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ANALYSIS AND EVALUATION OF DISCRETE RELIABILITY GROWTH MODELS WITH AND WITHOUT FAILURE DISCOUNTING

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ANALYSIS AND EVALUATION OF DISCRETE RELIABILITY GROWTH MODELS WITH AND WITHOUT FAILURE DISCOUNTING

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Abstract

A survey of some evaluation work on discrete reliability growth models is presented. Extension of an accurate exponential growth model is provided that uses regression analysis to fit the natural logarithm of the failure rate $1-p$ in the geometric distribution. Some useful theorems and relationships are developed that provide estimates of reliability which have better properties than the usual maximum likehood estimates. The effect of discounting is portrayed with graphs that allow comparison among different failure discounting methods and their affect on different models.

Keywords: Reliability Growth, Failure Discounting, Reliability Estimation.

1 SUMMARY

Reliability growth models and their use in the acquistion of systems by DOD agencies have become more important in recent years. They are used not only to track reliability growth, but also to verify reliability requirements. It is important that these models track reliability change accurately. This tracking

accuracy can be determined by application of computer simulation. The results of simulating several reliability models that utilize discrete test data are reviewed in this paper.

How to dispose of past failures after their cause has been removed by appropriate changes in design or production has been a problem of interest for many years. Removal of a failure, or a fraction of a failure, from the test data is called failure discounting.

The results show that discounting definitely affects the accuracy of a reliability growth model. Certain types of failure discounting improve the accuracy of some reliability growth models analyzed. One type of failure discounting, which is in use today, decreases the accuracy of some reliability growth models analyzed.

2 BACKGROUND AND FOUNDATIONS

Let $X(r)$ denote number of independent trials to r failures with probability p of success on each trial. The probability distribution of $X(r)$ is the well known Pascal distribution with probability mass function:

$$f(x) = \binom{x-1}{r-1} p^{x-r}(1-p)^r \qquad x = r,\ r+1, \ldots .$$

We say $X(r)$ is $P(p,r)$ to denote this probability. For $r = 1$, $X(1)$ denotes trials to first failure. If $X_j(1)$ denotes trials between failure numbers $j-1$ and j, $j = 1, 2, \ldots, r$, then,

$$X(r) = X_1(1) + X_2(1) + \cdots + X_r(1).$$

Mission testing is frequently performed on items until r failures occur, at which time some change is made that hopefully will increase the probability p of success for subsequent trials. Let p_k denote the probability of success (reliability) after modification number k. Thus, $p_0, p_1, \ldots$, denote reliabilities of the items

under test in phases $0, 1, 2, \ldots$. We can model p_k as follows:

$$\begin{aligned} p_k &= 1 - e^{-(a+bk)}, \qquad a + bk > 0, \ \ k = 0, 1, 2, \ldots \\ &\equiv 1 - e^{-A_k}. \end{aligned} \tag{1}$$

This model has become known as the discrete exponential reliability growth model. It first appeared in reference (1). It was first evaluated by computer simulations in reference (3). In each phase, we seek current estimates for $a + bk$ using test data from the previous and current phases. In this model $a + bk = -\ln(1 - p_k)$. Since $1 - p_k$ is the failure rate, the problem is one of fitting the negative natural logarithm of the failure rate. Regression methods have been used to make this fit. One such method is described in detail in Section 4.

Some readers may prefer to estimated p_k directly using well known estimation results for this class of probability distributions. Since $X(r)$ is a complete sufficient statistic for p_k , it can be shown that a uniformly minimum variance unbiased estimator for p_k is

$$\hat{p}_k = \begin{cases} \dfrac{X(r) - r}{X(r) - 1} & \text{if } X(r) > r \\ 0 & \text{if } X(r) = r \end{cases} .$$

In development programs it may be desireable to take $r = 1$. In this case the uniformly minimum variance unbiased estimator for p_k is

$$\hat{p}_k = \begin{cases} 1 & \text{if } X(1) > 1 \\ 0 & \text{if } X(1) = 0 \end{cases} .$$

This is hardly a desirable estimator since p_k will never be 0 or 1.

In any event some model is needed for p_k that reflects changes in hardware reliability due to changes in the acquisition process. Since changes can improve or degrade reliability, a model (including the estimation procedures) should have the ability to track both degrading reliability and reliabililty growth.

2.1 Point Estimation Using Trials Between Failures

Suppose X is $P(p,1)$ and $p = 1 - e^{-A}$. Then the probability mass function $f(x)$ is

$$f(x) = \begin{cases} p^{x-1}(1-p), & x = 1, 2, \ldots \\ 0, & \text{elsewhere.} \end{cases}$$

The maximum likehood estimator (MLE), $\hat{p}*$, for p is $\hat{p}* = (X-1)/X$. This estimator has negative bias. An estimator with less negative bias is $\hat{p} = 1 - e^{-\hat{A}}$, where $\hat{A}$ is unbiased for A. $\hat{A}$ is obtained as follows: From equation (1),

$$A = -\ln(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \ldots.$$

If $g(x)$ is unbiased for A then,

$$\sum_{x=1}^{\infty} g(x)p^{x-1}(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \ldots. \tag{2}$$

The left member of equation (2) yields

$$(1-p)\left[g(1) + g(2)p + g(3)p^2 + \ldots\right] = g(1) + p(g(2) - g(1)) + p^2(g(3) - g(2)) + \ldots.$$

Equating coefficients of this polynomial with those of the right member of equation (2) we obtain

$$g(X) = \begin{cases} 0, & X = 1 \\ 1 + \frac{1}{2} + \ldots + \frac{1}{X-1}, & X > 1 \end{cases}. \tag{3}$$

Therefore, $\hat{A} = g(X)$ and $\hat{p} = 1 - e^{-\hat{A}}$. Of course $\hat{p}$ is not unbiased for p, but the following theorem is useful.

Theorem 1: If X is $P(p,1)$, $\hat{A} = g(X)$, $\hat{p} = 1 - e^{-\hat{A}}$, $\hat{p}*$ the MLE for p then

1. $E(\hat{p}*) = 1 + \frac{1-p}{p}\ln(1-p)$
2. $\hat{p}* < \hat{p}$ for all $X > 1$

3. $E(\hat{p}*) < E(\hat{p}) < p$

where $g(X)$ is defined by equation (3). The proof follows:

1. $\underline{E(\hat{p}*) < p}$:

$$\hat{p}* = \frac{X-1}{X} = 1 - \frac{1}{X}$$

$$E\left(\frac{1}{X}\right) = \sum_{1}^{\infty} \frac{1}{x}\, p^{x-1}(1-p)$$

$$= (1-p) + \frac{1}{2}\, p(1-p) + \frac{1}{3}\, p^2(1-p)$$

$$= 1 - p + \frac{p}{2} - \frac{p^2}{2} + \frac{p^2}{3} - \frac{p^3}{3} \dots$$

$$= \ln(1-p) - \frac{1}{p}\ln(1-p). \qquad (4)$$

Thus,

$$E(\hat{p}*) = E\left(1 - \frac{1}{x}\right) = 1 - \ln\,(1-p) + \frac{1}{p}\ln(1-p)$$

$$= p - \frac{p}{2} + \frac{p^2}{2} - \frac{p^2}{3} \dots < p$$

2. $\underline{\hat{p}* < \hat{p} \text{ for all } x > 1}$: (by induction)

Since $e^{\frac{1}{x}} > 1 + \frac{1}{x}$ for all $x \geq 1$, then

$$\frac{1}{x} > \ln\,\left(1 + \frac{1}{x}\right) \qquad (5)$$

a) for $x = 2$, $\hat{p}* = \frac{1}{2}$, $\hat{p} = 1 - e^{-1} = .632$

b) suppose for some interger $x > 2$, $\hat{p}* < \hat{p}$. Then,

$$1 - \frac{1}{x} < 1 - \exp\left[-\left(1 + \frac{1}{2} + \dots + \frac{1}{x-1}\right)\right]; \qquad (6)$$

i.e.

$$\ln\ x < 1 + \frac{1}{2} + \ldots + \frac{1}{x-1}. \tag{7}$$

From equation (5)

$$\ln\ \left(1 + \frac{1}{x}\right) \equiv \ln\ (x+1) - \ln\ (x) < \frac{1}{x} \tag{8}$$

adding equation (7) and (8) we obtain

$$\ln\ (x+1) < 1 + \frac{1}{2} + \ldots + \frac{1}{x}. \tag{9}$$

This establishes (6) and (7) for all integers > 1. Therefore, $\hat{p}* < \hat{p}$ for all $x \geq 1$.

3. $\underline{E(\hat{p}*) < E(\hat{p}) < p}$:

$$\hat{p}* < \hat{p} \quad \text{implies} \quad E(\hat{p}*) < E(\hat{p})\ .$$

Also $\hat{p}(\widehat{A}) \equiv 1 - e^{-\widehat{A}}$ is convex up in $\widehat{A}$ and $\hat{p}(A) = p$. The derivative of $\hat{p}(\widehat{A})$ at $\widehat{A} = A$ is e^{-A} and is the slope of the tangent line to $\hat{p}(\widehat{A})$ at $\widehat{A} = A$. This tangent line is

$$L(\widehat{A}) = \hat{p}(A) + e^{-A}(\widehat{A} - A) \equiv p + e^{-A}(\widehat{A} - A).$$

From convexity of $\hat{p}(\widehat{A})$,

$$\hat{p}(\widehat{A}) \leq L(\widehat{A}).$$

with equality only at $\widehat{A} = A$. Therefore

$$E(\hat{p}(\widehat{A})) \leq E[L(\widehat{A})] = L(E(\widehat{A})) = p.$$

Suppose $X(r)$ is $P(p, r,)$ and X_i, $i = 1, 2, \ldots, r$ denotes the trials between failures. Then $\widehat{A}_i = 1 + \frac{1}{2} + \ldots + \frac{1}{X_i - 1}$ is unbiased for $A = -\ln\ (1-p)$. Consequently, $\overline{\widehat{A}} \equiv \frac{1}{r}\sum_1^r \widehat{A}_i$ is also unbiased for A. The estimator $\bar{\hat{p}} = 1 - \exp(-\overline{\widehat{A}})$ is an improved estimator for p. We shall see in section 2.4 that there is a better estimator for A using

$$\sum_{i=1}^{r} X_i \equiv X(r).$$

2.2 Characteristrics of $\hat{p}$ and $\hat{p}*$

Even though $E(\hat{p}*) < E(\hat{p}) < p$, $\hat{p}$ may not be preferable to $\hat{p}*$, unless $var(\hat{p}) \doteq var(\hat{p}*)$. More specifically, it would be desireable to have the mean squared errors compare favorably: i.e., we would like

$$E(\hat{p} - p)^2 \leq E(\hat{p} * -p)^2$$

for values of p in the ranges of interest. Expressions for the mean and the variance of $\hat{p}*$ are as follows:

$$E(\hat{p}*) = 1 + \left(\frac{1-p}{p}\right) \ln(1-p)$$

$$var\left(\frac{1}{x}\right) = var(\hat{p}*) = \frac{1-p}{p} \sum_{1}^{\infty} \frac{1}{x^2} p^x - \left(\frac{1-p}{p} \ln(1-p)\right)^2$$

The mean and variance of $\hat{p}$ will be approximated. It is of interest to note that

$$var\hat{A} = \sum_{1}^{\infty} \frac{1}{x^2} p^x$$

(see ref(1)). Table 1. presents some values of p, $E(\hat{p}*)$, $E(\hat{p})$, $var(\hat{p}*), var(\hat{p})$, $MSE(\hat{p}*)$, $MSE(\hat{p})$. Most of the values were obtained from computer simulations using 10,000 observations. Table 2. affords a comparison of $\hat{p*}$ with $\hat{p}$ for various values of X.

p	E($\hat{p}*$)	E($\hat{p}$)	$var(\hat{p}*)$	$var(\hat{p})$	MSE($\hat{p}*$)	MSE($\hat{p}$)
.5	.307	.357	.102	.137	.141	.157
.6	.389	.451	.112	.143	.157	.166
.7	.484	.547	.115	.140	.163	.163
.8	.598	.661	.107	.121	.149	.140
.9	.744	.797	.079	.080	.103	.090
.95	.842	.881	.051	.047	.062	.052
.97	..892	.921	.035	.031	.040	.033
.99	.953	.968	.014	.012	.015	.012

Table 1: Statistical Characteristics of $\hat{p}*$ and $\hat{p}$

$X(1)$	$\hat{p}*$	$\hat{p}$
5	.800	.875
10	.900	.941
20	.950	.971
150	.993	.996

Table 2: Values of $\hat{p}*$, $\hat{p}$ and $X(1)$

2.3 Point Estimation Using X(r)

When testing items until r fail, the number of trials between failures $X_1, \ldots,$ X_r,may or may not be recorded. Either way $X(r)$ is observed and it is a complete and sufficient statistic for p. It is also sufficient for $-\ln(1-p)$; i.e. for A. Therefore, there is an unbiased minimum variance estimator $g(X(r))$ for A. The unbiased estimator $\overline{\hat{A}}$ discussed after Theorem 1, is unbiased, but may have larger variance or mean squared error than $g(X(r))$. Since $X(r)$ is $P(p,r)$, we could construct $g(X(r))$ as before, by writing

$$\sum_{x=r}^{\infty} g(x) \binom{x-1}{r-1} p^{x-r}(1-p)^r = p + \frac{p^2}{2} + \frac{p^3}{3} + \ldots$$

and equate coeffecients of both members of this equation. Instead, we construct $g(X(r))$ using recursive relationships.

