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ON SINGULAR VALUES OF HANKEL
OPERATORS OF FINITE RANK

William Gragg
Lothar Reichel

November 1988

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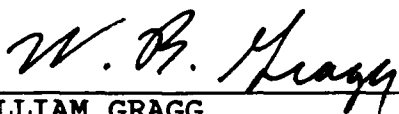
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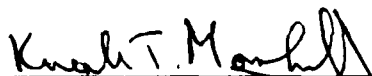
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On singular values of Hankel operators of finite rank

W. B. Gragg† and L. Reichel ‡

Abstract

Let H be a Hankel operator defined by its symbol $\rho = \pi/\chi$ where χ is a monic polynomial of degree n and π is a polynomial of degree less than n . Then H has rank n . We derive a generalized Takagi singular value problem defined by two $n \times n$ matrices, such that its n generalized Takagi singular values are the positive singular values of H . If ρ is real, then the generalized Takagi singular value problem reduces to a generalized symmetric eigenvalue problem. The computations can be carried out so that the Lanczos method applied to the latter problem requires only $O(n \log n)$ arithmetic operations for each iteration. If π and χ are given in power form, then the elements of all $n \times n$ matrices required can be determined in $O(n^2)$ arithmetic operations.

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Keywords

Hankel operator, singular values, generalized Takagi singular value problem, generalized eigenvalue problem, Lanczos iterations

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1. Introduction

Let $H = [\eta_{j+k}]_{j,k=0}^{\infty}$ be a Hankel operator defined by its rational symbol $\rho = \pi/\chi$, where

$$\pi(\lambda) := \sum_{j=0}^{n-1} \pi_j \lambda^j \quad \text{and} \quad \chi(\lambda) := \sum_{j=0}^n \chi_j \lambda^j, \quad \chi_n = 1. \quad (1.1)$$

We assume that π and χ have no common zeros. The elements η_j of H are then given by

$$\rho(\lambda) = \frac{\pi(\lambda)}{\chi(\lambda)} = \sum_{j=0}^{\infty} \eta_j \lambda^{-j-1}. \quad (1.2)$$

In order to simplify our presentation, we assume that the zeros $\{\lambda_k\}_{k=1}^n$ of χ are distinct. How our formulas need to be modified in order to remove this assumption is discussed in Remark 1.1 below. Hence ρ has a partial fraction decomposition

$$\rho(\lambda) =: \sum_{k=1}^n \frac{\alpha_k}{\lambda - \lambda_k}. \quad (1.3)$$

Expansion of the right hand side of (1.3) into a geometric series, and comparison with (1.2), yields

$$\eta_j = \sum_{k=1}^n \alpha_k \lambda_k^j. \quad (1.4)$$

We now express (1.4) in matrix form. Let

$$A := \text{diag}[\alpha_1, \alpha_2, \dots, \alpha_n] \in \mathbb{C}^{n \times n}, \quad (1.5)$$

$$\Lambda := \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n] \in \mathbb{C}^{n \times n}, \quad (1.6)$$

and introduce the Vandermonde matrix

$$V_0 := [\lambda_{k+1}^j]_{j,k=0}^{n-1} \in \mathbb{C}^{n \times n}. \quad (1.7)$$

Define

$$V := [V_j]_{j=0}^{\infty} \in \mathbb{C}^{\infty \times n}, \quad (1.8)$$

where

$$V_j := V_0 \Lambda^{jn}, \quad j \geq 1. \quad (1.9)$$

Then (1.4) can be written as

$$H = V A V^T. \quad (1.10)$$

Let l^2 denote the vector space $\mathbb{C}^{\infty}$ equipped with the Euclidean norm.

Proposition 1.1. $H : l^2 \rightarrow l^2$ bounded $\Leftrightarrow |\lambda_k| < 1$ for $1 \leq k \leq n$.

Proof. The proposition holds independent of the multiplicity of the λ_k . In the present proof we assume that the λ_k are distinct. The proof for confluent λ_k is commented on in Remark 1.1.

Let $e_1 = [\epsilon_j]_{j=0}^\infty \in \mathcal{C}^\infty$ be the axis vector with $\epsilon_0 = 1$. Then

$$h = [\eta_j]_{j=0}^\infty := H e_1 \in l^2 \Rightarrow \eta_j \rightarrow 0 \text{ as } j \rightarrow \infty \Rightarrow$$

$$|\lambda_k| < 1 \text{ for } 1 \leq k \leq n,$$

where the last implication follows from (1.4).

