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EVALUATION OF 15TH- AND 30TH-ORDER GEOPOTENTIAL HARMONIC COEFFICIENTS FROM 26 RESONANT SATELLITE ORBITS

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SUMMARY

The Earth's gravitational potential is usually expressed as an infinite series of harmonics, and the values of harmonic coefficients of order 15 and 30 can be determined most accurately by analysis of satellite orbits which experience 15th-order resonance. The results from two recent resonance analyses, for 1965-09A and 1968-70A, have here been added to those previously available, to produce an improved evaluation of 44 coefficients of order 15 and degree 15-36, and 12 coefficients of order 30 and even degree 30,32,...40.

Compared with previous results, the new evaluation shows a great improvement in the standard deviations of many of the 15th-order coefficients of even degree, thanks largely to the contribution of 1965-09A at inclination 31.8°; for the coefficients of degree 24, 26, 28 and 30, the standard deviation (sd) has been reduced by a factor of 3.1 on average; and for degree 32, 34 and 36 by a factor of 1.4 on average. For the other coefficients - those of 30th order, and odd-degree 15th order - the changes are relatively small. In the new 15th-order solution, all the 30 coefficients of degree 15-29 have sd $\leq 2.0 \times 10^{-9}$, and the average sd of these 30 values is equivalent to an error in geoid height of 0.7 cm. Comparison of our values with those in comprehensive geoid models, which usually have larger sd, lead us to conclude that, for orders 15 and 30, the nominal standard deviations of the comprehensive models are quite realistic.

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1 INTRODUCTION

Our previous evaluation[1] of individual harmonic coefficients of order 15 and 30 from analysis of 25 satellites at 15th-order resonance can now be improved in four ways. First, the recent analysis[2] of the resonant variation in eccentricity for 1965-09A, at inclination 31.8°, gives a much stronger hold on coefficients of even degree for degree ⩾24; previously the lowest inclination used was 43°. Second, the new and accurate analysis[3] of 1968-70A at inclination 56°, which includes many Hewitt camera observations, provides further good values of lumped harmonics of order 15 and 30. Third, we have corrected a mistake in the value of one of the lumped 30th-order S harmonics previously used. The fourth improvement[4], important in principle though hardly significant numerically, is the replacement of an approximation for the functions $G_{\ell pq}$ by more accurate values from the computer program GQUAD.

The format of this Report is similar to that of its predecessor[1]: sections 2 to 4 offer a very brief outline of the notation, the data used and the method of solution. Sections 5 to 9 give the solutions for the harmonic coefficients obtained on taking account of the first three of the four improvements mentioned above. In section 10 we take account of the fourth improvement - the use of more accurate forms for the G functions - and produce revised solutions, given in Tables 14, 15 and 17.

2 NOTATION

The longitude-dependent part of the geopotential at an exterior point (r, θ, λ) can be written in normalized form[5] as

$$\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R}{r}\right)^{\ell}P_{\ell}^{m}(\cos\theta)\left\{\bar{C}_{\ell m}\cos m\lambda + \bar{S}_{\ell m}\sin m\lambda\right\}N_{\ell m} , \tag{1}$$

where r is the distance from the Earth's centre, θ is co-latitude, λ is longitude (positive to the east), μ is the gravitational constant for the Earth ($398600\ km^3/s^2$) and R is the Earth's equatorial radius (6378.1 km). The $P_{\ell}^{m}(\cos\theta)$ are the associated Legendre functions of order m and degree ℓ, and $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ are the normalized tesseral harmonic coefficients: only those of order $m = 15$ and $m = 30$ are relevant here. The normalizing factor $N_{\ell m}$ is given by[5]

$$N_{\ell m}^{2} = \frac{2(2\ell + 1)(\ell - m)!}{(\ell + m)!} . \tag{2}$$

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Note that $\ell \geq m$, so that, if $m = 15$, then $\ell = 15, 16, 17, \ldots$.

When a satellite passes slowly through 15th-order resonance as its orbit contracts under the influence of air drag, it is possible to analyse the variations in some orbital parameters and to determine accurate values of 'lumped' geopotential harmonics, denoted by $\bar{C}_m^{q,k}$ and $\bar{S}_m^{q,k}$, which are linear functions of the individual coefficients $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$. By analysing both inclination i and eccentricity e , good values can usually be obtained for three pairs of lumped harmonics of 15th order, for $(q,k) = (0,1)$, $(1,0)$ and $(-1,2)$. The first of these is derived by analysis of the changes in inclination, and the linear equation for the lumped harmonic in terms of the individual coefficients may be written

$$\bar{C}_{15}^{0,1} = \bar{C}_{15,15} + Q_{17}^{0,1}\bar{C}_{17,15} + Q_{19}^{0,1}\bar{C}_{19,15} + Q_{21}^{0,1}\bar{C}_{21,15} + \cdots \tag{3}$$

and similarly for S . The $Q_\ell^{0,1}$ coefficients here are functions of inclination, eccentricity and semi major axis, but may be taken as constant for a particular satellite passing through resonance. By evaluating $\bar{C}_{15}^{0,1}$ for satellites at many different inclinations, the resulting equations of the form (3) can be solved to determine the individual coefficients $\bar{C}_{\ell,15}$ of odd degree (and similarly for S).

The coefficients $\bar{C}_{\ell,15}$ of even degree are linked with the other two lumped harmonics, $\bar{C}_{15}^{1,0}$ and $\bar{C}_{15}^{-1,2}$, via the equations

$$\left.\begin{aligned} \bar{C}_{15}^{1,0} &= \bar{C}_{16,15} + Q_{18}^{1,0}\bar{C}_{18,15} + Q_{20}^{1,0}\bar{C}_{20,15} + \cdots \\ \bar{C}_{15}^{-1,2} &= \bar{C}_{16,15} + Q_{18}^{-1,2}\bar{C}_{18,15} + Q_{20}^{-1,2}\bar{C}_{20,15} + \cdots \end{aligned}\right\} \tag{4}$$

and similarly for S . When these even-degree lumped harmonics, which are derived chiefly from the changes in orbital eccentricity, are evaluated for satellites at a variety of inclinations, the equations (4) can be solved for the individual coefficients of even degree.

Analysis of 15th-order resonance may also yield values of lumped harmonics of order 30, of which $\bar{C}_{30}^{0,2}$ and $\bar{S}_{30}^{0,2}$ are the best determined, being obtained from the changes in orbital inclination. The appropriate linear equations for these lumped harmonics are

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$$\bar{C}_{30}^{0,2} = \bar{C}_{30,30} + Q_{32}^{0,2}\bar{C}_{32,30} + Q_{34}^{0,2}\bar{C}_{34,30} + \ldots \qquad (5)$$

and similarly for S . Thus values of the individual coefficients of order 30 and even degree are obtainable if values of these lumped harmonics can be determined for satellites over a wide enough range of inclination. For further details of the theory, see Ref 4.

3 THE DATA

3.1 Introduction

We shall be using results from analysis of the orbits of 26 satellites which have experienced 15th-order resonance. As two separate analyses are included for one satellite (1964-52B), we have 27 equations of type (3) for the odd-degree C coefficients of order 15 (and 27 for S), obtained from analysis of the inclination of each satellite. These analyses also yielded 11 equations of type (5) for 30th-order C coefficients of even degree (and 11 for S).

Analyses of eccentricity have been made for 20 of the 26 satellites (again with two for 1964-52B), giving 21 equations for the lumped harmonic $\bar{C}_{15}^{1,0}$ and 21 for $\bar{C}_{15}^{-1,2}$, *ie* a total of 42 equations of the type (4) for C coefficients of 15th order and even degree (and 42 for S).

All but two of the resonance analyses have been described previously. The new ones are discussed in sections 3.2 and 3.3.

Table 1 gives the values of $(\bar{C},\bar{S})_{15}^{0,1}$ used in the solutions, together with the normalized inclination function[6] $\bar{F}_{\ell mp}$ for $\ell = m = 15$, where $2p = \ell - k$. It is useful to multiply the values of the lumped harmonics by $\bar{F}_{15,15,7}$ before plotting the results graphically, so as to avoid large changes in the values between different inclinations. Values of $\bar{F}\bar{C}$ and $\bar{F}\bar{S}$ are therefore also recorded in Table 1.

Table 2 gives the values of $(\bar{C},\bar{S})_{15}^{1,0}$ and $(\bar{C},\bar{S})_{15}^{-1,2}$ with the appropriate $\bar{F}$. The 30th-order lumped harmonics are given later (Table 7).

3.2 Pegasus 1, 1965-09A

The variation of inclination for this satellite at resonance was successfully analysed some years ago[7], but no useful results could be obtained at that time from the changes in eccentricity. In our previous determination of even-degree harmonics of order 15, the chief weakness was the lack of any satellite at inclination lower than 43°. Orbits at lower inclinations are much more strongly

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Table 1

Values of lumped harmonics $\left(\bar{C},\bar{S}\right)^{0,1}_{15}$ for the 26 satellites