Theorem 2: If $X(r)$ is $P(p,r)$, $p = 1 - e^{-A}$, then

$$g(X(r)) = \begin{cases} 0 & \text{if } X(r) = r \\ \frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{X(r)-1} & \text{if } X(r) > r \end{cases}$$

is a minimum variance unbiased estimator for $-\ln(1-p) \equiv A$.

Proof:

Suppose $X(r)$ is $P(p,r)$ and, let $g_r(X(r))$ be an unbiased estimator for $-\ln(1-p)$. Likewise, for $X(r+1)$, let $h_r(X(r+1)) = g_{r+1}(X(r+1))$. We shall drop subscripts and write

$$X \text{ is } P(p,r),\ Y \text{ is } P(p,r+1)$$

and let $g(X)$ and $h(Y)$ be unibiased estimators for $-\ln(1-p)$.

<u>*Case 1: $r = 1$*</u>

We know from equation (3) that

$$g(X) = \begin{cases} 0 & \text{if } X = 1 \\ 1 + \frac{1}{2} + \ldots + \frac{1}{x-1} & \text{if } X > 1 \end{cases}.$$

Since both $g(X)$ and $h(Y)$ are unbiased for the same quantity,

$$(1-p)\sum_{x=1}^{\infty} g(x)p^{x-1} = (1-p)^2 \sum_{x=2}^{\infty} h(x) \binom{x-1}{1} p^{x-2}. \tag{10}$$

Dividing out $(1-p)$, expanding the resulting $R.H.S.$ of (9), transposing one of the resulting series to the $L.H.S.$, and using the fact that $g(1) = 0$, we have

$$\sum_{x=2}^{\infty} [g(x) + (x-1)h(x)]\, p^{x-1} = \sum_{x=2}^{\infty} h(x)(x-1)p^{x-2}. \tag{11}$$

Writing x as $1 + j$, $(r + j$ where $r = 1)$, eq.(10) becomes

$$\sum_{j=1}^{\infty} [g(1+j) + jh(1+j)]\, p^j = \sum_{j=1}^{\infty} h(1+j)jp^{j-1}.$$

From equating coefficents,

1. $h(2) = 0$
2. $g(1+j) + jh(1+j) = h(1+j+1)(j+1)$,

or

$$h(1+j+1) = \frac{g(1+j) + jh(1+j)}{j+1}$$

If $j = 1$:

$$h(3) = \frac{g(2) + 0}{2} = \frac{1}{2}$$

If $j = 2$:

$$h(4) = \frac{g(3) + 2(\frac{1}{2})}{3} = \frac{1 + \frac{1}{2} + 1}{3} = \frac{5}{6} = \frac{1}{2} + \frac{1}{3}.$$

By induction, suppose

$$h(1+j) = \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j}.$$

Then

$$\begin{aligned} h(1+j+1) &= \frac{g(1+j)+jh(1+j)}{j+1} \\ &= \left[1+\frac{1}{2}+\ldots+\frac{1}{j}+j\left(\frac{1}{2}+\ldots+\frac{1}{j}\right)\right] \div (j+1) \\ &= \frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{j+1}. \end{aligned}$$

Thus,

$$h(1+j) = \frac{1}{2}+\frac{1}{3}\ldots+\frac{1}{j}, \qquad j=2,3,\ldots.$$

Since $h(X(2))$ is the unbiased estimator for A when $r=2$, it follows that

$$h(X(2)) = \begin{cases} 0 & X(2)=2 \\ \frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{X(2)-1} & X(2)=3,4,\ldots \end{cases}.$$

Case 2: $r>1$ (by induction)

Suppose

$$X \text{ is } P(p,r), Y \text{ is } P(p,r+1), g(X) \text{ and } h(Y)$$

are unbiased for $-\ln(1-p)$. Then

$$\sum_{x=r}^{\infty} g(x)\begin{pmatrix} x-1 \\ r-1 \end{pmatrix} p^{x-r}(1-p)^r = \sum_{x=r+1}^{\infty} h(x)\begin{pmatrix} x-1 \\ r \end{pmatrix} p^{x-r-1}(1-p)^{r+1}.$$

Dividing out $(1-p)^r$, expanding the resulting $R.S.H.$ of the equation and supposing $g(r)=0$, and $g(r+j)=\frac{1}{r}+\frac{1}{r+1}+\ldots+\frac{1}{r+j-1}$, we obtain,

$$\sum_{x=r+1}^{\infty}\left[\begin{pmatrix} x-1 \\ r-1 \end{pmatrix} g(x) + \begin{pmatrix} x-1 \\ r \end{pmatrix} h(x)\right] p^{x-r} = \sum_{x=r+1}^{\infty}\begin{pmatrix} x-1 \\ r \end{pmatrix} h(x)p^{x-(r+1)}. \tag{12}$$

Setting $x=r+j$, equation (12) becomes

$$\sum_{j=1}^{\infty}\left[\begin{pmatrix} r+j-1 \\ r-1 \end{pmatrix} g(r+j) + \begin{pmatrix} r+j-1 \\ r \end{pmatrix} h(r+j)\right] p^{j} = \sum_{j=1}^{\infty}\begin{pmatrix} r+j-1 \\ r \end{pmatrix} h(r+j)p^{j-1}.$$

Equating coefficients, we get

1. $h(r+1) = 0$

2.

$$h(r+j+1) = \frac{\binom{r+j-1}{r-1} g(r+j) + \binom{r+j-1}{r} h(r+j)}{\binom{r+j}{r}} \quad (13)$$

If $j = 1$:

$$h(r+1+1) = \frac{\binom{r}{r-1} g(r+1) + 0}{\binom{r+1}{r}} = \frac{r}{r(r+1)} \frac{1}{r} = \frac{1}{r+1}$$

If $j = 2$:

$$\begin{aligned} h(r+1+2) &= \frac{\binom{r+1}{r-1} g(r+2) + \binom{r+1}{r} h(r+1+1)}{\binom{r+2}{r}} \\ &= \frac{1}{r+1} + \frac{1}{r+2} \end{aligned} \quad (14)$$

By induction, suppose

$$h(r+1+j) = \frac{1}{r+1} + \frac{1}{r+2} + \ldots + \frac{1}{r+j}. \quad (15)$$

Then

$$h(r+1+j+1) = \frac{\binom{r+j}{r-1} g(r+j+1) + \binom{r+j}{r} h(r+j+1)}{\binom{r+j+1}{r}}. \quad (16)$$

Since,

$$\frac{\binom{r+j}{r-1}}{\binom{r+j+1}{r}} = \frac{r}{r+j+1}$$

$$\frac{\binom{r+j}{r}}{\binom{r+j+1}{r}} = \frac{j+1}{r+j+1}$$

equation (16) becomes

$$h(r+1+j+1) = \frac{rg(r+j+1)+(j+1)h(r+j+1)}{r+j+1}$$

$$= \frac{r\left[\frac{1}{r}+\frac{1}{r+1}+\ldots\frac{1}{r+j}\right]+(j+1)\left[\frac{1}{r+1}+\frac{1}{r+2}+\ldots+\frac{1}{r+j}\right]}{r+j+1} \cdot$$

$$= \frac{1}{r+1}+\frac{1}{r+2}+\ldots+\frac{1}{r+j}+\frac{1}{r+j+1}$$

Thus,

$$h(r+j+1) = \begin{cases} 0 & j=0 \\ \dfrac{1}{r+1}+\ldots+\dfrac{1}{r+j}, & j=1,2,\ldots \end{cases} .$$

Since $h(X(r+1))$ is the unbiased estimator for A_{r+1} it follows that

$$h(X(r+1)) = \begin{cases} 0 & X(r+1)=r+1 \\ \dfrac{1}{r+1}+\ldots+\dfrac{1}{X(r+1)-1}, & X(r+1)=r+2,\ r+3,\ldots. \end{cases}$$

The induction has been established for any r. Thus, if $X(r)$ is $P(p,r)$, $r \geq 1$, then an unbiased estimator, $g(X(r))$, for $-\ln(1-p)$ is

$$g(X(r)) = \frac{1}{r}+\frac{1}{r+1}+\ldots+\frac{1}{X(r)-1} \equiv \hat{A}_r.$$

The corresponding estimator for p is $\hat{p}_r = 1 - \exp(-\hat{A}_r)$. Since $\hat{A}_r$ is a function of X only, and since X is a complete sufficient statistic for p, then from the Lehman-Scheffe theorem, $\hat{A}_r$ is a uniformily minimum variance unbiased estimator for $A \equiv -\ln(1-p)$. Also $E(\hat{p}_r) \leq p$ by using a proof similar to the one used to show that $E(\hat{p}) \leq p$.

Table 3 provides a comparison of the statistical properties of $\hat{p}_3$ with those of $\bar{\hat{p}}$ described at the end of Theorem 1. The comparisons are the result of 10,000 computer replications of sampling from $P(p, r = 3)$, i.e. testing until 3 failures.

p	$E(\bar{\hat{p}})$	$E(\hat{p}_3)$	MSE($\bar{\hat{p}}$)	MSE($\hat{p}_3$)
.5	.450	.452	.0537	.0538
.6	.548	.551	.0516	.0493
.8	.757	.763	.0293	.0248
.9	.872	.876	.0120	.0094
.95	.933	.936	.0043	.0032
.99	.985	.986	.0025	.0017

Table 3: Statistical Characteristics of $\bar{\hat{p}}$ and $\hat{p}_3 (r = 3)$

It is interesting to compare the values of $\bar{\hat{p}}$ in Table 2 $(r = 3)$ with those of $\hat{p}$ in Table 1. $(r - 1)$.

3 RELIABILITY GROWTH IN ACQUISITION PROGRAMS

Reliability growth methods were developed as early as 1962, primarily for use in major DOD acquistion programs. Reference (1), presents several RG models, including the discrete exponential (DE) model. Reference (2) evaluated many reliability growth models in the open literature, and other sources, at the time of the publication. It showed the DE model had some of the most accurate characteristics among all growth models considered. Reference (3) presents the results of numerous simulations of a modification to the DE model. Reference (4) is a refinement and extension of Reference (3).

Reliability growth methods were used for many years in early development phases of acquisition programs as a management tool to indicate trends in growth patterns. They were not used to demonstrate reliability requirements, because numerous demonstration tests were performed for that reason. MILHDBK 189 (reference 7) was developed under that concept. In October 1986, DOD issued reference (5). This document, recent RFP's, and funding policies have diminished the amount of demonstration testing. "Reliability testing" is performed during the development phases to induce failures, make corrections, and test again. This type of testing program results in changes in the design and manufacturing processes. One consequence of such a program is the need to track growth in reliability. These new changes in testing philosphy and reliability measurement are evident in references (5) and(8).

4 EXPONENTIAL RELIABILITY GROWTH MODEL

4.1 Description of the Model

The model assumes that a type of component will see several phases of improvement during that portion of its life cycle to which the model is applied. R_K denotes reliability in the Kth phase or modification to the type of component. Specifically, the model is

$$R_K = 1 - \exp\{-(\alpha + \beta K)\} \tag{17}$$

for $K = 0, 1, 2, \ldots$. The $K = 0$ phase denotes the phase prior to any modification. The model assumes testing is performed in each phase until a specified number of failures are observed at which time a change may be made. No assumption is made about the distribution of the time to failure. Only attributes data are used with this model. Estimates $\hat{\alpha}$ and $\hat{\beta}$ are updated as new test data (new phase) becomes available due to testing subsequent to a modification in the design or manufacturing process. This data may, or may not, be mixed with test data from previous phases. All data from all previous phases are used in

the estimation methods presented here.

The estimates $\hat{\alpha}_K$ and $\hat{\beta}_K$ for a α and β at the end of the Kth phase are obtained using linear regression methods and an unbiased estimator for $(\alpha+\beta K)$. The data collected during testing in the Kth phase is the following: $F_K \equiv$ the total number of failures during the Kth phase, and $N_{j,K}$ = the number of test between the $(j-1)^{st}$ failure and the jth failure, including the jth failure, in the Kth phase, $j = 1, 2, \ldots, F_K$.

An unbiased estimator Y_{jK} of the quantity $(\alpha + \beta K)$ using the jth sequence of tests in phase K is given by (Reference 1).

$$Y_{jK} \equiv (\widehat{\alpha + \beta K})_{jK} = \begin{cases} 0 & \text{if } N_{j,K} = 1 \\ 1 + \frac{1}{2} + \ldots + \frac{1}{N_{j,K}} & \text{if } N_{j,K} \geq 2 \end{cases} \tag{18}$$

for $K = 0, 1, 2, \ldots$ and $j = 1, 2, 3, \ldots, F_K$. Since each Y_{jK} is unbiased,

$$\overline{Y}_K \equiv (Y_{1K} + Y_{2K} + \ldots + Y_{F_K K})/F_K$$

is unbiased. Then least squares estimates α_K and β_K for α and β at the Kth phase are

$$\hat{\beta}_K = \frac{\sum_{j=0}^{K}(j - \overline{K})\overline{Y}_j}{\sum_{j=0}^{K}(j - \overline{K})^2} \tag{19}$$

and

$$\hat{\alpha}_K = \overline{Y} - \hat{\beta}_K \overline{K} \tag{20}$$

for $K = 1, 2, 3, \ldots$, where $\overline{Y} = (Y_0 + Y_1 + \ldots + Y_K)/(K+1)$ and $\overline{K} = (1 + 2 + \ldots + K)/(K+1)$.