Conversely, assume that $|\lambda_k| < 1$ for $1 \leq k \leq n$. Then by (1.8) - (1.10) we obtain

$$\begin{aligned} \|H\|_2 &\leq \|A\|_2 \|V\|_2^2 \leq \|A\|_2 \|V_0\|_2^2 \left\| \sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj} \right\|_2^2 \\ &= \|A\|_2 \|V_0\|_2^2 \|(I - (\Lambda^H \Lambda)^n)^{-1}\|_2^2. \quad \blacksquare \end{aligned}$$

We assume henceforth that $|\lambda_k| < 1$ for $1 \leq k \leq n$. Introduce

$$U := V V_0^{-1}, \quad (1.11)$$

$$H_0 := V_0 A V_0^T. \quad (1.12)$$

Then H_0 has rank n . We note, by comparing (1.12) with (1.10), that H_0 is the leading principal $n \times n$ submatrix of H . From (1.10) - (1.12) it follows that

$$H = U H_0 U^T. \quad (1.13)$$

The leading $n \times n$ submatrix of U is I_n , the $n \times n$ identity matrix. U therefore is of rank n and can be factored

$$U = Q R, \quad Q \in \mathcal{C}^{\infty \times n}, \quad R \in \mathcal{C}^{n \times n},$$

where $Q^H Q = I_n$ and R is a nonsingular right triangular matrix. We obtain

$$\sigma_+(H) = \sigma_+(Q R H_0 R^T Q^T) = \sigma(R H_0 R^T), \quad (1.14)$$

where σ denotes the set of singular values and σ_+ denotes the subset of the positive ones.

The $n \times n$ matrix $R H_0 R^T$ is complex symmetric. Takagi [Ta1], [Ta2] showed the existence of a complex symmetric singular value decomposition

$$R H_0 R^T = W \Sigma W^T, \quad W \in \mathcal{C}^{n \times n}, \quad \Sigma = \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_n], \quad (1.15)$$

where $W^H W = I_n$ and $\sigma_j > 0$ are the singular values of $R H_0 R^T$. In Section 2 we present an elementary proof of the existence of this decomposition. Let $W = [w_1, w_2, \dots, w_n]$, $w_j \in \mathcal{C}^n$. Then (1.15) can be written as the Takagi singular value problem

$$R H_0 R^T \bar{w}_j = w_j \sigma_j, \quad w_j^H w_k = \delta_{jk}, \quad 1 \leq j, k \leq n, \quad (1.16)$$

where the bar denotes complex conjugation and δ_{jk} is Kronecker's δ function. The problems (1.15) - (1.16) could be solved by the algorithm described in [BGG], but this would require RH_0R^T to be explicitly computed. In order to avoid these matrix multiplications we let $v_j := R^H w_j$ and obtain from (1.16) the *generalized Takagi singular value problem*

$$H_0 \bar{v}_j = (R^H R)^{-1} v_j \sigma_j, \quad v_j^H (R^H R)^{-1} v_k = \delta_{jk}, \quad 1 \leq j, k \leq n. \quad (1.17)$$

The solution of (1.17) requires $(R^H R)^{-1}$ to be known. In Section 3 we show that

$$(R^H R)^{-1} = I - B_0 B_0^H, \quad (1.18)$$

where $B_0 \in \mathbb{C}^{n \times n}$ is a triangular Toeplitz matrix. The elements of B_0 and H_0 can be determined from the coefficients of π and χ in $O(n \log n)$ arithmetic operations by the fast Fourier transform (FFT) method. This is demonstrated in Section 4. Section 5 shows that

$$R^H R = \overline{T_1 M_0 T_1^H}, \quad T_1, M_0 \in \mathbb{C}^{n \times n}, \quad (1.19)$$

where T_1 and M_0 are Toeplitz matrices, and describes a numerical scheme for the computation of this factorization from (1.16) in $O(n^2)$ arithmetic operations. We also present a Hermitian factorization of $R^H R$ into $n \times n$ triangular matrices.

