{1,1/2}(V^+)} + |u^-|_{C_\mu^{1,1/2}(V^-)}.$$

Since  $G = (g_1^\pm, g_2^\pm, g) \in \Lambda_\mu$ , it is immediate from (3.19) that  $a_k(\cdot)t(\cdot) \in X$  for every  $k$ . Since the function  $\gamma_k$  has support on  $[3k - 2, 3k + 2]$ , only a finite number of the  $a_k$  are non-zero on  $[\zeta, \zeta + 1]$ . Hence

$$\sum_{k=-n}^m a_k(\cdot)t(\cdot)$$

converges in  $X$  as  $m \rightarrow \infty$  to an element  $a(\cdot)t(\cdot)$ , where  $a$  is given by (3.19) with  $\gamma_k$  replaced by unity. From this representation of  $a$  it is immediate that

$$(3.21) \quad |at|_{\zeta} \leq \text{const.} |G|_{\Lambda_{\mu}}$$

where the constant is independent of  $\zeta$  and  $G$ , though does grow as  $\mu$  approaches zero.

The rest of the proof is devoted to estimating  $b_k$ . In order to use elliptic estimates we require an equation for  $b_k$  in which all test functions  $\phi \in \dot{W}^{1,2}(T)$  are used. By using a splitting  $\phi = \Phi(x)t(y) + \psi(x, y)$  with  $\psi \in E$  we can appeal to (3.16)-(3.18), and the equations satisfied by  $a_k$  and  $b_k$  to write

$$(3.22) \quad \begin{aligned} \langle b_k, \phi \rangle_T &= \langle b_k, \psi \rangle_T \\ &= [\gamma_k G, \phi - \Phi t] \\ &= [\gamma_k G, \phi] - \langle a_k t, \Phi t \rangle_T \\ &= [\gamma_k G, \phi] - \int \int_T \rho_{\infty} a'_k t \phi_x \end{aligned}$$

where  $a_k$  is given by (3.19). One verifies that

$$\begin{aligned} \int \int_T \rho_{\infty} a'_k t \phi_x &= \int \int_T \rho_{\infty} t(y) [\gamma_k(x) \int_I \rho_{\infty} g_1(x, z) t(z) dz] |_0^x \phi_x(x, y) \\ &\quad + \int \int_T \rho_{\infty} t(y) [\gamma_k(x) t(0) (\rho g_2^+(x, 0) - g_2^-(x, 0) - g(x)) \\ &\quad + \int_I \rho_{\infty} g_2(x, z) t(z) dz] \phi(x, y). \end{aligned}$$

Let  $\tilde{k}$  denote the smallest integer with  $3\tilde{k} - 1 < \zeta < 3\tilde{k} + 3$ , so that  $3(\tilde{k} - 1) + 2 < \zeta < \zeta + 1 < 3(\tilde{k} + 2) - 2$ . Then the support of  $\gamma_k$  is outside  $[\zeta, \zeta + 1]$  unless  $k = \tilde{k}$  or  $k = \tilde{k} + 1$ . If  $k \neq \tilde{k}, \tilde{k} + 1$  and  $\phi$  is restricted to  $C_0^{\infty}(U)$  then from (3.22),

$$(3.23) \quad \langle b_k, \phi \rangle_T = 0, \quad \forall \phi \in C_0^{\infty}(U).$$

Standard elliptic estimates [2] give

$$(3.24) \quad |b_k|_{\zeta} \leq \text{const.} e^{-\mu|\zeta|} |b_k|_{L_2(U)}, \quad k \neq \tilde{k}, \tilde{k} + 1$$

where the constant is independent of  $\mu$ ,  $\zeta$ , and  $k$ . When  $k = \tilde{k}$  or  $\tilde{k} + 1$  the regularity theory gives

$$(3.25) \quad \begin{aligned} |b_k|_{\zeta} &\leq \text{const.} e^{-\mu|\zeta|} \left\{ |b_k|_{L_2(U)} + \sum_{i=1}^2 [|\gamma_k g_i^+|_{C^{0,1/2}(\bar{U}^+)} + |\gamma_k g_i^-|_{C^{0,1/2}(\bar{U}^-)}] \right. \\ &\quad \left. + |g|_{C^{0,1/2}([\zeta, \zeta+1])} \right\} \\ &\leq \text{const.} e^{-\mu|\zeta|} |b_k|_{L_2(U)} + \text{const.} |G|_{\Lambda_{\mu}}. \end{aligned}$$

The theory for (3.24)-(3.25) differs from the standard Schauder estimates in two ways. First, one obtains  $C^{1,1/2}$  estimates rather than the usual  $C^{2,1/2}$  estimates, and second, boundary conditions are posed on the line  $\{y = 0\}$  between two regions. The bounds are obtained by applying the theory in section 8 of [2] for equations in variational form, to the functions  $\rho b_k^+(x, y) + b_k^-(x, -y)$  and  $b_k^+(x, y) - b_k^-(x, -y)$  for  $y > 0$ . The arguments are technical, but straightforward.

The next objective is to estimate  $b_k$  in  $L_2(U)$ . Restrict attention first to the case  $k < \tilde{k}$ . It follows from (3.22) that  $b_k$  satisfies (3.23) for all  $\phi \in C_0^\infty((3k+2, \infty) \times I)$  and the regularity theory just cited gives  $b_k^\pm \in C^\infty((3k+2, \infty) \times \bar{I}^\pm)$  as well as the pointwise decay of  $b_k$  and  $\nabla b_k$  to zero as  $x \rightarrow +\infty$ , since  $b_k$  is known to be in  $\dot{W}^{1,2}(T)$ . In particular,  $b_k$  satisfies the homogeneous version of (3.1)-(3.3) on  $(3k+2, \infty) \times I$ . Multiply equation (3.1) for  $b_k$  by  $\rho_\infty b_k$  and integrate over  $(x, \infty) \times I$  to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dx} \int_I \rho_\infty b_k^2(x, y) dy &= - \int_x^\infty \int_I \rho_\infty |\nabla b_k(s, y)|^2 dy ds \\ &\quad + \lambda_d (1 - \rho) \int_x^\infty b_k^2(s, 0) ds, \quad x \geq 3k + 2. \end{aligned}$$

The use of (3.10) and (3.12) yields

$$- \int_I \rho_\infty \left| \frac{\partial b_k}{\partial y}(s, y) \right|^2 dy + \lambda_d (1 - \rho) b_k^2(s, 0) \leq -\nu_2 \int_I \rho_\infty b_k^2(s, y) dy$$

for all  $s \in \mathbb{R}$ . It follows that

$$\frac{d}{dx} \int_I \rho_\infty b_k^2(x, y) dy \leq -2\nu_2 \int_x^\infty \int_I \rho_\infty b_k^2(s, y) dy$$

whence

$$- \int_x^\infty \int_I \rho_\infty b_k^2 \leq e^{2\bar{\mu}(3k+2-x)} \int_{3k+2}^\infty \int_I \rho_\infty b_k^2$$

for all  $x \geq 3k + 2$ , where  $\bar{\mu} = \sqrt{\nu_2/2}$ . In particular,

$$(3.26) \quad \int_\zeta^{\zeta+1} \int_I \rho_\infty b_k^2 \leq \text{const.} e^{2\bar{\mu}(3k-\zeta)} \int \int_T \rho_\infty b_k^2.$$

The use of  $\phi = b_k$  in (3.18) and the Poincaré inequality yield

$$(3.27) \quad \begin{aligned} \int \int_T \rho_\infty b_k^2 &\leq \text{const.} \int_{3k-2}^{3k+2} \{g(x, 0)^2 + g_2^-(x, 0)^2 + g_2^-(x, 0)^2 + \int_I (g_1^2 + g_2^2) dy\} dx \\ &\leq \text{const.} e^{6\mu|k|} |G|_{\Lambda_\mu}^2 \end{aligned}$$

and this estimate holds for all  $k \in \mathbf{Z}$ . Its use in (3.26) gives

$$|b_k|_{L_2(U)} \leq \text{const. } e^{(\bar{\mu}(3k-\zeta)+\mu^3|k|)} |G|_{\Lambda_\mu}, \quad k < \tilde{k}.$$

A similar argument yields

$$|b_k|_{L_2(U)} \leq \text{const. } e^{(\bar{\mu}(-3k+\zeta)+\mu^3|k|)} |G|_{\Lambda_\mu}, \quad k > \tilde{k} + 1.$$

Since (3.27) holds for  $k = \tilde{k}, \tilde{k} + 1$  the unified estimate:

$$|b_k|_{L_2(U)} \leq \text{const. } e^{(-\bar{\mu}|3k-\zeta|+\mu^3|k|)} |G|_{\Lambda_\mu}$$

holds for all  $k \in \mathbf{Z}$ . The sum over  $k$  is tantamount to a convolution by a kernel with a decay parameter  $\bar{\mu}$ , which is known to take a space with weight  $0 < \mu < \bar{\mu}$  into itself. The result is

$$(3.28) \quad \sum_{k=-\infty}^{\infty} |b_k|_{L_2(U)} \leq \text{const. } e^{\mu|\zeta|} |G|_{\Lambda_\mu}.$$

It is then immediate from (3.24) and (3.25) that  $\sum_{|k| \leq m} b_k$  converges in  $X$  to an element  $b$  with  $|b|_\zeta \leq \text{const.} |G|_{\Lambda_\mu}$ , so that  $w(x, y) = a(x)t(y) + b(x, y) \in X$  satisfies (3.6) for all  $\phi \in C_0^\infty(V)$ . Since  $\zeta$  is arbitrary, the theorem is proved. q.e.d.

