

UNCLASSIFIED

2

SECURITY CLASSIFICATION OF THIS PAGE

AD-A204 841

DRY DOCUMENTATION PAGE

DTIC FILE COPY

1a. REPORT SECURITY CLASSIFICATION Unclassified		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE			
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR-89-0075	
6a. NAME OF PERFORMING ORGANIZATION University of Maryland Baltimore County	6b. OFFICE SYMBOL (If applicable) UMBC	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research	
6c. ADDRESS (City, State and ZIP Code) Department of Mathematics Catonsville, Maryland 21228		7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematics and Information Sciences BCI 410 Bolling AFB, D.C. 20332-6448	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR	8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR -85-0322	
8c. ADDRESS (City, State and ZIP Code) Directorate of Mathematics and Information Sciences Bolling AFB BCI 410 D. C. 20332 - 6448		10. SOURCE OF FUNDING NOS.	
11. TITLE (Include Security Classification) Analysis of the performance of Mixed Finite Element Methods (unclassified)		PROGRAM ELEMENT NO. 61102 F	PROJECT NO. 2304
12. PERSONAL AUTHOR(S) Manil Suri		TASK NO. A 3	WORK UNIT NO.
13a. TYPE OF REPORT Final Scientific	13b. TIME COVERED FROM 10-1-85 TO 9-30-88	14. DATE OF REPORT (Yr., Mo., Day) 1988, Nov 28	15. PAGE COUNT 14
16. SUPPLEMENTARY NOTATION			
17. COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD 12.01	GROUP	Finite Elements, p-version; h-p version; approximation theory; integral equations; reaction-diffusion problems.	
19. ABSTRACT (Continue on reverse if necessary and identify by block number) The main focus of this project has been the investigation of various questions related to p- and h - p versions of the finite element method, including mixed methods. These new versions differ from the classical h-version where the degree p of polynomials used is kept fixed (usually p + 1 or 2) and accuracy is achieved by, and p is increased for accuracy. The h-p version combines the two approaches.			
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>		21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL Dave Nelson, Lt. Col., USAF.		22b. TELEPHONE NUMBER (Include Area Code) (202) 767-5025	22c. OFFICE SYMBOL AFOSR UNCLASSIFIED

DTIC
FEB 17 1989
D CS

Analysis of the Performance of Mixed Finite Element Methods

Principal Investigator: Manil Suri
Grant Number: AFOSR-85-0322

Final Scientific Report**Abstract**

The main focus of this project has been the investigation of various questions related to the p - and $h-p$ versions of the finite element method, including mixed methods. These new versions differ from the classical h -version where the degree p of polynomials used is kept fixed (usually $p = 1$ or 2) and accuracy is achieved by decreasing the mesh-size h . In the p -version, a fixed mesh with constant h is used and p is increased for accuracy. The $h-p$ version combines the two approaches.

We have studied various theoretical and computational questions related to these methods. The specific areas we have covered are as follows:

1. Optimal approximation results for the p - and $h-p$ versions.
2. Approximation of boundary conditions by the p and $h-p$ versions.
3. The p and $h-p$ versions for mixed methods.
4. The p and $h-p$ versions for integral equations.
5. An implementation of the p and $h-p$ versions for MODULEF.
6. Numerical methods for a class of reaction-diffusion problems.

Some of our theoretical results have already been incorporated into the commercial $h-p$ version code PROBE that is being used by several industries in the U.S., including those involved in aeronautical engineering.

Final Scientific Report

1. Introduction

The main focus of this project has been the investigation of various questions related to the p - and h - p versions of the finite element method (FEM), including mixed finite element methods.

The p - and h - p versions are relatively new methods, the first theoretical papers for which appeared in 1981 [A4], [A3]. One of the main differences between these methods and the standard h -version (which is the classical form of the FEM) is the use of higher order elements. In the h -version, accuracy is achieved by decreasing the mesh size h while keeping the degree p of elements used fixed at a low level, $p = 1, 2$. In the p -version, the mesh is kept fixed (i.e. h is constant) and accuracy is achieved by increasing the degree p of elements. The h - p version uses simultaneous increase of the degrees and mesh refinement to obtain the desired accuracy.

