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→ This project was a continuation of an ongoing study of numerical/analytic techniques for the identification of periodic solutions to functional differential equations. The techniques developed apply to very general classes of equations, and have been implemented on a variety of specific model problems.

The areas investigated involve techniques and information not attainable by standard simulation methods. The work completed can roughly be subdivided according to the local (Hopf bifurcation) analysis in the neighborhood of equilibria, and global tracking methods for following 1-parameter families of periodic orbits and examining their secondary bifurcation structure.

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FUNCTIONAL DIFFERENTIAL EQUATIONS

AFOSR-87-0268

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SUMMARY

This project was a continuation of an ongoing study of numerical/analytic techniques for the identification of periodic solutions to functional differential equations. The techniques developed apply to very general classes of equations, and have been implemented on a variety of specific model problems.

"Local" techniques refer to methods that apply to the problem of analyzing the Hopf bifurcation structure of small periodic orbits of multiparameter systems. A FORTRAN code, BIFDE, was modified to allow analysis of certain nongeneric bifurcations in general systems with infinite delay. Related questions have been investigated.

"Global" tracking methods have been developed to study the growth and parameter dependence of global Hopf bifurcations. Investigations have centered on the development of spline-based approximation techniques and their implementation in a FORTRAN code FDETRAK. The capabilities of the code have been expanded, and applied to scalar and two-dimensional delay difference systems.

RESEARCH OBJECTIVES

This project, along with the grant AFOSR-86-0071 with the same name (budgeted through the Virginia Polytechnic Institute) initiated the a thorough study of the use of numerical techniques in the analysis of general parameter-dependent functional differential systems. The research of this grant was simultaneously supported by the National Science Foundation grant DMS-8701456, and a grant from the DARPA Program in Applied and Computational Mathematics (subcontracted through Michigan State University.)

The areas investigated involve techniques and information not attainable by standard simulation methods. The work completed can roughly be subdivided according to the local (Hopf bifurcation) analysis in the neighborhood of equilibria, and global tracking methods for following 1-parameter families of periodic orbits and examining their secondary bifurcation structure.

At this early stage of the research, emphasis was placed on the establishment of the numerical characteristics of some of the proposed methods. An algorithm of the PI for the analysis of the bifurcation structure of Hopf bifurcations was chosen to be implemented in a general purpose code BIFDE. A graduate student, Archana Sathaye was supported by AFOSR-86-0071 to assist in the implementation. Her work was continued by a graduate student, Nihal Aboud, under the support of this grant. Alternate numerical implementations (symbolic manipulation) and techniques for obtaining the necessary bifurcation data were addressed, as well.

Complimentary to an understanding of the local (Hopf bifurcation) structure of multi-parameter FDE is the use of global tracking methods to gain insight into the secondary bifurcation structure of these problems. A goal of this study has been to design a general purpose FORTRAN-based code for the analysis of scalar delay-difference equations, as well as certain multi-dimensional generalizations. The code has been tested on a wide variety of scalar equations and a system modeling "chugging" in liquid-propellant fuel rockets.

RESEARCH STATUS

Concerning the local problem, various extensions were made on an Liapunov-Schmidt based algorithm of the PI. The algorithm, which leads to a direct means of determining the stability and direction of bifurcation in functional differential equations requires substantial computation. A FORTRAN-based implementation of the algorithm called BIFDE was designed to provide a numerical method for resolving generic and 1st order nongeneric bifurcations in functional differential equations (not necessarily of delay -difference, or even finite delay type). The code has been checked on specific equations from the literature, and has been shown to be useful in analyzing a relatively unstudied model of chugging in liquid-propellant fuel rockets. See [1], [2], and [3].

A second numerical implementation investigates the feasibility of executing the algorithm with symbolic-manipulation software. The use of MACSYMA for such purposes has been undertaken, and results

indicate that this algorithm succeeds where symbolic implementation of alternate approaches (eg, center manifold approximation followed by the use of Poincare transforms) have been unsuccessful. The investigations involve both the algebraic calculation of the bifurcation function for specific classes of delay-difference equations, and the derivation of general formulae (applicable to general FDE) for the resolution of the bifurcation structure at points of second order nongenericity. See [4].