An operator  $M$  defined on a function space over  $T$  is called **translation invariant** if

$$M(u_\tau) = (Mu)_\tau$$

where for all  $\tau \in \mathbf{R}$  and  $(x, y) \in T$ ,  $u_\tau(x, y) = u(x + \tau, y)$ . If  $\gamma_k G$  is replaced by  $\gamma_k G_\tau$  in (3.6), the resulting solution will be  $(b_k)_\tau$ , and these sum to  $b_\tau$ . Hence the linear map which takes  $G \in \Lambda_\mu$  to  $b \in X_\mu^1$  is translation invariant.

The next theorem summarizes the analysis of this section and introduces some notation useful for the nonlinear problem in the next section. Throughout the rest of the paper it is assumed that  $0 < \mu < \bar{\mu} = \sqrt{\nu_2/2}$ .

**Theorem 3.3.** *There exist bounded linear maps  $L_1, L_2 : \Lambda_\mu \rightarrow C_\mu^{0,1/2}(\mathbf{R})$  and  $L_3 : \Lambda_\mu \rightarrow X_\mu^1$  such that, given  $G \in \Lambda_\mu$  and  $(\xi_1, \xi_2) \in \mathbf{R}^2$ , the weak form (3.6) of (3.1)-(3.3) has a unique solution  $w(x, y) = Q(x)t(y) + R(x, y)$ , with  $(Q, Q', R) \in U_\mu$  satisfying*

$$(3.29) \quad \begin{aligned} Q(x) &= \xi_1 + \int_0^x Q', \quad x \in \mathbf{R}, \\ Q'(x) &= \xi_2 + L_1 G|_0^x + \int_0^x L_2 G, \quad x \in \mathbf{R}, \\ R(x, y) &= (L_3 G)(x, y), \quad (x, y) \in \bar{T}. \end{aligned}$$

Here

$$(3.30) \quad (L_1 G)(x) = \int_I \rho_\infty(y) t(y) g_1(x, y) dy,$$

$$(3.31) \quad (L_2 G)(x) = t(0) \{g(x) + g_2^-(x, 0) - \rho g_2^+(x, 0)\} - \int_I \rho_\infty(y) t'(y) g_2(x, y) dy,$$

$L_3$  is translation invariant, and

$$\int_I \rho_\infty(y) t(y) (L_3 G)(x, y) dy = 0, \quad x \in \mathbf{R}.$$

**Remark 3.4.** Theorem 3.3 is stated for  $\mu > 0$ , and it will be important in section 4 to have results for  $G \in \Lambda_0$ . Since  $\Lambda_0 \subset \Lambda_\mu$  for  $\mu > 0$ , the solution  $w = at + b$  has  $at$  and  $b$  in  $X_\mu^1$  for all  $\mu \in (0, \bar{\mu})$ . As observed, the constant in (3.21) grows as  $\mu \rightarrow 0$ . However, the constants in (3.25)-(3.28) are independent of  $\mu$  so there is the additional estimate

$$(3.32) \quad |L_3 G|_{X_0^1} \leq \text{const.} |G|_{\Lambda_0}.$$

#### 4. THE NONLINEAR PROBLEM

The nonlinear equations (1.11) and (1.12) can be expressed as

$$(4.1) \quad \Delta w^\pm = \text{div}(g_1(\nabla w^\pm), g_2(\nabla w^\pm)) \quad \text{in } T^\pm,$$

$$(4.2) \quad \rho(w_y^+ - \lambda_d w^+) - (w_y^- - \lambda_d w^-) = g(\nabla w^\pm, w, \lambda) \quad \text{on } y = 0,$$

where

$$(4.3) \quad g_1 = \frac{w_x w_y}{1 + w_y}, \quad g_2 = \frac{w_y^2}{1 + w_y} + \frac{|\nabla w|^2}{2(1 + w_y)^2},$$

and

$$(4.4) \quad g(\nabla w^\pm, w, \lambda) = \rho g_2(\nabla w^+) - g_2(\nabla w^-) + (\lambda_d - \lambda)(1 - \rho)w.$$

The intention is to show that solutions of (3.1)-(3.3) in  $X_\mu^1$  for which  $|\lambda_d - \lambda|$  is small and for which  $w$  and its derivatives are small uniformly in  $T$ , have the decomposition

$w(x, y) = Q(x)t(y) + R(x, y)$ , where  $R$  and its derivatives are pointwise functions of  $Q(x)$  and  $Q'(x)$ . This 'center-manifold' reduction is then used to derive a differential equation for  $Q(x)$  to be analyzed in section 5.

#### 4.1. A transformed problem

Set  $Q_1 = Q$ ,  $Q_2 = Q'/\beta$ , where  $\beta > 0$  is a constant to be chosen in due course and let  $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$  denote the value of  $(Q_1(0), Q_2(0))$ . Let  $\Omega = (Q_1, Q_2, R)$  and  $p = \lambda_d - \lambda$ . Equation (3.29) can be expressed as

$$(4.5) \quad \begin{aligned} Q_1(x) &= \xi_1 + \beta \int_0^x Q_2, \quad x \in \mathbf{R}, \\ Q_2(x) &= \xi_2 + \beta^{-1} L_1 G|_0^x + \beta^{-1} \int_0^x L_2 G, \quad x \in \mathbf{R}, \\ R(x, y) &= (L_3 G)(x, y), \quad (x, y) \in \bar{T} \end{aligned}$$

or in symbolic form  $\Omega = N_0(G, \xi)$  where  $N_0$  is linear. If one replaces  $G = (g_1^+(x, y), \dots)$  from (3.6) by  $G(\Omega, p) = (g_1(\nabla(Q_1(x)t(y) + R(x, y))^+, \dots)$  from (4.3)-(4.4) and defines  $N(\Omega, \xi, p)$  to be  $N_0(G(\Omega, p), \xi)$ , one obtains a nonlinear equation  $\Omega = N(\Omega, \xi, p)$  whose fixed points formally provide solutions of (4.1)-(4.2).

As is standard in an analysis leading small solutions in a center manifold we shall use a truncation. Let  $\eta : \mathbf{R} \rightarrow \mathbf{R}$  be an odd, nondecreasing,  $C^\infty$  function which is the identity for  $-1 \leq t \leq 1$ , equal to 2 for  $t \geq 2$ , and satisfies  $\eta' \leq 2$ . Set  $\eta_r(t) = r\eta(t/r)$  for  $r > 0$ . The terms  $g_1$ ,  $g_2$ , and  $g$  in (4.3)-(4.4) can be expressed in terms of  $Q$ ,  $Q'$ ,  $R$  and their derivatives, or in terms of  $Q_1$ ,  $Q_2$ ,  $R$ , and their derivatives. In such an expression we replace each occurrence of these latter variables by its composition with  $\eta_r$ . Thus, for example,  $g_1 = w_x w_y / (1 + w_y)$  is replaced by

$$(4.6) \quad g_1(\Omega, r, \beta) = \frac{\{\beta \eta_r(Q_2(x))t(y) + \eta_r(R_x(x, y))\} \{\eta_r(Q_1(x))t'(y) + \eta_r(R_y(x, y))\}}{1 + \eta_r(Q_1(x))t'(y) + \eta_r(R_y(x, y))}$$

and similar expressions  $g_2(\Omega, r, \beta)$ ,  $g(\Omega, p, r, \beta)$  are defined. Now, with  $G$  in equation (4.5) replaced by  $G(\Omega) = G(\Omega, p, r, \beta) = (g_1^+(\Omega, r, \beta), \dots, g(\Omega, p, r, \beta))$  one obtains an equation

$$(4.7) \quad \Omega = N(\Omega, \xi, p, r, \beta)$$

where  $N$  has components

$$\begin{aligned}
 (4.8) \quad N_1(\Omega, \xi, p, r, \beta) &= \xi_1 + \beta \int_0^x \eta_r(Q_2)(s) ds, \\
 N_2(\Omega, \xi, p, r, \beta) &= \xi_2 + \beta^{-1} L_1 G(\Omega)|_0^x + \beta^{-1} \int_0^x L_2 G(\Omega)(s) ds, \\
 N_3(\Omega, \xi, p, r, \beta) &= (L_3 G)(x, y), \quad (x, y) \in \bar{T}.
 \end{aligned}$$

Before proceeding to an existence result for (4.7), some remarks are in order. A solution  $w$  corresponding to a wave will be small in  $X_0^1$ ; that is,  $w$  and its derivatives will be uniformly small. However, the operators  $N_1$  and  $N_2$  do not map bounded functions into bounded functions. They do map spaces with growth parameter  $\mu > 0$  into themselves, and that is where we are obliged to begin. In [3] operators similar to the  $N_i$  arose through a solution process involving Fourier transforms. There, however, the maps had sufficient regularizing properties that it was not necessary to introduce the 'small' parameter  $\beta$  nor to introduce the sets  $C_{\mu, \delta}^{j, \alpha}$  used here. In the ensuing subsections we shall repeatedly use results from [6] regarding the behavior of nonlinear maps in weighted spaces.