Early theoretical results showed that for smooth solutions, the rates of convergence obtained by these new methods were not bounded by the degree of polynomials used (as in the h -version) and consequently could be significantly higher. Moreover, the rates of convergence obtained for solutions of linear elliptic problems (where r^α type singularities arise at the corners of the domain) were also established to be higher (roughly twice that obtained from the h -version). However, none of these results established the optimal rates of convergence that could be expected. (Our first goal was to establish such rates).

The p - and h - p versions formed the basis of the industrial program PROBE [A10], developed by Noetic Tech, St. Louis [with a first release in 1985 and a second one in 1986]. This program implements these versions for two-dimensional problems (plane elasticity and heat transfer). In a relatively short period of time, PROBE has become popular in several industries in the U.S., including those involved in aeronautical engineering. One reason for this is the fact that the p - and h - p versions are the most efficient known finite element methods for the analysis of linear elliptic problems over polygonal domains (see [A2] for details and description of industrial usage feedback). Some of our theoretical results, particularly those on the treatment of inhomogeneous

7000	
Codes	
Dist	and/or Special
A-1	

essential boundary conditions by the p -version have found direct application in the implementation of PROBE.

The p -version has also been used in other programs like FIESTA, developed at ISMES, Bergamo, Italy. A three-dimensional p -version code called STRIPE [A1] has been developed at the FFA (Aeronautical Research Institute of Sweden) and can currently be implemented on CRAY computers. Moreover, a p -version implementation for MODULEF [B16] (INRIA, France) has been developed under this project and described in the next section.

2. Results achieved

In this section, we describe the results achieved under the grant over the past three years. A list of categories under which results have been obtained may be found on the cover page.

2.1. Optimal Approximation Results for the p - and $h - p$ versions

One of the basic pieces of desirable information for any finite element method is the asymptotic rate of convergence that may be expected and whether or not this rate is optimal. For the standard conforming finite element method based on the p -version, previous results established in [A4] by Babuska, Szabo and Katz and in [A7] by Dorr were non-optimal. More precisely, for a two-dimensional polygonal region Ω , the error e in approximating a function $u \in H^k(\Omega) \cap H_0^1(\Omega)$ using the space $S_p \subset H_0^1(\Omega)$ of conforming piecewise polynomial functions of degree $\leq p$ was shown to satisfy

$$\|e\|_{H^1(\Omega)} \leq C(\epsilon)p^{-(k-1)+\epsilon} \|u\|_{H^k(\Omega)} \quad (2.1)$$

for $\epsilon > 0$ arbitrarily small.

The proof of (2.1) indicated that the term $C(\epsilon)$ could grow quickly with $\epsilon \rightarrow 0$. Nevertheless, computational experience suggested that (2.1) holds without the term ϵ , i.e.

$$\|e\|_{H^1(\Omega)} \leq Cp^{-(k-1)} \|u\|_{H^k(\Omega)} \quad (2.2)$$

Therefore, the first problem considered by us was to solidify the foundations of the p - and $h - p$ versions by proving optimal estimates like (2.2) for the expected rates of convergence. These results are summarized below.

- (a) In [B2] we have proven optimal approximation results using the p -version with C^0 elements. This includes the proof of (2.2) for the case when the solution is smooth and a separate estimate for the case of a singular solution.

(b) In [B3] similar results are proven for the $h - p$ version using quasiuniform meshes. Numerical experiments are described which show the higher rates of convergence obtained with the $h - p$ (and p) version over the h -version.

(c) In [B11] the results from [B2] are extended to the case when elements possessing C^{m-1} continuity are used. These are necessary for problems in elasticity, like those involving plates and shells. Our results improve upon previous work by Katz and Wang (for $m = 2$) [A8] and by Dorr (for general m) [A7] in two ways. First, like (2.1), the results in [A8], [A7] are optimal up to an arbitrary $\epsilon > 0$, while ours are optimal. Second, in order to prove estimates for the case of elements satisfying a homogeneous essential boundary condition in $H_0^m(\Omega)$, the authors in [A8] and [A7] are forced to make an assumption involving the interpolation of Sobolev spaces ([A8] eqn. 2.37 and implicitly in [A7] Theorem 3.4). We are able to prove our estimate without using this assumption provided our solution lies in $H^k(\Omega)$, $k > 2m - \frac{1}{2}$. Since we have a separate estimate to treat the singular portion of the solution, this restriction is not a severe one in practice. (Using the interpolation assumption allows us to extend our results for the case $m < k \leq 2m - \frac{1}{2}$).

