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On the Equation $X^a + Y^a = Z^a$

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ON THE EQUATION $X^a + Y^a = Z^a$

INTRODUCTION

In this report we study a problem related to Fermat's last theorem. Suppose that X , Y , and Z are positive numbers where

$$X^a + Y^a = Z^a. \quad (1)$$

We show that we can solve this equation for a ; that is, we find a unique $a = a(X, Y, Z)$ in closed form. The method of solution is rather elementary, and we employ Wright's generalized hypergeometric function in one variable [1], as defined below:

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} \right] z \equiv \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(\alpha_i + A_i n)}{\prod_{i=1}^q \Gamma(\beta_i + B_i n)} \frac{z^n}{n!}.$$

When $p = q = 1$, we see that

$${}_1\Psi_1 \left[\begin{matrix} (\alpha, A); \\ (\beta, B); \end{matrix} \right] z = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + An)}{\Gamma(\beta + Bn)} \frac{z^n}{n!}, \quad (2)$$

which is a generalization of the confluent hypergeometric function ${}_1F_1[\alpha; \beta; z]$.

AN EQUIVALENT FORM OF EQUATION (1)

In Eq. (1), the case $X = Y$ is not interesting since clearly

$$a = \frac{\ln(1/2)}{\ln(X/Z)}.$$

Therefore we shall assume without loss of generality that

$$Z > Y > X > 0,$$

and write Eq. (1) as

$$e^{a \ln(X/Z)} + e^{a \ln(Y/Z)} - 1 = 0.$$

Now making the transformation

$$e^{a \ln(Y/Z)} \equiv y, \quad (3)$$

we obtain

$$y^{\frac{\ln(X/Z)}{\ln(Y/Z)}} + y - 1 = 0,$$

and since

$$\frac{\ln(X/Z)}{\ln(Y/Z)} = \frac{\ln(Z/X)}{\ln(Z/Y)} > 1,$$

we arrive at

$$y^{\frac{\ln(Z/X)}{\ln(Z/Y)}} + y - 1 = 0. \quad (4)$$

Equation (4) is then equivalent to Eq. (1), and our aim is to solve this equation for y , thereby obtaining a . We note that it is not difficult to verify that Eq. (4) has a unique positive root y in the interval $(1/2, 1)$.

SOLUTION OF EQUATION (4)

In 1915, Mellin [2,3] investigated certain transform integrals named after him in connection with his study of the trinomial equation

$$y^N + xy^P - 1 = 0, \quad N > P, \quad (5)$$

where x is a real number and N, P are positive integers. Mellin showed that for appropriately bounded x , a positive root of Eq. (5) is given by

$$y = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z) x^{-z} dz, \quad 0 < c < 1/P, \quad (6)$$

where

$$F(z) = \frac{\Gamma(z)\Gamma\left(\frac{1}{N} - \frac{P}{N}z\right)}{N\Gamma\left[1 + \frac{1}{N} + \left(1 - \frac{P}{N}\right)z\right]}$$

and

$$|x| < (P/N)^{-P/N} (1 - P/N)^{P/N-1} \leq 2. \quad (7)$$

The inverse Mellin transform, Eq. (6), is evaluated by choosing an appropriate closed contour and using residue integration to find that

$$y = \frac{1}{N} \sum_{n=0}^{\infty} \frac{\Gamma \left[\frac{1}{N} + \frac{P}{N} n \right]}{\Gamma \left[1 + \frac{1}{N} + \left(\frac{P}{N} - 1 \right) n \right]} \frac{(-x)^n}{n!}. \quad (8)$$

Under the condition shown in Eq. (7), Mellin, in fact, found all of the roots of Eq. (5). However, suppose we relax the restriction that N and P are positive integers. Instead, let N and P be positive numbers. We then observe that Eq. (8) gives *a fortiori* a positive root of Eq. (5) for positive numbers N and P . Further, without loss of generality we set $P = 1$, $N = \omega$. Then, using the Wright function defined by Eq. (2), we arrive at the following. The unique positive root of the transcendental equation

$$y^{\omega} + xy - 1 = 0, \quad \omega > 1, \quad (9)$$

where

$$|x| < \omega/(\omega - 1)^{1-1/\omega}$$

is given by

$$y = \frac{1}{\omega} {}_1\Psi_1 \left[\begin{matrix} \left(\frac{1}{\omega}, \frac{1}{\omega} \right) \\ \left(\frac{1}{\omega} + 1, \frac{1}{\omega} - 1 \right) \end{matrix} ; -x \right]. \quad (10)$$

We observe that for any $|x| < \infty$, Eq. (9) has a unique positive root y . Equations (9) and (10) may also be obtained from Ref. 4, p. 713, Eq. (30).