#### 4.2. Existence with truncation

The basic nonlinear map of interest is the composition of an element  $u = (u_1, \dots, u_n)$  with a smooth function  $g$  from  $\mathbf{R}^n$  to  $\mathbf{R}$ . More generally we consider a smooth function  $g = g(u_1, \dots, u_n, p_1, \dots, p_n)$  depending on a parameter  $p = (p_1, \dots, p_n)$ , though the dependence on the parameter  $p$  is not central in the discussion and will sometimes be suppressed. We use  $D^k$  for a  $k$ th derivative of a map between Banach spaces and  $|D^k(\cdot)|$  for its norm (cf. [1]). It will be assumed that the derivatives with respect to  $u$  satisfy  $|D_u^k g| \leq c_k < \infty$ ,  $k = 0, 1, 2, \dots$  for all  $(u, p) \in \mathbf{R}^n \times \mathbf{R}^n$ . We use  $C$  for a generic constant depending on  $j, n$ , and other quantities indicated in the context.

Suppose  $g$  is smooth and that  $g(0, \dots, 0, p_1, \dots, p_n) = 0$ . Let  $M_g$  be the map defined by

$$(4.9) \quad M_g(u, p)(x, y) = g(u(x, y), p).$$

If  $\mu \geq 0$ , then  $M_g$  maps  $C_{\mu, \delta}^{j, \alpha}(T, \mathbf{R}^n) \times \mathbf{R}^n$  into  $C_{\mu, \sigma}^{j, \alpha}$  where  $\sigma \leq C \sup_{1 \leq k \leq j+1} c_k \delta^k$ . Of course, we shall be interested in truncated functions as, for example, in (4.6). We define  $g(u, p, r)$  to be  $g(\eta_r(u_1), \dots, \eta_r(u_n), p_1, \dots, p_n)$ . For such a composition  $g$ , results regarding  $M_g$  needed for an existence theorem are now summarized (cf. [6]).

**Proposition 4.1.** Suppose  $h(u, p)$  is smooth and that  $h(0, p) = 0$ . Let  $g$  denote the truncated function  $h(u, p, r)$ . Then  $M_g$  maps  $C_{\mu, \delta}^{j, \alpha}(T, \mathbf{R}^n) \times \mathbf{R}^n$  into  $C_{\mu, \sigma}^{j, \alpha}$  with

$$\sigma \leq BCr(1 + |p|),$$

where  $B = \sup_{1 \leq k \leq j+1} (\delta/r)^k$  and  $C$  depends on  $h$ . Further,  $|M_g|_{L^\infty} \leq Cr$  and

$$(4.10) \quad |M_g(u, p) - M_g(u', p')| \leq BC(|u - u'| + r|p - p'|),$$

where the norms of functions are those in  $C_{\mu}^{j, \alpha}$ . If, in addition,  $g$  is flat (i.e., satisfies  $D_{u, p_1} g(0, \dots, 0, 0, p_2, \dots, p_n) = 0$  for all  $p$ ), then  $M_g$  maps  $C_{\mu, \delta}^{j, \alpha}(T, \mathbf{R}^n) \times \mathbf{R}^n$  into  $C_{\mu, \sigma}^{j, \alpha}$  where

$$\sigma \leq BCr(r + |p_1|).$$

Further,  $|M_g|_{L^\infty} \leq Cr(r + |p_1|)$  and

$$(4.11) \quad |M_g(u, p) - M_g(u', p')| \leq BC(r + |p_1|)|u - u'| + BCr|p - p'|.$$

Because of the explicit nature of the map  $N_1$  we can, and will, choose  $\delta = r$  in what follows, so that  $B = 1$ . Recall that  $\Omega$  lies in  $U_{\mu, r} = C_{\mu, r}^{1, 1/2}(\mathbf{R}) \times C_{\mu, r}^{0, 1/2}(\mathbf{R}) \times X_{\mu, r}^1$ . In (4.8),  $p = p_1 = \lambda_d - \lambda$  and we drop the subscript. To see that  $N$  maps  $U_{\mu, r}$  into itself for small  $r$  and is contractive, we can ignore the additive terms  $\xi_1, \xi_2$ . The remaining terms all have the form of compositions: linear maps (derivatives and injections), followed by nonlinear functions of the type covered by Proposition 4.1, followed by other linear maps. We use pairs or triples of numbers to indicate the supremum bounds on a function. Thus for  $N_1$  in (4.8), from Proposition 4.1 or by direct computation, it follows that the map  $Q_2 \rightarrow \eta_r(Q_2)$  takes a function with bounds  $(\infty, r)$  on the function and its Hölder estimate, respectively, to one with bounds  $(2r, 2r)$ . The linear map  $\beta \int_0^x$  produces bounds  $(\infty, 2\beta r, 2\beta r)$ , the third term indicating the Hölder bound on the first derivative. The Lipschitz estimate is

$$|N_1(\Omega) - N_1(\Omega')|_{C_{\mu}^{1, 1/2}} \leq \beta C |\Omega - \Omega'|_{U_{\mu}}$$

where, because of the integration,  $C$  depends inversely on  $\mu$ .

In  $N_2$  the linear maps  $\partial/\partial x$  and  $\partial/\partial y$  take a function with bounds  $(\infty, r, r)$  to one with bounds  $(r, r)$ . The injection of  $C_{\mu}^{1, 1/2}$  into  $C_{\mu}^{0, 1/2}$  is a bounded linear map and produces

functions with bounds  $(\infty, r)$ . Hence the nonlinear flat functions in  $G(\Omega)$  act on functions with bounds  $(\infty, r)$  and produce  $(Cr(r + |p|), Cr(r + |p|))$ , according to Proposition 4.1. Both  $L_1$  and  $L_2$  preserve such bounds with a new constant  $C$ . The integration following  $L_2$  gives  $(\infty, Cr(r + |p|))$ . In all,  $N_2$  produces bounds  $(\infty, \beta^{-1}Cr(r + |p|))$  and, from (4.11), satisfies

$$(4.12) \quad |N_2(\Omega, p) - N_2(\Omega', p')|_{C_\mu^{0,1/2}} \leq \frac{1}{\beta} [C(r + |p|)|\Omega - \Omega'|_{U_\mu} + Cr|p - p'|]$$

In  $N_3$ , as earlier, the term  $G(\Omega)$  produces terms with bounds  $(Cr(r + |p|), Cr(r + |p|))$ . As  $L_3$  acts from  $\Lambda_0$  to  $X_0^1$  (cf. Remark 3.4), there is a bound  $C'r(r + |p|)$  for the resulting function, its derivatives, and their Hölder estimates. The right-hand side of the Lipschitz estimate for  $N_3$  in  $X_\mu^1$  is the same as that for  $N_2$ , but without the factor  $\beta^{-1}$  so that

$$(4.13) \quad |N(\Omega, p) - N(\Omega', p')|_{U_\mu} \leq C_2(\beta + (1 + \beta^{-1})(r + |p|))|\Omega - \Omega'|_{U_\mu} + C(1 + \beta^{-1})r|p - p'|.$$

In order to carry out a contractive scheme for  $N$  we require (i)  $r \in (0, r_1]$  so that all denominators in terms  $g_1(u, p, r)$ , etc. are at least  $1/2$ ; (ii)  $2\beta \leq 1$  and  $C_1\beta^{-1}(r + |p|) \leq 1$  so that  $N$  maps into  $U_{\mu, r}$ ; and (iii)  $C_2(\beta + (1 + \beta^{-1})(r + |p|)) < 1$  so that it is contractive in the argument  $\Omega$ . Here  $C_1$  and  $C_2$  arise from combining terms from the last paragraph. To satisfy these conditions with a contraction factor of  $1/2$ , first choose  $\beta = \min((4C_2)^{-1}, 1/2)$  and restrict  $p$  by  $|p| < r$ . Then there exists an  $r_0 > 0$  such that (i)-(iii) hold for any  $r \in (0, r_0]$ .