The factorization (1.19) may be of interest for the numerical solution of (1.17). Assume that the coefficients of π and χ are real valued. Then H_0 , $(R^H R)^{-1} \in \mathbb{R}^{n \times n}$, and (1.17) reduces to a generalized symmetric eigenvalue problem. The Lanczos method ([Pa, Section 15.11], [ER]) would appear suitable for solving this eigenproblem for the following reason. Let $C \in \mathbb{C}^{n \times n}$ be a Hankel or Toeplitz matrix and let $v \in \mathbb{C}^n$ be arbitrary. It is well known that Cv can be computed in $O(n \log n)$ arithmetic operations using FFTs. Hence $H_0 v$, $(R^H R)^{-1} v$ and $(R^H R)v$ can be computed in $O(n \log n)$ arithmetic operations, where we use (1.18) - (1.19). Each iteration of the Lanczos algorithm given in [Pa, p.324] therefore requires only $O(n \log n)$ arithmetic operations.

The computation of singular values of H is important in Hankel norm approximation problems of systems theory, such as the model reduction problem [Gl]. The approximation of functions by the Carathéodory - Fejér method yields another application [Gu], [Tr].

Other methods for reducing the singular value problem for H to a finite dimensional one have been described by Kung and Gutknecht [Gu] and Young [Yo]. These methods, however, do not preserve symmetry. Moreover, Young's approach requires generally $O(n^3)$ arithmetic operations to compute the matrices required.

Remark 1.1. Formulas (1.3) - (1.8) and the proof of Proposition 1.1 require distinct λ_k . This restriction can be removed. Assume first that $\lambda_1 = \lambda_2 = \dots = \lambda_n$. Then (1.3) - (1.4) have to be replaced by

$$\rho(\lambda) =: \sum_{k=1}^n \frac{\alpha_k}{(\lambda - \lambda_1)^k}, \quad (1.3')$$

$$\eta_j = \sum_{k=1}^n \frac{\alpha_k}{\lambda^k} \left[\sum_{j=0}^{\infty} \left(\frac{\lambda_1}{\lambda} \right)^j \right]^k. \quad (1.4')$$

In (1.5) A has to be substituted by the upper triangular Hankel matrix

$$A = [\alpha_{j+k+1}]_{j,k=0}^{n-1} \in \mathbb{C}^{n \times n}; \quad \alpha_p := 0, \quad p > n.$$

The matrix Λ in (1.6) has to be replaced by the Jordan matrix with all diagonal elements equal to λ_1 and all superdiagonal elements equal to one. The matrix V_0 in (1.7) need be replaced by the confluent Vandermonde matrix. For instance, we obtain for $n = 3$

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 1 & & \\ \lambda_1 & 1 & \\ \lambda_1^2 & 2\lambda_1 & 1 \end{bmatrix}.$$

With A , Λ and V_0 modified as described, we define V_j and V by (1.8) - (1.9), U by (1.11) and H_0 by (1.12). Then (1.10) and (1.13) hold and H_0 is the leading principal $n \times n$ submatrix of H . Also (1.14) - (1.19) remain valid. Proposition 1.1 can be shown by replacing (1.4) by (1.4'), and by bounding the sum

$$\left\| \sum_{j=0}^{\infty} (\Lambda^H \Lambda)^{nj} \right\|_2^2$$

where Λ now is a Jordan matrix. This sum is bounded if $|\lambda_1| < 1$, and the proposition remains valid.

In general, when the λ_k are of arbitrary multiplicity, A in (1.5) has to be replaced by a block diagonal matrix, where each block is an upper triangular Hankel matrix. The blocks are of the same sizes as the multiplicities of the λ_k , and the number of blocks equals the number of distinct λ_k . Λ in (1.6) is replaced by a Jordan matrix with Jordan boxes of the same sizes as the multiplicities of the λ_k , and the number of boxes equal to the number of distinct λ_k . V_0 in (1.7) is replaced by an appropriate confluent Vandermonde matrix. With these changes (1.10) - (1.19) are valid, and so is Proposition 1.1. We omit the details since the numerical computations are independent of the multiplicity of the λ_k . ■