2.2. Approximation of Boundary Conditions by the h - and h - p versions

Previous results for the p -version in [A4] and [A7] and for the $h - p$ version in [A3] did not deal with the case of inhomogeneous essential boundary conditions. Consequently, the first version of PROBE assumed that essential boundary data was either zero or a piecewise polynomial of fixed low degree on the grid being used. As a result, for general inhomogeneous conditions, when higher degree polynomials were used inside, the accuracy of the approximation on the boundary did not increase correspondingly, leading to a loss in the rate of convergence in this case. What was needed was a method that would simultaneously increase the degree of approximation inside Ω and on the boundary. The results obtained by us to address this problem are summarized below.

(a) In [B2], we presented and analyzed a method to approximate the boundary data based on a projection in the H^1 norm. A similar method for the $h - p$ version was analyzed in [B3]. Our results had a quick practical payoff, since these methods were incorporated into the new version of PROBE.

(b) In [B8], we showed that the H^1 projection may not work as well for problems with rough Dirichlet data on the boundary. An alternate method, based on a projection in the $H^{\frac{1}{2}}$ norm was introduced and shown to be more robust. This

method generalizes easily to systems of equations and can be implemented using Fast Fourier Transforms.

(c) In [B13], we analyzed a family of methods for inhomogeneous Dirichlet conditions based on the use of Jacobi polynomials with varying indices. The H^1 and $H^{\frac{1}{2}}$ projections analyzed in [B2] and [B8] are special members of this family. We considered the comparison of various projections by looking at both their theoretical and computational properties. It was shown numerically that for certain problems, the use of an incorrect boundary data approximation method could lead to a complete lack of convergence for the p -method.

(d) In [B9], we considered the implementation of inhomogeneous conditions for elliptic systems of equations, particularly "boundary constraint problems". A mixed-type method using the p -version for this problem is described and analyzed in the above reference.

A detailed summary of the results in Section 2.1 and 2.2 has been presented by us in the expository paper [B4]. We have also presented some of these results in [B1] and [B7].

2.3. The p -version for Mixed Methods

Mixed finite element methods are useful for direct approximation of physical quantities of interest (which may not be the primary unknown in the usual formulation of the problem). They have obtained an enormous amount of attention in the literature in the context of the classical h -version. Unlike standard methods, where convergence only depends on *approximability* of the subspaces used, the convergence of mixed methods also depends upon the *stability* of the subspaces. Usually, this amounts to ensuring that a compatibility condition (the Babuska-Brezzi or inf-sup condition) is satisfied by the pair of surfaces used.

We have analyzed the p -version formulations of three mixed methods which have been previously investigated in the context of the h -version. In each case, the question of approximability is settled by using the results described in Section 2.1. The question of stability requires a different approach.

(a) Our first result concerns the solution of typical elliptic problems like the Poisson equation over a polygonal domain with boundary $\Gamma = \bigcup_{i=1}^n \Gamma_i$. We may write this as a mixed method by introducing a Lagrange multiplier ϕ for the Dirichlet boundary data. It may be shown that ϕ is formally equivalent to the normal component of the flux along the boundary (i.e. $\phi = -\frac{\partial u}{\partial n}$).

To approximate our problem, we introduce finite-dimensional subspaces $V_p \subset H^1(\Omega)$, $S_p \subset H^{-\frac{1}{2}}(\Gamma)$, parametrized by p , the degree of polynomials used. Here, functions in V_p are piecewise polynomials of degree $\leq p$ on a fixed grid on Ω , while functions in S_p are piecewise polynomials of degree $\leq p-2$ on each Γ_i . Then we seek a pair $(u_p, \phi_p) \in V_p \times S_p$ satisfying

$$\begin{aligned} \int_{\Omega} \nabla u^p \cdot \nabla v^p + \int_{\Gamma} v^p \phi^p &= \int_{\Gamma} f v^p & \forall v^p \in V_p^0 \\ \int_{\Gamma} u^p \psi^p &= \int_{\Gamma} g \psi^p & \forall \psi^p \in S_p \\ u^p(N_i) &= g(N_i) & \text{for each node } N_i \in \Gamma \end{aligned} \quad (2.3)$$

where $V_p^0 \subset V_p$ consists of those functions vanishing at each $N_i \in \Gamma$.