**Theorem 4.2.** *There exist a positive constants  $\beta$  and  $r_0$  such that if  $r \in (0, r_0]$ , the equation  $\Omega = N(\Omega, \xi, p, r, \beta)$  has a unique solution  $\Omega \in U_{\mu, r}$  for each choice of  $\xi \in \mathbf{R}^2$  and  $p = \lambda_d - \lambda \in (-r, r)$ . Moreover,  $\Omega$  is Lipschitz continuous as a function of  $\xi$  and  $p$ .*

We fix  $r = r_0$  and  $\beta$  from the theorem and suppress them when convenient. Let  $\Omega(\xi, p) = \Omega(\xi, p)(x, y) = (Q_1(\xi, p)(x), Q_2(\xi, p)(x), R(\xi, p)(x, y))$  denote the solution given by Theorem 4.2.

**Remark 4.3.** *If  $\xi_1 = \xi_2 = 0$ , then the trivial solution satisfies the fixed point equation, whence  $\Omega(0, p) \equiv 0$ ,  $p \in (-r, r)$ .*

**Remark 4.4.** *The constants  $C_1$  and  $C_2$  in the discussion preceding Theorem 4.2 depend on  $\mu$  and will approach  $\infty$  as  $\mu \rightarrow 0$ . Suppose  $\tilde{I} = [\mu_1, \mu_2]$  where  $0 < \mu_1 < \mu_2 < \bar{\mu}$  and let*

$\tilde{C}_i$ ,  $i = 1, 2$  be the supremum of  $C_i(\mu)$  for  $\mu \in \tilde{I}$ . A choice of  $\beta$  and  $r_0$  based on these new constants provides a solution for any  $\mu \in \tilde{I}$  and, by uniqueness, a solution in  $U_\mu$  coincides with that for any larger parameter in  $\tilde{I}$ .

### 4.3. Reduction and regularity

Here the special structure of the third component,  $R(x, y)$ , is examined. It is the dependence on  $x$  that is central here and we can naturally suppress the variable  $y$  by regarding  $R(x, \cdot)$ ,  $R_x(x, \cdot)$ , and  $R_y(x, \cdot)$  as taking values, respectively, in

$$V_0 = \{f \in C(I) : f \in C^{1,1/2}(\overline{I^\pm})\},$$

$$V_1 = \{f \in C(I) : f \in C^{0,1/2}(\overline{I^\pm})\},$$

$$V_2 = \{f \in C^{0,1/2}(\overline{I^\pm})\}.$$

Let  $x_0$  be fixed. As for ordinary differential equations, one defines  $\Omega^*(x) \equiv \Omega(\xi, p)(x + x_0)$ ,  $\Omega^{**}(x) \equiv \Omega(Q_1(\xi, p)(x_0), Q_2(\xi, p)(x_0), p)(x)$  and shows (cf. [3],[6]) that they satisfy

$$\Omega^*(x) - \Omega^{**}(x) = N(\Omega^*(x)) - N(\Omega^{**}(x)).$$

As they have the same data at  $x = 0$ , they coincide, yielding the following important result.

**Lemma 4.5.** *For all  $\xi \in \mathbf{R}^2$ ,  $p \in (-r, r)$ ,  $x \in \mathbf{R}$ , and  $x_0 \in \mathbf{R}$*

$$\Omega(\xi, p)(x + x_0) = \Omega(Q_1(\xi, p)(x_0), Q_2(\xi, p)(x_0), p)(x).$$

By taking derivatives with respect to  $x$  and  $y$  in the third component of this last relation, setting  $x = 0$ , and then replacing  $x_0$  by  $x$  one obtains the local dependence of  $R$  and its derivatives on  $Q_1(x)$  and  $Q_2(x)$  expressed in the following 'center manifold' result.

**Theorem 4.6.** *There are Lipschitz continuous functions  $J_0$ ,  $J_1$ , and  $J_2$  from  $\mathbf{R}^2 \times (-r, r)$  into  $V_0$ ,  $V_1$ , and  $V_2$ , respectively, such that*

$$(4.14) \quad \begin{aligned} R(\xi, p)(x, \cdot) &= J_0(Q_1(\xi, p)(x), Q_2(\xi, p)(x), p) \\ R_x(\xi, p)(x, \cdot) &= J_1(Q_1(\xi, p)(x), Q_2(\xi, p)(x), p) \\ R_y(\xi, p)(x, \cdot) &= J_2(Q_1(\xi, p)(x), Q_2(\xi, p)(x), p) \end{aligned}$$

At this point we indicate a difficulty that arises in the context of weighted spaces, where the analysis eventually requires some degree of smoothness in the resulting solutions.

Let  $g(u) = \eta_r^2(u)$  for  $r > 0$ . As a map of  $C_1^0$  into itself the map  $M_g$  is not differentiable at  $u = 0$ . Were there a derivative, it would take each uniformly bounded function in  $C_1^0$  to zero. However, if  $h \in C^0(\mathbf{R})$  is a function with compact support, the function  $h_k = h(x-k)$  has  $C_1^0$  norm  $|h_k| \leq Ce^{-k} \rightarrow 0$  as  $k \rightarrow \infty$ . However, one verifies that the norm  $|\eta_r^2(h_k)|$  does not approach zero faster than  $|h_k|$ . One can show that  $M_g$  is Lipschitz continuous from  $C_1^0$  into itself (cf. Remark 4.8 below). However, it is not Lipschitz from  $C_\mu^{j,\alpha}$  into itself for  $\mu > 0$  when either  $j$  or  $\alpha$  is positive. The lack of regularity of  $M_g$  is attributable to the presence of multiplicative factors which can have exponential growth in expressions for derivatives. One can control these factors if one views the target of  $M_g$  as a space allowing more rapid growth than that of the domain and for applications, differentiability in this sense will suffice.

For fixed  $(x, y) \in T$  and  $q = 1, 2, \dots$   $D^q g = D_{(u,p)}^q g(u(x, y), p)$  is a symmetric  $q$ -linear map from  $\mathbf{R}^n \times \mathbf{R}^n$  to  $\mathbf{R}$ . It induces a symmetric  $q$ -linear map on  $C_\mu^{j,\alpha} \times \mathbf{R}^n$  defined by

$$M_{D^q g}(h_1, \pi_1) \dots (h_q, \pi_q) = D^q g(u(x, y), p)(h_1(x, y), \pi_1) \dots (h_q(x, y), \pi_q).$$

The following result is shown in [6].

**Theorem 4.7.** *Suppose  $g$  is smooth and  $\mu \geq 0$ . Then  $M_g$  defined in (4.9) is  $q$  times differentiable as a map from  $C_\mu^{j,\alpha}(T, \mathbf{R}^n) \times \mathbf{R}^n$  to  $C_{(j+q+2)\mu}^{j,\alpha}$  and the derivatives, defined by*

$$D^k M_g = M_{D^k g}, \quad 1 \leq k \leq q$$

*are Lipschitz continuous.*

**Remark 4.8.** *While  $DM_g$  does not generally exist as a map of  $C_\mu^{j,\alpha}$  into itself, the map  $M_{Dg}$  is a bounded linear map of  $C_\mu^{j,\alpha}$  into itself when evaluated at a  $u \in C_{\mu,\delta}^{j,\alpha}$*

From Theorem 4.7 one can obtain:

**Corollary 4.9.** *For  $\mu > 0$  the map  $N(\Omega, \xi, p)$  has  $q$  Lipschitz continuous derivatives acting from  $U_\mu \times \mathbf{R}^2 \times \mathbf{R}$  to  $U_{(q+2)\mu}$ .*

In the spirit of Remark 4.8 one can show that the formal derivative of  $N$  with respect to  $\Omega$ , evaluated at a solution, is a bounded map of  $U_\mu$  into itself with a bound equal to the Lipschitz constant in (4.13), at most  $1/2$  under our working hypotheses. It is known [11] that functions describing center manifolds, such as those in Theorem 4.6, are not  $C^\infty$  in general. However, using Remark 4.4 and the previous corollary it can be shown inductively (cf. [6]) starting with a sufficiently small  $\mu$  that the following holds.

**Theorem 4.10.** Given  $q \geq 1$  there is an integer  $k(q)$  such that if  $\tilde{\mu} = k(q)\mu < \bar{\mu}$ , then the solution  $\Omega \in U_{\mu,r}$  from Theorem 4.2 is  $q$  times differentiable with respect to  $(\xi, p)$  as a map into  $U_{\tilde{\mu}}$ .

A final result will be needed for the analysis in section 5.

**Theorem 4.11.** Under the hypotheses of the previous theorem the functions  $J_i$ ,  $i = 0, 1, 2$  in Theorem 4.6 are  $q$  times differentiable and satisfy

$$(4.15) \quad |J_i(\xi, p)|_{V_i} \leq C|\xi|(|\xi| + |p|).$$

*Proof.* Let  $E_0$  be the evaluation map taking  $R$ ,  $R_x$ , or  $R_y$  in  $X_\mu^1$ , etc. to its value at  $x = 0$  in  $V_0$ ,  $V_1$ , or  $V_2$ , respectively. Since  $E_0$  is linear and hence infinitely differentiable, the smoothness of the  $J_i$  follows from Theorem 4.10.