The success of the above algorithm points out the need for effective means of computing the associated bifurcation data. An important step in that direction is the determination of the global convergence properties of various rootfinding techniques when applied to quasipolynomials of the type that arise as characteristic equations for linearized problems. Such a study was initiated with a grant of CPU time on the Minnesota Supercomputer Center Cray - 2. While fractal convergence basin boundaries were anticipated, their precise form was found to share certain geometric characteristics independent of the rootfinding method in use. Comparison of Newton's method to various third order methods (eg., Halley's) indicate that the global convergence properties can vary greatly from method to method. Description of the results and a vector algorithm for their generation are given in [5] and [6].

The work involving global tracking techniques has centered on the design of a general purpose FORTRAN - based code for following periodic solutions in one - parameter families of FDE, identifying secondary bifurcations, and following certain branches. A general purpose code for the analysis of scalar delay - difference equations was completed and applied to various equations from the literature. See [7]. A redesign and extension of the code to certain systems has been completed as well [3]. A first step in designing a parallel processing version of the code was taken in [8].

In addition to the FDE taken from the literature to test the effectiveness of these methods, work has been initiated in the analysis of a class of new problems from modern electrodynamics. In particular, the general equations of motion of charged particles (including relativistic and radiation effects) lead -- depending on the simplifying assumptions made -- to either retarded FDE, FDE of neutral type, or retarded FDE with state-dependent time delays. The model's derivation and the results of certain simulation studies are given in [9] and [10].

PROJECT PUBLICATIONS:

- * [1] BIFDE: Software for the Investigation of the Hopf Bifurcation Problem in Functional Differential Equations (with N. Aboud and A. Sathaye), Proceedings for the 27th IEEE Conference on Decision and Control, to appear.
- [2] "Contributions to the Computer-Aided Analysis of Functional Differential Equations," Master's Thesis by N. Aboud, University of Minnesota, August, 1988.
- [3] Periodicity in a Model of Liquid-Propellant Fuel Rockets, in preparation.
- [4] "Symbolic Hopf Bifurcation Calculations for Functional Differential Equations," Master's Thesis by J. Franke, University of Minnesota, in progress.

[5] Graphical Analysis of Newton's Method, University of Minnesota-Duluth Mathematics Technical Report # 88-4, J. Reichenborn, 1988.

[6] Fractal Convergence Basins for Rootfinding Techniques Applied to Transcendental Equations, University of Minnesota-Duluth Mathematics Technical Report, E. Swieringa, in progress.

* [7] Computer-Aided Analysis of Periodic Solutions of Functional Differential Equations, Differential Equations and Applications, to appear.

[8] Parallel Scientific Programming on the Encore Multimax, University of Minnesota - Duluth Mathematics Technical Report, D. Kingsley, in progress.

[9] Time Delays in Models from Electrodynamics, (with N. Jähren and T. Jordan), in progress.

[10] FDESLV: An ODE-based Software Package for the Simulation of Functional Differential Equations, University of Minnesota - Duluth Mathematics Technical Report #88-3, K. Hill, 1988.

* 6 preprints enclosed with final technical report.

RESEARCH PERSONNEL

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Jeffrey Franke, graduate student

Keith Hill, graduate student

David Kingsley, graduate student

Janet Reichenborn, undergraduate student

Erin Swieringa, undergraduate student

Neil Jahren, undergraduate student

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COUPLING ACTIVITIES

Conference Participation:

First International Conference on Industrial and Applied Mathematics, June, 1987, Paris, France.

Annual meeting of the Applied and Computational Mathematics Program (DARPA), October, 1987, Washington, DC.