We have proven that if $u \in H^k(\Omega)$ with $k > \frac{3}{2}$, then the error in u satisfies the optimal estimate

$$\|u - u^p\|_{H^1(\Omega)} \leq Cp^{-(k-1)} \|u\|_{H^k(\Omega)} \quad (2.4)$$

Moreover, the inf-sup condition takes the following form:

$$\begin{aligned} \inf_{\phi \in S_p} \sup_{v \in V_p} \int_{\Gamma} v \phi &\geq Cp^{-\frac{1}{4}} \\ \|\phi\|_{H^{-\frac{1}{2}}(\Gamma)} = 1 \quad \|\psi\|_{H^1(\Omega)} = 1 & \end{aligned} \quad (2.5)$$

which leads to the error estimate

$$\|\phi - \phi_p\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Cp^{-(k+\frac{1}{4})} \|\phi\|_{H^k(\Gamma)} \quad (2.6)$$

(Here $\|\cdot\|_{H^r(\Gamma)} := \sum_{i=1}^n \|\cdot\|_{H^r(\Gamma_i)}$)

The result (2.4) has been presented by us in [B9], where we have used it in the analysis of the boundary constraint problem. The results (2.5)-(2.6) have not been published yet.

(b) An alternative mixed formulation of the Poisson equation involves writing it as a first order system so that u and $\bar{\sigma} = \text{grad } u$ are the unknowns in the spaces $L^2(\Omega)$ and $\bar{H}(\text{div}, \Omega)$ respectively. The most well known finite element method spaces for this method are the Raviart-Thomas spaces [A9] $V_h^p \subset L^2(\Omega)$, $H_h^p \subset \bar{H}(\text{div}, \Omega)$ which have been defined for arbitrary polynomial degree p . These spaces have been thoroughly analyzed in the context of the h -version. In particular, it has been shown

that when $p = p_0$ is fixed, they satisfy the inf-sup condition for the h version (see [A9]), i.e.

$$\inf_{v \in V_h^{p_0}} \sup_{\bar{\sigma} \in H_h^{p_0}} \int_r \operatorname{div}_{\sigma} v \geq C \quad (2.7)$$

$$\|v\|_{L^2(\Omega)} = 1 \quad \|\bar{\sigma}\|_{H(\operatorname{div}, \Omega)} = 1$$

where C is a constant independent of h but depending upon p_0 . (2.7) leads to optimal error estimates for the error in terms of h .

Suppose now that we use a fixed grid of mesh size h_0 and increase p , the degree of the spaces used instead, i.e., we consider the pairs $\{H_{h_0}^p, V_{h_0}^p\}$ with increasing p . We would like to know whether the method based on this p -type extension process is stable or not.

In [B15], we have analyzed the question of stability for arbitrary combinations of h and p . In particular, we have shown that (2.7) holds for the p -version as well, so that C is a constant independent of p and h . We have also derived estimates that are optimal in both h and p (up to an arbitrary $\epsilon > 0$). Our results indicate in addition that the use of the h -version with $p \geq 2$ is to be recommended over the h -version with $p = 1$.

(c) The Brezzi-Douglas-Marini elements [A6] are an alternative to the Raviart-Thomas elements. We have shown in [B15] that the inf-sup condition for these elements satisfies (2.7) where for arbitrary $\epsilon > 0$,

$$C(p) \geq Cp^{-\epsilon} \quad (2.8)$$

Here C depends upon ϵ but is independent of p and h . This once again leads to optimal estimates (up to $\epsilon > 0$) in both p and h [i.e. for the h -, p -, and h - p versions]

2.4. The p - and h - p versions for Integral Equations

Many problems that are posed originally over a region Ω with boundary $\partial\Omega$ may be re-formulated, using integral equations, as problems on just $\partial\Omega$ alone. We believe that the p - and h - p versions will prove to be valuable tools in solving Galerkin formulations of integral equations.