2. The Symmetric Singular Value Decomposition

In this section we present an elementary proof of Takagi's theorem, i.e. we show the existence of a symmetric singular value decomposition of a complex symmetric matrix. Let $C = C^T \in \mathbb{C}^{n \times n}$, and define $A, B \in \mathbb{R}^{n \times n}$ by $C := A + iB$, $i := \sqrt{-1}$. Then $A = A^T$ and $B = B^T$, so the matrix

$$\tilde{C} := \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is real and symmetric. Let $\{\sigma_j\}_{j=1}^r$ be the positive eigenvalues of $\tilde{C}$ and form

$$\Sigma := \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_r].$$

Let

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \quad (2.1)$$

with

$$U, V \in \mathbb{R}^{n \times r}$$

and

$$U^T U + V^T V = I_r.$$

Write (2.1) as

$$\begin{cases} AU + BV = U\Sigma \\ BU - AV = V\Sigma \end{cases} \quad (2.2)$$

and note that (2.2) also can be written as

$$\begin{cases} AV + B(-U) = V(-\Sigma) \\ BV - A(-U) = (-U)(-\Sigma), \end{cases}$$

i.e.

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} V \\ -U \end{bmatrix} = \begin{bmatrix} V \\ -U \end{bmatrix} (-\Sigma) \quad (2.3)$$

with

$$V^T V + (-U)^T (-U) = I_r.$$

Hence $\tilde{C}$ has at least r negative eigenvalues. We could also have let σ_j be the negative eigenvalues of $\tilde{C}$ and then (2.3) would have given us positive ones. We therefore may assume that $\pm\sigma_1, \pm\sigma_2, \dots, \pm\sigma_r$ are all the nonzero eigenvalues of $\tilde{C}$.

Since eigenvectors associated with distinct eigenvalues of a real symmetric matrix are orthogonal, we have

$$0 = [V^T, -U^T] \begin{bmatrix} U \\ V \end{bmatrix} = V^T U - U^T V.$$

The spectral resolution of $\tilde{C}$ is thus

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} = \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} \Sigma & \\ & -\Sigma \end{bmatrix} \begin{bmatrix} U^T & V^T \\ V^T & -U^T \end{bmatrix},$$

which yields

$$\begin{cases} A = U\Sigma U^T - V\Sigma V^T \\ B = V\Sigma U^T + U\Sigma V^T. \end{cases}$$

Therefore

$$\begin{aligned} C = A + iB &= U\Sigma U^T - V\Sigma V^T + i(V\Sigma U^T + U\Sigma V^T) \\ &= (U + iV)\Sigma(U^T + iV^T) = W\Sigma W^T = \sum_{k=1}^r \sigma_k w w_k^T, \end{aligned}$$

where

$$U + iV =: W = [w_1, w_2, \dots, w_r], \quad w_k \in \mathbb{C}^n.$$

Moreover

$$W^H W = (U^T - iV^T)(U + iV) = (U^T U + V^T V) + i(U^T V - V^T U) = I_r.$$

If $r < n$ then one may replace Σ by

$$\Sigma_0 := \text{diag}[\sigma_1, \sigma_2, \dots, \sigma_r, 0, \dots, 0] \in \mathbb{R}^{n \times n}$$

and W by

$$W_0 := [w_1, w_2, \dots, w_r, w_{r+1}, \dots, w_n] \in \mathbb{C}^{n \times n},$$

where $w_{r+1}, \dots, w_n \in \mathbb{C}^n$ are chosen so that $W_0^H W_0 = I_n$. ■

3. A Simple Expression for $(R^H R)^{-1}$

In this section we derive (1.18). Introduce the Frobenius matrix

$$F := [e_2, e_3, \dots, e_n, -f] \in \mathbb{C}^{n \times n},$$

where

$$\begin{aligned} e_j &:= [\delta_{1j}, \delta_{2j}, \dots, \delta_{nj}]^T \in \mathbb{R}^n, \quad 2 \leq j \leq n, \\ f &:= [\chi_0, \chi_1, \dots, \chi_{n-1}]^T \in \mathbb{C}^n. \end{aligned} \quad (3.1)$$

Then F is the companion matrix of χ and

$$F^T V_0 = V_0 \Lambda. \quad (3.2)$$

Throughout this section V_0 and Λ are defined by (1.6) - (1.7) if the λ_k are distinct. For confluent λ_k we modify V_0 and Λ according to Remark 1.1. The following lemma shows that

$$G := \overline{R^H R} \quad (3.3)$$

satisfies a Stein equation. This will enable us to obtain a simple expression for G^{-1} by an application of the Sherman-Morrison-Woodbury formula.

