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WITH INCOMPLETE INFORMATION

by

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# Finite Approximations to a Zero-sum Game with Incomplete Information

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Abstract: In this paper, we investigate a scheme for approximating a two-person zero-sum game  $G$  of incomplete information by means of a natural system  $G_{mn}$  of its finite subgames. The main question is: For large  $m$  and  $n$ , is an optimal strategy for  $G_{mn}$  necessarily an  $\epsilon$ -optimal strategy for  $G$ ? *Keywords: theorems, computation, ...*

*epsilon*

Section 1. Introduction.

To formalize our idea of approximating a two-person zero-sum game of incomplete information by its subgames, we introduce what we shall call a game structure. A game structure is a system of the form  $(\Omega, U, \mathcal{F}_m, \mathcal{G}_n)_{m,n=1}^{\infty}$ . Here  $\Omega = (\Omega, \mathcal{B}, P)$  is a probability space,  $U = (U_{ij} : i = 1, \dots, M; j = 1, \dots, N)$  is a matrix of random variables on  $\Omega$  (the payoff matrix), and  $\mathcal{F}_m$  and  $\mathcal{G}_n$  are sub- $\sigma$ -fields of the  $\sigma$ -field  $\mathcal{B}$  such that  $\mathcal{F}_{m+1} \supseteq \mathcal{F}_m$  and  $\mathcal{G}_{n+1} \supseteq \mathcal{G}_n$ . We put  $\mathcal{F} = \mathcal{F}_{\infty}$  = the  $\sigma$ -field generated by  $\bigcup_m \mathcal{F}_m$ .  $\mathcal{G} = \mathcal{G}_{\infty}$  = the  $\sigma$ -field generated by  $\bigcup_n \mathcal{G}_n$ .

For  $m, n = 1, 2, \dots, \infty$ , let  $G_{mn}$  be the two-person, zero-sum game in which a strategy for player I is an  $\mathcal{F}_m$ -measurable  $\alpha: \Omega \rightarrow S^M$ , and a strategy for player II is a  $\mathcal{G}_n$ -measurable  $\beta: \Omega \rightarrow S^N$ . (Here  $S^M$  is the simplex  $\{x \in R^M : \sum_i x_i = 1, x_i \geq 0\}$ .) If player I plays  $\alpha$  and player II plays  $\beta$ , then the payoff to I is  $\Gamma(\alpha, \beta) = E(\sum_{ij} U_{ij} \alpha_i \beta_j)$ .

Thus in the game  $G_{mn}$ ,  $\mathcal{F}_m$  and  $\mathcal{G}_n$  embody the information available to I and II, respectively. If  $\mathcal{F}_m$  and  $\mathcal{G}_n$  are finite, then  $G_{mn}$  is a finite approximation to the game  $G = G_{\infty}$ .

By standard minimax theorems [Sion], each game  $G_{mn}$  (and a fortiori  $G$ ) has a saddle point. Let  $V_{mn}$  denote the value of the game  $G_{mn}$  to player I, and let  $V = V_{\infty}$ .

If  $\mathcal{F}_m$  and  $\mathcal{G}_n$  are finite, then the game  $G_{mn}$  is, at least in principle, solvable by finite methods. The question we shall study is: To what extent is an optimal strategy for  $G_{mn}$  a useful substitute for an optimal strategy for  $G$ ? An ideal result along these lines would be

- (1) Fix  $\epsilon > 0$ . Suppose that, for  $m, n = 1, 2, \dots$ ,  $\alpha^{mn}$  is an optimal strategy for I in  $G_{mn}$ . Then, for all sufficiently large  $m$  and  $n$ ,  $\alpha^{mn}$  is an  $\epsilon$ -optimal strategy for I in  $G$ .

As we shall see, (1) is, alas, in general false. The best we can do is a weaker version of (1) (Theorem 1), and a special case of (1) (Theorem 2). We shall state these

theorems presently. For a strategy  $\alpha$  for player I in the game  $G$ , let  $\text{Val}_n(\alpha) = \inf_{\beta} \Gamma(\alpha, \beta)$ , where  $\beta$  ranges over  $\bar{G}_n$ -measurable strategies for II. (Thus if  $\alpha$  is  $\bar{F}_m$ -measurable, then  $\text{Val}_n(\alpha)$  is the value to I of the strategy  $\alpha$  in the game  $G_{mn}$ .) We shall write  $\text{Val}_G(\alpha)$  for  $\text{Val}_\infty(\alpha)$ .