One has  $D_{\xi,p}\Omega = (I - D_\Omega N)^{-1}D_{\xi,p}N$  and calculates that  $D_\Omega N$  evaluated at  $\Omega = 0$ ,  $p = 0$  takes  $(u_1, u_2, u_3)$  to  $(\beta \int_0^x u_2(s)ds, 0, 0)$ . Since neither  $\xi_1$  nor  $\xi_2$  appears in  $N_3$  and since  $p$  intervenes through flat functions, one finds

$$D_{\xi_1, \xi_2, p}\Omega(0, 0, 0)(\sigma_1, \sigma_2, \pi) = (\sigma_1 + \beta\sigma_2x, \sigma_2, 0)$$

and thus the third component is the zero function. It follows that the derivatives of  $R$ ,  $R_x$ , and  $R_y$  with respect to  $\xi, p$  vanish at  $(0, 0)$ . From Remark 4.3 each of these functions vanishes identically when  $\xi = 0$  and  $p \in (-r, r)$  so that, for example,  $R(\xi, p) = S(\xi, p)\xi$ . Since  $D_\xi R(0, 0) = 0$ ,  $S(0, 0) = 0$ , implying that  $S$  vanishes at least linearly in  $|\xi| + |p|$ . Similar considerations apply to  $R_x$  and  $R_y$ , yielding the desired bounds on  $J_i$ ,  $i = 0, 1, 2$ . q.e.d.

## 5. THE REDUCED EQUATIONS

### 5.1. Structure of the fixed-points

In this section we analyze solutions of (4.7) for  $\lambda$  in a neighborhood of  $\lambda_d$  and it will be convenient to express all quantities in terms of  $p = \lambda_d - \lambda$  throughout most of the section. For fixed  $\xi$  and  $p$ , write  $\Omega(x, y)$  for  $\Omega(\xi_1, \xi_2, p)(x, y)$ ,  $\Omega^*(x, y)$  for  $\Omega(\xi_1, -\xi_2)(x, y)$  and let

$$\tilde{\Omega}(x, y) \equiv (Q_1(\xi_1, -\xi_2, p)(-x), -Q_2(\xi_1, -\xi_2, p)(-x), R(\xi_1, -\xi_2, p)(-x, y)).$$

The particular form of the truncated  $g_1$  in (4.6) and its analogues for  $g_2$  and  $g$  give the relations  $g_1(\tilde{\Omega})(x, y) = -g_1(\Omega^*)(-x, y)$ ,  $g_2(\tilde{\Omega})(x, y) = g_2(\Omega^*)(-x, y)$ , and  $g(\tilde{\Omega})(x, y) = g(\Omega^*)(-x, y)$ . Use of the explicit forms of  $L_1$  and  $L_2$  from (3.30)-(3.31) in (4.8) yields  $N_i(\tilde{\Omega}, \xi_1, \xi_2, p)(x) = \tilde{\Omega}_i(x)$  for  $i = 1, 2$ , while the linear analysis of Section 3 (cf. (3.6)-(3.7)) proves the analogous relation for  $N_3$ . Hence,  $N(\tilde{\Omega}, \xi_1, \xi_2, p)(x) = \tilde{\Omega}(x)$ . Since  $\Omega$  satisfies the same equation and  $N$  is a contraction, there results  $\Omega(x, y) = \tilde{\Omega}(x, y)$ , giving the following useful structure relations:

$$\begin{aligned}
 (5.1) \quad & Q_1(\xi_1, \xi_2, p)(x) = Q_1(\xi_1, -\xi_2, p)(-x), \\
 & Q_2(\xi_1, \xi_2, p)(x) = -Q_2(\xi_1, -\xi_2, p)(-x), \\
 & R(\xi_1, \xi_2, p)(x, y) = R(\xi_1, -\xi_2, p)(-x, y), \\
 & R_x(\xi_1, \xi_2, p)(x, y) = -R_x(\xi_1, -\xi_2, p)(-x, y), \\
 & R_y(\xi_1, \xi_2, p)(x, y) = R_y(\xi_1, -\xi_2, p)(-x, y),
 \end{aligned}$$

for all  $(\xi_1, \xi_2) \in \mathbf{R}^2$ ,  $p = \lambda_d - \lambda \in (-r_0, r_0)$ , and  $(x, y) \in \bar{T}$ .

## 5.2. A differential equation for $Q$

Let  $s, s_1 \in (0, r_0)$ , where  $r_0$  arises in Theorem 4.2, and *assume* that for some  $\xi \in \mathbf{R}^2$  and  $p \in [-s, s]$  there holds

$$(5.2) \quad |Q_1(\xi, p)(\cdot)|_{L_\infty(\mathbf{R})}, |Q_2(\xi, p)(\cdot)|_{L_\infty(\mathbf{R})} \leq s_1.$$

The use of this in (4.15) gives the following estimate whenever  $s_1$  is sufficiently small:

$$(5.3) \quad |R(\xi; p)(\cdot)|_{L_\infty(T)}, |R_x(\xi; p)(\cdot)|_{L_\infty(T)}, |R_y(\xi; p)(\cdot)|_{L_\infty(T)} < r_0.$$

These estimates allow one to remove the truncations present in (4.8). Write  $Q(x)$  for  $Q_1(\xi; \lambda)(x)$ , so that  $\beta^{-1}Q'(x) = Q_2(\xi, p)(x)$  by (4.5). Differentiating the second equation in (4.8) yields

$$(5.4) \quad Q''(x) = \beta \frac{d}{dx} Q_2(\xi, p)(x) = \frac{d}{dx} L_1(G(\Omega(\xi, p)))(x) + L_2(G(\Omega(\xi, p)))(x).$$

Wherever  $R, R_x$ , or  $R_y$  occurs in the right-hand side, we use (4.14) to replace it by a function of  $Q(x)$  and  $Q'(x)$ . There results

$$(5.5) \quad Q''(x) = \frac{d}{dx} E_1(Q(x), Q'(x), p) + E_2(Q(x), Q'(x), p)$$

where  $E_1$  and  $E_2$  are smooth functions of their arguments, by Theorem 4.11. One may check easily that  $E_i$  and its gradient vanishes at  $(0,0,0)$  for  $i = 1, 2$ . If  $s$  and  $s_1$  are sufficiently small, then by solving for  $Q''$  in the equations resulting from (5.5) we are led to an equation of the form

$$(5.6) \quad \begin{aligned} Q''(x) &= F(Q(x), Q'(x), p), \quad x \in \mathbf{R} \\ Q(0) &= \xi_1, \quad Q'(0) = \beta \xi_2. \end{aligned}$$

We have shown that if (5.2) holds for some suitably small  $s_1$ , then a solution  $Q(x) = Q_1(\xi, p)(x)$  to (5.6) exists. Conversely, if one can construct a solution  $(p, Q)$  of (5.6) with

$$(5.7) \quad |Q|_{L^\infty(\mathbf{R})}, \beta^{-1}|Q'|_{L^\infty(\mathbf{R})} \leq s_1, \quad |p| \leq s,$$

then setting  $Q_1(\xi, p)(x) = Q(x)$ ,  $Q_2(\xi, p)(x) = \beta^{-1}Q'(x)$ , and (cf. (4.14))

$$(5.8) \quad R(x, y) \equiv R(\xi, p)(x, y) = R(Q(x), \beta^{-1}Q'(x), p)(0, y)$$

yields  $\Omega(\xi, p)$  satisfying (4.8) without any truncation. In particular, setting  $w(x, y) = Q(x)t(y) + R(x, y)$  yields a small solution of (1.11)-(1.13). Hence, the study of small solutions to (1.11)-(1.13) for  $p = \lambda_d - \lambda$  small is reduced to studying (5.6).