Lemma 3.1. G is the unique solution of the Stein equation

$$X - F^n X F^{nH} = I_n, \quad X \in \mathbb{C}^{n \times n}. \quad (3.4)$$

Proof. By (1.8), (1.9) and (1.11) we obtain

$$R^H R = U^H U = \sum_{k=0}^{\infty} V_0^{-H} (\Lambda^{nk})^H V_0^H V_0 \Lambda^{nk} V_0^{-1}, \quad (3.5)$$

and (3.2) yields now

$$G = \sum_{k=0}^{\infty} F^{nk} (F^{nk})^H. \quad (3.6)$$

The series in (3.5) - (3.6) converge because $|\lambda_k| < 1$ for all k . Substitution of (3.6) into (3.4) shows that G solves (3.4). The unicity follows from $|\lambda_k| < 1$ for all k . The latter can be seen by a similarity transform of F^n to Schur triangular form. ■

Introduce the cyclic downshift operator in $\mathbb{C}^{2n}$

$$E := [e_2, e_3, \dots, e_n, e_1] \in \mathbb{C}^{2n \times 2n},$$

where

$$e_j := [\delta_{1j}, \delta_{2j}, \dots, \delta_{2n,j}]^T \in \mathbb{R}^{2n}. \quad (3.7)$$

Let

$$t := [\chi_0, \chi_1, \dots, \chi_n, 0, 0, \dots, 0]^T \in \mathbb{C}^{2n},$$

and define the Toeplitz matrix T of parallelogram form

$$T := [t, Et, E^2t, \dots, E^{n-1}t] \in \mathbb{C}^{2n \times n}. \quad (3.8)$$

Let T_0 be the leading $n \times n$ submatrix of T , and let T_1 be the trailing $n \times n$ submatrix of T . Then T_0 is a left triangular Toeplitz matrix, and T_1 is a unit right triangular Toeplitz matrix.

Example 3.1. Let $n = 3$. Then

$$T = \begin{bmatrix} \chi_0 & & & \\ \chi_1 & \chi_0 & & \\ \chi_2 & \chi_2 & \chi_0 & \\ \chi_3 & \chi_2 & \chi_1 & \\ & \chi_3 & \chi_2 & \\ & & \chi_3 & \end{bmatrix}, \quad T_0 = \begin{bmatrix} \chi_0 & & \\ \chi_1 & \chi_0 & \\ \chi_2 & \chi_1 & \chi_0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} \chi_3 & \chi_2 & \chi_1 \\ & \chi_3 & \chi_2 \\ & & \chi_3 \end{bmatrix},$$

where we note that $\chi_3 = 1$. ■

Lemma 3.2. Let T_0 and T_1 be defined as above. Then

$$T_0^H T_0 + T_1^H T_1 = T_0 T_0^H + T_1 T_1^H. \quad (3.9)$$

Proof. Let $N := T^H T = T_0^H T_0 + T_1^H T_1$. We first show that N is a Toeplitz matrix. Let e_j be defined by (3.1). Then by (3.8) we have for $1 \leq j, k \leq n$,

$$e_j^T N e_k = e_j^T T^H T e_k = t^H (E^H)^{j-1} E^{k-1} t = t^H E^{k-j} t,$$

where we have used that $E^H = E^{-1}$. We next define the reversal matrix

$$J := [e_n, e_{n-1}, \dots, e_1] \in \mathbb{R}^{n \times n}.$$

Toeplitz matrices are counter symmetric, i.e. $N = J N^T J$. Using that N is counter symmetric and Hermitian yields

$$\begin{aligned} T_0^H T_0 + T_1^H T_1 &= N = J N^T J = J \overline{N} J = J (T_0^T \overline{T_0} + T_1^T \overline{T_1}) J \\ &= J T_0^T J \cdot J \overline{T_0} J + J T_1^T J \cdot J \overline{T_1} J = T_0 T_0^H + T_1 T_1^H. \quad \blacksquare \end{aligned}$$

The next lemma presents a Gaussian factorization of F^n in terms of T_0 and T_1 . This will be used together with Lemma 3.1 to express G^{-1} in terms of T_0 and T_1 .