**Theorem 1.** For  $m, n = 1, 2, \dots$ , suppose that  $\alpha^{mn}$  is an optimal strategy for player I in  $G_{mn}$ , and that  $\bar{F}_m$  and  $\bar{G}_n$  are finite  $\sigma$ -fields. Then  $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \text{Val}_G(\alpha^{mn}) = V$ .

Moreover, this convergence is uniform in the choices  $\alpha^{mn}$  of optimal strategies, i.e.,

$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \inf_{\alpha \in A(m, n)} \text{Val}_G(\alpha) = V$ , where  $A(m, n)$  is the set of strategies optimal for player I in  $G_{mn}$ .

Theorem 2 says that, under an additional hypothesis, (1) does hold. This hypothesis, which we shall call (M), is a version of the "continuity of information" assumption first used in [Milgrom-Weber]. (M) says roughly that the joint probability on  $\bar{F}$  and  $\bar{G}$  is absolutely continuous with respect to the product probability on  $\bar{F} \times \bar{G}$ . A precise statement of (M) will be found in Sec. 2.

**Theorem 2.** Assume (M) holds. If, for  $m, n = 1, 2, \dots$ ,  $\alpha^{mn}$  is an optimal strategy for I in  $G_{mn}$ , then  $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \text{Val}_G(\alpha^{mn}) = V$ , uniformly in the choices  $\alpha^{mn}$  of optimal strategies.

In Sec. 2, we shall prove all of these various claims.

## Section 2. Results.

We first present an example which shows that assertion (1) of the introduction does not hold in general.

**Example :** A game structure in which (1) fails.

Let  $\Omega$  be the interval  $[0, 1)$  with Lebesgue measure,  $M=N=2$ , and the payoff

$$U_{ij} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases} \text{ for } i, j = 1, 2 \text{ (independent of } \omega). \text{ For } m = 1, 2, \dots,$$

let  $\bar{F}_m = \bar{G}_m =$  the  $\sigma$ -field on  $[0, 1)$  generated by the partition

$\{ [(k-1)/2^m, k/2^m) : k = 1, 2, \dots, 2^m \}$ . Thus  $\mathcal{F} = \mathcal{G} =$  the Borel  $\sigma$ -field on  $\Omega$ . It is easy to see that, for all  $m$  and  $n$ ,  $V_{mn} = 0$ , and in  $G_{mn}$  the players have the optimal strategies  $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1/2$ , for all  $\omega \in \Omega$ .

For finite  $m > 1$ , consider the game  $G_{m,m-1}$ . The strategy  $\alpha^{m,m-1}$  given by

$$\alpha_1^{m,m-1} = \begin{cases} 1 & \text{if } \omega \in [(n-1)/2, n/2], \text{ } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_2^{m,m-1} = 1 - \alpha_1^{m,m-1}$$

is easily seen to be optimal for player I in the game  $G_{m,m-1}$ , i.e.,  $\text{Val}_{m-1}(\alpha^{m,m-1}) = 0$ . On the other hand,  $\text{Val}_G(\alpha^{m,m-1}) = -1$ ;  $\alpha^{m,m-1}$  is a very poor strategy for player I in  $G$ . Thus in any system  $(\alpha^{mn} : m, n = 1, 2, \dots)$  of optimal strategies for player I in which  $\alpha^{m,m-1} = \alpha^{m,m-1}$  for all  $m > 1$ ,  $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \text{Val}_G(\alpha^{mn}) < V$ .

We shall next prove Theorems 1 and 2. We first require a series of lemmas. Our first lemma is a special case of Theorem 2 of [Blackwell- Dubins].

**Lemma 2.1** Let  $(X_k)$  be a uniformly bounded sequence of random variables, and suppose that  $X_k \rightarrow X_\infty$  a.s. as  $k \rightarrow \infty$ . Then  $E(X_k | \mathcal{G}_k) \rightarrow E(X_\infty | \mathcal{G})$  a.s. as  $k \rightarrow \infty$ .

Our second lemma computes  $\text{Val}_n(\alpha)$ .

**Lemma 2.2** Fix a strategy  $\alpha$  for I in  $G$ , and fix  $n \in \{1, 2, \dots, \infty\}$ . Define the random variable  $\xi$  by :  $\xi =$  the  $j \in \{1, 2, \dots, N\}$  which minimizes  $E(\sum_j U_{ij} \alpha_j | \mathcal{G}_n)$ .