Finally, these $\hat{\alpha}_K$ and $\hat{\beta}_K$ estimates are utilized in the discrete reliability model, equation (15), to produce sequentially, the model estimates of the modified component reliabilities $R_1, R_2, \ldots$ from the equation

$$\hat{R}_K = 1 - \exp\{-(\hat{\alpha}_K + \hat{\beta}_K K)\} \tag{21}$$

for $K = 1, 2, 3, \ldots$. Note that since the regression procedure requires a minimum of two observations, model reliability estimates are produced from the first modification thru the last modification. A reliability estimate for original version of the component is

$$\hat{R}_0 = \exp\{-\overline{Y}_0\}. \tag{22}$$

4.2 FAILURE DISCOUNTING

Reduced testing support, new funding policies and new DOD documents, such as reference (5), in the acquisition process require maximum use of small amounts of test data to obtain reliability estimates. Human judgment by both contractor and customer personnel have played a larger role in reliability assessment – particularly in deciding what data to use to estimate reliability and track reliability growth. Failure analysis and review frequently generates a change that both contractor and customer feel have removed the cause of failure. It may be desireable to reduce the affect of the original failure on the reliability estimate since it is believed to have been "removed".

This raises the question of what to do with the recorded failure in the data. One method for failure removal is to remove fractions of a failure, repeatly, as more testing is accumulated without failure for the same cause. By some rule, or agreement, fractions of such a failure are removed until all, or nearly all, of the failure is removed. If the same failure cause reappears, the orginal failure is restored and, perhaps, the rule for removal of fractional failure for this or any other cause may be tightened due to the reduced confidence in ability to remove failure causes.

Accuracy of reliability growth models with or without failure discounting can be readily examined by computer simulations. Several reliability growth models have been simulated under a variety of testing and discounting plans by Drake (9) and Chandler (10).

4.3 Discounting Methodology

Once a system weakness has been corrected, and improvement validated through further testing, fractionally discounting the past failure is a method of reflecting the improved system reliability in the previously collected data. If a proper discounting methodology can be developed, failure data from all past phases could be discounted and made compatible with the data from the current phase. Regardless which discounting method is used, experience and good engineering judgment will be necessary to choose input parameters to the discounting method. If improper parameters are applied, the model may consistently over or underestimate system reliability. Computer programs have been written by Drake (9) and Chandler (10) that allow a user to examine the affect of discounting on specific reliability growth models and testing programs. The following sections describe the specific steps for applying two failure discounting methods, including examples of their use.

a. Straight Percent Discounting Method

The Straight Percent Discounting Method removes a fixed fraction of a failure each time a predetermined number of trials are observed without re-occurrence of the failure cause. The fraction of a failure removed, F, is referred to as the discount fraction. After the initial occurrence of a particular failure cause, any trial in which the failure cause is not repeated is considered a success. The number of successful trials between applications of the discount fraction is referred to as the discount interval, N. Both N and F are input parameters to be specified by the user of the simulation programs written by Drake (9) and Chandler (10). They remain constant throughout failure discounting computations. In this discounting method, the computer records the number of successful trials, M, since the last occurence of each observed failure cause. Failure discounting is performed at the end of each testing phase using the values for N, F, and M. Detailed explanations of this method are provided in references (9) and (10).

b. Upper Confidence Bound (UCB)

The underlying premise of the UCB Discounting Method states that the fraction of a failure removed by the discounting method should not be arbitrarily chosen. Some statistical basis should be used to determine how much of a past failure remains after each successful test subsequent to failure and corrective action. The UCB procedures takes the upper confidence limit for the probability of failure to be the fraction of the failure that remains. This confidence limit is recomputed after each successful test subsequent to the corrective action taken in response to the failure. This technique allows the user to control the amount by which failures are discounted by specifying the level of confidence bound. A single input parameter, the confidence level γ, is required to perform failure discounting. This discounting method was proposed by Lloyd (6).

The UCB discounting equation is given as

$$\text{ADJUSTED FAILURE; ie, UCB} \begin{array}{ll} = 1 - (1-\gamma)^{1/M} & \text{for } M > 0 \\ = 1.0 & \text{for } M = 0 \end{array}.$$

5 SUMMARY OF ACCURACY ANALYSIS

The simulations performed by Drake (9) allowed for 10 sequential testing phases. There were 5 possible statistically independent failure causes in each phases. Initial failure probabilities are assigned to each failure cause. A single discounting fraction is also assigned. A probability of correcting a failure causes whenever it occurs is assigned to each failure cause; consequently, the actual growth pattern on any one replication is random depending on the actual failures and fixes. The percent increase in reliability following a failure fix is also an input parameter. Simulations are performed with and without failure discounting. The number of successive success before discounting is applied is also an input variable.

Chandler (10) extended the computer program written by Drake to permit the user to set the actual reliability in each phase. This allows the user to examine the accuracy of the growth models for specific growth patterns; i.e., for

specific phase reliabilities $p_1, p_2, \ldots, p_{10}$ which the user specifies. Chandler extends Drakes's program in other ways and examines features of growth patterns for which specific discounting methods and growth models are reasonably accurate. Figures 1 through 19 have been selected from Drake (9) and Chandler (10) to provide a summary of the accuracy of the DE and other reliability growth models.

The following notation is used in the Figures 1 through 19 which are graphical representations of the accuracy of the growth models analyzed:

F: fraction of remaining failure removed each time a discount is made.

I: number of successful tests without failure before the next failure discount can be made.

Example: If F=.2 and I=5, then after a failure occurs and correction made, the one failure for that cause is reduced to .8 of a failure if no failure for the same cause occurs in the next 5 tests. It is reduced to .64 if 5 additional success are recorded. Anytime a failure for the same cause re-occurs the failure value is restored to 1.

MLE Single Phase: the name of the growth model that computes the reliability in each phase by (number of successes that phases) ÷ (number of tests in that phase).

MLEFD: the name of the growth model that computes the reliability in the kth phase by (total number successes in first k phases) ÷ (total number of tests in first k phases).

Exponential Regression: The discrete exponential regression model.

Actual Reliability: The reliability values assigned to each phase by the user.

The reliability values for each phase for a specific growth model (e.g. exponential regression) is the average of 500 reliability values obtained by replicating the 10 phases of testing 500 times via computer simulations. Consequently, the points on the growth "curves" represent the average or expected value of the reliability estimates for the actual reliabilities. Ideally the growth curves would be the same as the actual reliability curve and the standard deviation would be very small. The standard deviation curves are the sample standard deviations taken from the 500 replications.

The test plan in Figures 1 through 19 is to test until 1 failure occurs in each phase. This implies that a *change* (correction) is made after each failure. The number of failure–causes is the same from phases to phase. The *change* may not have totally removed the cause of the failure. It may have only reduced the likelihood that it will occur. Alternatively, it may mean that a new mode of failure, less likely to occur, was introduced with the corrective action.

Instead of 5 failure causes per phase, one may wish to think of a system with 5 components in series. The program will permit any number of phases and any number of causes per phase. If it is desired to simulate a case where each failure cause is removed with probability one, this can be done using Drake's program with probability of fix equal to one and the number of phases less than or equal to the number of causes.

Figures 1–19 present graphical results of simulations for 6 different growth patterns each with various combinations of I and F, and two values of confidence level for the Lloyd discounting method. Recall that I=1 in the Lloyd discounting method.

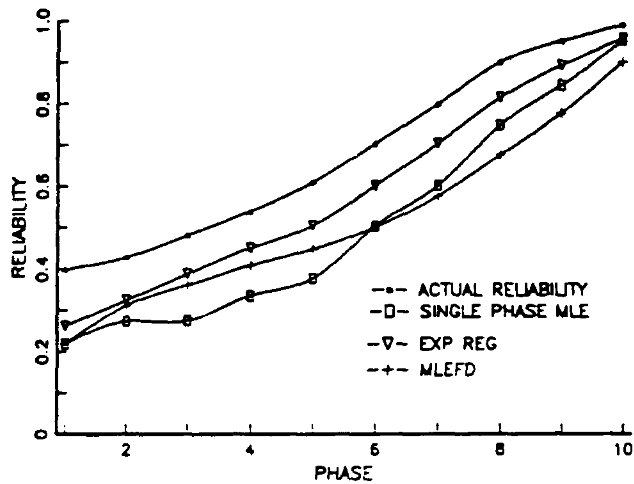
In general the following conclusions can be from Figures 1–19.

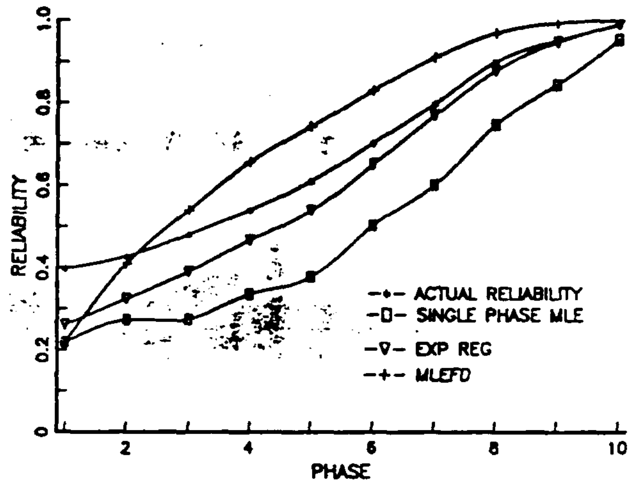
1) The MLEFD Method is highly sensitive to failure discounting. This method is commonly used in industry.

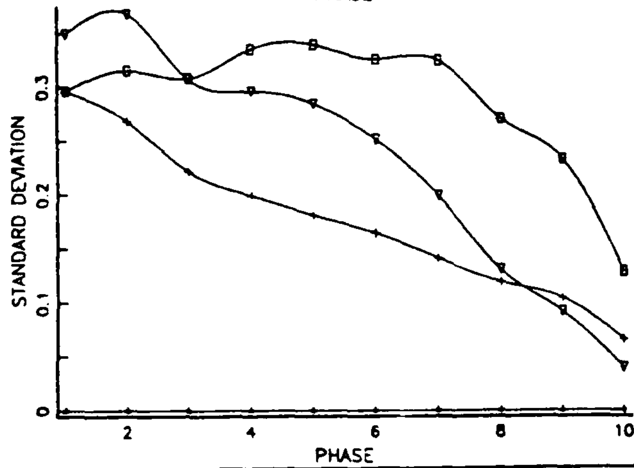
2) The exponential regression method is consistently more accurate than the

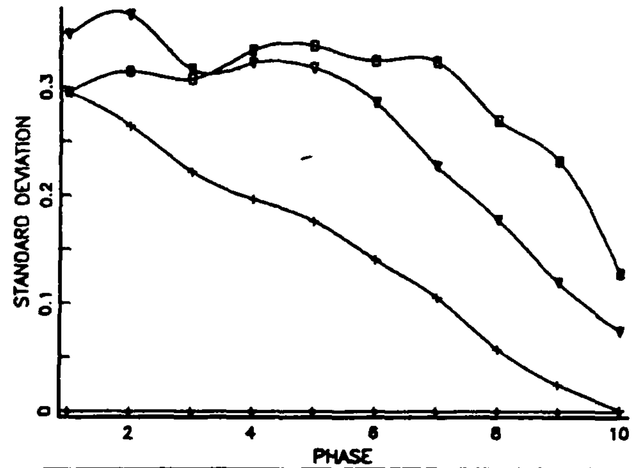
other models and is not very vulnerable to failure discounting.

3) Failure discounting improves the exponential regression method in many cases.





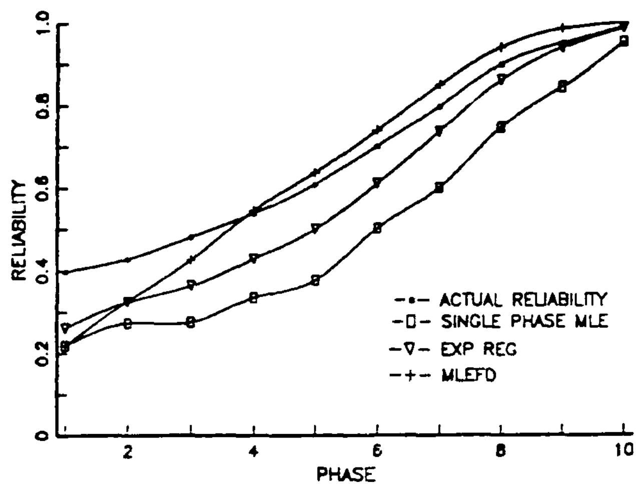


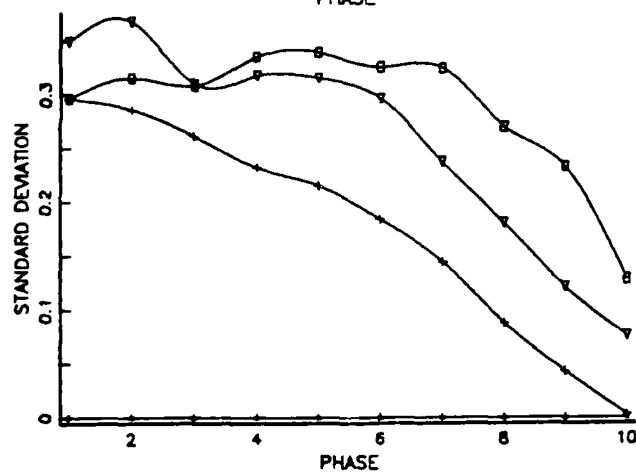


Pattern I, No Discounting

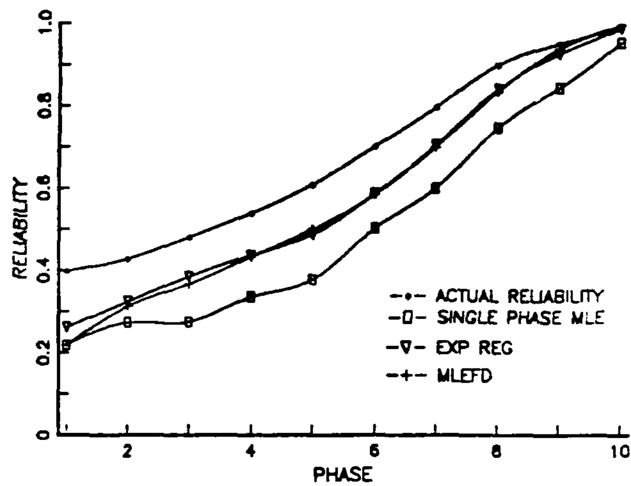
Pattern I, F = .25, I = 1

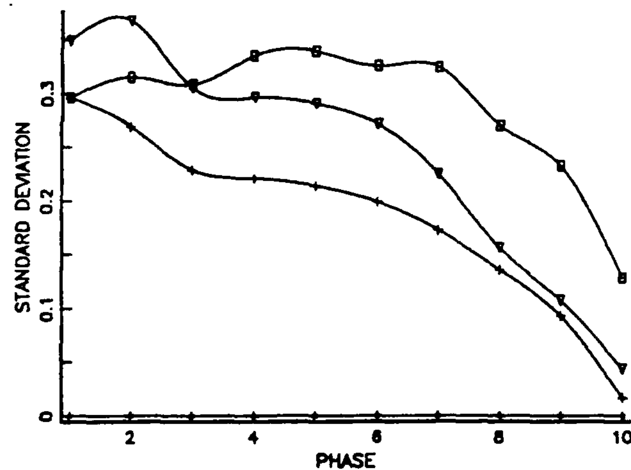
Figure 1.