Special Session on Functional Differential Equations, January, 1988, Atlanta, Georgia. (Invited Lecture: "Computer-Aided Analysis of Periodic Solutions of Functional Differential Equations").

Conference on the Differential Equations and Applications, March, 1988, Columbus, Ohio. (Invited Lecture: "Computer-Aided Analysis of Periodic Solutions of Functional Differential Equations").

Annual Meeting of SIAM, July, 1988, Minneapolis, Minnesota.

COMPUTER-AIDED ANALYSIS OF PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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I. INTRODUCTION

Computational techniques have played an important role in the advancement of our understanding of the qualitative nature of solutions to functional differential equations. Simulation studies often provide compelling evidence for the asymptotic behavior of models too complex for direct analysis. Stable equilibria, periodic or quasi-periodic solutions, or chaotic dynamics are often more easily "observed" through simulation than rigorously proven to exist.

As important as simulation studies are, they are often restricted in their effectiveness by the fact that they (in all but elementary situations) only identify the numerical analogue of "asymptotically stable" solutions of the associated differential equations. It is clear that unstable solutions play an integral role in understanding the underlying structure of chaotic dynamics. While computational methods now exist for the analysis of unstable Hopf bifurcations in autonomous functional differential equations (see [1], [3], [5]), the information so obtained is of limited value in understanding large periodic orbits.

In the case of autonomous ordinary differential equations, the code AUTO of E. Doedel [2] assists in the computation of large periodic orbits (regardless of their stability type) in parameter - dependent equations. The code allows (among many things) the tracking of parameter - dependent equilibria, location of bifurcation points, tracking parameter - dependent periodic solutions arising through Hopf bifurcations, calculation of Floquet multipliers and secondary bifurcation points, and the tracking certain secondary bifurcations. Clearly, it would be desirable to have at one's disposal a general purpose code that can accomplish these same tasks

AMS(MOS) Subject Classification: 34C99, 34K99.

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for systems of autonomous functional differential equations. The purpose of this paper is to report on recent progress made in that direction.

We discuss a modest "first step" in the development of a general purpose code for the numerical computation of the structure of periodic orbits in parameter - dependent functional differential equations. In particular, a FORTRAN code has been recently developed for the general scalar delay difference equation

$$x'(t) = f(a; x(t), x(t-1)), \quad (1.1)$$

where a is a real (bifurcation) parameter.

II. NUMERICAL METHODS

The model problem (1.1) has been chosen as one of clear importance in the mathematical literature, as well as one through which one can judge the feasibility of developing an analogous code for general systems of functional differential equations (not necessarily of delay - difference form). While ~~the~~ (1.1) is of special structure, the numerical algorithms employed have made little reliance on that fact, so as to provide a more valid prediction of the computational complexity to be expected for the general functional differential equation

$$x'(t) = f(a; x_t), \quad (2.1)$$

where for fixed a , $f : C([-1, 0]; R^n) \rightarrow R^n$, the usual space of continuous functions.

The development of a general purpose code for the tracking of periodic solutions can be roughly divided into 3 numerical issues.

1. Numerical Approximation of Periodic Solutions

It should be stressed here that both unstable and stable periodic solutions are of equal importance. Thus, simulation techniques, which would lend itself to finding stable orbits, are not used. As in the case of AUTO, collocation methods have been chosen.

In particular, one first scales time in (1.1) so that a periodic solution of (1.1) of period $T(= 1/w)$ is transformed into a periodic solution of period 1 in

$$wx'(t) - f(a; x(t), x(t-w)) = 0 \quad (2.2)$$

A partition of the interval $[0, 1]$ is then made into N subintervals of equal length. The 1-periodic solution $x(t)$ of (2.2) is then approximated in terms of periodic B-splines of order k ($3 \leq k \leq 9$). Using the superscript N to denote this approximation, we have

$$x^{(N)}(t) = \sum_{i=1}^N c_i B_i(t), \quad (2.3)$$

where the coefficients c_i , $i = 1, 2, \dots, N$ are to be determined. One now imposes N collocation constraints on $x^{(N)}$ by requiring (2.2) hold at N collocation nodes