In case of a tie, for definiteness, take the least such  $j$ . Then, for all strategies  $\beta$  for II in  $G_{\infty n}$ ,

- (i)  $E(\sum_j U_{ij} \alpha_j) \leq \Gamma(\alpha, \beta)$ , so
- (ii)  $\text{Val}_n(\alpha) = E(\min_j E(\sum_j U_{ij} \alpha_j | \mathcal{G}_n))$ .

**Proof :** (i) immediately implies (ii), so we prove (i). Let  $\beta$  be a strategy for II in  $G_{\infty n}$ , that is, a  $\mathcal{G}_n$ -measurable  $\beta : \Omega \rightarrow S^N$ . Then, since  $\sum_j \beta_j = 1$ , we have

$E(\sum_i U_{ij} \alpha_i | \tilde{G}_n) \leq \sum_j \beta_j E(\sum_i U_{ij} \alpha_i | \tilde{G}_n) = E(\sum_{ij} U_{ij} \alpha_i \beta_j | \tilde{G}_n)$  a.s. since  $\beta$  is  $\tilde{G}_n$ -measurable. Taking expected values on both sides yields (i).

**Lemma 2.3** Let  $(\alpha^k)$  be a sequence of strategies for I in G, and suppose that  $\alpha^k \rightarrow \alpha$  a.s. as  $k \rightarrow \infty$ . Then

- (i) for fixed  $n = 1, 2, \dots, \infty$ ,  $\text{Val}_n(\alpha^k) \rightarrow \text{Val}_n(\alpha)$  as  $k \rightarrow \infty$ , and
- (ii)  $\text{Val}_G(\alpha^k) \rightarrow \text{Val}_G(\alpha)$  as  $k \rightarrow \infty$ .

**Proof:** First note that applying (ii) in a system where  $\tilde{G}_n = \tilde{G}_{n+1} = \dots = \tilde{G}_\infty$  yields (i), so (i) is a special case of (ii). To prove (ii), let  $X_k = \sum_i U_{ij} \alpha_i^k$  in lemma 2.1; then we have  $E(\sum_i U_{ij} \alpha_i^k | \tilde{G}_k) \rightarrow E(\sum_i U_{ij} \alpha_i | \tilde{G})$  a.s. as  $k \rightarrow \infty$ , so by dominated convergence  $E(\min_j E(\sum_i U_{ij} \alpha_i^k | \tilde{G}_k)) \rightarrow E(\min_j E(\sum_i U_{ij} \alpha_i | \tilde{G}))$ . By lemma 2.2(ii), we are done.

- Lemma 2.4** (i) For  $m, n = 1, 2, \dots, \infty$ ,  $\lim_{m \rightarrow \infty} V_{mn} = V_{\infty n}$  and  $\lim_{n \rightarrow \infty} V_{mn} = V_{m\infty}$ .  
(ii)  $\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} V_{mn} = V$ .

**Proof:** Fix  $n$ , and let  $\alpha$  be an optimal strategy for I in  $G_{\infty n}$ . Now for  $m = 1, 2, \dots$ , put  $\alpha^m = E(\alpha | \tilde{F}_m)$ . Thus  $\alpha$  is a legal, though likely not optimal, strategy for I in  $G_{mn}$ . We have

$$\text{Val}_n(\alpha^m) \leq V_{mn} \leq V_{\infty n}.$$

By lemma 2.1,  $\alpha^m \rightarrow \alpha$  a.s. By lemma 2.3(i),  $\lim_{m \rightarrow \infty} \text{Val}_n(\alpha^m) = \text{Val}_n(\alpha) = V_{\infty n}$ . By the inequality directly above, we infer  $\lim_{m \rightarrow \infty} V_{mn} = V_{\infty n}$ . By symmetry, we also have  $\lim_{n \rightarrow \infty} V_{mn} = V_{m\infty}$  for all  $m$ . This proves (i). Finally, it is easy to see that

$V_{m\infty} \leq V_{mn} \leq V_{\infty n}$ . Claim (ii) now follows by letting  $m, n \rightarrow \infty$ .

We are now ready to prove Theorem 1.