The structure results (5.1) easily show that  $E_1$  is an odd function of its second argument  $Q'(x)$  while  $E_2$  is even. It follows immediately that  $F$  in (5.6) is an even function of its argument  $Q'(x)$ . This ensures that any solution of (5.6) is necessarily symmetric about a point at which  $Q'$  vanishes, whence

**Theorem 5.1.** *Let  $Q$  satisfy (5.6)-(5.7). Then exactly one of the following holds: (i)  $Q(x) = \text{constant}$  ( $Q'(x) \equiv 0$ ); (ii)  $Q$  is monotone ( $Q'(x) \neq 0$  on  $\mathbf{R}$ ); (iii)  $Q$  is even about some point  $x_0$  and is monotone on  $(x_0, \infty)$  ( $Q'(x)$  has a single zero on  $\mathbf{R}$ ); (iv)  $Q$  is a periodic function ( $Q'$  has more than one zero on  $\mathbf{R}$ ).*

This result has implications for any small solution  $w$ . Since  $w(x, y) = Q(x)t(y) + R(x, y)$ , where  $R(x, y) = R(Q(x), \beta^{-1}Q'(x), p)(0, y)$ , there results  $w_x(x, y) = Q'(x)t(y) + R_x(Q(x), \beta^{-1}Q'(x), p)(0, y)$  by (4.14). This may be rewritten as

$$(5.9) \quad w_x(x, y) = Q'(x)\{t(y) + Q'(x)^{-1}R_x(Q(x), \beta^{-1}Q'(x), p)(0, y)\}.$$

Now  $R_x(\xi_1, \xi_2; p)(0, y)$  is an odd function of  $\xi_2$  by (5.1), and so  $\xi_2^{-1}R_x(\xi_1, \xi_2, p)(0, y)$  is an even function of  $\xi_2$  and is smooth by Theorem 4.11. This quantity vanishes in  $V_1$  when

$\xi_1 = \xi_2 = 0$  and  $p = 0$ . It follows that the expression added to  $t(y)$  in the brackets in (5.9) is small in  $V_1$ , uniformly in  $x$ , whenever  $s$  and  $s_1$  are sufficiently small. Elliptic theory then shows the added term is small in a topology including derivatives near  $y = -h$  and  $y = 1 - h$ . Thus the whole bracketed term in (5.9) is positive on  $T$  whenever  $s$  and  $s_1$  are sufficiently small. Theorem 5.1 then ensures that  $w$  satisfies one of the following: (i)  $w(x, y)$  is a function of  $y$  alone; (ii)  $w_x(x, y)$  is one-signed on  $T$ ; (iii)  $w(x, y)$  is even in  $x$  about some  $x_0$  and  $w_x(x, y)$  is one-signed on  $(x_0, \infty) \times I$ ; (iv)  $w$  is a periodic function of  $x$ . In case (i),  $w$  is either identically zero or is a nontrivial  $x$ -independent solution of ((1.11)-(1.13)); these are termed conjugate solutions by Benjamin [7],[8]. In case (ii), we have a heteroclinic orbit connecting a conjugate flow at  $x = -\infty$  to a different one at  $x = +\infty$ . We shall show there are solutions connecting the trivial solution to a non-trivial conjugate flow; these are 'internal surges'. After a suitable translation, we may take  $x_0 = 0$  for case (iii); then the flow connects a conjugate flow at  $x = -\infty$  to itself at  $x = +\infty$ , and so is a homoclinic orbit. If the conjugate flow is the trivial solution, then the solutions are usually called solitary waves.

One may seek small solutions of (5.6)-(5.7) by expanding  $F$  to quadratic or cubic terms in  $(Q, Q', p)$ , scaling the resulting equations, and analyzing them with the aid of an implicit function theorem. This approach is used in [3] for a water-wave problem and in [5] for the case of internal waves. One may avoid these technicalities with the use of an important conserved quantity, and this is now shown.

### 5.3. A first integral

As noted in (1.17) or [7], the flow force is independent of  $x$ . In terms of the function  $w$ , this becomes

$$(5.10) \quad \int_I \rho_\infty \frac{w_y^2 - w_x^2}{(1 + w_y)} dy + \lambda(\rho - 1)w^2(x, 0) = C$$

for all  $x$ , where  $C$  is a constant. This result can also be derived from (1.11)-(1.13): one multiplies (1.11) by  $\rho_\infty w_x$  and integrates over  $(-h, 1 - h)$  using (1.12)-(1.13). Since  $w_y(x, y) = Q(x)t'(y) + R_y(Q(x), \beta^{-1}Q'(x), p)(0, y)$ , it follows from (5.1) that  $w_y$  is an even function in the argument  $Q'(x)$ . By Theorem 4.6, we may assume that  $w(x, y)$  and  $w_y(x, y)$  are functions of  $Q(x), Q'(x)^2$  and  $y$ . Equation (5.9) gives  $w_x^2(x, y)$  as a function of  $Q(x), Q'(x)^2$ , and  $y$ , which behaves like  $Q'(x)^2 t^2(y)$  if  $s$  and  $s_1$  are sufficiently small. Hence, (5.10) has the form

$$(5.11) \quad H(Q'(x)^2, Q(x), p) = C$$

where  $\partial H/\partial a_1(a_1, a_2, p)$  has the value:

$$-\int_I \rho_\infty(y)t^2(y)dy = -1$$

when  $a_1 = a_2 = 0$  and  $p = 0$ . To prove this one uses (4.15) in (5.10). It follows from (5.11) with the aid of the implicit function theorem that

$$(5.12) \quad Q'(x)^2 = h(Q(x), p), \quad x \in \mathbf{R},$$

where  $h$  depends on the constant  $C$ . Recall that  $F$  in (5.6) is an even function of  $Q'$ ; as  $F$  is smooth, it is a function of  $(Q')^2$ . The use of (5.12) in (5.6) yields

$$(5.13) \quad Q''(x) = \tilde{F}(Q(x), p).$$

The function  $h$  in (5.12) is related to the primitive of  $\tilde{F}$ .

#### 5.4. Small solutions of equations (5.12)-(5.13)

One may seek solutions of (5.12)-(5.13) satisfying (5.7), depending on the parameters  $p$  and  $C$ . Since the original interest of this paper was with internal surges (solutions connecting the trivial solution at  $x = -\infty$  to a non-trivial conjugate flow at  $x = +\infty$ ), we begin with the case  $Q(x) \rightarrow 0$  as  $x \rightarrow -\infty$ , whence  $C = 0$  in (5.10). Periodic solutions will be found later by a phase-plane argument.

Before the differential equation for  $Q$  is studied, a few remarks are necessary about the parameters in the problem. The numbers  $\rho$  and  $h$  have been fixed in the interval  $(0, 1)$ . All estimates, such as those needed for the contraction mapping hold uniformly for  $\rho$  and  $h$  varying in compact subsets of  $(0, 1)$ . Instead of working with  $\rho$  and  $h$ , we take  $e$  and  $\rho$  as independent parameters and define  $h(e, \rho)$  implicitly through (cf. (1.15))

$$(5.14) \quad e = \frac{1}{h^2} - \frac{\rho}{(1-h)^2}.$$

When convenient, we shall continue to write  $h$  rather than the function of  $e$  and  $\rho$ . Given a fixed positive number  $k$ , we shall restrict attention to  $e \in [-k, k]$  and  $\rho$  with  $\rho^{-1} \leq k$ ,  $|1 - \rho|^{-1} \leq k$ . The use of the representation  $w(x, y) = Q(x)t(y) + R(x, y)$ , where  $R(x, y) = R(Q(x), \beta^{-1}Q'(x), p)(0, y)$ , together with (4.15) in (5.10) with  $C = 0$  yields

$$(5.15) \quad Q'(x)^2 = t(0)^2 p(1 - \rho)Q^2(x) + O(pQ^2(x)) + O(Q^3(x)), \quad x \in \mathbf{R}$$

as  $Q, p \rightarrow 0$ . As noted earlier,  $R(\xi_1, \xi_2, p)(0, y)$  is a smooth function of  $(\xi_1, \xi_2, p)$  which is even in  $\xi_2$ . In addition,  $R(0, 0; p)(0, y) = 0$  for all  $p \in (-r_0, r_0)$  and the Frechet derivative of the map  $(\xi_1, \xi_2, p) \rightarrow R(\xi_1, \xi_2, p)(0, y)$  vanishes at  $\xi_1 = \xi_2 = p = 0$ . The use of this together with (5.15) gives

$$(5.16) \quad R(x, y) = pa(y, p)Q(x) + b(y, p)Q^2(x) + Q^3(x)d(Q(x), y, p),$$

where  $a, b$ , and  $d$  are smooth functions of their arguments. One can use this representation in (5.4), and after lengthy, but straightforward calculations, arrive at the equation

$$(5.17) \quad \begin{aligned} Q''(x) = & pa_1(p, e)Q(x) + \frac{3}{2}(ea_2(p, e) + pa_3(p, e))Q^2(x) \\ & + 2a_4(p, e)Q^3(x) + Q^4(x)a_5(Q(x), p, e) \end{aligned}$$

where  $a_1, \dots, a_5$  are smooth functions of their arguments. These functions also depend on  $\rho$ , but the dependence is suppressed. The following holds for the coefficients in (5.17):

$$(5.18) \quad \begin{aligned} a_1(0, e) &= m^2 h^2 (1 - \rho), \\ a_2(0, e) &= -m^3 h^3, \\ a_3(0, 0) &= \frac{(1 - \rho)^2}{3} m^5 h^5 \\ a_4(0, 0) &= m^4 h^4 \left( \frac{1}{h^3} + \frac{\rho}{(1 - h)^3} \right) \end{aligned}$$

where  $m$  is the normalization constant in (3.9):

$$m^2 = \frac{3}{h^2(h + \rho(1 - h))}.$$

Our task is to find the totality of solutions  $(p, Q)$  to (5.17) as  $e$  varies over  $[-k, k]$  with

$$(5.19) \quad Q(x) \rightarrow 0 \text{ as } x \rightarrow -\infty,$$

subject to the constraints (cf. (5.7))

$$(5.20) \quad |p| < s \text{ and } 0 < |Q|_{L^\infty(\mathbf{R})} < s_1.$$

The condition in (5.7) that  $|Q'(x)| < \beta s_1$  is implied by (5.15) and (5.20) whenever  $s$  and  $s_1$  are sufficiently small.