(one taken from each of the N subintervals.) Additionally, one must impose a phase shift constraint in order to factor out the indeterminacy due the fact that (2.1) is autonomous. (Thus a phase shift of any periodic solution generates another.) The resulting system of $N + 1$ equations in $N + 2$ unknowns $a, c_i, i = 1, 2, \dots, N$, and w is then expected to possess 1-parameter families of solutions. Thus, one is lead to the consideration of:

2. One-Parameter Curve Tracking Techniques

Methods for computing one-parameter families of solutions to nonlinear systems are now well known. It is most convenient to visualize the family locally parameterized in terms of arc length, s . Initial points on the solution curve can be obtained from Hopf bifurcation data near equilibria, or from spline approximations to stable periodic orbits as observed through simulation.

Given two nearby points on the periodic family, one "predicts" another by linear extrapolation to a point at some prescribed distance, ds , along the curve. One now imposes a final scalar constraint so as to insure that the next sought-after point on the solution curve is ds arclength units from the previous point of the curve. The resulting $N + 2$ equations in $N + 2$ unknowns are then solved by Newton's method (with numerically determined Jacobian).

The current code is designed to implement stepsize selection automatically. Input data states the maximum allowable stepsize, as well as the optimal and minimum choices. If Newton's method has not converged after a certain number of iterations (prescribed as input), the stepsize is reduced and continuation is attempted again. If convergence fails again, the process is repeated until either convergence is obtained or the stepsize is reduced to a value smaller than the selected minimum value of ds . The latter situation results in program termination.

3. Floquet Multiplier Approximation

Identification of the Floquet multipliers for the approximated periodic solution is important both for determining the solution's stability, and for identification of points of secondary bifurcation. These multipliers are the eigenvalues of the linearized Poincare map for the T -periodic orbit $x(t)$ of (1.1). It is convenient to consider the linearized equation

$$z'(t) = D_1 f(x(t), x(t-1))z(t) + D_2 f(x(t), x(t-1))z(t-1) \quad (2.4)$$

(suppressing the dependence on a) on the phase space $L^2[-1, 0] \times R$, rather than $C([-1, 0]; R)$. Without loss of generality, we may assume $T > 1$. Thus, the time T map for this equation is known to be compact. It is approximated by computing the action of the map on a finite dimensional approximation to the phase space. In particular, one numerically computes the solutions of the initial value problems for the linearized equation with initial conditions taken to be one of the $M + 1$ characteristic functions of the intervals $[-hi, -h(i-1)]$, $i = 1, \dots, M$, ($h = 1/M$), and $\{0\}$ for $i = 0$. Using the inner product structure of the phase space, solutions on $[T-1, T]$ are then projected onto the $M + 1$ dimensional subspace spanned by the above initial conditions.

As a result, one obtains a linear map on R^{M+1} , the eigenvalues of which are expected to correspond to Floquet multipliers for the periodic solution. Clearly, the accuracy of the approximation scheme relies both on the accuracy of the approximation to the periodic solution of (1.1), as well as the dimension of finite dimensional approximation to the phase space. It should be remarked that (2.1) being autonomous implies that 1. should be a Floquet multiplier, thus providing a convenient symptom of the overall accuracy of computed multipliers.

III. TWO EXAMPLES

Wright's Equation

$$x'(t) = -ax(t-1)[1. + x(t)], \quad a > 0 \quad (3.1)$$

satisfies the standard conditions for a Hopf bifurcation at $a = \pi/2$. It is well known that the bifurcation is generic, supercritical, and locally orbitally asymptotically stable (see [4] and the references therein.) The family of periodic orbits are easily calculated by the code described above. Figure 3.1 depicts the bifurcation diagram obtained from 75 curve tracking steps performed on the collocation approximation (5th order splines) with $N = 28$. The vertical axis is, roughly speaking, the L^2 norm of the periodic orbit (normalized to period 1.)