**Proof of Theorem 1:** Suppose that  $\alpha^{mn}$  is an optimal strategy for player I in G, for  $m, n = 1, 2, \dots$ . We claim that, for  $m = 1, 2, \dots$ ,  $\lim_{n \rightarrow \infty} \text{Val}_G(\alpha^{mn}) = V_{m\infty}$ . To prove this, fix  $m$ . We shall show that every subsequence of the sequence  $\text{Val}_G(\alpha^{m \cdot})$  has in turn a subsequence which converges to  $V_{m\infty}$ . Indeed, since each  $\alpha^{mn}$  is  $\tilde{F}_m$ -measurable,  $\tilde{F}_m$  being a finite  $\sigma$ -field, by the Bolzano-Weierstrass theorem every subsequence of  $\alpha^{m \cdot}$  has

a (pointwise) convergent subsequence; thus we may assume that  $\alpha^{mn} \rightarrow \alpha^m$  as  $n \rightarrow \infty$ . By lemma 2.3(ii), then,  $\text{Val}_n(\alpha^{mn}) \rightarrow \text{Val}_G(\alpha^m)$  as  $n \rightarrow \infty$ . By hypothesis,  $\text{Val}_n(\alpha^{mn}) = V_{mn}$ , so in fact  $V_{mn} \rightarrow \text{Val}_G(\alpha^m)$  as  $n \rightarrow \infty$ . Thus by lemma 2.4(i),  $\text{Val}_G(\alpha^m) = V_{m\infty}$ . On the other hand, since  $\alpha^{mn} \rightarrow \alpha^m$  as  $n \rightarrow \infty$ , by 2.3(i) we also have  $\text{Val}_G(\alpha^{mn}) \rightarrow \text{Val}_G(\alpha^m) = V_{m\infty}$ . This proves our claim.

Now by another use of lemma 2.3(i),  $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \text{Val}_G(\alpha^{mn}) = V$ . To prove the "moreover" clause in Theorem 1, let  $(\epsilon_{mn})$  be a sequence of numbers which converges to 0 as  $m, n \rightarrow \infty$ . For all  $m, n$ , note that there exists  $\alpha^{mn} \in A(m, n)$  such that  $\text{Val}_G(\alpha^{mn}) - \epsilon_{mn} \leq \inf_{\alpha \in A(m, n)} \text{Val}_G(\alpha) \leq \text{Val}_G(\alpha^{mn})$ . The "moreover" clause follows at once.

We now consider Theorem 2. We must first discuss hypothesis (M).

$\mathbb{F} \times \mathbb{G}$  is the  $\sigma$ -field on  $\Omega \times \Omega$  generated by sets of the form  $S \times T$ , where  $S \in \mathbb{F}$  and  $T \in \mathbb{G}$ . Let  $Q$  and  $R$  be the probability measures on  $(\Omega \times \Omega, \mathbb{F} \times \mathbb{G})$  defined by

$$Q(A) = P(\{ \omega : (\omega, \omega) \in A \}) \quad \text{and}$$

$$R(A) = \iint_{\{ (\omega, \eta) \in A \}} 1 P(d\omega) P(d\eta) \quad \text{for } A \in \mathbb{F} \times \mathbb{G}.$$

We now state assumption (M).

(M)  $Q$  is absolutely continuous with respect to  $R$ , that is, for all  $A \in \mathbb{F} \times \mathbb{G}$ , if  $R(A)=0$ , then  $Q(A)=0$ .

Assumption (M) is a version of a hypothesis introduced in [Milgrom-Weber]. It is easy to see that (M) is satisfied either if  $\mathbb{F}$  and  $\mathbb{G}$  are independent (in which case  $Q=R$ ), or if either  $\mathbb{F}$  or  $\mathbb{G}$  is atomic.

**Lemma 2.5** Suppose (M) is satisfied. Then if  $(X_k)$  is a uniformly bounded sequence of  $\mathbb{F}$ -measurable random variables which converges weakly to  $X_\infty$ , and if  $Z$  is any bounded random variable, then

- i)  $E(X_k Z | \mathbb{G}) \rightarrow E(X_\infty Z | \mathbb{G})$  a.s. and
- ii)  $E(X_k Z | \mathbb{G}_k) \rightarrow E(X_\infty Z | \mathbb{G})$  a.s. as  $k \rightarrow \infty$ .

**Proof:** First note that, since  $E(E(X_k Z | \mathbb{G}) | \mathbb{G}_k) = E(X_k Z | \mathbb{G}_k)$ , by lemma 2.1, (i) implies (ii). Next, note that we may assume without loss of generality that  $Z$  is measurable in  $\mathbb{F} \vee \mathbb{G}$  (the  $\sigma$ -field generated by  $\mathbb{F} \cup \mathbb{G}$ ). This is because  $E(X_k Z | \mathbb{G}) =$

$E(X_k \cdot E(Z | \mathcal{F} \vee \mathcal{G}) | \mathcal{G})$ , so we may replace  $Z$  by  $E(Z | \mathcal{F} \vee \mathcal{G})$  if necessary. We shall therefore prove (i), assuming that  $Z$  is  $\mathcal{F} \vee \mathcal{G}$ -measurable.