Lemma 3.3.

$$F^n = -T_0 T_1^{-1}. \quad (3.10)$$

Proof. We first show that

$$[T_0^T, T_1^T] \begin{bmatrix} V_0 \\ V_0 \Lambda^n \end{bmatrix} = 0. \quad (3.11)$$

Let e_j be defined by (3.7) and assume for the moment that the λ_k are distinct. Then

$$e_j^T [T_0^T, T_1^T] \begin{bmatrix} V_0 \\ V_0 \Lambda^n \end{bmatrix} e_k = \chi(\lambda_k) \lambda_k^{j-1} \quad (3.12)$$

and the right hand side vanishes for $1 \leq j, k \leq n$. If the λ_k are confluent, then the right hand side expression of (3.12) contains derivatives of $\chi(\lambda)$ evaluated at λ_k . The right hand side of (3.12), however, still vanishes and (3.11) holds.

We now write (3.11) as

$$T_0^T V_0 + T_1^T V_0 \Lambda^n = 0$$

and apply (3.2). This shows (3.10). ■

We are now in a position to show (1.18). By (3.4) G satisfies

$$G = I + F^n G F^{nH}$$

and an application of the Sherman-Morrison-Woodbury formula yields

$$G^{-1} = (I + F^n G F^{nH})^{-1} = I - F^n (G^{-1} + F^{nH} F^n)^{-1} F^{nH}. \quad (3.13)$$

We now determine an expression for

$$Y := I - G^{-1}. \quad (3.14)$$

Substitute Y and (3.10) into (3.13) to obtain

$$Y = T_0(T_0^H T_0 + T_1^H T_1 - T_1^H Y T_1)^{-1} T_0^H. \quad (3.15)$$

In order to determine a simple expression for Y from (3.15) we need the following observation, which is also central to Section 4. T_0 and T_1^{-H} are both left triangular $n \times n$ Toeplitz matrices. Multiplication of T_0 with T_1^{-H} can be identified with polynomial multiplication, see [He1, Section 1.3] and Section 4. Since multiplication of polynomials commutes, we obtain

$$T_0 T_1^{-H} = T_1^{-H} T_0. \quad (3.16)$$

From the correspondence between polynomials and left triangular Toeplitz matrices it also follows that $T_0 T_1^{-H}$ is a left triangular Toeplitz matrix.

Lemma 3.4. Equation (3.15) has the unique solution

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \quad (3.17)$$

Proof. Unicity follows from (3.14) and that (3.4) has a unique solution. From (3.16) we obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 T_1^{-H} T_1^{-1} T_0^H. \quad (3.18)$$

Now substitute

$$Y = T_1^{-H} T_0 T_0^H T_1^{-1}$$

into (3.15). We obtain

$$T_1^{-H} T_0 T_0^H T_1^{-1} = T_0 (T_0^H T_0 + T_1^H T_1 - T_0 T_0^H)^{-1} T_0^H. \quad (3.19)$$

An application of (3.9) reduces (3.19) to (3.18). The latter has already been shown to be valid. Therefore (3.17) solves (3.15). ■

Let

$$B_0 := \overline{T}_0 T_1^{-T} = T_1^{-T} \overline{T}_0. \quad (3.20)$$

Then B_0 is a left triangular $n \times n$ Toeplitz matrix. By (3.14) and (3.17)

$$G^{-1} = I - \overline{B}_0 B_0^T = I - B_0^T \overline{B}_0.$$

From (3.3) it now follows that

$$(R^H R)^{-1} = I - B_0 B_0^H. \quad (3.21)$$

4. Computation of H_0 and B_0

We summarize some results in [He 1, Section 1.3] and [He 2, Section 13.9] in order to show that the elements of H_0 and B_0 can be computed in $O(n \log n)$ arithmetic operations from the coefficients χ_j of χ and π_j of π , see (1.1). To a polynomial or power series

$$\varsigma(\lambda) := \sum_{j=0}^{n-1} \varsigma_j \lambda^j + O(\lambda^n)$$

we associate the left triangular $n \times n$ Toeplitz matrix

$$Z = [\varsigma_{j-k}]_{j,k=0}^{n-1}, \quad \varsigma_j = 0 \text{ for } j < 0,$$

and we write $\varsigma \rightarrow Z$. If $\xi(\lambda)$ is a polynomial and X a left triangular $n \times n$ Toeplitz matrix such that $\xi \rightarrow X$, then it is easily seen that $\varsigma\xi \rightarrow ZX$. In particular, ZX is a left triangular $n \times n$ Toeplitz matrix. From $\xi\varsigma = \varsigma\xi$ and $\xi\varsigma \rightarrow XZ$ it follows that $ZX = XZ$.