(a) In [B10], we have analyzed the p -version for integral equations of the first kind that arise from the Dirichlet crack problem and the Neumann crack problem in two-dimensional elasticity. Our results also apply to the corresponding screen problems in two-dimensional acoustics. The solutions for these problems are known to have an r^α type singularity, which leads to twice the rate of convergence (in the energy norm) as that observed using the h -version. We have also analyzed p -version methods

for some other integral equations arising from two- and three- dimensional problems arising in elasticity and acoustics. As expected, the p -version yields higher rates of convergence in most cases.

(b) In [B14], we extend our results to the h - p version. We consider integral equations on polygonal two-dimensional domains, for which the solution is known to possess singularities of the form r^α where α is general and depends upon the geometry of the domain. We consider the case with mixed boundary conditions and derive estimates that take into consideration both the degree p and mesh-size h of the quasiuniform mesh used.

Our results provide a partial theoretical basis to some computations reported by E. Rank on the p -version, where our predicted rate of convergence was observed experimentally for a range of p between 1 and 8. (See [B14] for details).

Summaries of the above results may be found in [B6], [B12].

2.5. An implementation of the p - and h - p versions for MODULEF

The MODULEF club, created by INRIA, Rocquencourt, France has 141 members worldwide and brings together universities and industrial companies with the goal of designing and implementing a library of finite element modules. The p - and h - p versions were implemented for scalar linear elliptic problems during an extended visit there, in the form of a module called 'HP'. This module uses polynomials of degree $1 \leq p \leq 8$ combined with geometrically refined meshes to attain approximations with a high degree of accuracy. A feature of HP allows the user to obtain (at low additional cost) a sequence of solutions for various p , providing an effective and easy method to decide whether the convergence of a solution is acceptable or not. (See [B16])

HP is the only module in MODULEF that is not based on the classical h -version. The availability of this code to the large body of MODULEF users is expected to greatly enhance the awareness, use and understanding of these new versions and stimulate further research in this field. The code will be thoroughly tested before being released in 1989.

2.6. Numerical methods for a class of reaction-diffusion problems

We consider a class of steady-state semilinear reaction-diffusion problems with non-differentiable kinetics whose analytical properties have received considerable attention in the literature, see e.g. [A5]. More precisely, we wish to solve the following equations for the concentration u :

$$\Delta u = \lambda f(u(x)) \quad x \in \Omega \quad (u \geq 0 \text{ in } \Omega) \quad (2.9)$$

$$u = 1 \quad x \in \partial\Omega \quad (2.10)$$

where $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) is a bounded domain with smooth boundary $\partial\Omega$ and λ is a positive constant measuring the ratio of reaction to diffusion rates. We are interested in the case of p th order isothermal reactions, for the case $0 < p < 1$, where f is explicitly given by

$$\begin{aligned} f(t) &= t^p \quad \text{for } t > 0 \\ &= 0 \quad \text{for } t \leq 0 \end{aligned} \quad (2.11)$$

For $p \geq 1$ this nonlinear problem may be approximated by several methods. However, when $p < 1$, f is not differentiable at the origin and the usual techniques of analysis do not work.

In [B5], we have considered a finite element approximation for (2.9)-(2.11). We have shown that the approximate problem has a unique solution v^h , that when linear functions are used, $v^h \geq 0$, and that the following error estimate can be obtained:

$$\begin{aligned} \|v - v^h\|_{H^1(\Omega)} &\leq Ch \quad \text{if } \frac{1}{2} \leq p < 1 \\ &\leq Ch^{2p} \quad \text{if } 0 < p < \frac{1}{2} \end{aligned} \quad (2.12)$$

Numerical experiments reported by us show, however, that one gets $O(h)$ convergence for any p so that the result (2.12) is pessimistic for the case $p < \frac{1}{2}$.

We also analyze a finite difference scheme for (2.9)-(2.11), proving existence, uniqueness and convergence of the approximation using the theory of M -functions. Details of the analysis together with numerical experiments in one- and two-dimensions may be found in [B5].