Floquet multiplier approximation is reserved (in this case) to every 10th tracking step. Because the multipliers are known to cluster at the origin of the complex plane, we tabulate only those multipliers with modulus larger than one half. The dimension of the phase space approximation is $M = 41$.

step	multipliers larger than .5	
10	0.9988	0.9104
20	0.9991	0.6521
30	0.9992	
40	0.9995	
50	1.0007	
60	1.0085	
70	1.0346	

Observe that (except for the numerical equivalent of the unit multiplier) all multipliers lie within the unit circle, thus confirming the stability type of the periodic orbit. Note also that accuracy of the unit multiplier degenerates as the periodic orbit grows in size. This can be explained by the fact that the periodic orbits to Wright's equation possess regions of fast variation of x . Accuracy of the approximation scheme can be maintained if one increases the collocation parameter N as one tracks along the curve.

The above computation was performed on the four - processor Cray-2 of the University of Minnesota Supercomputer Center, and required 47.75 seconds CPU time. Based on benchmark comparisons with a VAX 11/750, one would expect the computation to take approximately 25 minutes on this more common machine.

As a final example, we consider the delayed negative feedback equation

$$x'(t) = -2ax(t) - 2bx(t-1)/[1. + x^{10}(t-1)], \quad (3.3)$$

where a and b are positive parameters. If one fixes $a = 1.00$, then the spectral conditions for a Hopf bifurcation from $x = 0$. are satisfied at $b = 1.5219$. Due to the choice of nonlinearity the bifurcation is not generic. However, the stability algorithm of [5] can be applied to show the periodic orbits to exist supercritically, and to be locally orbitally asymptotically stable. *delete*
C.

Figure 3.2 depicts the computed bifurcation diagram. Fifth order splines are used, with N and M varied so as to maintain acceptable accuracy of the unit multiplier. The period along the stable leg of the primary Hopf bifurcation is approximately 2.743. The first secondary bifurcation (at $b = 1.953$) corresponds to a simple branch, with a Floquet multiplier leaving the unit disk at 1. on the real axis. The stability of the periodic orbit is observed to switch to this secondary family. This secondary family itself loses stability to a stable period doubling orbit of period 5.2961 at $b = 2.385$. These periodic orbits lose stability at another period doubling (near $b = 2.453$), and the tracking is terminated when the resulting period 10.569 orbits undergo yet another period doubling bifurcation near $b = 2.479$ to an orbit of period 21.113. See Figure 3.3 for a simultaneous plot of existing periodic orbits near 2.470. Dashed curves denote unstable orbits.

IV. SUMMARY

The examples of the previous section indicate the types of information that can be now routinely obtained through the described code. However, the benchmark data for Wright's equation make it clear that additional work must be done if a code of this type is to be effectively implemented on standard mainframe computers.

Ongoing research concerns the effects of variable mesh collocation methods and alternate Floquet multiplier approximation methods on improving the performance of the above code. Tayloring a code for delay difference equations is expected to improve code performance, as well.

REFERENCES

- [1] S.-N. Chow and J. Mallet-Paret, Integral averaging and Hopf bifurcation, *J. of Differential Equations*, Vol. 26 (1977), 112-159.
- [2] E.J. Doedel, AUTO: A program for the automatic bifurcation analysis of autonomous systems, *Cong. NUM*, Vol. 30 (1980), 265-284.
- [3] B.D. Hassard, N.D. Kazarinoff, and Y.-H. Wan, Theory and Applications of Hopf Bifurcation, *London Mathematical Society Lecture Notes*, No. 41, Cambridge Univ. Press, Cambridge, 1981.
- [4] H.W. Stech, Hopf bifurcation calculations for functional differential equations, *J. of Math. Anal. and Appl.*, Vol. 109, No. 2 (1985), 472-491.
- [5] H.W. Stech, Nongeneric Hopf bifurcations in functional differential equations, *SIAM J. of Math. Anal.*, Vol 16., No. 6 (1985), 1134-1151.