Let us now apply the latter result to Eq. (4). On setting

$$x = 1$$

$$\omega^{-1} = \frac{\ln(Z/Y)}{\ln(Z/X)} \equiv \lambda,$$

and noting that $1 < \omega/(\omega - 1)^{1-1/\omega}$, we find

$$y = \lambda {}_1\Psi_1 \left[\begin{matrix} (\lambda, \lambda) \\ (\lambda + 1, \lambda - 1) \end{matrix} ; -1 \right], \quad 0 < \lambda < 1. \quad (11)$$

SOLUTION OF EQUATION (1)

We now solve Eq. (1) for a . From the transformation Eq. (3), we see that

$$a \ln (Y/Z) = \ln y. \quad (12)$$

Then, using Eq. (11), we arrive at the following. If $Z > Y > X > 0$ are such that

$$X^a + Y^a = Z^a,$$

then

$$a = \frac{\ln \left\{ \lambda {}_1\Psi_1 \left[\begin{matrix} (\lambda, \lambda) \\ (\lambda + 1, \lambda - 1) \end{matrix} ; -1 \right] \right\}}{\ln (Y/Z)}, \quad (13)$$

where

$$\lambda \equiv \frac{\ln (Z/Y)}{\ln (Z/X)}, \quad 0 < \lambda < 1. \quad (14)$$

We now prove the following. Consider for $X < Y$, $M \geq 1$, the diophantine equation

$$X^M + Y^M = Z^M.$$

Then the positive integers X , Y , and Z must satisfy

$$X^\lambda Y^{-1} Z^{1-\lambda} = 1, \quad (15)$$

where λ is an irrational number such that $0 < \lambda < 1$.

From Eq. (12) we have

$$(Y/Z)^M = y, \quad (16)$$

so that y is a rational number in the interval $1/2 < y < 1$, as we noted earlier. If λ is rational, there exist relatively prime integers s and t such that

$$\lambda = \omega^{-1} = s/t.$$

Hence, y is the unique positive root of

$$y^{t/s} + y - 1 = 0.$$

Now since $\lambda < 1$, then $s < t$, and we obtain the polynomial equation of degree t with integer coefficients:

$$y^t + (-1)^s y^s + \dots + 1 = 0.$$

The only positive rational root that this equation may have is $y = 1$ [5, p. 67]. But $y < 1$, so the assumption that λ is rational leads to a contradiction. We have then that λ is irrational, and Eq. (15) follows from Eq. (14). This proves our result. Another proof of this result [6] is given in the appendix of this report.

The Wright function ${}_1\Psi_1$ appearing in Eq. (13) depends only on the parameter λ . Thus, for brevity, we define

$$\Psi(\lambda) \equiv {}_1\Psi_1 \left[\begin{matrix} (\lambda, \lambda) \\ (\lambda + 1, \lambda - 1) \end{matrix} ; -1 \right], \quad 0 < \lambda < 1.$$

From our previous result, we see that if Fermat's theorem* is false, then there exist positive integers $X < Y < Z$ such that λ is irrational.

Therefore, Fermat's theorem is false if and only if there exist positive integers $Y < Z$, $M > 2$ and an irrational number λ ($0 < \lambda < 1$) such that

$$(Y/Z)^M = \lambda\Psi(\lambda).$$

Thus Fermat's conjecture may be posed as a problem involving the special function $\lambda\Psi(\lambda)$. We remark that recently, Fermat's conjecture has been given in combinatorial form [7].

SOME ELEMENTARY PROPERTIES OF $\lambda\Psi(\lambda)$

The series representation Eq. (17) for $\lambda\Psi(\lambda)$ does not converge for $\lambda = 0, 1$. Nevertheless, it is natural to define

$$\lambda\Psi(\lambda) \Big|_{\lambda=1} = 1/2, \quad \lambda\Psi(\lambda) \Big|_{\lambda=0} = 1.$$

Below we give a brief table of values for $\lambda\Psi(\lambda)$ correct to five significant figures:

λ	$\lambda\Psi(\lambda)$	λ	$\lambda\Psi(\lambda)$
0.0	1.00000	0.6	0.58768
0.1	0.83508	0.7	0.56152
0.2	0.75488	0.8	0.53860
0.3	0.69814	0.9	0.51825
0.4	0.65404	1.0	0.50000
0.5	0.61803		

Observe that we may write the inverse relation

$$\lambda = \ln \lambda\Psi(\lambda) / \ln [1 - \lambda\Psi(\lambda)].$$

*Fermat's theorem states that there are no integers $x, y, z > 0$, $n > 2$ such that $x^n + y^n = z^n$.