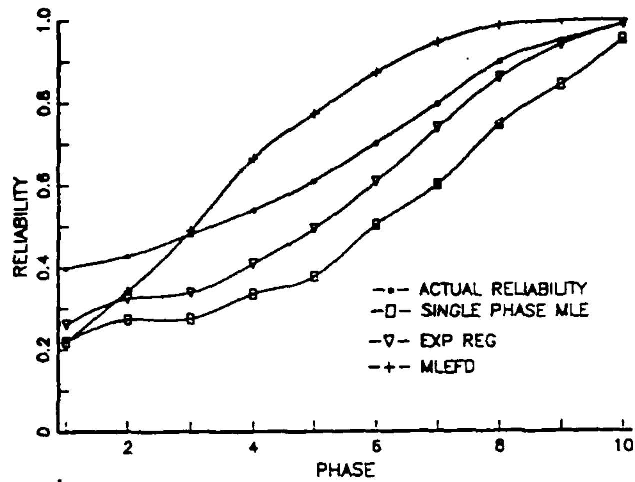
Pattern I, F = .50, I = 3

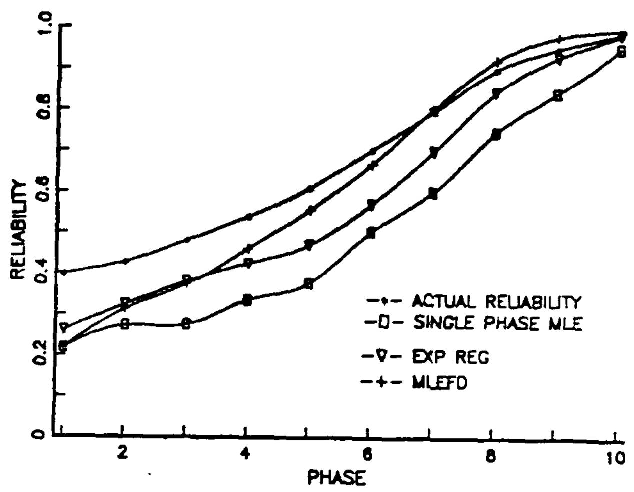


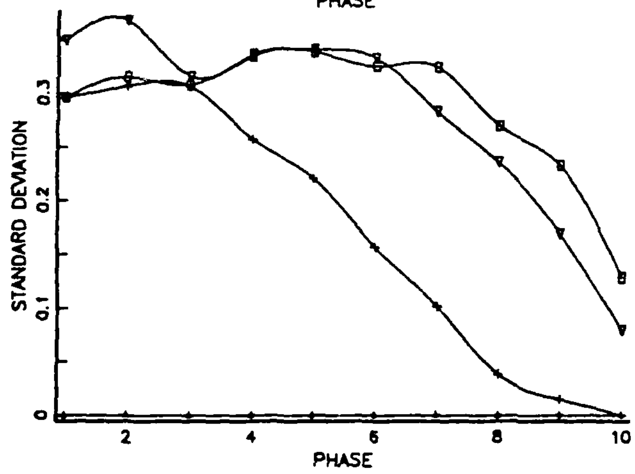


Pattern I, F = .50, I = 6

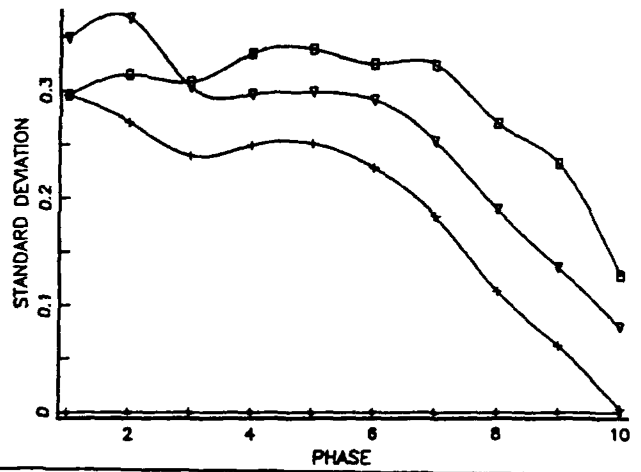
Figure 2.





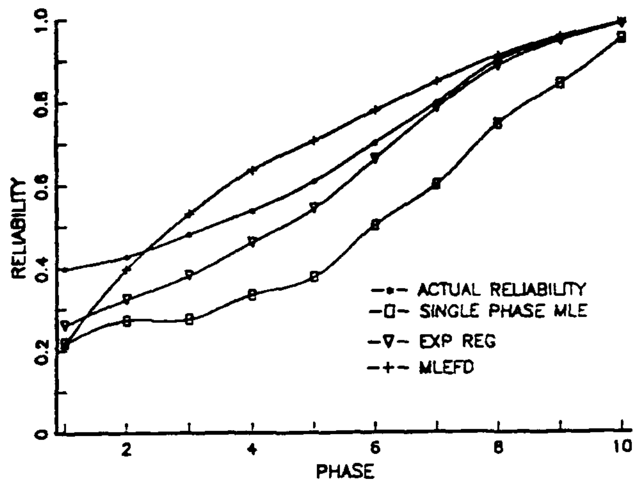


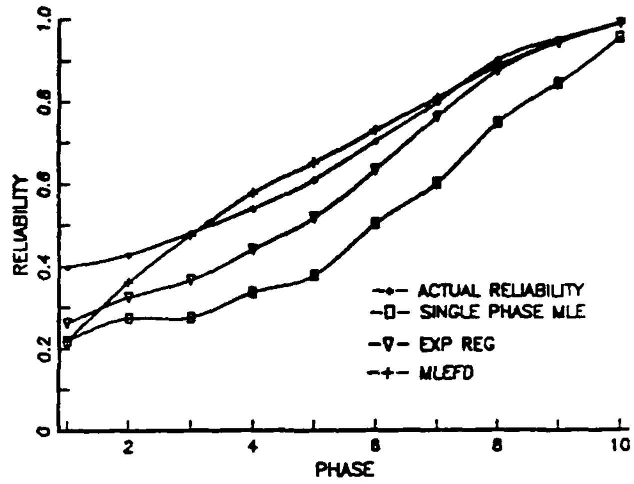
Pattern I, F = .75, I = 3

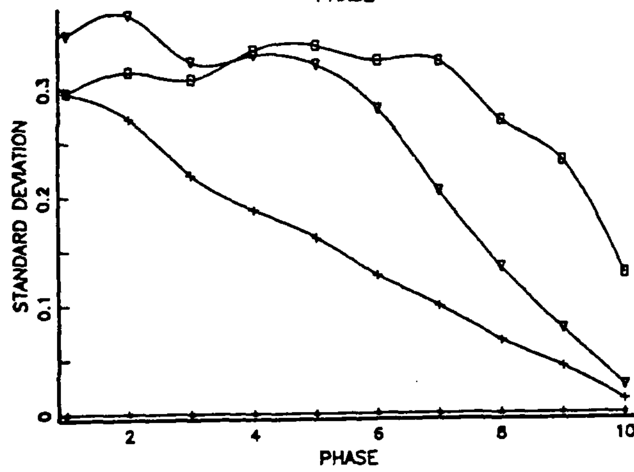


Pattern I, F = .75, I = 6

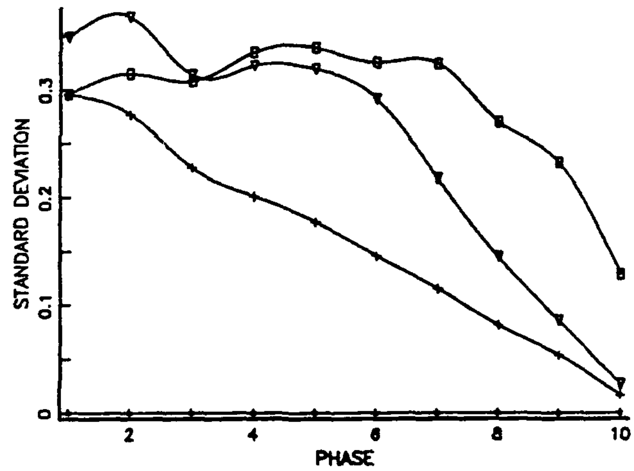
Figure 3.





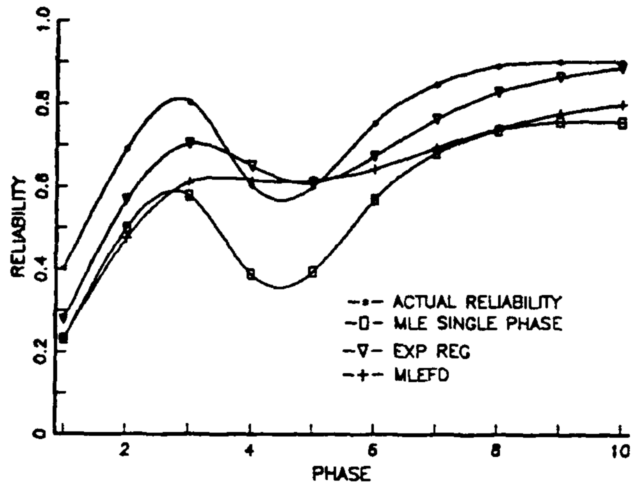


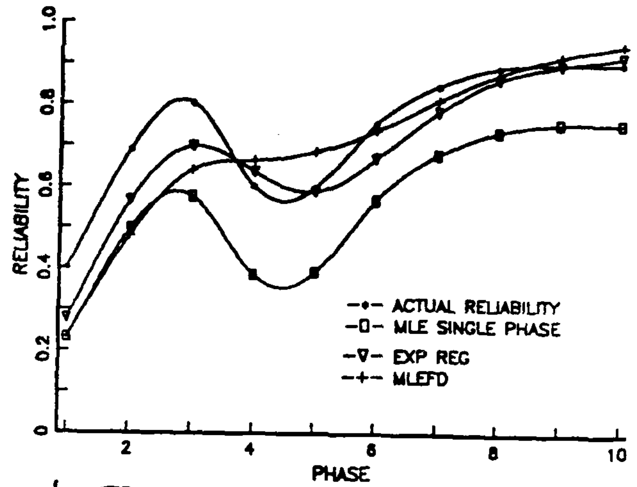
Pattern I, Lloyd, CI = .8

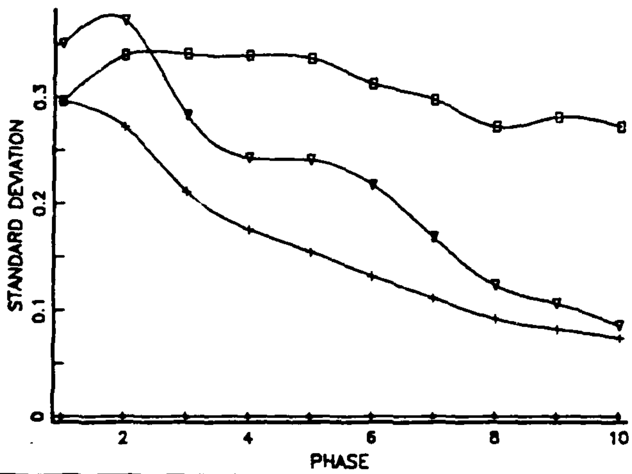


Pattern I, Lloyd, CI = .9

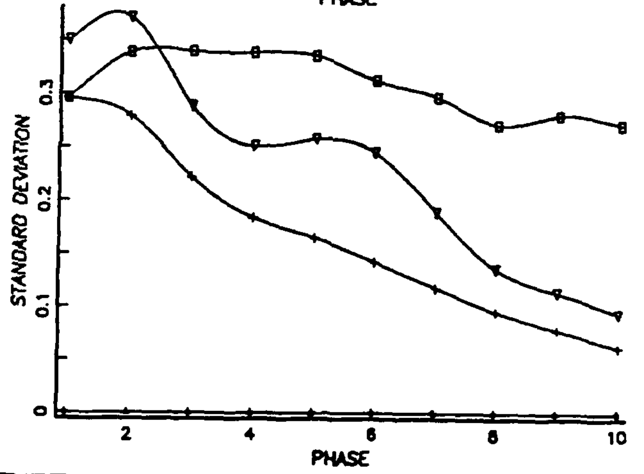
Figure 4.