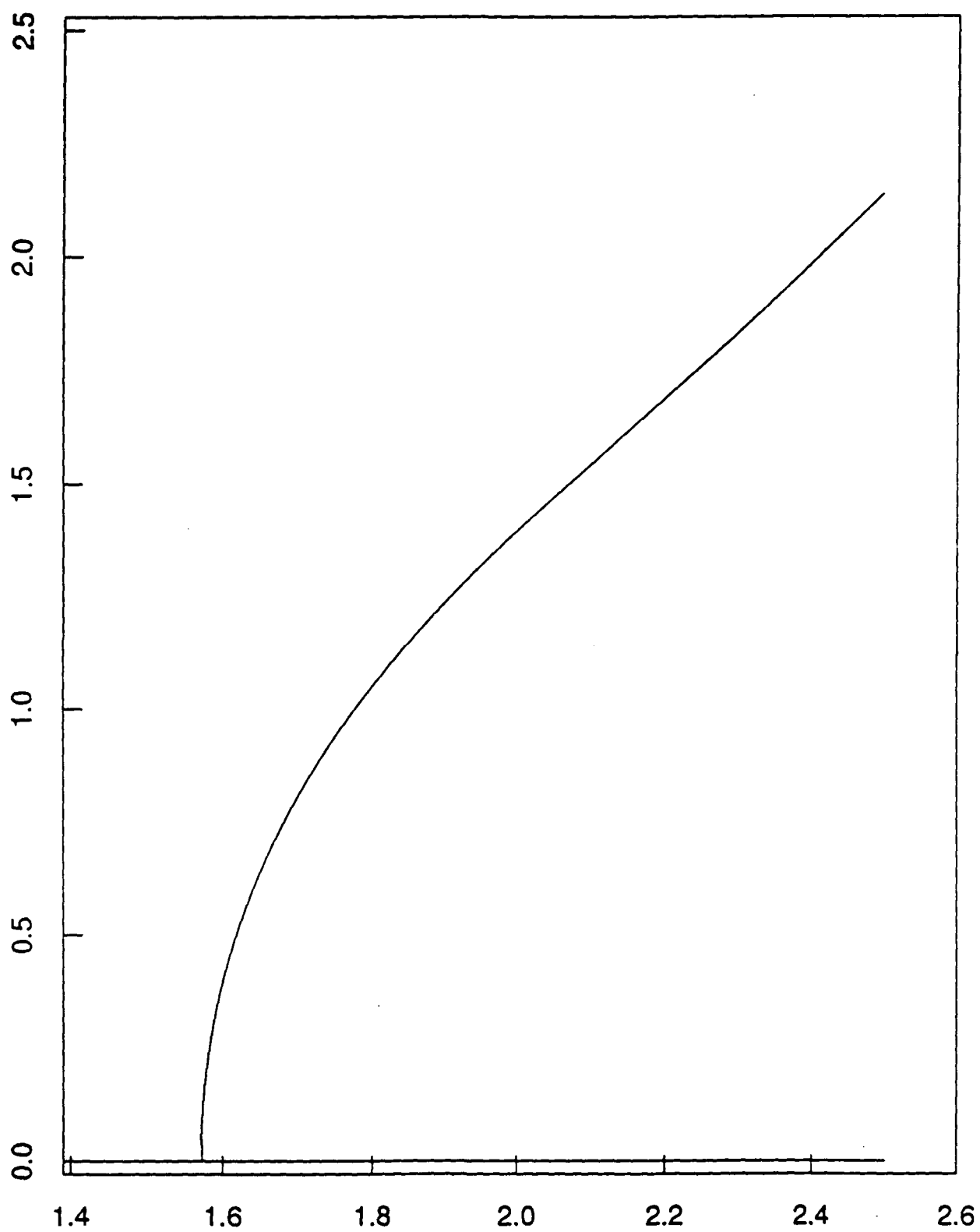


figure 3.2