No.	Satellite	i (deg)	Semi major axis (km)	e	$10^9\bar{C}^{0,1}_{15}$	$10^9\bar{S}^{0,1}_{15}$	$\bar{F}_{15,15,7}$	$10^9\bar{F}_{15,15,7}\bar{C}^{0,1}_{15}$	$10^9\bar{F}_{15,15,7}\bar{S}^{0,1}_{15}$
1	65-09A	31.76	6857	0.007	30980 ± 1960	13540 ± 960	136.3×10^{-6}	4.22 ± 0.27	1.85 ± 0.13
2	69-68B	32.97	6857	0.004	20340 ± 750	6280 ± 910	216.0×10^{-6}	4.39 ± 0.16	1.36 ± 0.20
3	64-84A	37.80	6860	0.042	560 ± 580	-2000 ± 1450	1.111×10^{-3}	0.62 ± 0.64	-2.22 ± 1.61
4	79-82A	43.60	6862	0.001	-467 ± 34	-767 ± 424†	5.576×10^{-3}	-2.55 ± 0.19	-4.28 ± 2.36
5	71-30B	46.36	6869	0.011	-398 ± 50	-797 ± 30	0.01074	-4.27 ± 0.54	-8.56 ± 0.32
6	74-34A	50.64	6872	0.002	-430.2 ± 10.0	-320.9 ± 8.3	0.02620	-11.27 ± 0.26	-8.41 ± 0.22
7	71-58B	51.05	6874	0.011	-354 ± 94*	-248 ± 45	0.02834	-10.03 ± 2.66	-7.03 ± 1.28
8	62-15A	53.82	6876	0.022	-370 ± 14	-114 ± 31	0.04657	-17.23 ± 0.65	-5.31 ± 1.44
9	65-53B	56.04	6879	0.003	-233.4 ± 6.6*	-103 ± 34†	0.06681	-15.59 ± 0.44	-6.88 ± 2.27
10	68-70A	56.08	6880	0.002	-213.5 ± 10.8*	-91.1 ± 16.8†	0.06727	-14.36 ± 0.73	-6.13 ± 1.13
11	63-24B	58.20	6883	0.002	-110.6 ± 5.6	-41.6 ± 4.5	0.09207	-10.18 ± 0.52	-3.83 ± 0.41
12	70-87A	62.92	6888	0.007	-5.4 ± 3.6	-31.4 ± 11.2†	0.1683	-0.91 ± 0.61	-5.28 ± 1.88
13	65-14A	65.02	6892	0.003	4.8 ± 2.1	-7.1 ± 2.5	0.2114	1.01 ± 0.44	-1.50 ± 0.53
14	77-12B	65.49	6894	0.029	13.4 ± 12.4*	0.7 ± 13.3	0.2217	2.97 ± 2.75	0.16 ± 2.95
15	71-106A	65.70	6895	0.045	-36 ± 42*	9 ± 17	0.2264	-8.15 ± 9.51	2.04 ± 3.85
16	71-10B	65.83	6893	0.002	-0.7 ± 4.1	2.4 ± 3.9	0.2293	-0.16 ± 0.94	0.55 ± 0.89
17	71-18B	69.84	6900	0.040	-37 ± 24†	10 ± 6	0.3261	-12.07 ± 7.83	3.26 ± 1.96
18	70-111A	74.00	6905	0.001	-26.0 ± 1.0	-5.2 ± 1.3	0.4312	-11.21 ± 0.43	-2.24 ± 0.56
19	71-13B	74.05	6903	0.002	-24.6 ± 1.3	-6.1 ± 1.0	0.4324	-10.63 ± 0.56	-2.64 ± 0.43
20	77-95B	75.82	6908	0.029	-22.5 ± 5.1	-3.0 ± 5.4	0.4745	-10.68 ± 2.42	-1.42 ± 2.56
21	67-42A	80.17	6918	0.007	-23.1 ± 1.6	-8.6 ± 1.3	0.5594	-12.92 ± 0.90	-4.81 ± 0.73
22	70-19A	81.16	6916	0.005	-21.0 ± 1.6	-1.1 ± 5.2†	0.5736	-12.05 ± 0.92	-0.63 ± 2.98
23	67-73A	85.98	6925	0.025	-13.9 ± 2.3	-6.4 ± 3.3	0.6076	-8.45 ± 1.40	-3.89 ± 2.01
24	71-54A	90.21	6930	0.002	-16.05 ± 0.21	-6.90 ± 0.21	0.5855	-9.40 ± 0.12	-4.04 ± 0.12
25	64-52B(H)	98.68	6945	0.023	-28.3 ± 2.0	1.5 ± 8.0†	0.4247	-12.02 ± 0.85	0.64 ± 3.40
26	64-52B(B)	98.68	6945	0.023	-13.2 ± 4.6*	-4.4 ± 3.2	0.4247	-13.25 ± 1.95	-1.87 ± 1.36
27	66-63A	144.16	7009	0.003	36900 ± 9700	12200 ± 9400*	$61.95 \cdot 10^{-6}$	2.28 ± 0.60	0.76 ± 0.58

Key: * Standard deviation < 2
† Standard deviation < 4

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Table 2

Values of lumped harmonics $(\bar{C},\bar{S})_{15}^{1,0}$ and $(\bar{C},\bar{S})_{15}^{-1,2}$

Satellite	i (deg)	$10^9\bar{C}_{15}^{1,0}$	$10^9\bar{C}_{15}^{-1,2}$	$10^9\bar{S}_{15}^{1,0}$	$10^9\bar{S}_{15}^{-1,2}$	$\bar{F}_{16,15,8}$	$\bar{F}_{16,15,7}$	$10^9\bar{F}_{16,15,8}\bar{C}_{15}^{1,0}$	$10^9\bar{F}_{16,15,7}\bar{C}_{15}^{-1,2}$	$10^9\bar{F}_{16,15,8}\bar{S}_{15}^{1,0}$	$10^9\bar{F}_{16,15,7}\bar{S}_{15}^{-1,2}$
65-09A	31.76	53000 ± 18200*	-340 ± 510	-5000 ± 9700	200 ± 790	1.886×10^{-4}	1.768×10^{-3}	10.0 ± 3.4	-0.60 ± 0.90	-0.94 ± 1.83	0.35 ± 1.40
79-82A	43.60	-860 ± 150	-234 ± 34	-1930 ± 160	185 ± 268†	9.277×10^{-3}	42.65×10^{-3}	-8.0 ± 1.4	-10.0 ± 1.5	-17.9 ± 1.5	7.9 ± 11.4
71-30B	46.36	-420 ± 93	-201 ± 60*	-1025 ± 670∇	205 ± 80†	18.22×10^{-3}	72.35×10^{-3}	-7.7 ± 1.7	-14.5 ± 4.3	-18.7 ± 12.2	14.8 ± 5.8
74-34A	50.64	-211.2 ± 24.9	-3.0 ± 8.6	128.4 ± 20.2	63.5 ± 3.4	45.17×10^{-3}	0.1440	-9.5 ± 1.1	-0.4 ± 1.2	5.8 ± 0.9	9.1 ± 0.5
71-58B	51.05	-466 ± 232†	-50 ± 92†	253 ± 120*	45 ± 14	48.86×10^{-3}	0.1526	-22.8 ± 11.3	-7.6 ± 14.0	12.4 ± 5.9	6.9 ± 2.1
62-15A	53.82	-76 ± 18	151 ± 60†	172 ± 76*	11 ± 34	80.16×10^{-3}	0.2180	-6.1 ± 1.4	32.9 ± 13.1	13.8 ± 6.1	2.4 ± 7.4
65-53B	56.04	18 ± 17	106.9 ± 8.7	57 ± 23	2.4 ± 8.1	0.1141	0.2780	2.1 ± 1.9	29.7 ± 2.4	6.5 ± 2.6	0.7 ± 2.3
68-70A	56.08	20.4 ± 10.2	74.4 ± 23.2†	39.3 ± 15.2*	38 ± 44∇	0.1149	0.2792	2.3 ± 1.2	20.8 ± 6.5	4.5 ± 1.7	10.6 ± 12.3
63-24B	58.20	59.3 ± 10.2	101.5 ± 13.0*	26.8 ± 7.4	12.9 ± 16.4†	0.1551	0.3395	9.2 ± 1.6	34.5 ± 4.4	4.2 ± 1.1	4.4 ± 5.6
65-14A	65.02	74.8 ± 5.8	-12.4 ± 3.7	-9.6 ± 4.4	-29.4 ± 3.0	0.3268	0.5035	24.4 ± 1.9	-6.2 ± 1.9	-3.1 ± 1.4	-14.8 ± 1.5
71-106A	65.70	51 ± 24	-67 ± 40†	-55 ± 70∇	-19 ± 24	0.3455	0.5129	17.6 ± 8.3	-34.4 ± 20.5	-19.0 ± 24.2	-9.7 ± 12.3
71-10B	65.83	35.1 ± 23.4*	-20.0 ± 10.7	-13.8 ± 11.0	-18.9 ± 10.3	0.3491	0.5145	12.3 ± 8.2	-10.3 ± 5.5	-4.8 ± 3.8	-9.7 ± 5.3
70-111A	74.00	-18.0 ± 3.3	-46.5 ± 2.7	-44 ± 25∇	-40.5 ± 4.0	0.5145	0.4402	-9.3 ± 1.7	-20.5 ± 1.2	-22.6 ± 12.9	-17.8 ± 1.8
71-13B	74.05	-19.8 ± 1.8	-45.5 ± 2.0	-24.8 ± 0.7	-35.2 ± 1.0	0.5149	0.4386	-10.2 ± 0.9	-20.0 ± 0.9	-12.8 ± 0.4	-15.4 ± 0.4
77-95B	75.82	-3 ± 26*	-63 ± 15	-4 ± 22*	-46 ± 18	0.5200	0.3732	-1.6 ± 13.5	-23.5 ± 5.6	-2.1 ± 11.4	-17.2 ± 6.7
67-42A	80.17	-5[illegible]7 ± 6.4*	-131 ± 21*	-37.2 ± 2.6	-97 ± 18*	0.4617	0.1552	-25.3 ± 3.0	-20.3 ± 3.3	-17.2 ± 1.2	-15.1 ± 2.8
70-19A	81.16	-26 ± 41∇	-128 ± 29	-15 ± 20†	-130 ± 37	0.4337	0.09806	-11.3 ± 17.8	-12.6 ± 2.8	-6.5 ± 8.7	-12.7 ± 3.6
67-73A	85.98	-87 ± 38	-122 ± 65	84 ± 120*	-175 ± 172*	0.2281	-0.1827	-19.8 ± 8.7	22.3 ± 11.9	19.2 ± 27.4	32.0 ± 31.4
71-54A	90.21	-92 ± 48	-62.9 ± 2.6	-170 ± 112*	-53.4 ± 1.6	-0.01237	-0.3833	1.1 ± 0.6	24.1 ± 1.0	2.1 ± 1.4	20.5 ± 0.6
64-52B(H)	98.68	-88 ± 28†	-3 ± 10*	-37 ± 8	-34 ± 11	-0.4287	-0.5139	37.7 ± 12.0	1.5 ± 5.1	15.9 ± 3.4	17.5 ± 5.7
64-52B(B)	98.68	-79 ± 28†	-42 ± 35∇	-69 ± 21†	-26.1 ± 8.9	-0.4287	-0.5139	33.9 ± 12.0	21.6 ± 18.0	29.8 ± 9.0	13.4 ± 4.6

Key: * Standard deviation × 2
† Standard deviation × 4
∇ Standard deviation × 10

affected by harmonic coefficients of high degree, and are therefore powerful in determining the values for high degree. The inclination of 1965-09A was 31.8°, and an orbit determination was undertaken with the aim of analysing the variations in eccentricity at resonance. Orbits were determined[2] at 73 epochs from 4057 observations, chiefly US Navy, at the times when the effects of resonance on eccentricity were expected to be greatest - between November 1973 and September 1974 (37 orbits), and between April 1975 and January 1976 (36 orbits). Good values of the four relevant lumped harmonics were determined and are given in Table 2, with the standard deviations increased as specified in Ref 2, to allow for the neglect of harmonics of degree greater than 36.