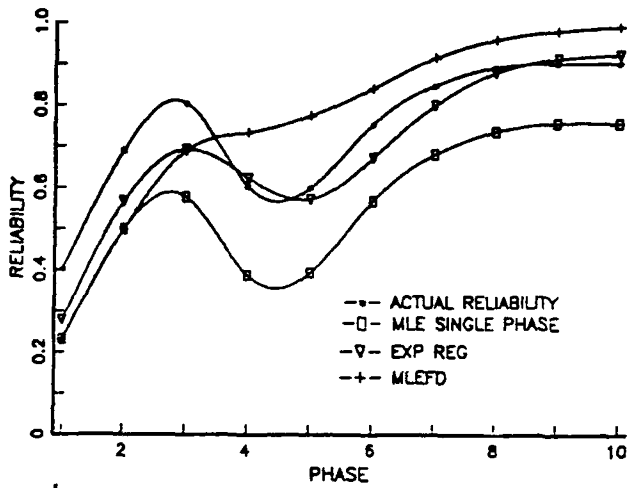


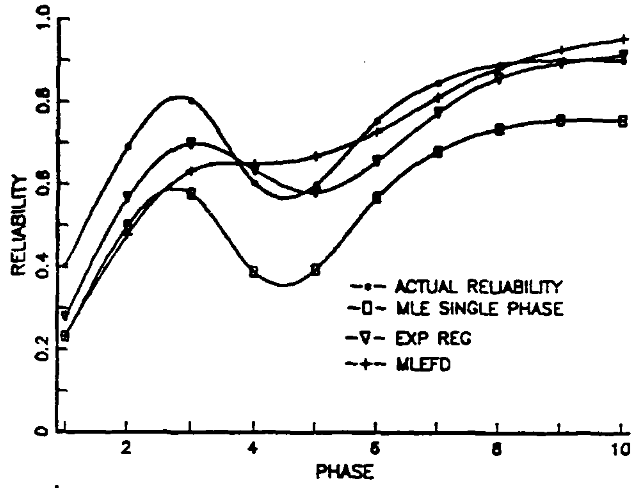
Pattern II, No Discounting

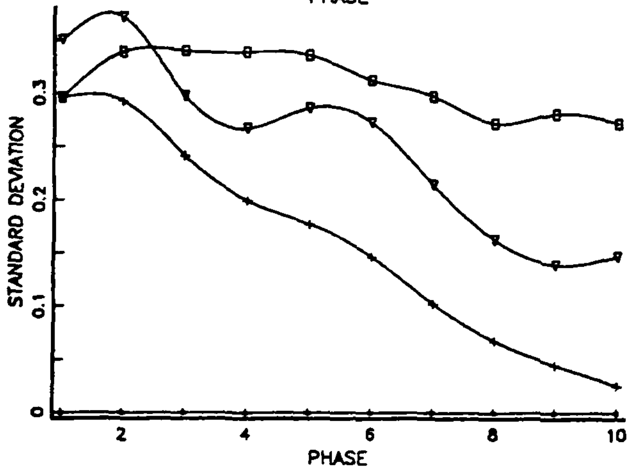


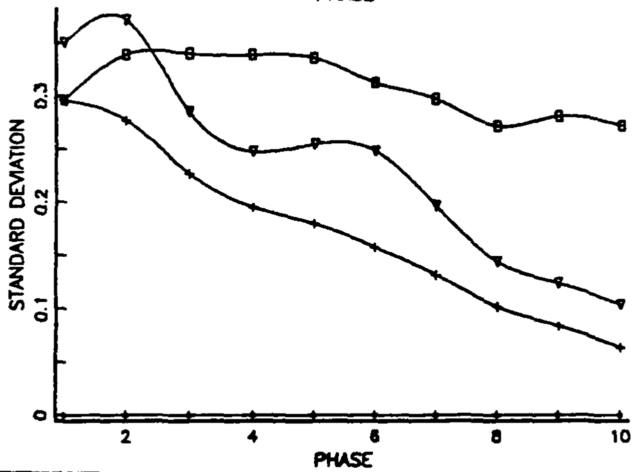
Pattern II, F = .25, I = 3

Figure 5.





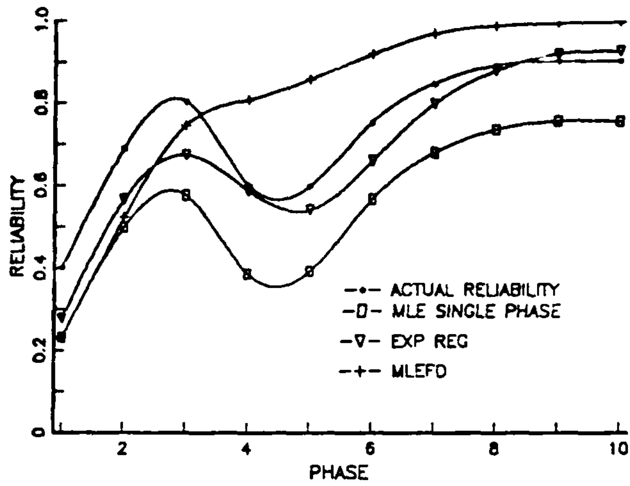


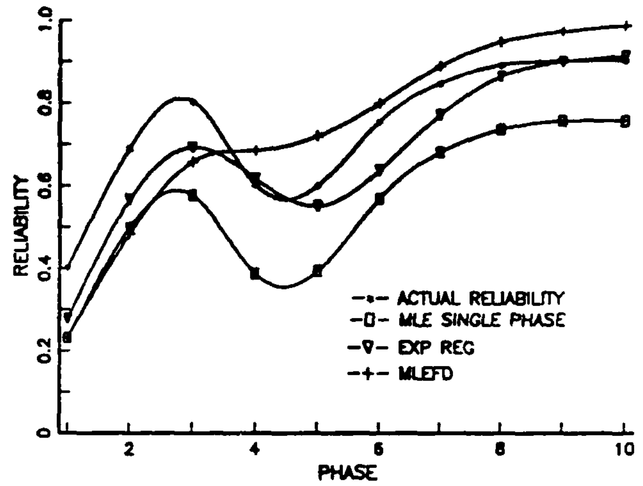


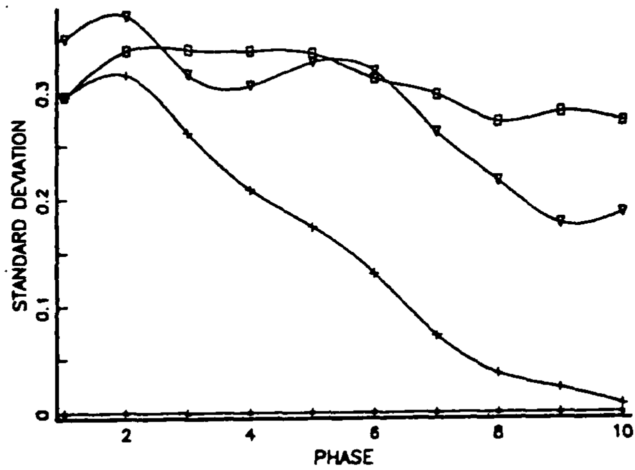
Pattern II, F = .50, I = 3

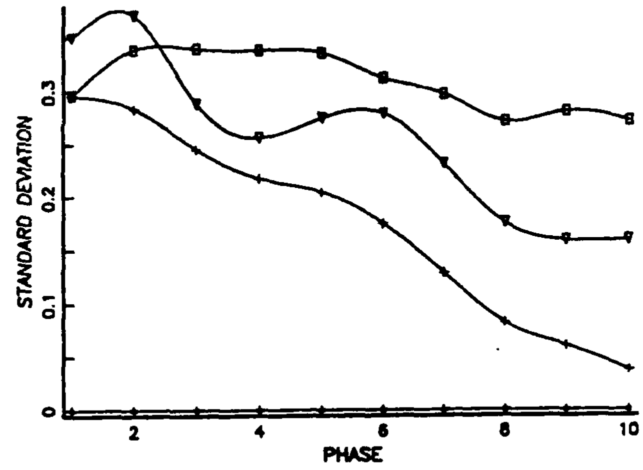
Pattern II, F = .50, I = 6

Figure 6.





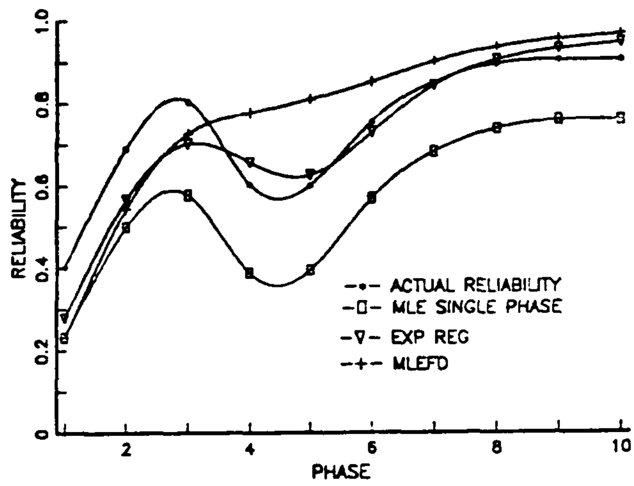


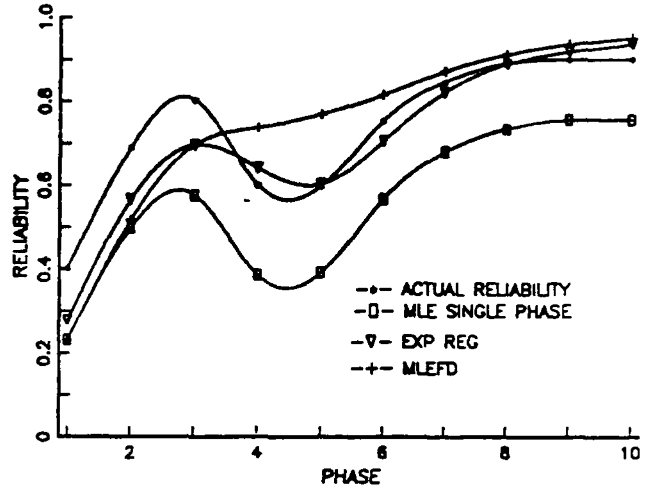


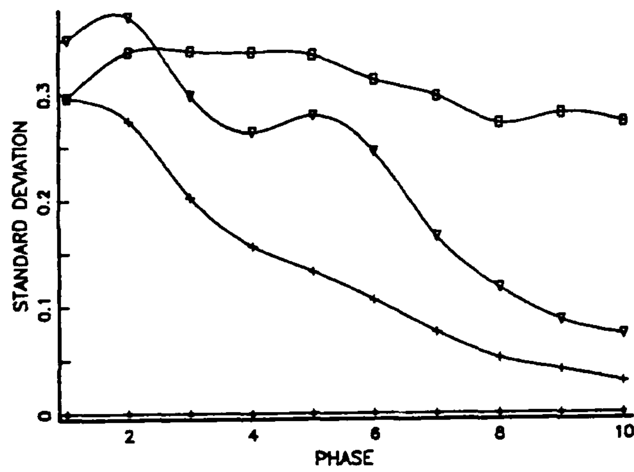
Pattern II, $F = .75$, $I = 3$

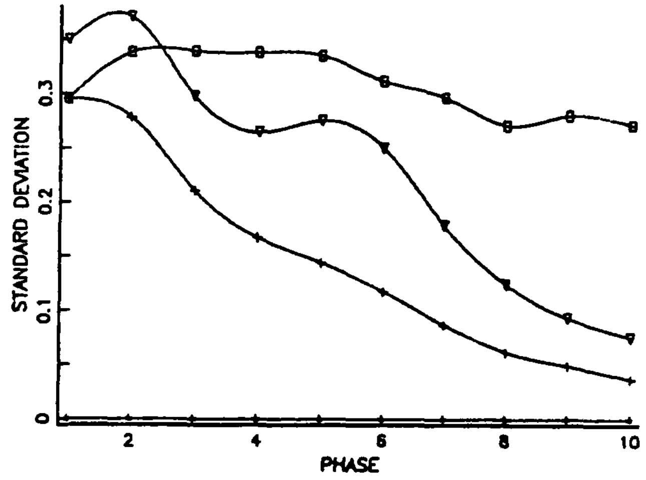
Pattern II, $F = .75$, $I = 6$

Figure 7.