Assume that $\varsigma_0 \neq 0$ and let $1/\varsigma \rightarrow Z'$. Then $1/\varsigma \cdot \varsigma \rightarrow I$, $Z'Z$ and ZZ' . We obtain $Z' = Z^{-1}$ and therefore Z^{-1} is a left triangular Toeplitz matrix.

Example 4.1. We have $\chi \rightarrow T_0$. Let

$$\tilde{\chi}(\lambda) := \lambda^n \bar{\chi}(1/\lambda) = \sum_{j=0}^n \bar{\chi}_{n-j} \lambda^j. \quad (4.1)$$

Then $\tilde{\chi} \rightarrow T_1^H$ and the Blaschke product

$$\frac{\chi}{\tilde{\chi}} \rightarrow T_0 T_1^{-H} = \bar{B}_0. \quad (4.2)$$

Now let $\xi(\lambda)$ and $\varsigma(\lambda)$ be arbitrary polynomials such that $\varsigma(0) \neq 0$. Henrici [He2, Theorem 13.9e] shows that the first n coefficients in the MacLaurin expansion of $\xi(\lambda)/\varsigma(\lambda)$ can be computed in $O(n \log n)$ multiplications. The proof uses FFT. It is easily seen that the number of additions also is $O(n \log n)$.

From $\chi_n = 1$ and (4.1) we obtain $\tilde{\chi}(0) \neq 0$. Hence, the first n terms in the MacLaurin expansion of $\chi/\tilde{\chi}$ can be computed in $O(n \log n)$ arithmetic operations. By (4.2) therefore $T_0 T_1^{-H} = \bar{B}_0$ can be computed in $O(n \log n)$ arithmetic operations.

Because $\lambda^n \chi(1/\lambda) \neq 0$ for $\lambda = 0$, we can compute the first n terms in the MacLaurin expansion of

$$\frac{\lambda^n \pi(1/\lambda)}{\lambda^n \chi(1/\lambda)} = \sum_{j=0}^{n-1} \eta_j \lambda^{j+1} + O(\lambda^n)$$

in $O(n \log n)$ arithmetic operations. This shows that H_0 can be computed in $O(n \log n)$ arithmetic operations.

5. A Factorization of $R^H R$

It follows from (3.3) and (3.20) - (3.21) that

$$G^{-1} = \overline{(R^H R)}^{-1} = I - \overline{B_0 B_0^H} = I - T_1^{-H} T_0 T_0^H T_1^{-1}, \quad (4.1)$$

and therefore

$$T_1^H G^{-1} T_1 = T_1^H T_1 - T_0 T_0^H =: M_0^{-1}. \quad (4.2)$$

The expression defining M_0^{-1} is a Gohberg-Semencul formula for the inverse of an $n \times n$ Toeplitz matrix, see, e.g., [Io, Theorem 18.2, p. 152]. We denote this Toeplitz matrix by M_0 . From the left hand expression of (4.2) and the nonsingularity of T_1 and R it follows that M_0 is Hermitian and positive definite. The desired factorization of $R^H R$ is

$$R^H R = \overline{T_1} M_0 T_1^H.$$

We will now show how M_0 can be computed. The computation involves running the Levinson algorithm backwards.