3.3 Cosmos 236, 1968-70A

This satellite, at an orbital inclination of 56.1°, passed slowly through 15th-order resonance between July 1983 and October 1984, and the orbit has recently[3] been determined at 77 epochs by A.N. Winterbottom from 4744 observations, including 284 Hewitt camera observations. Well-defined values were determined for the six relevant lumped harmonics of 15th order and for two of 30th order; so this satellite contributes to all the solutions, for 15th and 30th order.

4 METHOD OF SOLUTION FOR 15th ORDER

The method is a modified weighted-least-squares, with extra equations and with rules for relaxing the standard deviations of ill-fitting values of the lumped harmonics.

The main equations for 15th order are of the form (3) or (4), and the extra equations are constraints of the form

$$\bar{C}_{\ell,15} = 0 \pm 10^{-5}/\ell^2 \quad , \tag{6}$$

and similarly for S . These extra equations express the expectation[5] that $\bar{C}_{\ell,15}$ will be of order $10^{-5}/\ell^2$, as is confirmed in a general way by the Goddard Earth Model 10C (Ref 8). As in our previous evaluation[1], we discarded all the constraint equations for $\ell \leqslant 23$, because these coefficients were so well determined that the constraints seemed unhelpful: the equations embody the instruction 'keep this coefficient as small as possible', an instruction which is undesirable when (for example) we have determined that $10^9\bar{C}_{15,15} = -20.4 \pm 0.4$, so that its value is very unlikely to be numerically less than 18 or 19. The constraint equations were retained for $\ell \geqslant 24$, with the proviso that, if the value for the coefficient of degree ℓ exceeded $10^{-5}/\ell^2$, the constraint was relaxed until the weighted residual was 1.0. The occasions when this happened are indicated in the tables giving the residuals.

The rules for the relaxation of standard deviations on ill-fitting values of the lumped harmonics were the same as adopted previously. As usual, the weighted residual is defined as the residual divided by the assumed standard deviation. If the weighted residual exceeds a chosen value - and 1.4 proved to be a convenient choice - the standard deviation is doubled. If the weighted residual still exceeds 1.4, the standard deviation is doubled again. Occasionally the weighted residual even then exceeds 1.4, and as a last resort the standard deviation is increased to ten times its original value. Relaxation by a factor of 10 is tantamount to rejection, but there is no harm in retaining the values, because their weighted residuals are near the average (which is 0.8). The relaxations necessary under these rules are indicated in Tables 1, 2 and 7.

Though successful in the past, this procedure is empirical, and there can be no guarantee that the solution obtained will be stable or unique. Despite the past successes, we did have problems here with the 15th-order harmonics of odd degree, in that several somewhat different fittings were possible: we chose the one that was least oscillatory.

5 SOLUTION FOR COEFFICIENTS OF ORDER 15 AND ODD DEGREE

As before, we have evaluated 11 coefficients, of degree 15,17,19,...35. There are three reasons for this choice. First, the Q coefficients are large up to degree 35 for some of the low-inclination satellites. Second, the measure of fit ε improves for up to 11 coefficients, but is not appreciably better for 12. Third, this choice facilitates comparisons with two comprehensive models of the gravity field that go to degree 36, namely GEM 10B (Ref 8) and the European GRIM3-L1 (Ref 9).

The solution for the odd-degree coefficients of order 15 is given, with standard deviations, in Table 3.

For the C coefficients in Table 3, comparison with our previous solution[1] shows only small changes, always less than 1 standard deviation. However, the new value of $\bar{C}_{15}^{0,1}$ for 1968-70A, namely $(-213.5 \pm 5.4) \times 10^{-9}$, was somewhat in conflict with the existing value for 1963-53B, namely $(-233.4 \pm 3.3) \times 10^{-9}$ at nearly the same inclination. It was necessary to double the standard deviation for both, as shown in Table 1, and consequently the standard deviations of the values in Table 3 are slightly larger than before, on average 11% higher. The

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new value of ε (where ε^2 is the sum of the squares of the weighted residuals divided by the number of degrees of freedom) is 0.89, as against 0.79 previously. (We suspected previously that the standard deviation for 1965-53B was too small, but there was no justification for changing it, as it satisfied the procedure for solution that we specified.) The new fitting is shown graphically in Fig 1, where the lumped harmonics are multiplied by $\bar{F}_{15,15,7}$.

Table 3

Solution for odd-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$

ℓ	$10^9\bar{C}_{\ell,15}$	$10^9\bar{S}_{\ell,15}$
15	-20.5 ± 0.4	-6.7 ± 0.5
17	6.5 ± 0.6	3.4 ± 0.6
19	-16.5 ± 0.7	-14.4 ± 0.7
21	18.5 ± 0.5	12.3 ± 1.1
23	21.4 ± 1.1	-1.6 ± 1.5
25	-5.3 ± 1.8	2.6 ± 2.3
27	-3.7 ± 1.4	9.7 ± 2.0
29	-7.9 ± 1.3	-5.7 ± 1.5
31	16.6 ± 2.5	-2.7 ± 3.4
33	-1.8 ± 2.8	-10.0 ± 3.5
35	-8.2 ± 3.7	1.1 ± 4.6

For the S coefficients in Table 3, the solution was not straightforward, because it was possible to arrive at different fittings depending on the order in which the relaxations were made. After many trials, we chose the solution which was least oscillatory: it is shown in Fig 1 and is smoother than the fitting in Ref 1. The uncomfortable feature of Ref 1 was a relaxation by a factor of 10 for 1965-53B. This 'rejection' is not now acceptable, because the value is given some support by the new result from 1968-70A: the new solution uses both satellites, but with standard deviation quadrupled. The only other change was a further doubling of the standard deviation on 1979-82A, for which the original fitting[7] was unsatisfactory. The alternative solutions involved keeping smaller standard deviations for 1965-53B and 1968-70A at the expense of inducing oscillations elsewhere and requiring relaxation of other apparently accurate values (*eg* 1963-24B).

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The new S solution of Table 3 has larger standard deviations than the old, on average 10% larger, and the value of ε is 0.90, as compared with 0.82 previously. The new values of the S coefficients differ from the previous set by 0.8 sd on average (using the new standard deviation as the measure). The largest change is 1.4 sd for $\bar{S}_{31,15}$, and the largest change among the early coefficients is for $\bar{S}_{21,15}$ which goes from $(10.8 \pm 0.9) \times 10^{-9}$ to $(12.3 \pm 1.1) \times 10^{-9}$.

Table 4

Weighted residuals in the equations for odd-degree harmonics

Satellite equations			Constraint equations		
Satellite	$\bar{C}^{0,1}_{15}$	$\bar{S}^{0,1}_{15}$	ℓ	$\bar{C}_{\ell,15}$	$\bar{S}_{\ell,15}$
65-09A	-0.08	0.18	25	0.33	-0.16
69-68B	0.34	-0.03	27	0.27	-0.71
64-84A	-0.39	-0.21	29	0.66	0.48
79-82A	0.15	0.97	31	-1.00R	0.26
71-30B	-0.75	-0.19	33	0.19	1.00R
74-34A	0.47	-0.05	35	1.00	-0.14
71-58B	0.84	0.86			
62-15A	-1.37	0.43			
65-53B	-0.79	-1.02			
68-70A	1.10	-1.40			
63-24B	0.88	0.70			
70-87A	-1.12	-1.20			
65-14A	0.77	-0.71			
77-12B	0.88	0.27			
71-106A	-0.90	0.64			
71-10B	-0.59	0.95			
71-18B	-0.99	0.91			
70-111A	-0.53	0.06			
71-13B	0.72	-0.71			
77-95B	0.98	0.97			
67-42A	0.07	-0.55			
70-19A	0.15	1.19			
67-73A	0.11	0.10			
71-54A	-0.04	0.01			
64-52B(H)	-0.15	1.14			
64-52B(B)	-0.70	1.01			
66-63A	-1.18	-0.81			

The weighted residuals for each lumped harmonic in the solution, and for the constraint equations, are given in Table 4. The symbol 'R' indicates that the constraint $10^{-5}/\ell^2$ was relaxed to give a weighted residual of 1.0. In accordance with the rules specified in section 4, the standard deviations were

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relaxed to ensure that none of the weighted residuals exceeded 1.4 (or 1.0 for the constraint equations). For 1964-52B the symbols (H) and (B) refer to the two analysts, Hiller[10] and Boulton[11,12].

Fig 1 shows the fittings graphically, with the curves given by GEM 10B (Ref 8) for comparison.

Further comments on the solutions will be found in section 7.

6 SOLUTION FOR COEFFICIENTS OF ORDER 15 AND EVEN DEGREE

As with the coefficients of odd degree, and for the same reasons, we have evaluated 11 coefficients of even degree 16,18,20,...36 . As mentioned in section 3, there are 42 equations of type (4) and there are also seven constraint equations of type (6). Thus we have 49 equations for the C coefficients, and 49 for the S coefficients. The solution is given in Table 5.