3. References [authors other than P.I]

- A1. J. Andersson and U. Falk, *Finite element analysis of three-dimensional structures using adaptive p-extensions*, Tech. Rep. FFA 5-H-800, The Aeronautical Research Institute of Sweden, 1986.
- A2. I. Babuška, *The p- and h-p versions of the finite element method. The state of the art*, Technical Note BN-1156, Institute for Phy. Sci. and Tech., 1986.
- A3. I. Babuška and M. R. Dorr, *Error estimates for the combined h and p version of the finite element method*, Numer. Math., **37** (1981), pp. 252-277.
- A4. I. Babuška, B. A. Szabo, and I. N. Katz, *The p-version of the finite element method*, SIAM J. Numer. Anal., **18** (1981), pp. 515-545.
- A5. C. Bandle, R. B. Sperb, and I. Stakgold, *Diffusion and reaction with monotone kinetics*, Nonlinear Analysis, **8** (1984), pp. 321-333.
- A6. F. Brezzi, J. Douglas, and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Preprint.
- A7. M. R. Dorr, *The approximation theory for the p-version of the finite element method*, SIAM J. Numer. Anal., **21** (1984), pp. 1180-1207.
- A8. I. N. Katz and D. W. Wang, *The p-version of the finite element method for problems requiring C^1 -continuity*, SIAM J. Numer. Anal., **22** (1985), pp. 1082-1106.
- A9. P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, in Proc Conf. on Math. Aspects of Finite Element Methods, Lecture Notes in Math., Vol. 606, Springer-Verlag, Berlin, 1977.
- A10. B. A. Szabo, *PROBE: Theoretical Manual*, NOETIC Technologies, St. Louis, 1985.

4. Chronological list of written publications by P.I.

- B1. "Some Optimal Approximation Results with Applications to the h-, p- and h-p Versions of the Finite Element Method". *Methods of Functional Analysis in Approximation Theory*. C. A. Micchelli, D. S. Bai, B. A. Limaye editors, 245-259, Birkhauser Verlag (1986).

- B2. "The Optimal Convergence Rate of the p -Version of the Finite Element Method" (with I. Babuska), *SIAM Journal of Numerical Analysis*, 24, No. 4 (1987).
- B3. "The h - p Version of the Finite Element Method with Quasiuniform Meshes", (with I. Babuska), *RAIRO Mathematical Modelling and Numerical Analysis*, 21, 1987.
- B4. "The p -Version of the Finite Element Method for Elliptic Problems", *Advances in Computer Methods for Partial Differential Equations - VI*, IMACS (1987).
- B5. "Numerical Methods for Reaction-Diffusion Problems with Non-Differentiable Kinetics" (with A. K. Aziz and A. B. Stephens), *Numer. Math.* 53, 1-11 (1988).
- B6. "An h - p method with quasiuniform mesh for integral equations on polygons" (with E. Stephan), *Computational Mechanics '88*, Vol I, Proceedings of International Conference on Computational Engineering Science, April 10-14, 1988, Atlanta, GA, Editors: S. P. Atluri and G. Yaganra, Springer-Verlag, pg 5.IV.1 - 5.IV.5.
- B7. "The Approximations of smooth and singular essential boundary conditions by the p -version of the finite element method," proceedings of 12th IMACS World Congress, Paris, 1988, pp 499-501.
- B8. "The Treatment of Nonhomogeneous Dirichlet Boundary Conditions by the p -Version of the Finite Element Method" (with I. Babuska), Technical Note BN-1063, IPST, University of Maryland, College Park, Maryland (1987). (To appear in *Numerische Mathematik*)
- B9. "The p -Version of the Finite Element Method for Constraint Boundary Conditions", (with I. Babuska), Technical Note BN-1064, IPST, University of Maryland, College Park, Maryland (1987). (To appear in *Math. Comp.*)
- B10. "On the Convergence of the p -Version of the Boundary Element Galerkin Method" (with E. P. Stephan), Research Report No. 87-16, Department of Mathematics, University of Maryland Baltimore County (1987). (To appear in *Math. Comp.*)
- B11. "The p -Version of the Finite Element Method for Elliptic Equations of Order 2ℓ ", Research Report No. 87-17, Department of Mathematics, University of Maryland Baltimore County (1987). (To appear in *RAIRO*).
- B12. Integral equation methods for screen and crack problems (with E. Stephan), (to appear in *Rivista di Matematica*).