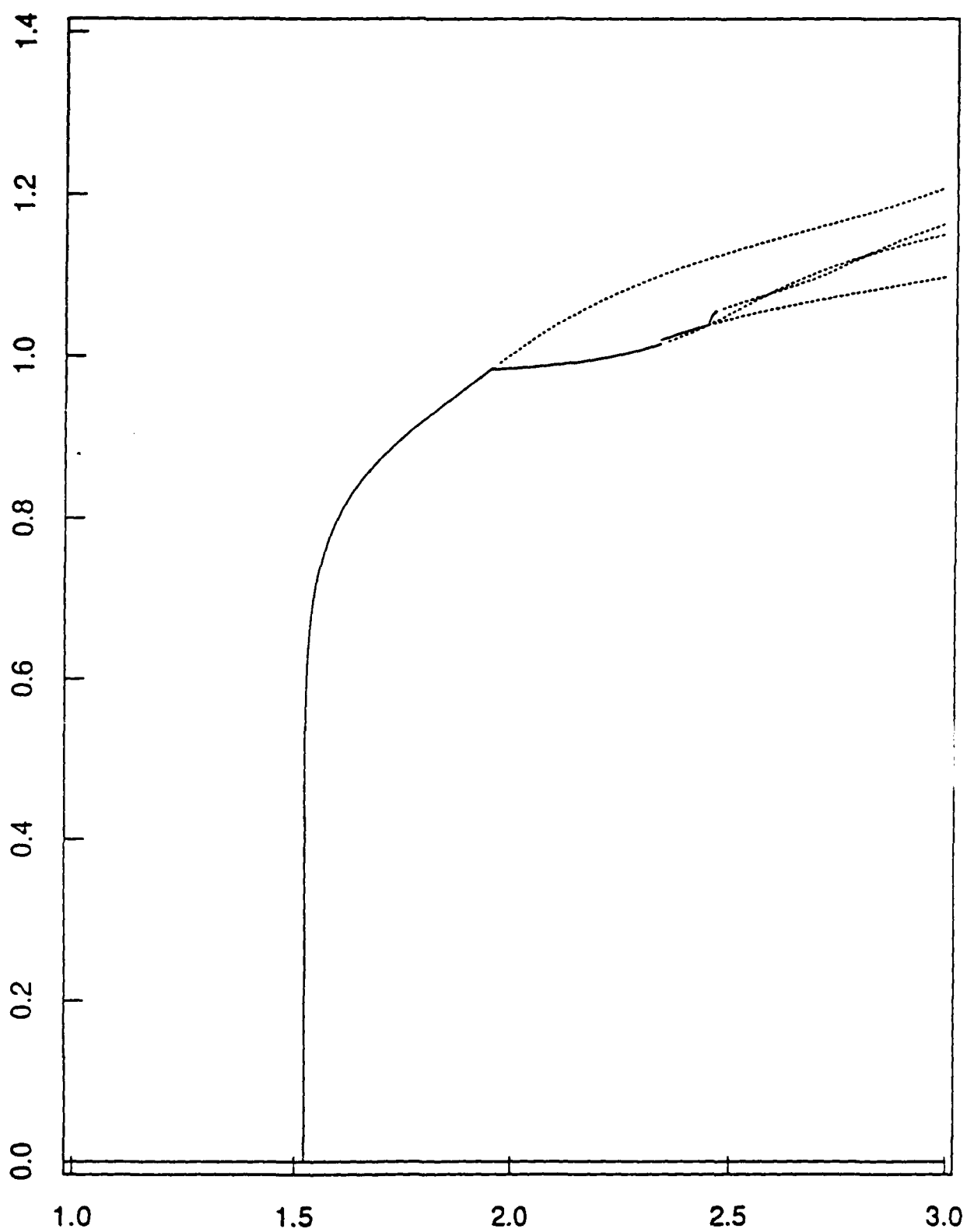


figure 3.4 ✓