If (5.17) is multiplied by  $Q'$  and integrated with the aid of (5.19) then

$$(5.21) \quad Q'(x)^2 = Q^2(x)\Pi(Q(x), p, e),$$

where

$$\Pi(Q, p, e) = pa_1(p, e) + (ea_2(p, e) + pa_3(p, e))Q + a_4(p, e)Q^2 + Q^3\tilde{a}_5(Q, p, e)$$

and  $\tilde{a}_5$  is related to  $a_5$  from (5.17) in an obvious way.

The next step is to find the zeros of  $\Pi(\cdot, p, e)$  and their dependence on  $(p, e)$ . Note that

$$(5.22) \quad \begin{aligned} \Pi(0, 0, e) &= 0, \quad e \in [-k, k], \\ \Pi_p(0, 0, e) &= a_1(0, e) > 0, \quad e \in [-k, k], \\ \Pi_Q(0, 0, e) &= ea_2(0, e) = -m^3h^3e \neq 0 \text{ for } e \neq 0, \\ \Pi_{QQ}(0, 0, 0) &= 2a_4(0, 0) > 0. \end{aligned}$$

Assume for the moment that there is a triple  $(\tilde{Q}, \tilde{p}, e) \in [-s_1, s_1] \times [-s, s] \times [-k, k]$  for which  $\Pi = \Pi_Q = 0$ . Then  $Q(\cdot) \equiv \tilde{Q}$  satisfies (5.17) and (5.21), whence  $w(x, y) = \tilde{Q}t(y) + R(\tilde{Q}, 0, \tilde{p})$  is a solution of (1.11)-(1.13) satisfying (5.10) with  $C = 0$ . Such solutions were considered in section 8 of [4] and the following shown:

$$(5.23) \quad \begin{aligned} \tilde{\lambda} &= \frac{(1 + \sqrt{\rho})^2}{1 - \rho} \\ \tilde{p}(e, \rho) &\equiv \lambda_d - \tilde{\lambda} = \frac{1 - h}{h(1 - \rho)}(1 - \sqrt{1 - eh^2})^2 \\ \tilde{Q}(e, \rho) &= \frac{1 - h}{m(1 + \sqrt{\rho})} \frac{e}{(\frac{1}{h} + \frac{\sqrt{\rho}}{1-h})} \cong \frac{\sqrt{3}}{6} \frac{\sqrt{\rho}\sqrt{1 + \rho^{3/2}}}{(1 + \sqrt{\rho})^{9/2}} e \end{aligned}$$

where  $h$  is considered a function of  $\rho$  and  $e$  (cf. (5.14)). Since  $|p| = |\lambda_d - \tilde{\lambda}| \leq s$ , and  $s$  is to be small, we have that  $e$  is small and

$$(5.24) \quad \tilde{p}(e, \rho) \cong \left[ \frac{1}{4} \frac{\sqrt{\rho}}{(1 - \rho)} \frac{1}{(1 + \sqrt{\rho})^4} \right] e^2,$$

so that  $\tilde{Q}$  is linear in  $e$  and  $\tilde{p}$  is quadratic, to lowest order.

Since the function  $\Pi(Q, p, e)$  has  $\Pi_p(0, 0, e) > 0$ , by (5.18) and (5.22), there is a surface  $S$  given by a function  $p = P(Q, e)$  for  $(Q, e)$  in a neighborhood of  $(0, 0)$ , which is precisely

the locus of zeros of  $\Pi$  in a neighborhood of  $(0,0,0)$ . The surface contains the curve of conjugate solutions

$$\Gamma = \{(\tilde{Q}(e), \tilde{p}(e), e) : -k' < e < k'\}$$

for some  $0 < k' < k$ , where now we suppress the dependence on  $\rho$ . On  $\Gamma$  one has  $\Pi = 0$ ,  $\Pi_Q = 0$ , and  $\Pi_{QQ} \neq 0$  from (5.22). It follows that in a neighborhood of  $(0,0,0)$  the following derivative relations hold:  $P_Q = 0$  on  $\Gamma$  and  $P_{QQ} = -\Pi_{QQ}/\Pi_p < 0$  on  $S$ . Hence the surface  $S$ , viewed in terms of  $p$  and  $e$  as parameters, is composed of two smooth sheets,  $Q = Q_1(p, e)$  lying above  $Q = Q_2(p, e)$ , and the curve  $\Gamma$  along which they meet and become singular. Since  $\tilde{Q}(e)$  has the same sign as  $e$ , the upper sheet is distinguished for  $p = 0$  by having  $Q_1 = 0$  for  $e < 0$ , while  $Q_2 = 0$  for  $p = 0$  and  $e > 0$ . Since  $\Pi_Q(0,0,e) \neq 0$  for  $e \neq 0$ , it is also clear that the distinguished sheet can be continued to nonzero  $p$  in a neighborhood of each  $e \neq 0$ . To summarize these results in a convenient form we introduce the following notation: For sufficiently small  $s$ , let  $e_1 \in (-k, 0)$ ,  $e_2 \in (0, k)$  be the small roots of  $s = \tilde{p}(\cdot, \rho)$ , so that

$$e_i^2 \cong \frac{4s(1-\rho)(1+\sqrt{\rho})^4}{\sqrt{\rho}}.$$

Let  $\delta$  be in  $(|e_1|, k) \cap (e_2, k)$ , and define

$$A_1^\pm = \{(p, e) : \pm e \in [\delta, k], p \in [-s, s]\}$$

$$A_2 = \{(p, e) : e \in [e_1, e_2], \tilde{p}(e, \rho) \leq p \leq s\}$$

$$A_0 = \{[-s, s] \times [-k, k]\} \setminus (A_1^+ \cup A_1^- \cup A_2)$$

The sets  $A_1^\pm, A_2, A_0$  are mutually disjoint, and their union is  $[-k, k] \times [-s, s]$ .

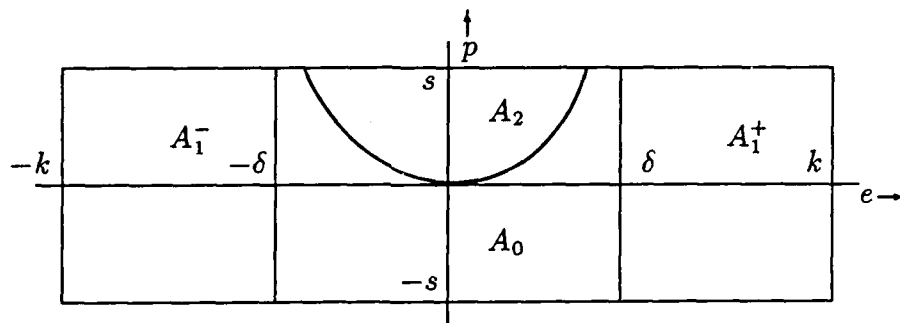


Figure 1

**Theorem 5.2.** *If  $s$  and  $s_1$  are sufficiently small, then there is a  $\delta \in (|e_1|, k) \cap (e_2, k)$  so that the following hold for  $|Q| \leq s_1$ :*

(a)  $\Pi(Q, p, e) > 0$  if  $(p, e) \in A_2$ . There exist smooth functions  $Q_1$  and  $\Pi^- : A_1^- \cup A_0 \rightarrow [-s_1, s_1]$ ;  $Q_2$  and  $\Pi^+ : A_1^+ \cup A_0 \rightarrow [-s_1, s_1]$ , and  $\Pi_0 : A_0 \rightarrow [-s_1, s_1]$ , such that

(b)  $\Pi(Q, p, e) = (Q - Q_1(p, e))\Pi^-(Q, p, e)$  if  $(p, e) \in A_1^-$ , where  $\Pi^- > 0$ ;  $\Pi(Q, p, e) = (Q - Q_2(p, e))\Pi^+(Q, p, e)$  if  $(p, e) \in A_1^+$ , where  $\Pi^+ < 0$ .

(c)  $Q_1(\tilde{p}(e), e) = Q_2(\tilde{p}(e), e)$ ,  $e \in [e_1, e_2]$ ;  $Q_1(p, e) > Q_2(p, e)$ ,  $(p, e) \in A_0$ ;  $Q_1(0, e) = 0$ ,  $e \in [-k, 0]$ ; and  $Q_2(p, e) = 0$ ,  $e \in [0, k]$

(d)  $\Pi(Q, p, e) = (Q - Q_1(p, e))(Q - Q_2(p, e))\Pi_0(Q, p, e)$  if  $(p, e) \in \bar{A}_0$ , where  $\Pi_0 > 0$ .