Consider the related Gohberg-Semencul formula, see, e.g., [Io, Theorem 18.1, p. 148] or [AG],

$$M_1^{-1} = \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 & & \\ & \ddots & & \vdots & & \\ & & \ddots & \vdots & & \\ & & & \chi_{n-1} & & \\ & & & \chi_n & & \end{bmatrix}^H \begin{bmatrix} \chi_n & \chi_{n-1} & \cdots & \chi_0 & & \\ & \ddots & & \vdots & & \\ & & \ddots & \vdots & & \\ & & & \chi_{n-1} & & \\ & & & \chi_n & & \end{bmatrix} \quad (4.3)$$

$$- \begin{bmatrix} 0 & & & & & \\ \chi_0 & & & & & \\ \chi_1 & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ \chi_{n-1} & \cdots & \chi_1 & \chi_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & & & & & \\ \chi_0 & & & & & \\ \chi_1 & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ \chi_{n-1} & \cdots & \chi_1 & \chi_0 & 0 \end{bmatrix}^H$$

where the four triangular Toeplitz matrices define the inverse of an $(n+1) \times (n+1)$ Hermitian Toeplitz matrix. Denote this Toeplitz matrix by M_1 . Then M_0 is the leading principal $n \times n$ submatrix of M_1 , see [Io, Theorems 18.1 - 18.2].

Let $R_1 := [\rho_{jk}]_{j,k=0}^n \in \mathcal{C}^{(n+1) \times (n+1)}$ be the unit right triangular matrix, and let $D_1 := \text{diag}[\delta_0, \delta_1, \dots, \delta_n]$ be the diagonal matrix such that

$$R_1^H M_1 R_1 = D_1. \quad (4.4)$$

Given $M_1 = [\mu_{j-k}]_{j,k=0}^n$, the matrices R_1 and D_1 can be computed by the Levinson algorithm, and by comparing R_1 with (4.3) one finds that

$$\rho_{jn} = \chi_j, \quad 0 \leq j \leq n \text{ and } \delta_n = \chi_n,$$

see, e.g., [AG]. We now apply the recursion formula in Levinson's algorithm backwards in order to determine R_1 and D_1 from the last column of R_1 and δ_n . Then the recursion formula is used forwards to determine M_0 . We will also obtain a Hermitian factorization of $R^H R$ into triangular matrices.

Backward Levinson algorithm

input: $[\rho_{jn}]_{j=0}^n, \delta_n$; output: R_1, D_1 , Schur parameters $\{\gamma_j\}_{j=1}^n$ of M_0 ;

for $k := n, n-1, n-2, \dots, 1$ do

$$\gamma_k := \rho_{0k}; \quad \rho_{k-1,k-1} := 1;$$

for $j := 1, 2, \dots, \text{integer part}(\frac{k}{2})$ do

solve for $\rho_{j-1,k-1}$ and $\rho_{k-1-j,k-1}$ the linear system of equations

$$\begin{bmatrix} 1 & \gamma_k \\ \bar{\gamma}_k & 1 \end{bmatrix} \begin{bmatrix} \rho_{j-1,k-1} \\ \bar{\rho}_{k-1-j,k-1} \end{bmatrix} = \begin{bmatrix} \rho_{j,k} \\ \bar{\rho}_{k-j,k} \end{bmatrix};$$

$$\delta_{k-1} := (\delta_k / (1 - |\gamma_k|)) / (1 + |\gamma_k|);$$

Levinson recursion for computing $M_0 = [\mu_{j-k}]_{j,k=0}^{n-1}$

input: $R_1, D_1, \{\gamma_j\}_{j=1}^n$; output: $\{\mu_j\}_{j=0}^{n-1}$;

$$\mu_0 := \delta_0; \quad \mu_1 := -\delta_0 \bar{\gamma}_1;$$

for $k := 1, 2, \dots, n-1$ do

$$\mu_{k+1} := -\delta_k \bar{\gamma}_{k+1} - \sum_{j=1}^k \mu_j \bar{\rho}_{j-1,k};$$

Hence M_0, R_1 , and D_1 are computed in $O(n^2)$ arithmetic operations from the coefficients of χ . Let R_0 and D_0 denote the $n \times n$ leading principal submatrices of R_1 and D_1 respectively. Similarly to (4.4) we have

$$R_0^H M_0 R_0 = D_0. \quad (4.5)$$

Because M_0 is positive definite, so is D_0 . $D_0^{1/2}$ can therefore easily be computed. We obtain from

(4.1) - (4.2) and (4.5), with $\hat{R} := D_0^{1/2} R_0^{-1}$,

$$R^H R = (\hat{R} T_1^H)^T (\overline{\hat{R} T_1^H}). \quad (4.6)$$

The right hand side of (4.6) is a Hermitian factorization into triangular matrices. It can be computed in $O(n^2)$ arithmetic operations from the coefficients of χ .

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