Table 5

Solution for even-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$

ℓ	$10^9\bar{C}_{\ell,15}$	$10^9\bar{S}_{\ell,15}$
16	-13.2 ± 1.2	-26.5 ± 0.8
18	-41.5 ± 1.3	-17.2 ± 0.9
20	-23.3 ± 1.1	-1.9 ± 0.9
22	23.3 ± 1.4	6.7 ± 1.2
24	-1.3 ± 1.6	-23.5 ± 1.4
26	-14.7 ± 1.7	5.2 ± 1.5
28	-10.7 ± 1.6	1.0 ± 1.4
30	-8.5 ± 2.5	-14.9 ± 1.7
32	19.7 ± 4.1	2.5 ± 2.6
34	-2.4 ± 4.1	14.0 ± 3.2
36	10.7 ± 4.5	-9.2 ± 2.9

A preliminary new solution for even-degree coefficients was derived in Ref 2 by adding the results from 1965-09A. The further addition of the four lumped harmonics from 1968-70A considerably modifies this preliminary solution, so we shall ignore it and refer back to the 'previous solution' of Ref 1, as for the odd-degree coefficients.

Table 5 shows that, for the higher-degree harmonics, the new solution has much smaller standard deviations than the previous solution, due largely to the new results from 1965-09A at low inclination (31.8°). For the coefficients of degree 24-30, the standard deviations have been reduced by a factor of 3.1 on average; for degree 32-36 the improvement is by a factor of 1.4. Consequently, if we loosely define 'well-determined' values as those with standard deviation $\leqslant 1.7 \times 10^{-9}$ (equivalent to an error less than about 1 cm in geoid height), the well-determined values now extend up to degree 28, instead of up to degree 20 as before. This is a considerable advance on our previous evaluation.

The new fittings are shown graphically in Figs 2 and 3, and the weighted residuals for the 84 lumped harmonics are given in Table 6. The value of ε is 0.92 for C, and 0.83 for S.

Table 6

Weighted residuals in the equations for even-degree harmonics

Satellite equations					Constraint equations		
Satellite	$\bar{C}_{15}^{1,0}$	$\bar{C}_{15}^{-1,2}$	$\bar{S}_{15}^{1,0}$	$\bar{S}_{15}^{-1,2}$	ℓ	$\bar{C}_{\ell,15}$	$\bar{S}_{\ell,15}$
65-09A	0.76	-0.08	0.19	-0.17	24	0.07	1.00R
79-82A	-0.15	-0.40	-0.19	1.02	26	1.00	-0.35
71-30B	0.33	-1.27	-0.63	0.90	28	0.84	-0.08
74-34A	0.55	-1.31	-0.29	0.24	30	0.76	1.00R
71-58B	-1.11	-0.75	0.92	-0.54	32	-1.00R	-0.25
62-15A	0.47	1.20	0.89	-0.00	34	0.27	-1.00R
65-53B	0.72	1.05	0.14	0.15	36	-1.00R	1.00R
68-70A	1.31	-1.01	-0.88	0.82			
63-24B	-1.30	1.00	-0.09	1.07			
65-14A	-0.02	0.35	-0.32	-0.19			
71-106A	-0.49	-1.18	-0.61	0.43			
71-10B	-1.09	0.10	-0.10	1.02			
70-111A	0.21	-0.21	-0.77	-1.31			
71-13B	-0.56	0.40	0.01	0.27			
77-95B	0.78	-0.09	0.99	-0.12			
67-42A	-0.88	-1.26	-0.05	-1.02			
70-19A	0.74	-0.34	1.31	-0.89			
67-73A	-0.31	-0.83	1.22	-0.70			
71-54A	-0.35	0.52	-0.88	0.22			
64-52B(H)	-1.12	1.35	0.60	-1.01			
64-52B(B)	-0.79	-0.72	-1.30	-0.36			

In the solution for the C coefficients, it was necessary to relax the accuracy of $\bar{C}_{15}^{-1,2}$ for the new satellite, 1968-70A. Erring on the side of caution, we decided to keep the doubled standard deviation for $\bar{C}_{15}^{1,0}$ from 1965-09A recommended in Ref 2: a solution with this value unrelaxed is possible, but leads to large values for the coefficients of degree 32 and 36.

In the solution for the S coefficients it was obvious that the new value of $\bar{S}_{15}^{-1,2}$ from 1968-70A would not fit, and in the end it has to be relaxed by a factor of 10. (This is believed[3] to result from difficulties in fitting the resonant variation in eccentricity for orbits of exceptionally low eccentricity, $e < 0.001$.) Unfortunately, the residual for $\bar{S}_{15}^{1,0}$ from 1968-70A remained just above the specified limit of 1.4 and so, under our rules, this value had to be relaxed by a factor of 2. No other changes were needed.

The new solution in Table 5, as well as having much lower standard deviations than before for the high-degree coefficients, shows some significant changes in the numerical values of the coefficients. For degree 16-22, the values are altered on average by only 0.4 sd, and none changes by more than 0.8 sd (where the standard deviation is that of the previous solution). The largest change is for $\bar{C}_{22,15}$ which goes from $(24.9 \pm 1.9) \times 10^{-9}$ to $(23.3 \pm 1.4) \times 10^{-9}$. For degree 24-36, the average change is 1.3 sd and the greatest is 2.1 sd: the larger changes are concentrated at the highest degrees, where the values were, and still are, poorly defined. The change of 2.1 sd occurs with $\bar{C}_{34,15}$ which goes from $(7.9 \pm 4.9) \times 10^{-9}$ to $(-2.4 \pm 4.1) \times 10^{-9}$.

Fig 2 shows good agreement for C between the curves fitted to our values and the curves from GEM 10B. In Fig 3 the GEM 10B curves for S agree well with ours except at low inclinations, where the curve for GEM 10B is sure to be close to zero because all the high-degree $\bar{S}_{\ell,15}$ coefficients in GEM 10B are small; our curve, however, has to fit the rather large negative value of $\bar{S}_{15}^{1,0}$ near $i = 44^{\circ}$ and exhibits quite a deep minimum there. As we showed in Fig 4 of Ref 1, the curve for $\bar{F}\bar{S}_{15}^{-1,2}$, when moved to the right by 6° in inclination, almost coincides with that of $\bar{F}\bar{S}_{15}^{1,0}$. So there is a minimum in the curve for $\bar{F}\bar{S}_{15}^{-1,2}$ near inclination 39° - a minimum which is apparently uncalled for as there are no values to provoke it: it can be regarded as a 'reflection' of the dip in $\bar{F}\bar{S}_{15}^{1,0}$ at inclinations near 44°.

It is a pity that there is no independent check on the value of $\bar{S}_{15}^{1,0}$ at 44°, which obviously influences the course of the curve. Its weighted residual is small, -0.19, but we tried the effect of relaxing this standard deviation by a factor of 10. The value of ε inevitably decreases, but the decrease is surprisingly small, from 0.83 to 0.82, and there is still a deep minimum in $\bar{F}\bar{S}_{15}^{1,0}$ near $i = 44^{\circ}$, although of course it is less pronounced than before, the drop between 52° and 44° being 20% less. We concluded that we had no justification for relaxing the standard deviation of the value at 44°.

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7 DISCUSSION OF THE 15TH-ORDER SOLUTIONS

Our aim in these evaluations has been to derive reliable and accurate values of 15th-order coefficients, firstly for their own sake, so as to establish their values, and secondly so as to provide a test of comprehensive gravity models[8,9,13,14], the accuracy of which is extremely difficult to assess because of the complexity of the solutions.

It should be said first that when testing comprehensive gravity models it is best to work with the values of the lumped harmonics, which constitute the primary data: this procedure has been adopted by Klokočník[15,16] and Wagner[17], who have made the most illuminating analyses.

However, the individual coefficients should in principle be capable of evaluation if lumped values are available over a wide enough range of inclinations. Why then do we need to relax a number of the standard deviations to achieve good fittings?

The first answer to this question is that several of our 26 satellites were either of rather high drag or had rather poorly-determined orbits, and we had no hesitation in relaxing the accuracy of the values for these satellites when necessary. Over-accurate fitting of the orbital variations, giving lumped harmonics with over-optimistic standard deviations, will inevitably occur by chance in a proportion of analyses where the data is poor. The alternative to relaxation would have been to remove the offending satellite altogether: but this is virtually the same as the relaxation by a factor of 10 which was applied when necessary, and, as it happened that such relaxations were never needed for both the C and S coefficients, the complete removal of any satellite would have removed some useful data. A good example is 1971-18B at inclination near 70° in Fig 1: the C value is relaxed by a factor of 4 and is almost useless; but the S value in Fig 1 fills a gap in inclination and, when it was relaxed as a trial, the fit was not altered, so that the value is *prima facie* reliable.

The second answer to the question is paradoxically the exact opposite of the first: some of the lumped harmonics may be *too* accurate for a fitting with only 11 coefficients. Imagine that we had highly-accurate lumped harmonics at 1° intervals in inclination. This would define the true variation with i, which would probably turn out to be very irregular. Any attempt to fit the variation with a set of 11 coefficients would be doomed to failure, and great relaxations of many of the very small standard deviations would be essential to achieve a credible fitting. More than 11 coefficients could of course be used

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in this imaginary scenario, but here we do not have that option (a) because we do not have enough orbits, and (b) because coefficients of very high degree (>36) have appreciable effects only on orbits at inclinations less than 30°, of which we have none. Using too many coefficients in our fitting would reduce the reliability of the solution by introducing spurious oscillations.

To summarize these answers, we should expect to have to make relaxations for orbits with high drag or poor data, where over-accurate fittings have arisen by chance, and also for a few of the most accurate values which may not all be amenable to a fitting with only 11 coefficients.

8 SOLUTION FOR COEFFICIENTS OF ORDER 30 AND EVEN DEGREE

8.1 Introduction

In our previous evaluation we had results from 9 satellites. With the addition of 1968-70A we now have 10 satellites, and, as the orbital inclination of 1968-70A is in a region where coverage was previously weak, the solution is considerably strengthened.

In our previous evaluation we commented that the solution for the C coefficients was very good, but that the solution for the S coefficients was not satisfactory, with conflicting values of lumped harmonics at inclinations near 60°. The source of this conflict has now been identified as an error in sign in the value taken for the lumped harmonic $\bar{S}^{0,2}_{30}$ for 1963-24B at 58° inclination.