Since  $Z$  is  $\mathcal{F} \vee \mathcal{G}$ -measurable, there exist a bounded  $\mathcal{F}$ -measurable random variable  $\hat{X}$ , a bounded  $\mathcal{G}$ -measurable random variable  $\hat{Y}$ , and a bounded, Borel measurable function  $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $Z = f(\hat{X}, \hat{Y})$ .

By (M) and the Radon-Nikodym theorem, there exists a bounded function  $g: \Omega \times \Omega \rightarrow \mathbb{R}$  such that, for all  $A \in \mathcal{F} \times \mathcal{G}$ ,  $P(\{\omega: (\omega, \omega) \in A\}) = \int_A g(\omega, \eta) P(d\omega) P(d\eta)$ . It follows by standard methods that, for any vector  $X$  of  $\mathcal{F}$ -measurable random variables, any vector  $Y$  of  $\mathcal{G}$ -measurable random variables, and any Borel-measurable  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ , we have

$$(*) \quad E(h(X, Y) | \mathcal{G})(\eta) = \int_{\Omega} h(X(\omega), Y(\eta)) g(\omega, \eta) P(d\omega) \quad \text{a.s. } [\eta].$$

Now by (\*) we have

$$E(X_k Z | \mathcal{G})(\eta) = \int_{\Omega} X_k(\omega) f(\hat{X}(\omega), \hat{Y}(\eta)) g(\omega, \eta) P(d\omega) \quad \text{a.s. } [\eta].$$

Since, by assumption,  $X_k \rightarrow X_{\infty}$  weakly, we have  $E(X_k Z | \mathcal{G})(\eta) \rightarrow E(X_{\infty} Z | \mathcal{G})(\eta)$  a.s., as desired.

In exact analogy to lemma 2.3, we have

**Lemma 2.6** Assume that (M) holds. If  $(\alpha^k)$  is a sequence of strategies for I in  $G$  which converges weakly to a strategy  $\alpha$ , then

- (i) for fixed  $n = 1, 2, \dots, \infty$ ,  $\text{Val}_n(\alpha^k) \rightarrow \text{Val}_n(\alpha)$ , and
- (ii)  $\text{Val}_k(\alpha^k) \rightarrow \text{Val}_G(\alpha)$  as  $k \rightarrow \infty$ .

**Proof of Theorem 2:** Suppose that  $\alpha^{mn}$  is an optimal strategy for player I in  $G_{mn}$  for all finite  $m$  and  $n$ . We shall prove that every sequence  $(m_k, n_k)$  of pairs of integers such that  $m_k \rightarrow \infty$  and  $n_k \rightarrow \infty$  has a subsequence  $(m'_k, n'_k)$  such that  $\text{Val}_G(\alpha^{m'_k, n'_k}) \rightarrow V$  as  $k \rightarrow \infty$ . To conserve notation, let us write  $\alpha^k$  for  $\alpha^{m'_k, n'_k}$  and  $V_k$  for  $\text{Val}_{m'_k, n'_k}$ . By weak compactness, we may choose the sequence  $(\alpha^k)$  to converge weakly to a strategy  $\alpha$  as  $k \rightarrow \infty$ . By lemma 2.6(ii),  $\text{Val}_{n'_k}(\alpha^k) \rightarrow \text{Val}_G(\alpha)$  as  $k \rightarrow \infty$ . By assumption,  $\text{Val}_{n'_k}(\alpha^k) = V_k$ , and by lemma 2.4(ii),  $V_k \rightarrow V$ ; thus  $\text{Val}_G(\alpha) = V$ .

On the other hand, since  $(\alpha^k)$  converges weakly to  $\alpha$ , by lemma 2.6(i),  $\text{Val}_G(\alpha^k) \rightarrow \text{Val}_G(\alpha) = V$  as  $k \rightarrow \infty$ . Uniformity follows just as in the proof of Theorem 1. This completes the proof of Theorem 2.

## References

Blackwell, D., and Dubbins, L., "Merging of Opinions with Increasing Information," *Annals of Mathematical Statistics* 33 (1962), pp. 882 - 886.

Milgrom, P., and Weber, R. "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research* 10 (1985), pp. 619 - 632.

Sion, M. "On General Minimax Theories," *Pacific Journal of Mathematics* 8 (1958), pp. 171 - 176.