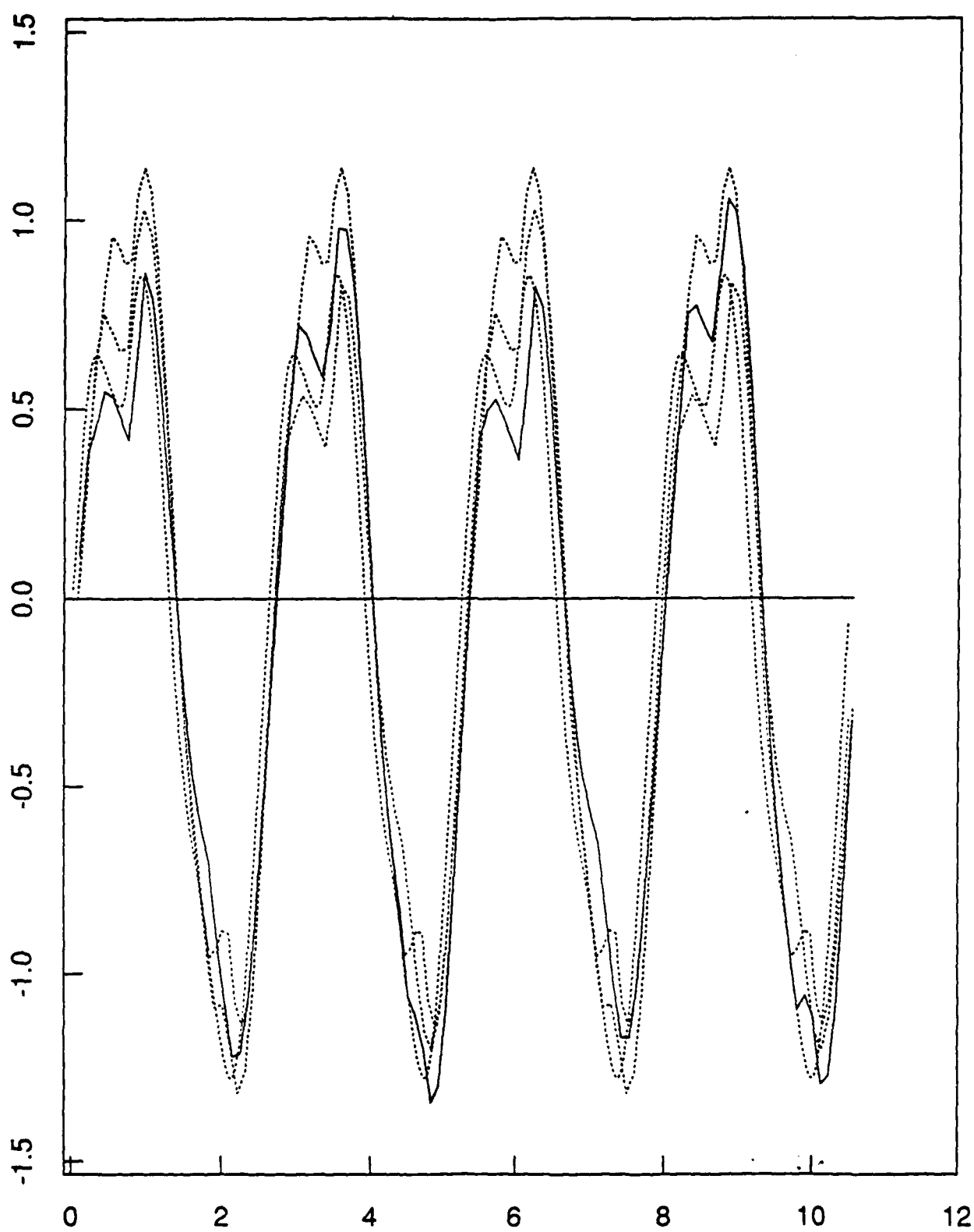


figure 3.5