**Remark 5.3.** *For  $e \neq 0$  the root vanishing at  $p = 0$  behaves approximately as  $-a_1 p/a_2 e$  while in  $A_0$  the two sheets are approximated by solutions of the quadratic:  $a_4 Q^2 + (ea_2 + pa_3)Q + pa_1 = 0$ .*

With Theorem 5.2 in hand, the analysis of (5.21) is straightforward. Theorem 5.1 ensures that either (i)  $Q'(x)$  is one-signed on  $(-\infty, \infty)$ , or (ii) (after a suitable translation)  $Q$  is even about  $x = 0$  and monotone on  $(-\infty, 0)$  and on  $(0, \infty)$ . Equation (5.15) ensures  $p \geq 0$ ; that is  $\lambda \leq \lambda_d$ . We now show that the inequality is strict for a nontrivial solution.

If  $p = 0$  and  $(p, e) \in A_1^\pm$ , then

$$Q'(x)^2 = Q^3(x)\Pi^\pm(Q(x), 0, e).$$

If (i) occurs, then  $Q(x)$  approaches a constant  $\hat{Q}$  as  $x \rightarrow \infty$ , whence  $\Pi^\pm(\hat{Q}, 0, e) = 0$ . However, this contradicts Theorem 5.2(b). If (ii) occurs, then  $Q'(0) = 0$  and  $Q(0) \neq 0$ , which is again impossible. If  $p = 0$ ,  $(p, e) \in A_0$ , and  $e > 0$ , then

$$(5.25) \quad Q'(x)^2 = Q^3(x)(Q(x) - Q_1(0, e))\Pi_0(Q(x), p, e).$$

For the case (i), we have  $Q(x) \rightarrow Q_1(0, e) > 0$  as  $x \rightarrow \infty$ , whence  $Q(x) > 0$ ,  $x \in \mathbf{R}$ , and  $Q^3(x)(Q(x) - Q_1(0, e)) < 0$ ,  $x \in \mathbf{R}$ . Theorem (5.2)(d) gives  $\Pi_0 > 0$ , and this contradicts (5.25). If  $e > 0$  and (ii) holds, then  $Q(0) = Q_1(0, e)$ ,  $Q(x) > 0$ ,  $x \in \mathbf{R}$ , and a contradiction follows as before. A similar argument holds for  $e < 0$ . If  $e = 0$ , then  $Q_2(0, 0) = 0$ , whence  $Q'(x)^2 = Q^4(x)\Pi_0(Q(x), 0, 0)$ , with  $\Pi_0 > 0$  which is again impossible. Hence, nontrivial solutions of (5.17)-(5.20) necessarily have  $p > 0$  or

$$\lambda < \lambda_d,$$

so they are supercritical in terms of the speed  $c = \sqrt{g/\lambda}$ .

Next, we note that if  $(p, e) \in A_2$ , then  $Q'(x)^2 = Q(x)^2 \Pi(Q, p, e)$  where  $\Pi(Q, p, e) > 0$  by Theorem 5.2(a). Hence, there are no solutions for  $(p, e) \in A_2$ . Define

$$A^+ = \{(0, s] \times [-k, k]\} \setminus A_2$$

and

$$\Gamma_0 = \{(\tilde{p}(e), e) : e \in [e_1, e_2] \setminus \{0\}\}.$$

which is a projection of  $\Gamma$  minus the origin.

For  $(p, e) \in A^+$ , there is a 'solitary wave' solution which is zero at  $x = \pm\infty$  and has a value at  $x = 0$  equal to the simple root of  $\Pi$  nearest zero. When  $(p, e) \in \Gamma_0$ ,  $\Pi$  has a double root at  $Q_1 = Q_2$  and there is a surge solution which approaches this value as  $x \rightarrow +\infty$ . These results are now summarized.

**Theorem 5.4.** (a) If  $(p, e) \in A^+$ , there is a unique solution  $Q(p, e)(x)$  to (5.17)-(5.20) with  $Q(p, e)$  even about  $x = 0$ . The functions  $Q(p, e)(x)$  and  $Q'(p, e)(x)$  have the same sign as  $e$  on  $(-\infty, \infty)$  and  $(-\infty, 0)$ , respectively. Furthermore  $Q(p, e)(0)$  equals  $Q_2(p, e)$  if  $e > 0$  and equals  $Q_1(p, e)$  if  $e < 0$ .

(b) If  $(p, e) \in \Gamma_0$ , there is a solution  $Q(p, e)(x)$  to (5.17)-(5.20) with  $Q(p, e)(x) \rightarrow Q_1(p, e) = Q_2(p, e) \neq 0$  as  $x \rightarrow \infty$ . The functions  $Q$  and  $Q'$  have the same sign as  $e$  on  $(-\infty, \infty)$ . Any other solution of (5.21) is a translation of  $Q(p, e)$ .

The solutions in (a) give rise to solitary wave solutions  $w(x, y)$  while those in (b) give surges. The solutions in Theorem 5.4 can be analyzed with a scaling argument. For the case  $(p, e) \in \Gamma_0$ ,

$$Q'(x)^2 = Q^2(x)(Q(x) - Q_1(p, e))^2 \Pi_0(Q(x), p, e).$$

Let  $Q_1(\tilde{p}(e), e) = q$  and define  $Y(x)$  by the relation  $Q(p, e)(x) = qY(|q|x)$  whence

$$Y'(x)^2 = Y^2(x)(Y(x) - 1)^2 \Pi_0(qY, p, e).$$

Note that  $Y'(x) > 0$  and  $Y(x) \rightarrow 1$  as  $x \rightarrow \infty$ . As  $e \rightarrow 0$ ,  $q \rightarrow 0$ , and so  $Y(x)$  will behave like a solution to

$$\begin{aligned} y'(x)^2 &= y^2(x)(y(x) - 1)^2 \Pi_0(0, 0, 0) \\ &= y^2(x)(y(x) - 1)^2 a_4(0, 0). \end{aligned}$$

Using (5.18) one has

$$y(x) = \frac{1}{2} \left( 1 + \tanh\left(\frac{a(\rho)x}{2}\right) \right),$$

where

$$a(\rho) = \frac{3(1 + \sqrt{\rho})^3}{\rho^{1/4}(1 + \rho^{3/2})}.$$

For small  $e$ ,  $Q_1(p, e)$  is given by (5.23), whence

$$(5.26) \quad Q(p, e)(x) \cong \frac{\sqrt{3}}{12} \frac{\sqrt{\rho} \sqrt{1 + \rho^{3/2}}}{(1 + \sqrt{\rho})^{9/2}} e \left\{ 1 + \tanh\left[ \frac{\sqrt{3}}{4} \frac{\rho^{1/4}}{(1 + \sqrt{\rho})^{3/2}} \frac{|e|x}{\sqrt{1 + \rho^{3/2}}} \right] \right\}.$$

Note by (5.23) that  $|e| = O(\sqrt{p})$  and, using (5.16)),  $R(x, y) = O(|p||Q| + |Q|^2) = O(e^2)$ . Hence, (5.26) gives an expansion for the bore solution  $w(x, y) = Q(x)t(y) + R(x, y)$  accurate to order  $e$ , or equivalently, order  $\sqrt{\lambda_d - \lambda}$ .

Suppose  $(p, e) \in A^+$ , and consider the behavior of  $Q(p, e)$  as  $p \rightarrow 0$  for fixed  $e$ . Now one sets  $Q(p, e)(x) = qY(\sqrt{|q|x})$ , where  $q = Q_1(p, e)$  for  $e < 0$  and  $q = Q_2(p, e)$  for  $e > 0$ , and deduces that  $Y$  behaves like a solution to

$$y'(x)^2 = m^3 h^3 |e| y^2(x) (1 - y(x)),$$

that is,

$$y(x) = \operatorname{sech}^2\left(\frac{(mh)^{3/2} \sqrt{|e|x}}{2}\right).$$

It follows that

$$Q(p, e)(x) \cong \frac{p(1 - \rho)}{emh} \operatorname{sech}^2\left(\frac{(mh)^{3/2} \sqrt{|e|x}}{2}\right).$$

Note that this solution tends to zero like  $|p| = |\lambda_d - \lambda|$  for fixed  $e$ , while the bore solution in (5.27) has the slower decay  $O(\sqrt{|p|})$ .

For a fixed  $(p, e) \in A^+$ , Theorem 5.4 gives the existence of a homoclinic orbit  $Q(p, e)$  to (5.17)-(5.20) with  $Q(p, e)(0) = Q_1(p, e)$ . This is a closed loop in the  $(Q, Q')$  phase-plane. One can show easily that the trajectory encircles only one critical point  $(\hat{Q}, 0)$  (that is, a non-zero constant for which the right-hand side of (5.17) vanishes). It is then immediate that the interior of the trajectory is filled with small periodic solutions of (5.17) which give rise to periodic (in  $x$ ) solutions  $w(x, y)$ .

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