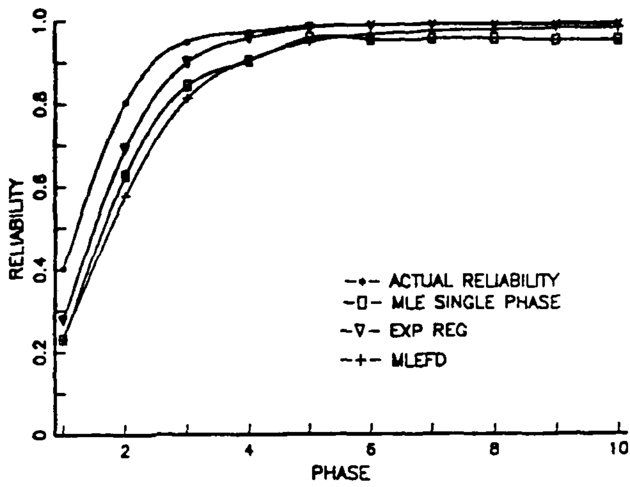


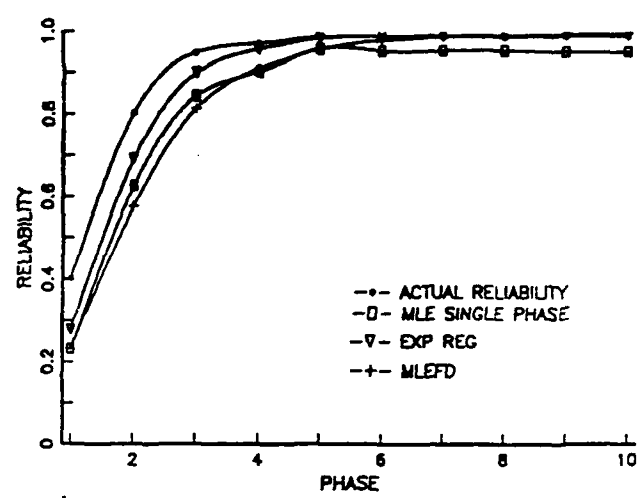


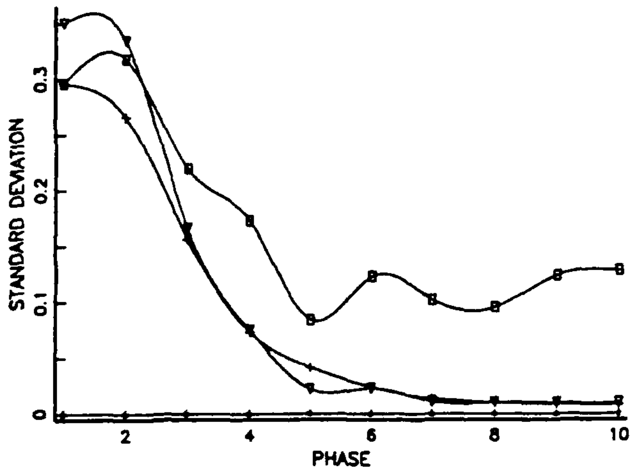
Pattern II, Lloyd, CI = .8

Pattern II, Lloyd, CI = .9

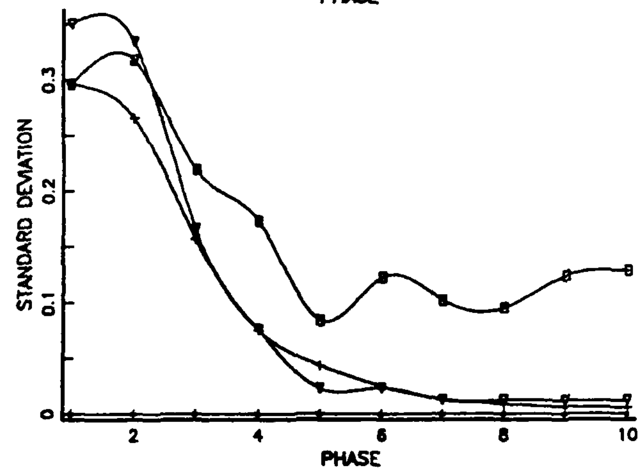
Figure 8.





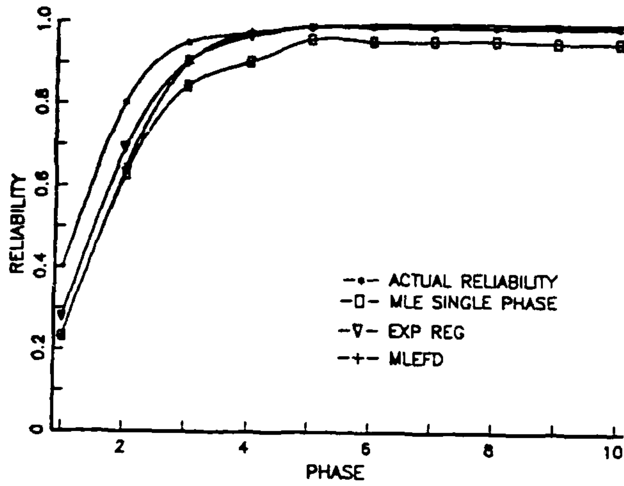


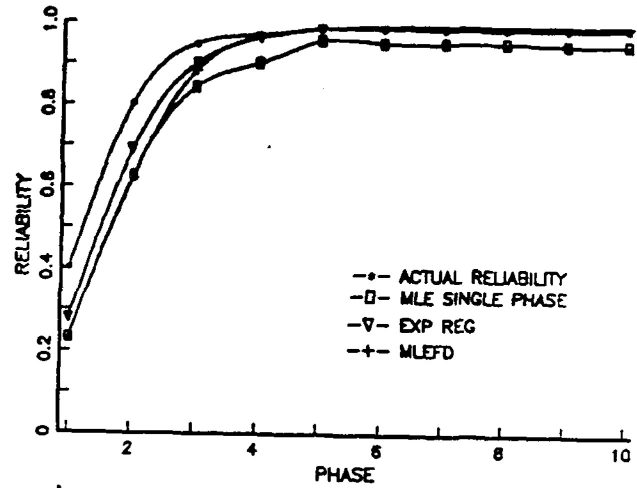
Pattern IV, No Discounting

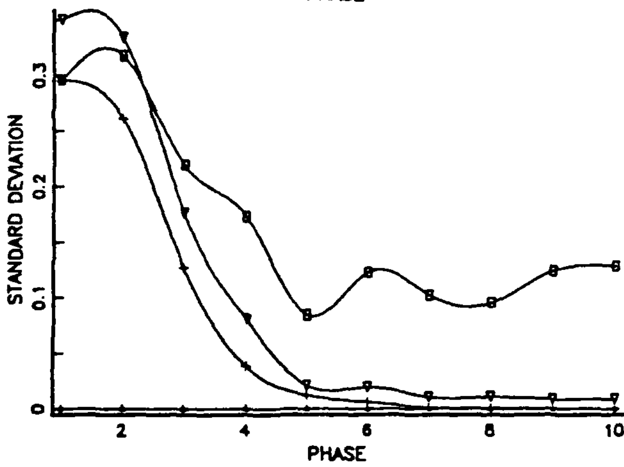


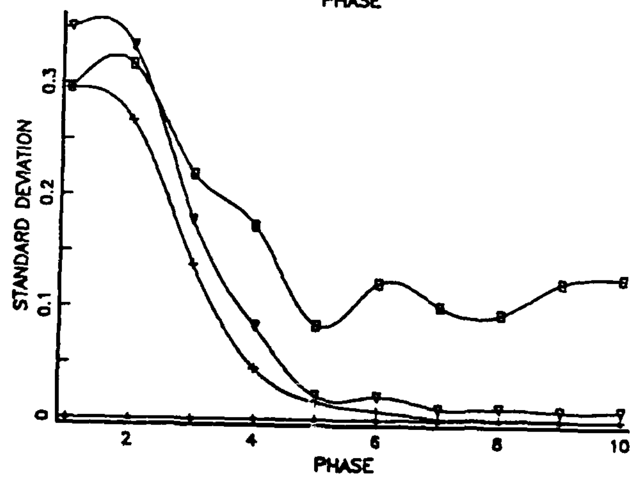
Pattern IV, F = .25, I = 25

Figure 9.





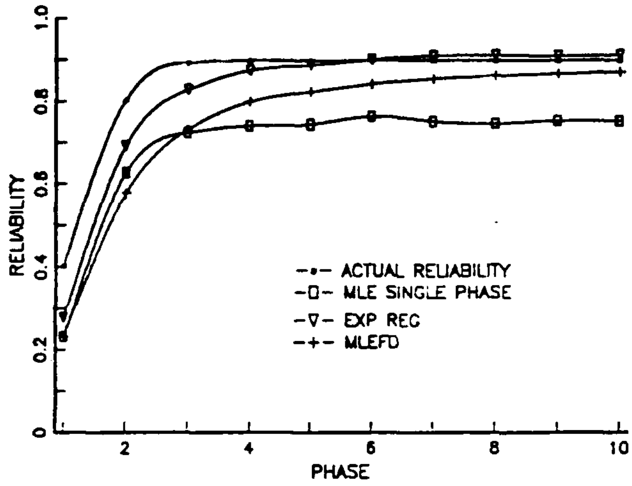


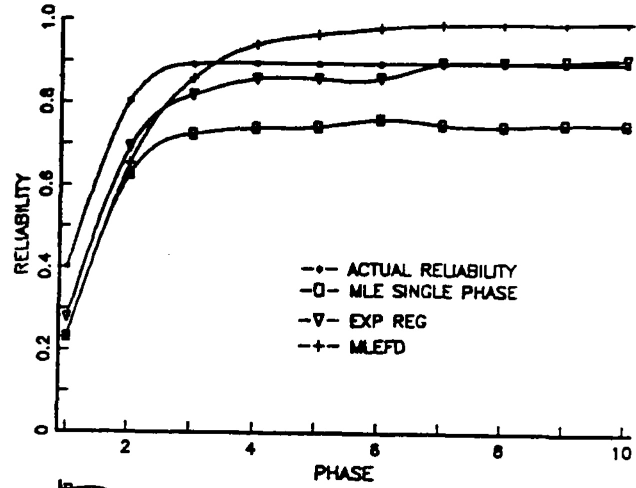


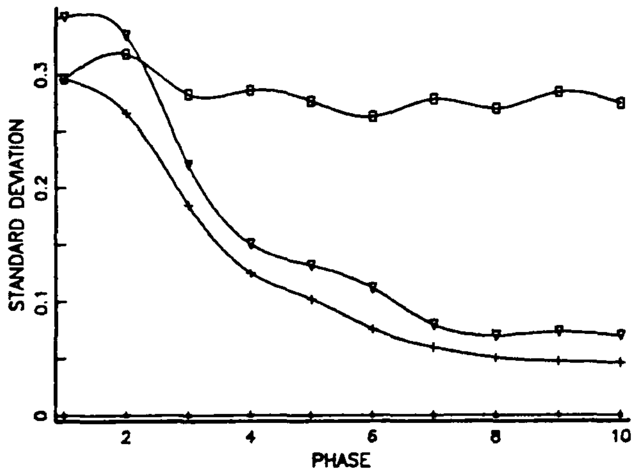
Pattern IV, Lloyd, CI = .8

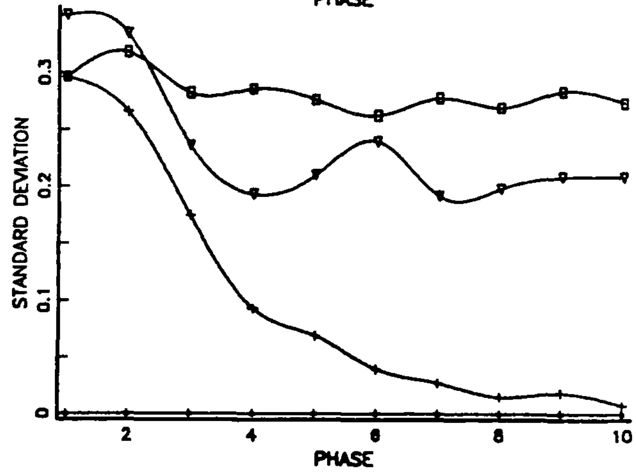
Pattern IV, Lloyd, CI = .9

Figure 10.