- B13. "Implementation of Non-homogeneous Dirichlet Boundary Conditions in the p -Version of the Finite Element Method" (with I. Babuska and B. Guo). (Submitted for publication)
- B14. The h - p Version of the Boundary Element Method on Polygonal Domains with quasiuniform Meshes (with E. Stephan) (Submitted to RAIRO)
- B15. On the Stability and Convergence of Higher Order Mixed Finite Element Methods for Second Order Elliptic Problems. (Submitted to Math Comp.)
- B16. Theoretical users' manual for MODULEF module HP. (in preparation)

5. Professional personnel associated with research

The following graduate students contributed to various computational aspects of the project, for which they received partial summer support from the grant:

Jinn-Liang Liu [Dept. of Math and Stat, UMBC]

T. Shen [Dept. of Math and Stat, UMBC]

Other researchers associated with the work include coauthors A. K. Aziz (UMBC), I. Babuska (UMCP), B. Guo (UMCP), E. Stephan (Georgia Inst. of Tech) and B. Stephens (UMBC).

The work on MODULEF was conducted in collaboration with M. Vridascu and P. Huynh [INRIA, Rocquencourt, France].

6. Talks given to disseminate information on funded research

Note: A large number of talks on the funded research were given at various universities in Europe during the stay at INRIA [Feb'88 through July'88]. This trip, which was sponsored jointly by INRIA and AFOSR was invaluable in popularizing the p - and h - p versions there.

(a) Invited Talks at Universities

- Fall '85 University of Maryland, College Park, MD, "The optimal convergence rate of the p -version of the finite element method"
- Jan '88 Technische Hochschule, Darmstadt, W. Germany, "The p -version for integral equations"
- Mar '88 University of Paris VI, Paris, France, "Implementational aspects of the p and h - p versions of the finite element method"
- Apr '88 INRIA, Rocquencourt, France, "Implementational aspects of the p and h - p versions of the finite element method"
- Apr '88 University of Stuttgart, Stuttgart, W. Germany, "The p - and h - p versions for integral equations"
- June '88 Ecole Polytechnique Federale de Lausanne, Lausanne, Switzerland, "The p - and h - p versions of the Finite Element Method"
- June '88 Istituto di Analisi Numerica del CNR, Pavia, Italy, "Some results on the p - and h - p versions of the Finite Element Method"
- June '88 University of Parma, Parma, Italy, "An introduction to the p - and h - p versions of the Finite Element Method"

(b) Invited Talks at Conferences

- June '87 Advances in Computer Methods for PDES-VI, (IMACS) Bethlehem, PA, "The p -version of the Finite Element Method for elliptic problems"
- May '88 Comett Course on Finite Elements (sponsored by E.E.C), CEMUL, Lisbon, Portugal (23-27 May, 1988). 2 day course presented on "Adaptive Finite Elements and the p -version"
- June '88 Annual MODULEF users' conference, INRIA, Rocquencourt, France, "The MODULEF implementation of the p - and h - p versions of the Finite Element Method."

(c) Contributed Talks at Conferences

- Spring '85 Semi-annual FEM Conference, University of North Carolina, Chapel Hill, NC, "The optimal convergence rate of the p -version of the FEM"
- July '85 Annual SIAM meeting, Pittsburgh, PA, "Optimal error estimates for the p -version of the FEM"
- Fall '85 Semi-annual FEM conference , Brookhaven Laboratories, NY, "The h - p version of the FEM"
- Dec '85 Methods of Functional Analysis in Approximation Theory, Bombay, India, "Some optimal approximation results with applications to the h -, p - and h - p versions of the FEM"
- Spring '86 Semi-annual FEM Conference, Rutgers University, NJ, "A numerical method for some reaction-diffusion problems"
- Aug '86 International Congress of Mathematicians '86, Berkeley, CA, "An optimal error estimate for the h - p version of the FEM"
- Fall '86 Semi-annual FEM conference, University of Tennessee, Knoxville, TN, "The orthogonal projection in the $H^{1/2}$ norm with applications to the p -version of the FEM"
- July '87 ICIAM 1987, Paris, France, "The mixed p -version of the FEM for elliptic problems"
- Fall '87 Semi-annual FEM conference, Cornell University, Ithaca, NY, "The p -extension using Raviart-Thomas elements"
- Dec '87 Ramanujan Birth Centenary Year International Symposium in Analysis, Pune, India, "Integral equation methods for screen and crack problems"
- July '88 12th IMACS World Congress, Paris, France, "The approximation of smooth and singular essential boundary conditions by the p -version of the finite element method"