8.2 The lumped harmonics and the equations to be solved

Table 7 gives the 11 values of the lumped harmonics on which the new solution is based. To these 11 equations of the form (5) we add, as usual, constraint equations of the form

$$\bar{C}_{\ell,30} = 0 \pm 10^{-5}/\ell^2 \qquad (7)$$

(and similarly for the S coefficients), so that we have 11 + N equations to solve for N coefficients. As before, we relax (by a factor of 2 or 4 as necessary) the standard deviations of lumped harmonics for which the weighted residual in the solution exceeds 1.4, and we also make one relaxation of the $10^{-5}/\ell^2$ to ensure that the weighted residual does not exceed 1.0.

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Table 7

Values of even-degree lumped harmonics $(\bar{C},\bar{S})^{0,2}_{30}$

Satellite	i (deg)	$10^9\bar{C}^{0,2}_{30}$	$10^9\bar{S}^{0,2}_{30}$	$\bar{F}_{30,30,14}$	$10^9\bar{F}_{30,30,14}\bar{C}^{0,2}_{30}$	$10^9\bar{F}_{30,30,14}\bar{S}^{0,2}_{30}$
1974-34A	50.64	597 ± 558	679 ± 651	0.000952	0.57 ± 0.53	0.65 ± 0.62
1968-70A	56.08	-34 ± 149	-624 ± 212*	0.006277	-0.21 ± 0.94	-3.92 ± 1.33
1963-24B	58.20	46 ± 106	-253 ± 88	0.01176	0.54 ± 1.25	-2.98 ± 1.03
1965-14A	65.02	-46 ± 23	-37 ± 19	0.06196	-2.85 ± 1.43	-2.29 ± 1.18
1971-10B	65.83	-54 ± 27	59 ± 80*	0.07292	-3.94 ± 1.97	4.30 ± 5.83
1970-111A	74.00	19.2 ± 4.9	4.1 ± 4.4	0.2579	4.95 ± 1.26	1.06 ± 1.13
1971-13B	74.05	27.1 ± 5.5	6.0 ± 3.3	0.2594	7.03 ± 1.43	1.56 ± 0.86
1967-42A	80.17	-9.1 ± 4.6	-5.0 ± 11.0*	0.4340	-3.95 ± 2.00	-2.17 ± 4.77
1971-54A	90.21	-9.81 ± 0.58	9.00 ± 0.75	0.4755	-4.66 ± 0.28	4.28 ± 0.36
1964-52B(H)	98.68	22.8 ± 7.9	38 ± 41†	0.2502	5.7 ± 2.0	9.5 ± 10.3
1964-52B(B)	98.68	39 ± 21*	52 ± 40†	0.2502	9.8 ± 5.3	13.0 ± 10.0

Key: * Standard deviation × 2
† Standard deviation × 4

When the C and S equations are solved for N coefficients, the values of ε are as follows, for $3 \leqslant N \leqslant 6$.

N	3	4	5	6
ε for C equations	1.47	1.06	0.99	0.88
ε for S equations	1.15	1.11	0.96	0.89

The 6-coefficient solutions are chosen, because ε decreases substantially between $N = 5$ and $N = 6$, and because the Q coefficients are quite large up to $\ell = 40$ for 1974-34A and 1968-70A.

8.3 The 6-coefficient solution

The values obtained in the 6-coefficient solution are listed in Table 8, with their standard deviations.

Table 8

Solution for even-degree $\bar{C}_{\ell,30}$ and $\bar{S}_{\ell,30}$

ℓ	$10^9\bar{C}_{\ell,30}$	$10^9\bar{S}_{\ell,30}$
30	-3.2 ± 0.9	7.4 ± 1.0
32	-8.6 ± 1.8	4.7 ± 1.7
34	-13.3 ± 2.2	-5.6 ± 2.4
36	-3.7 ± 3.1	5.5 ± 4.0
38	6.8 ± 3.2	3.8 ± 4.0
40	4.6 ± 2.6	-4.0 ± 3.1

The residuals for each of the lumped harmonics and the constraint equations are given in Table 9.

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Table 9

Weighted residuals in the equations for 30th-order harmonics

Satellite equations			Constraint equations		
Satellite	$\bar{C}_{30}^{0,2}$	$\bar{S}_{30}^{0,2}$	ℓ	$\bar{C}_{\ell,30}$	$\bar{S}_{\ell,30}$
74-34A	0.24	0.47	30	0.29	-0.67
68-70A	-0.63	-1.28	32	0.88	-0.48
63-24B	0.87	0.31	34	1.00R	0.64
65-14A	0.07	-0.33	36	0.48	-0.72
71-10B	-0.75	0.97	38	-0.99	-0.55
70-111A	-0.53	-0.30	40	-0.74	0.65
71-13B	0.99	0.18			
67-42A	-1.00	-1.13			
71-54A	0.04	0.10			
64-52B(H)	0.23	0.81			
64-52B(B)	0.84	1.19			

Fig 4 shows the fitting of the lumped harmonics, multiplied by the appropriate F factor, with the values from GEM 10B plotted as broken lines.

8.4 Discussion of the solution

The values for the C coefficients are close to those in the previous solution, and the curve (Fig 4) fits the lumped harmonics very well, The largest change in the value of an individual coefficient is for $\bar{C}_{40,15}$, which changes from (6.0 ± 2.8) to (4.6 ± 2.6) × 10^{-9}. On average the standard deviations decrease by 5%.

We commented that the previous S solution was unsatisfactory, firstly because the fitting was poor at inclinations between 55° and 70°, and secondly because it was necessary to relax the accuracy of the lumped harmonic from 1965-14A, although this satellite was thought to be more reliable than its 'competitor' 1971-10B. This conflict has been resolved by the reversal in the sign of the lumped harmonic for 1963-24B: we much regret this error, which arose from an unnoticed misprint in the original paper. The new fit for the S coefficients is shown by a full line in Fig 4: the curve differs greatly from the previous one at inclinations less than 70°, but is scarcely changed for inclinations greater than 70°. In the new fitting, it is satisfactory that the lumped harmonic from 1965-14A has its standard deviation restored to the original value (see Table 7); the standard deviation for 1971-10B has to be doubled, but this satellite was expected to be less reliable. The fitting for S, though

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improved, is still not completely satisfactory because the new value from 1968-70A requires a doubled standard deviation and even then does not fit well. Consequently, the standard deviations of the S values for degree 36,38 and 40 in Table 8 are appreciably worse than those for the C values: indeed, as these three S values are also quite small, they cannot be regarded as determinate. In the previous solution they were also small and indeterminate, so the changes in these values are of no consequence in themselves; but they do imply changes in the earlier coefficients which are significant. In particular, $10^9\bar{S}_{32,30}$ is now 4.7 ± 1.7 instead of 0.6 ± 2.3 and $10^9\bar{S}_{34,30}$ is now -5.6 ± 2.4 instead of 5.4 ± 2.9. The value of $\bar{S}_{30,30}$ is not significantly changed.

8.5 Coefficients of 30th order and odd degree

In theory it is possible to derive lumped harmonics of order 30 and odd degree from analysis of the eccentricity. In practice satellites of very low drag are needed to ensure that unmodelled atmospheric perturbations do not spoil the fitting. So far only three satellites have yielded values of these coefficients - 1971-54A, 1965-14A and 1968-70A - and this is not enough to allow evaluation of individual coefficients.

9 COMPARISONS WITH VALUES IN COMPREHENSIVE GEOPOTENTIAL MODELS

9.1 Coefficients of 15th order

Several comprehensive geopotential models, such as Rapp's 1981 model[13] and GRIM3-L1 (Ref 9), have utilized our previous values of 15th-order coefficients; so comparisons are not helpful. However, it is believed that GEM 10B and the recent GEM-T1 are independent of our values; so their accuracy can be tested if our values are the more accurate. The nominal accuracy of GEM-T1 is given[14] as between 3 and 5×10^{-9} for most of the relevant coefficients, whereas our standard deviations do not exceed 1.5×10^{-9} up to degree 23; so a comparison seems worth making and is shown in Table 10. For degree 15-23, the mean difference between our 18 values and the corresponding values in the GEMs is 3.1×10^{-9} for GEM 10B and 2.4×10^{-9} for GEM-T1. This strongly suggests that the standard deviations given for these coefficients in GEM-T1 (on average 3.1×10^{-9}) are realistic. It should be noted, however, that the agreement is not so good for higher degrees: going up to degree 24, rather than degree 23, gives mean differences of 3.9×10^{-9} for GEM 10B and 3.2×10^{-9} for GEM-T1.

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Table 10

Comparison of our 15th-order values with GEM 10B and GEM-T1 up to degree 24

ℓ	$10^9\bar{C}_{\ell,15}$			$10^9\bar{S}_{\ell,15}$		
	GEM 10B	GEM-T1	Our values	GEM 10B	GEM-T1	Our values
15	-19.7	-18.1 ± 3	-20.5 ± 0.4	-6.4	-8.1 ± 3	-6.7 ± 0.5
16	-14.4	-12.5 ± 4	-13.2 ± 1.2	-27.8	-32.3 ± 4	-26.5 ± 0.8
17	2.5	4.9 ± 1	6.5 ± 0.6	4.8	5.7 ± 1	3.4 ± 0.6
18	-48.3	-37.8 ± 4	-41.5 ± 1.3	-18.6	-19.8 ± 4	-17.2 ± 0.9
19	-20.6	-18.3 ± 3	-16.5 ± 0.7	-15.3	-12.8 ± 3	-14.4 ± 0.7
20	-23.9	-22.7 ± 3	-23.3 ± 1.1	4.8	-0.4 ± 3	-1.9 ± 0.9
21	16.2	16.6 ± 3	18.5 ± 0.5	9.5	15.0 ± 3	12.3 ± 1.1
22	24.1	27.9 ± 3	23.3 ± 1.4	-1.3	3.1 ± 3	6.7 ± 1.2
23	15.4	17.7 ± 4	21.4 ± 1.1	4.1	-2.3 ± 4	-1.6 ± 1.5
24	3.1	9.8 ± 4	-1.3 ± 1.6	-5.1	-13.5 ± 4	-23.5 ± 1.4

For the recent WGS 84 model[18], values of the coefficients are available only up to degree 18, and it is possible that the model utilizes our earlier values; so comparisons are of dubious worth. The eight values of coefficients of degree 15-18 in WGS 84 differ from the corresponding values in our solutions by 2.5×10^{-9} on average.