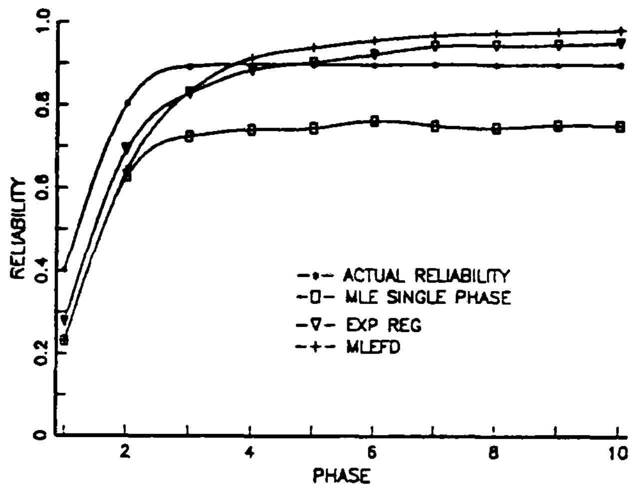


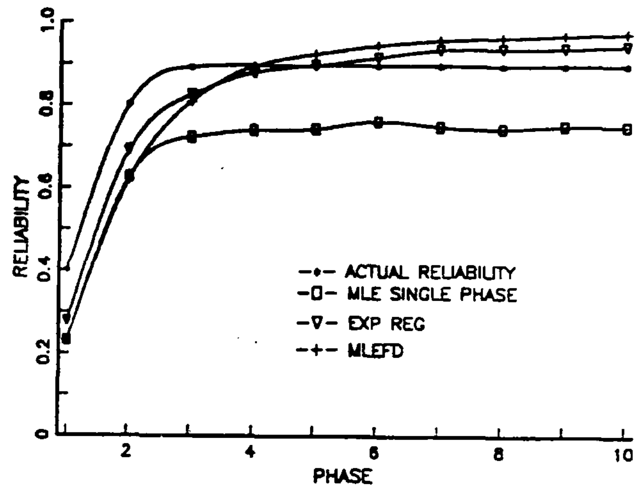


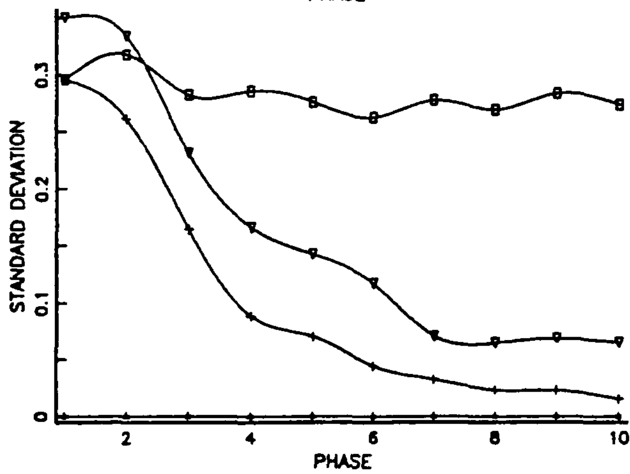
Pattern V, No Discounting

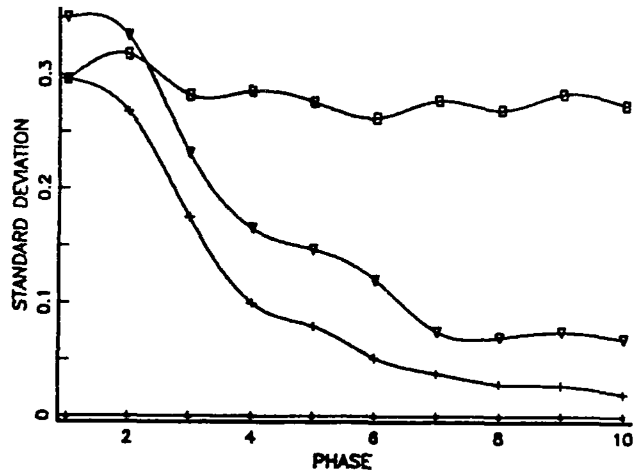
Pattern V, F = .25, I = 1

Figure 11.





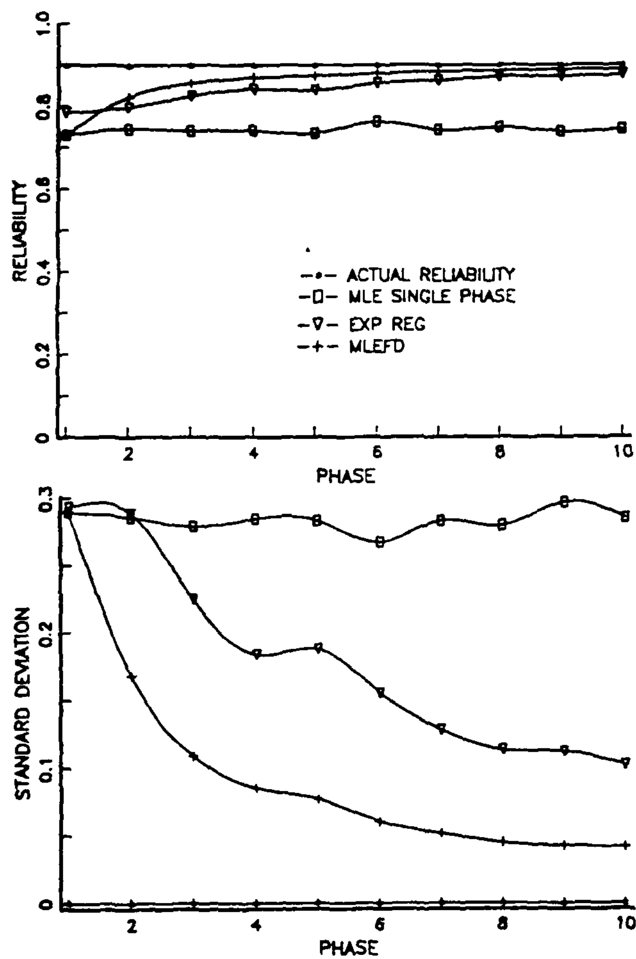




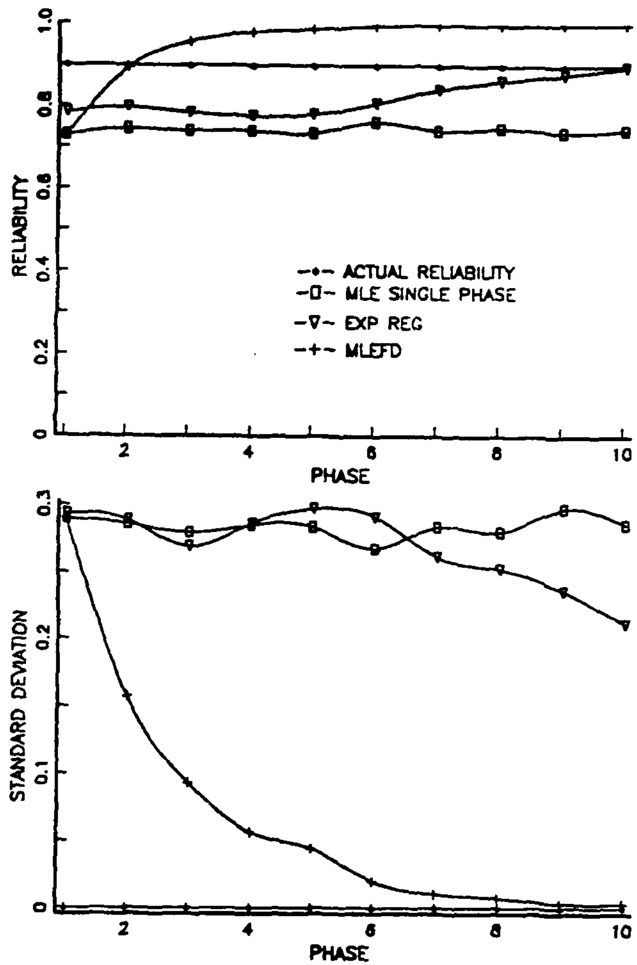
Pattern V, Lloyd, CI = .8

Pattern V, Lloyd, CI = .9

Figure 12.

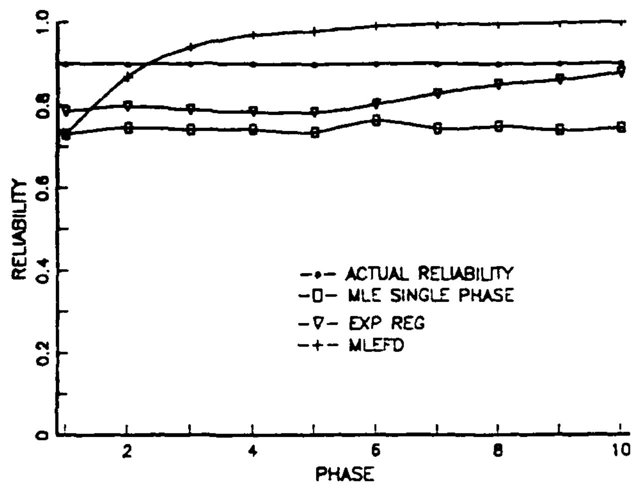


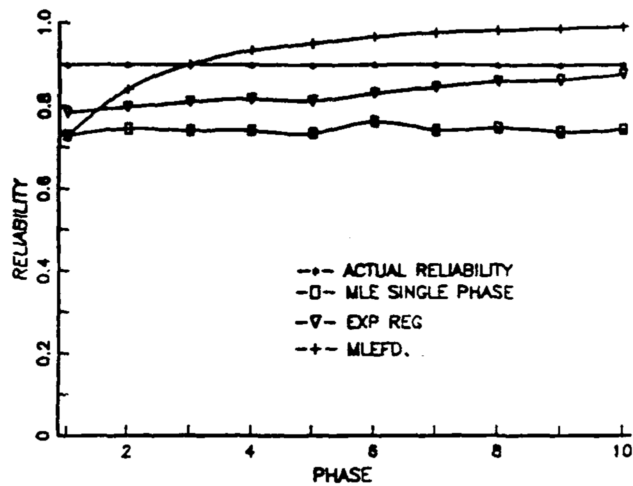
Pattern VI, No Discounting

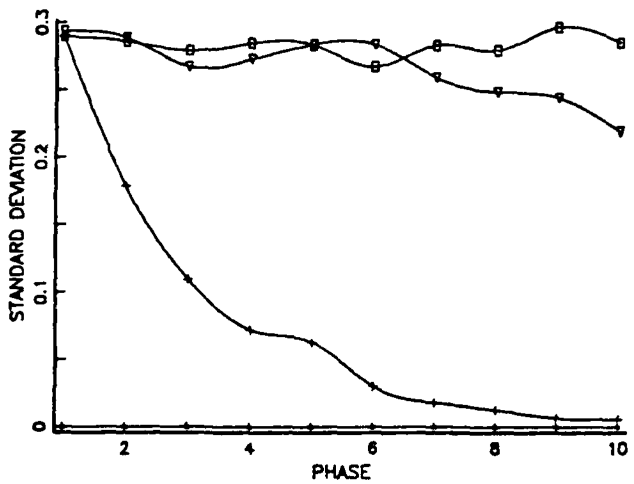


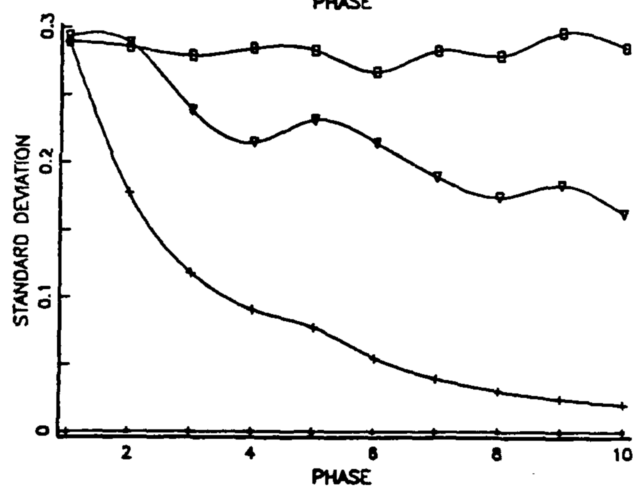
Pattern VI, F = .25, I = 1

Figure 13.





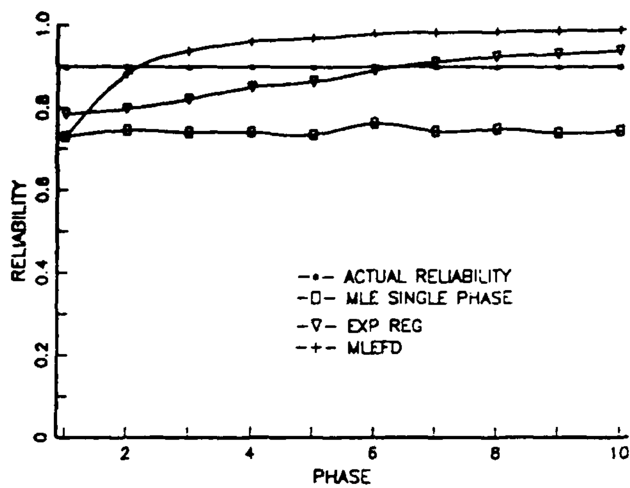


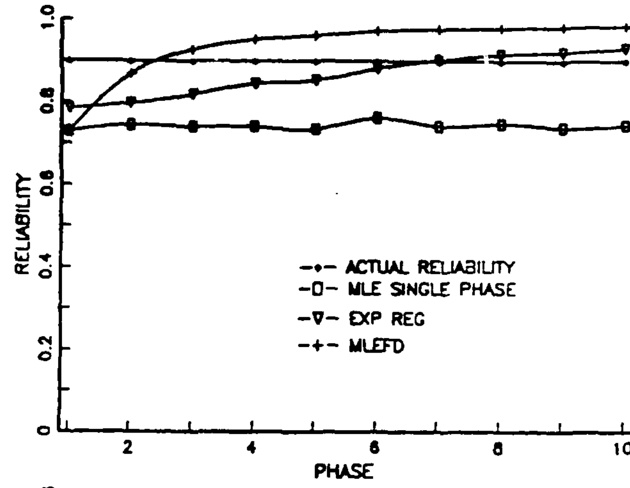


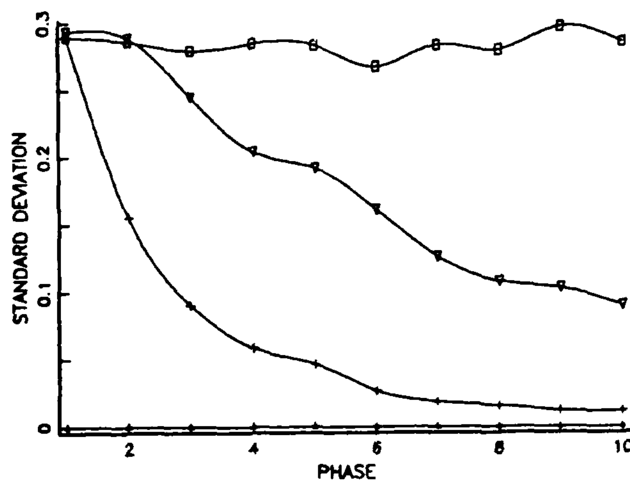
Pattern VI, F = .50, I = 3

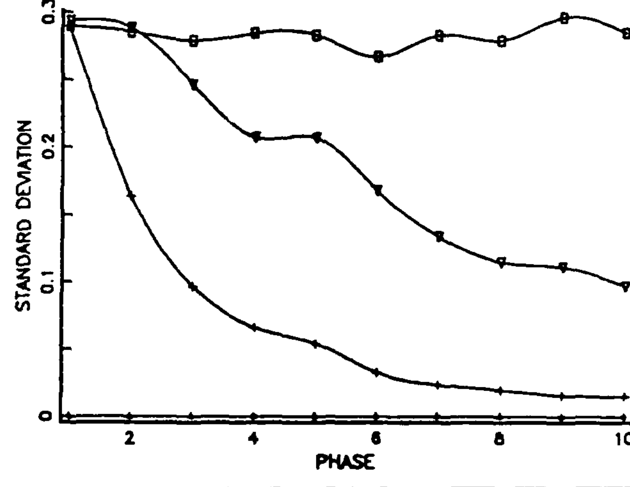
Pattern VI, F = .50, I = 6

Figure 14.