The following series representations for $\lambda\Psi(\lambda)$, $0 < \lambda < 1$ may easily be derived from the first one below:

$$\lambda {}_1\Psi_1 \left[\begin{matrix} (\lambda, \lambda) \\ (\lambda + 1, \lambda - 1) \end{matrix} ; -1 \right] = \lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\lambda + \lambda n)}{\Gamma(\lambda + 1 + (\lambda - 1)n)} \quad (17)$$

$$= \frac{\lambda}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(1 - \lambda)n - 1} \sin [\pi(1 - \lambda)n] B(\lambda n, n - \lambda n) \quad (18)$$

$$= 1 - \lambda \sum_{n=0}^{\infty} (-1)^n {}_2F_1[-n, (1 - \lambda)(n + 2); 2; 1] \quad (19)$$

$$= 1 + \lambda \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[\begin{matrix} \lambda(1 + n) - 1 \\ n - 1 \end{matrix} \right]. \quad (20)$$

Equation (18) follows from Eq. (17) by using $\Gamma(z)\Gamma(-z) = -\pi/z \sin \pi z$; $B(x, y)$ is the beta function. Equation (19) follows from Eq. (17) by using Gauss's theorem for ${}_2F_1[a, b; c; 1]$. And Eq. (20) follows from Eq. (17) by using $\left[\begin{matrix} \alpha \\ m \end{matrix} \right] = \Gamma(1 + \alpha)/m! \Gamma(1 + \alpha - m)$. Equation (20) for $1/\lambda$, an integer greater than one, is due to Lagrange [2, p. 56].

CONCLUSION

The equation $X^a + Y^a = Z^a$ has been solved for a as a function of X , Y , and Z in terms of a Wright function ${}_1\Psi_1$ with negative unit argument. An equivalent form of Fermat's last theorem has been given using this function. Further, some elementary properties of ${}_1\Psi_1$ have been stated.

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Appendix

Theorem: Consider for $X < Y$, $M \geq 1$, the diophantine equation

$$X^M + Y^M = Z^M. \quad (\text{A1})$$

Then the positive integers X , Y , and Z must satisfy

$$X^\lambda Y^{-1} Z^{1-\lambda} = 1, \quad (\text{A2})$$

where λ is an irrational number such that $0 < \lambda < 1$.

Proof: Clearly $0 < X < Z$ and $0 < Y < Z$. Define $f(\lambda) \equiv (X/Z)^\lambda$ which is a decreasing continuous function of λ on $[0, 1]$, since $(X/Z) < 1$. Since $f(0) = 1$, and $f(1) = X/Z$, by the intermediate value theorem there is a λ in the interval $(0, 1)$ such that

$$f(\lambda) = \left(\frac{X}{Z} \right)^\lambda = \frac{Y}{Z} \quad (\text{A3})$$

if and only if

$$\frac{X}{Z} < \frac{Y}{Z} < 1.$$

We know $Y/Z < 1$, so such a λ exists if and only if $X < Y$. Hence Eq. (A3) implies Eq. (A2).

To show λ is irrational, suppose p is a prime dividing X and Y . Then Eq. (A1) implies p divides Z . Similarly, if p divides any two of X , Y , or Z , it divides all three, and p^k must divide all three with the same maximum exponent k . Since $X < Y$, there must be some p^k that divides Y but does not divide X . Hence, p^k also does not divide Z . Suppose $\lambda = a/b$ is rational where a and b are relatively prime. Then by Eq. (A3)

$$\left(\frac{X}{Z} \right)^{a/b} = \frac{Y}{Z}$$

which implies

$$X^a Z^b = Y^b Z^a. \quad (\text{A4})$$

Since p^k divides Y , it divides the right side of Eq. (A4). But p^k not dividing X or Z implies p^k does not divide the left side of Eq. (A4), and we have a contradiction. Thus λ must be irrational.

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