9.2 Coefficients of 30th order and even degree

For 30th order, the models GEM 10B, GEM-T1 and GRIM3-L1 apparently do not make use of our previous values, and Rapp (1981) uses only the values for $\ell = 30$; so it seems legitimate to make comparisons, which are shown in Table 11 for coefficients up to degree 36. The respective authors estimate the average standard deviation of GEM-T1, GRIM3-L1 and Rapp (1981) as 5, 3 and 2.5×10^{-9} respectively, while our standard deviations range between 0.9 and 4.0×10^{-9}.

Our standard deviations are too large to allow any firm conclusions: but the mean differences between our values and the corresponding values in GEM 10B, GEM-T1, GRIM and Rapp are 4.2, 6.5, 3.9 and 4.0×10^{-9} respectively (excluding $\ell = 30$ for Rapp). These differences would be consistent with accuracies of about 5×10^{-9} in GEM-T1 and about 3×10^{-9} in the other models, *ie* in line with the authors' estimates given in the previous paragraph*.

* The greatest difference is for $\bar{C}_{32,30}$ with GEM-T1, which is a satellite-only model. A more recent model (PGS 3325), with surface gravity and altimeter data added, gives $10^9\bar{C}_{32,30} = -5.0 \pm 1.5$ (J.G. Marsh, Private Communication). This is much closer to our value.

Table 11

Comparison of our 30th-order values with comprehensive models, for $\ell \leqslant 36$

ℓ	$10^9\bar{C}_{\ell,30}$					$10^9\bar{S}_{\ell,30}$				
	GEM 10B	GEM-T1	GRIM3-L1	Rapp	Our values	GEM 10B	GEM-T1	GRIM3-L1	Rapp	Our values
30	-5.2	-1.5	-0.6	(-3.3)	-3.2 ± 0.9	11.1	-0.4	7.1	(7.5)	7.4 ± 1.0
32	-0.6	8.3	-6.9	-6.7	-8.6 ± 1.8	-0.2	1.7	-1.0	0.5	4.7 ± 1.7
34	-11.9	-6.1	-23.0	-22.9	-13.3 ± 2.2	1.2	0.0	0.7	-0.6	-5.6 ± 2.4
36	-3.9	-1.6	-7.6	-6.0	-3.7 ± 3.1	-0.9	-2.0	6.4	4.8	5.5 ± 4.0

There may be merit in comparing the mean of the four with the corresponding values in our solution: the average difference is 3.3×10^{-9}, which is better than for any individual model. Though it is impossible to allot a nominal standard deviation to the mean of the models, a value near 3×10^{-9} would be plausible, while the mean standard deviation of our values is 2.1×10^{-9}; so the observed mean difference of 3.3×10^{-9} is very satisfactory.

10 REVISED SOLUTIONS AFTER MORE ACCURATE COMPUTATION OF G-FUNCTIONS

10.1 Correction of the lumped harmonics

In all our previous evaluations of the individual harmonic coefficients, we have worked with the computer programs THROE, SIMRES and PROF, in which the values of the eccentricity function $G_{\ell pq}$ (defined in Ref 4) are obtained from an approximation $\hat{G}_{\ell pq}$ that ignores terms of order e^2 relative to the main term. Most of the orbits analysed have $e \leqslant 0.011$, and the error in using $\hat{G}_{\ell pq}$ instead of $G_{\ell pq}$ is very small; but there are some orbits with $e > 0.02$ for which the approximation is significantly in error. Recently, A.W. Odell has written the program GQUAD (described in Ref 4), in which G is evaluated accurately by numerical integration. We now take the opportunity of replacing $\hat{G}$ by G whenever necessary and recalculating the values of the coefficients derived in section 5 to 8.

The theory for G is given in Ref 4, and it is not appropriate to go into detail here. Briefly, the fitting of the changes in inclination at resonance with THROE leads to a numerical value of $G_{15,7,0}\bar{C}^{0,1}_{15}$. Previously we have calculated a (slightly incorrect) lumped harmonic value, $\hat{\bar{C}}^{0,1}_{15}$ say, by taking this numerical value equal to $\hat{G}_{15,7,0}\hat{\bar{C}}^{0,1}_{15}$. The correct value for the lumped harmonic is obviously given by:

$$\bar{C}^{0,1}_{15} = \hat{\bar{C}}^{0,1}_{15}\left(\hat{G}_{15,7,0}/G_{15,7,0}\right) . \tag{8}$$

Thus the previously derived values of $\bar{C}^{0,1}_{15}$ and their standard deviations must all be multiplied by $\hat{G}_{15,7,0}/G_{15,7,0}$. A similar procedure applies for the other lumped harmonics, with appropriate changes of suffix. For example,

$$\bar{C}^{-1,2}_{15} = \hat{\bar{C}}^{-1,2}_{15}\left(\hat{G}_{16,7,-1}/G_{16,7,-1}\right) . \tag{9}$$

The corrected values of the lumped harmonics, to replace those in Tables 1 and 2, are given in Tables 12 and 13. The correction factor, f say, is always less than 1. In Table 12 there are 18 orbits with $e \leqslant 0.011$, and for these $f \geqslant 0.99$; for the orbits that are most accurate, $e < 0.005$ and $f \geqslant 0.999$, so that the changes are negligible. For the orbits of higher eccentricity the corrections are larger but still not significant by comparison with the standard deviation: the largest is for 1971-106A, but is still less than 10% of the sd. For the values in Table 13 the correction factor for $(q,k) = (1,0)$ is the same as for $(q,k) = (-1,2)$. Again the largest change is for 1971-106A, but is less than 0.2 sd.

10.2 Correction of the Q coefficients

It is not only the lumped harmonics that are affected by the use of $\hat{G}$ instead of G : each Q coefficient also needs to be adjusted, because each Q is the ratio of two G functions: see Ref 4 for details. Specifically $Q^{q,k}_{\ell}$ is proportional to $G_{\ell pq}/G_{\ell_o p_o q}$, where ℓ_o and p_o are the lowest values of ℓ and p that appear. We have previously taken the ratio of these G functions as $\hat{G}_{\ell pq}/\hat{G}_{\ell_o p_o q}$. Thus we need to multiply each Q_{ℓ} by a factor

$$\xi = \frac{G_{\ell pq}}{\hat{G}_{\ell pq}} \cdot \frac{\hat{G}_{\ell_o p_o q}}{G_{\ell_o p_o q}} = f\,\frac{G_{\ell pq}}{\hat{G}_{\ell pq}} , \tag{10}$$

where f is the correction factor applied to the lumped harmonic, as before. It turns out that $\xi \geqslant 1$ and that its value increases with ℓ and with e ; so its effect is greatest for low inclination and high eccentricity. Among the satellites in Table 1, 1964-84A is by far the most strongly affected, as $i = 38^{\circ}$ and $e = 0.042$. For this satellite the Q factor for the largest term in the lumped harmonic, namely $Q^{0,1}_{23}\bar{C}_{23,15}$, is increased by 15% and so some decrease in the (rather large) value of $\bar{C}_{23,15}$ is to be expected. We have corrected the Q factors for all the orbits with $e > 0.011$, and the new values are listed in Tables 18 to 21 (pages 29 and 30). The values of Q for the other satellites are given in Ref 19.

Table 12

Values of lumped harmonics $(\bar{C},\bar{S})_{15}^{0,1}$ for the 26 satellites with $G_{15,7,0}$ corrected

No.	Satellite	i (deg)	Semi-major axis (km)	e	$\left(10^9\bar{C}_{15}^{0,1}\right)_{corr}$	$\left(10^9\bar{S}_{15}^{0,1}\right)_{corr}$	Correction factor f
1	65-09A	31.76	6857	0.007	30890 ± 1950	13500 ± 960	0.997
2	69-68B	32.97	6857	0.004	20320 ± 750	6270 ± 910	0.999
3	64-84A	37.80	6860	0.042	510 ± 520	-1810 ± 1310	0.903
4	79-82A	43.60	6862	0.001	-467 ± 34	-767 ± 424†	1.000
5	71-30B	46.36	6869	0.011	-395 ± 50	-791 ± 30	0.993
6	74-34A	50.64	6872	0.002	-430.2 ± 10.0	-320.9 ± 8.3	1.000
7	71-58B	51.05	6874	0.011	-352 ± 93*	-246 ± 45	0.993
8	62-15A	53.82	6876	0.022	-360 ± 14	-111 ± 30	0.972
9	65-53B	56.04	6879	0.003	-233.4 ± 6.6*	-103 ± 34†	1.000
10	68-70A	56.08	6880	0.002	-213.5 ± 10.8*	-91.1 ± 16.8†	1.000
11	63-24B	58.20	6883	0.002	-110.6 ± 5.6	-41.6 ± 4.5	1.000
12	70-87A	62.92	6888	0.007	-5.4 ± 3.6	-31.3 ± 11.2†	0.997
13	65-14A	65.02	6892	0.003	4.8 ± 2.1	-7.1 ± 2.5	1.000
14	77-12B	65.49	6894	0.029	12.8 ± 11.8*	0.7 ± 12.7	0.952
15	71-106A	65.70	6895	0.045	-3.2 ± 37†	8 ± 15	0.890
16	71-10B	65.83	6893	0.002	-0.7 ± 4.1	2.4 ± 3.9	1.000
17	71-18B	69.84	6900	0.040	-34 ± 22†	9 ± 5	0.911
18	70-111A	74.00	6905	0.001	-26.0 ± 1.0	-5.2 ± 1.3	1.000
19	71-13B	74.05	6905	0.002	-24.6 ± 1.3	-6.1 ± 1.0	1.000
20	77-95B	75.82	6908	0.029	-21.4 ± 4.9	-2.9 ± 5.1	0.952
21	67-42A	80.17	6918	0.007	-23.0 ± 1.6	-8.6 ± 1.3	0.997
22	70-19A	81.16	6916	0.005	-21.0 ± 1.6	-1.1 ± 5.2†	0.999
23	67-73A	85.98	6925	0.025	-13.4 ± 2.2	-6.2 ± 3.2	0.964
24	71-54A	90.21	6930	0.002	-16.05 ± 0.21	-6.90 ± 0.21	1.000
25	64-52B(H)	98.68	6945	0.023	-27.4 ± 1.9	1.5 ± 7.8†	0.969
26	64-52B(B)	98.68	6945	0.023	-30.2 ± 4.5*	-4.3 ± 3.1	0.969
27	66-63A	144.16	7009	0.003	36900 ± 9700	12200 ± 9400*	1.000