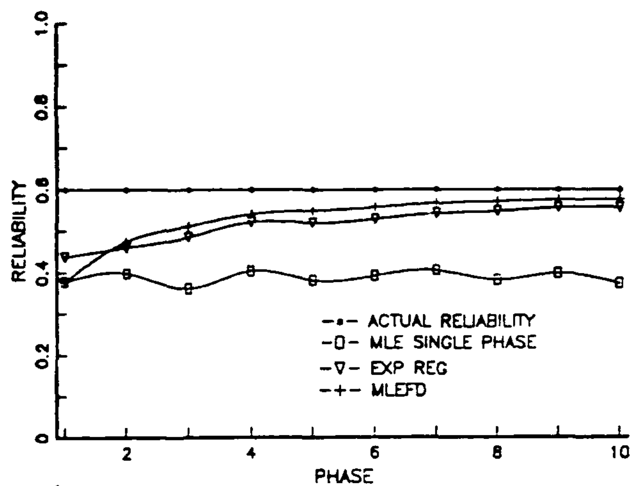


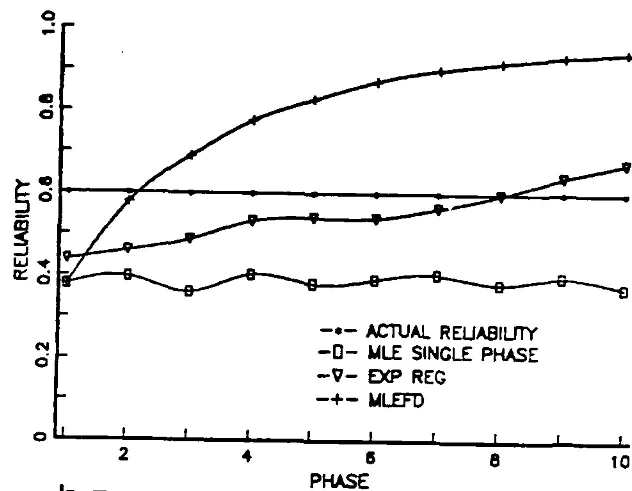


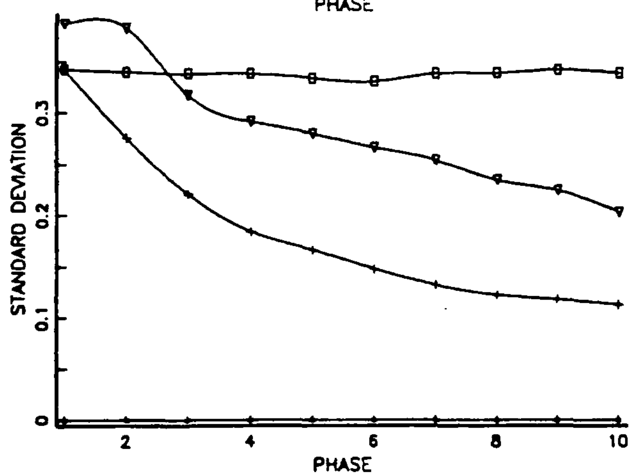
Pattern VI, Lloyd, CI = .8

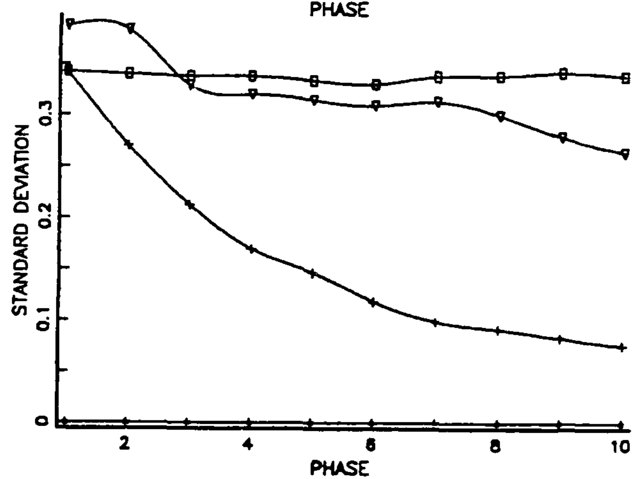
Pattern VI, Lloyd, CI = .9

Figure 15.





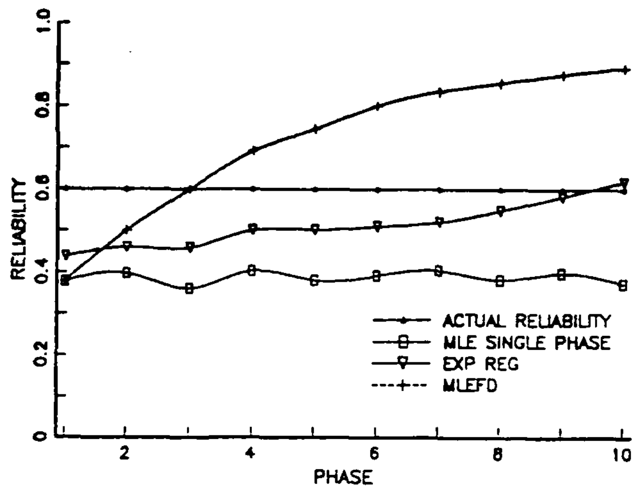


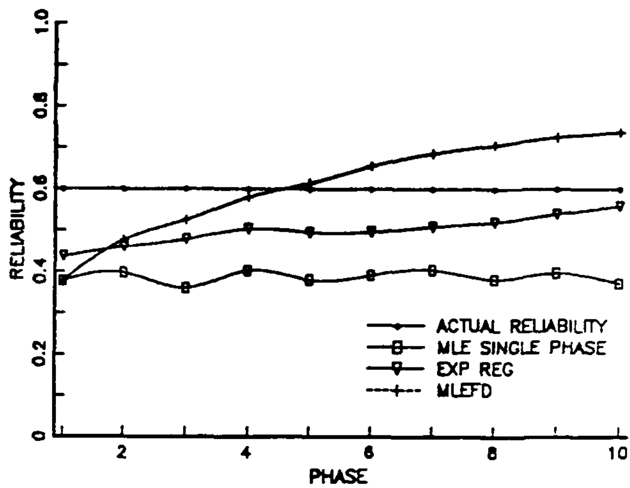


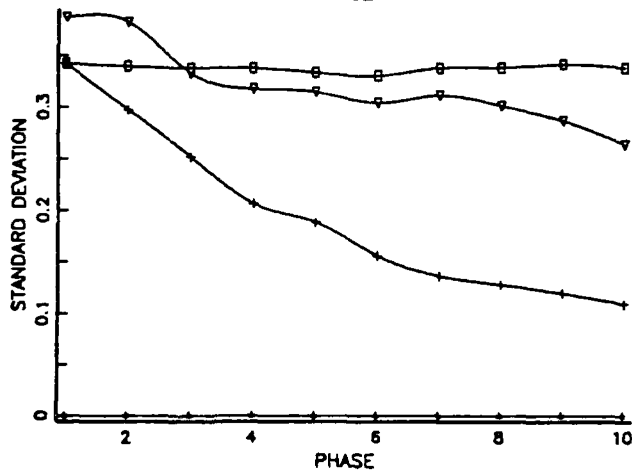
Pattern VII, No Discounting

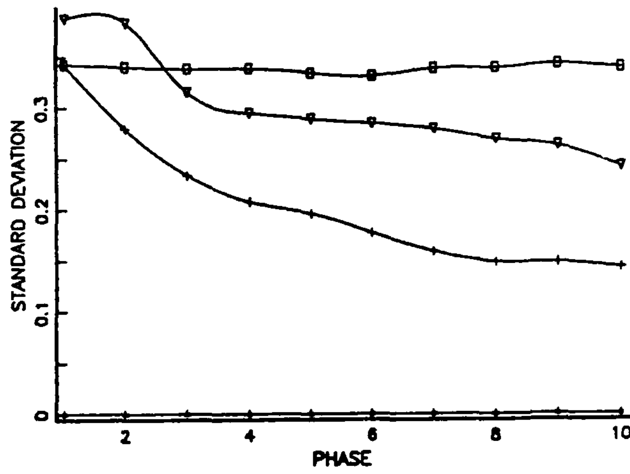
Pattern VII, F = .25, I = 1

Figure 16.





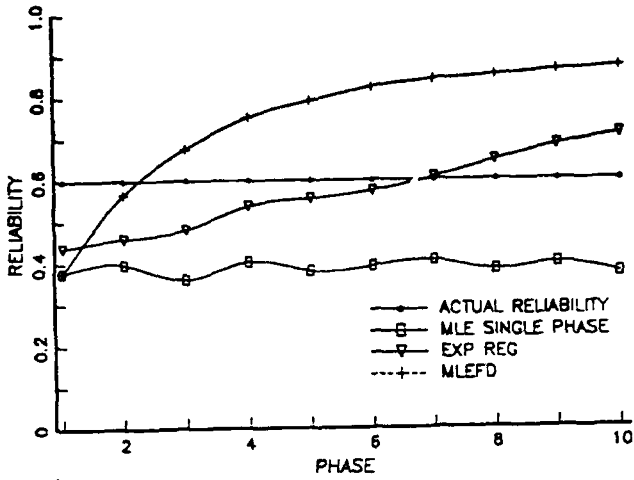


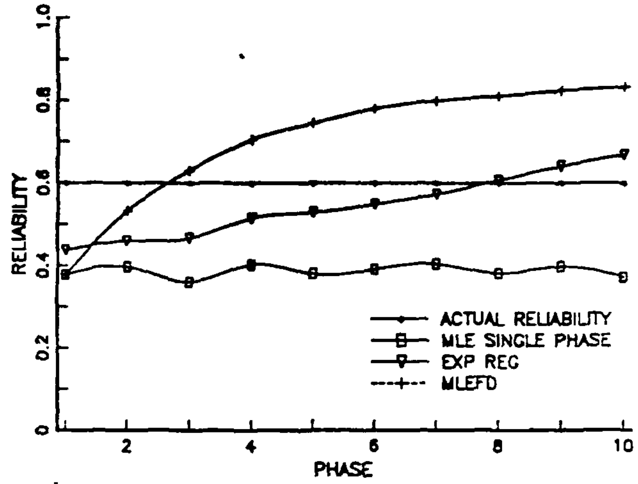


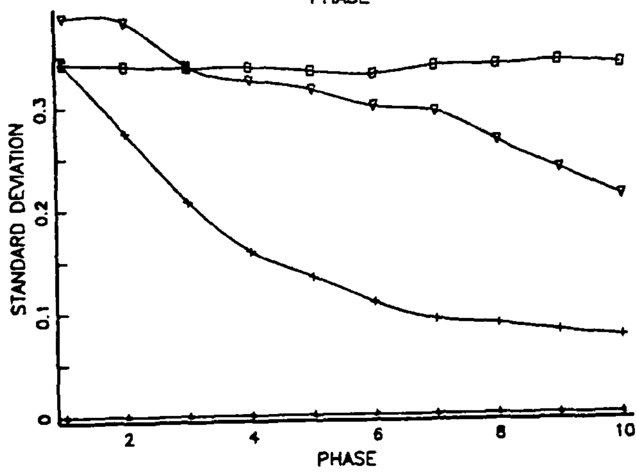
Pattern VII, F = .50, I = 3

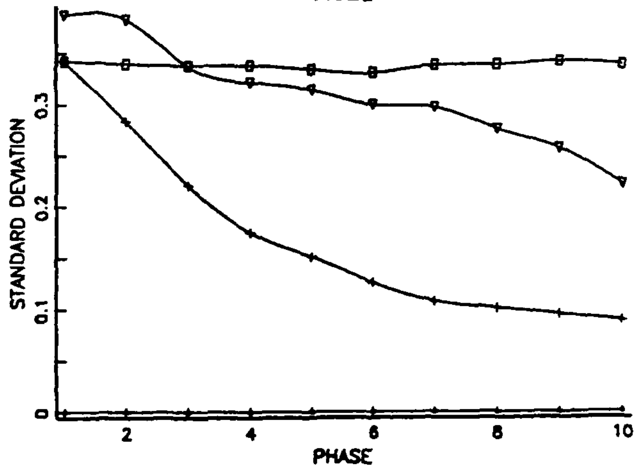
Pattern VII, F = .50, I = 6

Figure 17.









Pattern VII, Lloyd, CI = .8

Pattern VII, Lloyd, CI = .9

Figure 18.

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