Key: * Standard deviation × 2
† Standard deviation × 4

Table 13

Values of lumped harmonics $\left(\bar{C},\bar{S}\right)^{1,0}_{15}$ and $\left(\bar{C},\bar{S}\right)^{-1,2}_{15}$ after correction of G

Satellite	$\left(10^9\bar{C}^{1,0}_{15}\right)_{corr}$	$\left(10^9\bar{C}^{-1,2}_{15}\right)_{corr}$	$\left(10^9\bar{S}^{1,0}_{15}\right)_{corr}$	$\left(10^9\bar{S}^{-1,2}_{15}\right)_{corr}$	Correction factor f
65-09A	52900 ± 18200*	-340 ± 510	-5000 ± 9700	200 ± 790	0.998
79-82A	-860 ± 150	-234 ± 34	-1930 ± 160	185 ± 268†	1.000
71-30B	-418 ± 93	-200 ± 60*	-1020 ± 670∇	204 ± 80†	0.995
74-34A	-211.2 ± 24.9	-3.0 ± 8.6	128.4 ± 20.2	63.5 ± 3.4	1.000
71-58B	-464 ± 231†	-50 ± 92†	252 ± 119*	45 ± 14	0.995
62-15A	-75 ± 18	148 ± 59†	169 ± 75*	11 ± 33	0.982
65-53B	18 ± 17	106.9 ± 8.7	57 ± 23	2.4 ± 8.1	1.000
68-70A	20.4 ± 10.2	74.4 ± 23.2†	39.3 ± 15.2*	38 ± 44∇	1.000
63-24B	59.3 ± 10.2	101.5 ± 13.0*	26.8 ± 7.4	12.9 ± 16.4†	1.000
65-14A	74.8 ± 5.8	-12.4 ± 3.7	-9.6 ± 4.4	-29.4 ± 3.0	1.000
71-106A	47 ± 22	-62 ± 37†	-51 ± 65∇	-18 ± 22	0.927
71-10B	35.1 ± 23.4*	-20.0 ± 10.7	-13.8 ± 11.0	-18.9 ± 10.3	1.000
70-111A	-18.0 ± 3.3	-46.5 ± 2.7	-44 ± 25∇	-40.5 ± 4.0	1.000
71-13B	-19.8 ± 1.8	-45.5 ± 2.0	-24.8 ± 0.7	-35.2 ± 1.0	1.000
77-95B	-3 ± 25*	-61 ± 15	-4 ± 21*	-45 ± 17	0.969
67-42A	-54.6 ± 6.4*	-131 ± 21*	-37.1 ± 2.6	-97 ± 18*	0.998
70-19A	-26 ± 41∇	-128 ± 29	-15 ± 20†	-130 ± 37	0.999
67-73A	-85 ± 37	-119 ± 64	82 ± 117*	-171 ± 168*	0.977
71-54A	-92 ± 48	-62.9 ± 2.6	-170 ± 112*	-53.4 ± 1.6	1.000
64-52B(H)	-86 ± 27†	-3 ± 10*	-36 ± 8	-33 ± 11	0.980
64-52B(B)	-77 ± 27†	-41 ± 34∇	-68 ± 21†	-25.6 ± 8.7	0.980

Key: * Standard deviation × 2
† Standard deviation × 4
∇ Standard deviation × 10

10.3 Revised solutions for 15th order

With these corrected Q factors and the revised values of all the lumped harmonics given in Tables 12 and 13, we have derived revised solutions, taking the same relaxations in the standard deviations as before. The new solutions for odd degree are given in Table 14: the values of ε are 0.84 for C and 0.90 for S . Comparison with Table 3 shows that the most significant change, by 0.7 sd, is for $10^9\bar{C}_{23,15}$, which decreases from 21.4 ± 1.1 to 20.6 ± 1.0. There is also a change of about 0.8 sd in the poorly-determined $\bar{C}_{35,15}$. The average change is 0.23 sd. The standard deviations in Table 14 are on average 10% lower than in Table 3, none being higher: this can be read either as luck or as an indication that the corrections have led to better values for the coefficients.

The revised solutions for even degree are given in Table 15; the values of ε are 0.92 for C and 0.82 for S. The values of the coefficients and their standard deviations differ only trivially from those in Table 5, the largest change being 0.2×10^{-9} for $\bar{S}_{36,15}$. The average change is 0.03 sd.

The changes are too small to produce significant changes in Figs 1 to 3, and the weighted residuals are similar to those in Tables 4 and 6.

Table 14

Revised solutions for odd-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$

ℓ	$10^9\bar{C}_{\ell,15}$	$10^9\bar{S}_{\ell,15}$
15	-20.4 ± 0.4	-6.7 ± 0.4
17	6.6 ± 0.5	3.4 ± 0.6
19	-16.4 ± 0.6	-14.2 ± 0.7
21	18.3 ± 0.5	12.0 ± 1.0
23	20.6 ± 1.0	-1.4 ± 1.4
25	-5.8 ± 1.6	1.9 ± 2.0
27	-3.9 ± 1.3	9.7 ± 1.8
29	-8.3 ± 1.2	-5.4 ± 1.4
31	17.1 ± 2.2	-2.7 ± 3.0
33	-1.5 ± 2.4	-9.4 ± 3.0
35	-5.3 ± 3.4	2.4 ± 4.3

Table 15

Revised solutions for even-degree $\bar{C}_{\ell,15}$ and $\bar{S}_{\ell,15}$

ℓ	$10^9\bar{C}_{\ell,15}$	$10^9\bar{S}_{\ell,15}$
16	-13.2 ± 1.2	-26.5 ± 0.8
18	-41.4 ± 1.3	-17.2 ± 0.9
20	-23.2 ± 1.1	-1.9 ± 0.9
22	23.2 ± 1.4	6.7 ± 1.2
24	-1.4 ± 1.6	-23.5 ± 1.4
26	-14.7 ± 1.7	5.2 ± 1.5
28	-10.6 ± 1.6	1.1 ± 1.4
30	-8.4 ± 2.5	-14.9 ± 1.7
32	19.6 ± 4.1	2.4 ± 2.6
34	-2.3 ± 4.1	13.9 ± 3.2
36	10.7 ± 4.5	-9.4 ± 2.9

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10.4 Revised solutions for 30th order

Nearly all the orbits which yielded values of lumped 30th-order harmonics have $e \leqslant 0.007$, and the corrections to the lumped harmonics are scarcely significant. For 1964-52B with $e = 0.023$, however, there is a reduction of 11%. The corrected values of the lumped harmonics are given in Table 16.

The revised 30th-order solutions, with the Q values corrected for 1964-52B only, are given in Table 17. The values of $\bar{S}$ in Table 17 are the same as in Table 8, and so is ε, though one standard deviation has decreased. For $\bar{C}$, the value of ε decreases from 0.88 to 0.86, and the standard deviations are on average 3% lower than in Table 8. The changes in the residuals and in Fig 4 are too small to be worth recording.

Table 16

Values of even-degree lumped harmonics $\left(\bar{C},\bar{S}\right)^{0,2}_{30}$ with G corrected

Satellite	$\left(10^9\bar{C}^{0,2}_{30}\right)_{corr}$	$\left(10^9\bar{S}^{0,2}_{30}\right)_{corr}$	e	Correction factor f
1974-34A	596 ± 557	678 ± 650	0.002	0.999
1968-70A	-34 ± 149	-623 ± 212*	0.002	0.999
1963-24B	46 ± 106	-253 ± 88	0.002	0.999
1965-14A	-46 ± 23	-37 ± 19	0.003	0.998
1971-10B	-54 ± 27	59 ± 80*	0.002	0.999
1970-111A	19.2 ± 4.9	4.1 ± 4.4	0.001	1.000
1971-13B	27.1 ± 5.5	6.0 ± 3.3	0.002	0.999
1967-42A	-9.0 ± 4.5	-4.9 ± 10.9*	0.007	0.989
1971-54A	-9.80 ± 0.58	8.99 ± 0.75	0.002	0.999
1964-52B(H)	20.3 ± 7.0	34 ± 36†	0.023	0.890
1964-52B(B)	35 ± 19*	46 ± 36†	0.023	0.890

Key: * Standard deviation × 2
† Standard deviation × 4

Table 17

Revised solutions for even-degree $\bar{C}_{\ell,30}$ and $\bar{S}_{\ell,30}$

ℓ	$10^9\bar{C}_{\ell,30}$	$10^9\bar{S}_{\ell,30}$
30	-3.3 ± 0.9	7.4 ± 1.0
32	-8.4 ± 1.7	4.7 ± 1.7
34	-13.0 ± 2.1	-5.6 ± 2.4
36	-3.5 ± 3.1	5.5 ± 3.9
38	7.0 ± 3.1	3.8 ± 4.0
40	4.7 ± 2.5	-4.0 ± 3.1

11 CONCLUSIONS

The addition of two new orbit analyses, particularly that for the satellite 1965-09A, has led to great improvements in the standard deviations of many of the 15th-order coefficients of even degree: for degree 24, 26, 28 and 30 the standard deviation is reduced by a factor of 3.1 on average; and for degree 32, 34 and 36 by a factor of 1.4 on average. The 15th-order coefficients of odd degree are not significantly changed as a result of the addition of the one new satellite, 1968-70A. The new values for the 15th-order coefficients with G corrected, as given in Tables 14 and 15, have sd $\leqslant 2.0 \times 10^{-9}$ for all the 30 coefficients of degree 15,16,17,...29, and the average standard deviation of these 30 coefficients is 1.15×10^{-9}, equivalent to an error of 0.7 cm in geoid height. This precision is much better than has been achieved for any other order for so many coefficients. Nominally the most accurate of our coefficients is $\bar{C}_{15,15} = (-20.4 \pm 0.4) \times 10^{-9}$, where the standard deviation is equivalent to less than 0.3 cm in geoid height.

With the 30th-order coefficients we only have solutions for even degree, and only those of degree 30 and 32 are up to the standard of the previous paragraph. These four values have a mean standard deviation of 1.3×10^{-9}, equivalent to an error of 0.8 cm in geoid height.

Comparison with comprehensive geoid models, in which the coefficients are nominally less accurate than our values, shows satisfactory agreement. For 15th order, only the GEM models[8,14] are thought to be independent of ours; for degree 15-23, GEM 10B differs from the corresponding values in our solution by 3.1×10^{-9} on average, and for GEM-T1 the corresponding value is 2.4×10^{-9}. This suggests that the nominal standard deviation of GEM-T1, 3.1×10^{-9} on average, is realistic. For 30th order and degree 30,32,34 and 36, our values differ from the corresponding values given by the mean of four models by 3.3×10^{-9}; but for the models individually the differences are near 4×10^{-9} on average. Again this suggests that the nominal standard deviations of the models, mostly between 3 and 5×10^{-9}, are quite realistic.

Table 18

Revised values of $Q_{17}^{0,1}$, $Q_{19}^{0,1}$, ... $Q_{35}^{0,1}$ for orbits with $e > 0.011$

Satellite	$Q_{17}^{0,1}$	$Q_{19}^{0,1}$	$Q_{21}^{0,1}$	$Q_{23}^{0,1}$	$Q_{25}^{0,1}$	$Q_{27}^{0,1}$	$Q_{29}^{0,1}$	$Q_{31}^{0,1}$	$Q_{33}^{0,1}$	$Q_{35}^{0,1}$
64-84A	-10.3	43.1	-109.5	190.0	-229.8	177.1	-44.1	-82.9	112.4	-37.4
62-15A	-5.043	7.666	-2.991	-3.451	2.259	2.290	-1.235	-1.790	0.426	1.371
77-12B	-1.931	-0.286	0.857	0.738	0.042	-0.447	-0.437	-0.109	0.198	0.276
71-106A	-1.918	-0.345	0.855	0.807	0.099	-0.466	-0.514	-0.172	0.204	0.343
71-18B	-1.040	-0.801	-0.101	0.383	0.477	0.280	-0.004	-0.204	-0.247	-0.158
77-95B	-0.088	-0.423	-0.449	-0.327	-0.156	-0.002	0.102	0.146	0.142	0.103
67-73A	0.594	0.429	0.324	0.247	0.189	0.143	0.107	0.077	0.054	0.036
64-52B	-0.270	-0.524	-0.480	-0.337	-0.185	-0.058	0.031	0.084	0.106	0.106

Table 19

Revised values of $Q_{18}^{1,0}$, $Q_{20}^{1,0}$, ... $Q_{36}^{1,0}$ for orbits with $e > 0.011$

Satellite	$Q_{18}^{1,0}$	$Q_{20}^{1,0}$	$Q_{22}^{1,0}$	$Q_{24}^{1,0}$	$Q_{26}^{1,0}$	$Q_{28}^{1,0}$	$Q_{30}^{1,0}$	$Q_{32}^{1,0}$	$Q_{34}^{1,0}$	$Q_{36}^{1,0}$
62-15A	-3.69	5.32	-1.98	-2.90	2.02	2.08	-1.41	-1.80	0.73	1.58
71-106A	-1.181	-0.555	0.538	0.760	0.201	-0.410	-0.545	-0.212	0.219	0.401
77-95B	0.358	-0.173	-0.439	-0.458	-0.314	-0.107	0.083	0.203	0.238	0.199
67-73A	1.118	1.119	1.061	0.972	0.864	0.748	0.632	0.519	0.414	0.320
64-52B	0.865	0.611	0.345	0.108	-0.073	-0.194	-0.258	-0.274	-0.254	-0.210

Table 20

Revised values of $Q_{18}^{-1,2}$, $Q_{20}^{-1,2}$, ... $Q_{36}^{-1,2}$ for orbits with $e > 0.011$

Satellite	$Q_{18}^{-1,2}$	$Q_{20}^{-1,2}$	$Q_{22}^{-1,2}$	$Q_{24}^{-1,2}$	$Q_{26}^{-1,2}$	$Q_{28}^{-1,2}$	$Q_{30}^{-1,2}$	$Q_{32}^{-1,2}$	$Q_{34}^{-1,2}$	$Q_{36}^{-1,2}$
62-15A	-2.20	1.16	1.20	-0.91	-1.02	0.51	0.93	-0.12	-0.77	-0.20
71-106A	-0.106	-0.669	-0.496	0.034	0.429	0.434	0.129	-0.211	-0.345	-0.221
77-95B	1.040	0.833	0.494	0.121	-0.197	-0.404	-0.476	-0.425	-0.290	-0.118
67-73A	0.994	0.890	0.755	0.614	0.477	0.349	0.234	0.133	0.047	-0.025
64-52B	0.237	-0.264	-0.476	-0.480	-0.368	-0.211	-0.056	0.066	0.147	0.185

Table 21

Revised values of $Q_{32}^{0,2}$, $Q_{34}^{0,2}$, ... $Q_{40}^{0,2}$ for 1964-52B

$Q_{32}^{0,2}$	$Q_{34}^{0,2}$	$Q_{36}^{0,2}$	$Q_{38}^{0,2}$	$Q_{40}^{0,2}$
-1.130	-0.875	-0.295	0.136	0.341

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19	D.G. King-Hele D.M.C. Walker	Individual geopotential coefficients of order 15 and 30, from resonant satellite orbits. RAE Technical Report 84021 (1984)

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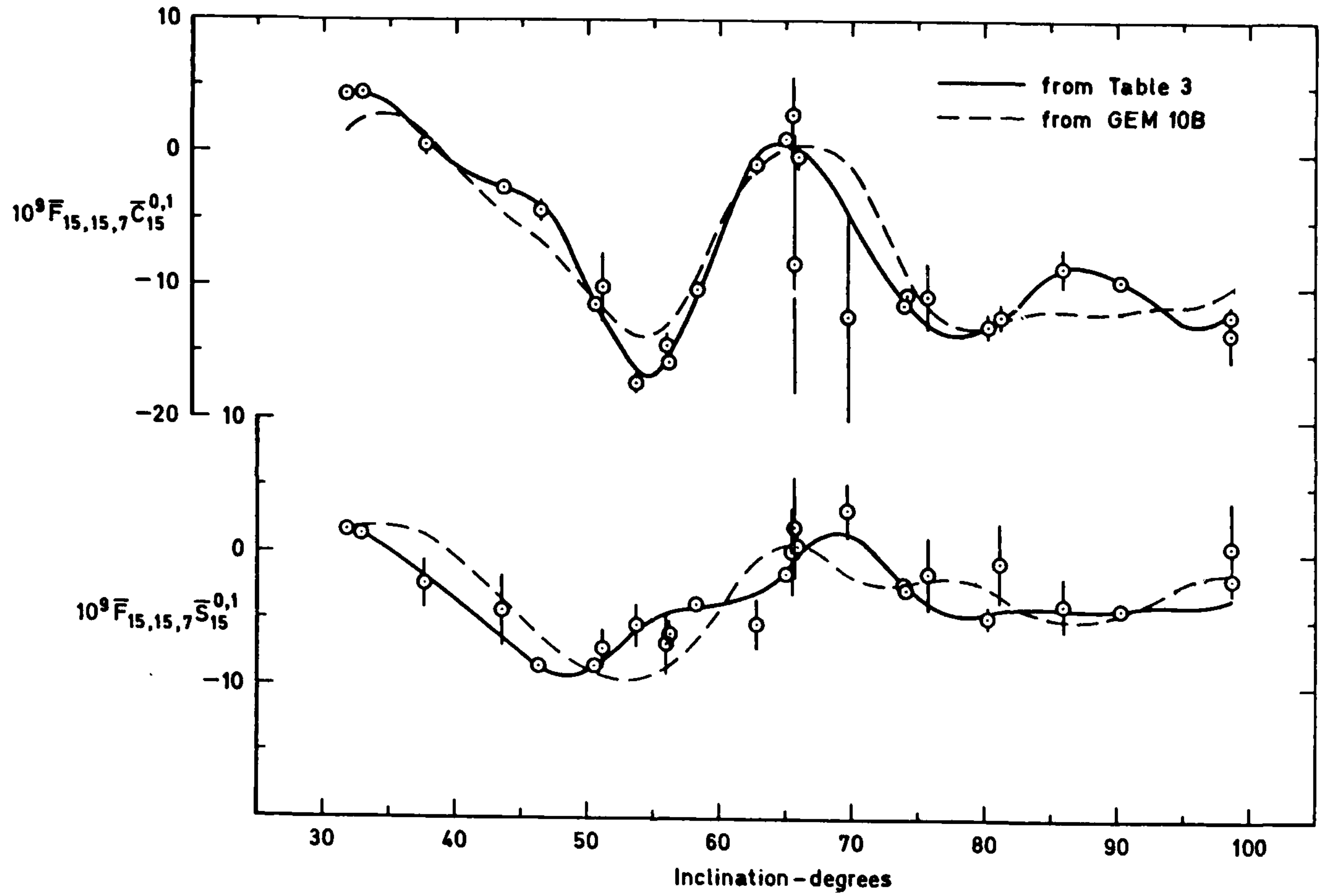

Fig 1 Values of $\bar{F}_{15,15,7}\bar{C}_{15}^{0,1}$ and $\bar{F}_{15,15,7}\bar{S}_{15}^{0,1}$ from Table 1, with the curves given by the coefficients in Table 3 and by GEM 10B

Fig 2

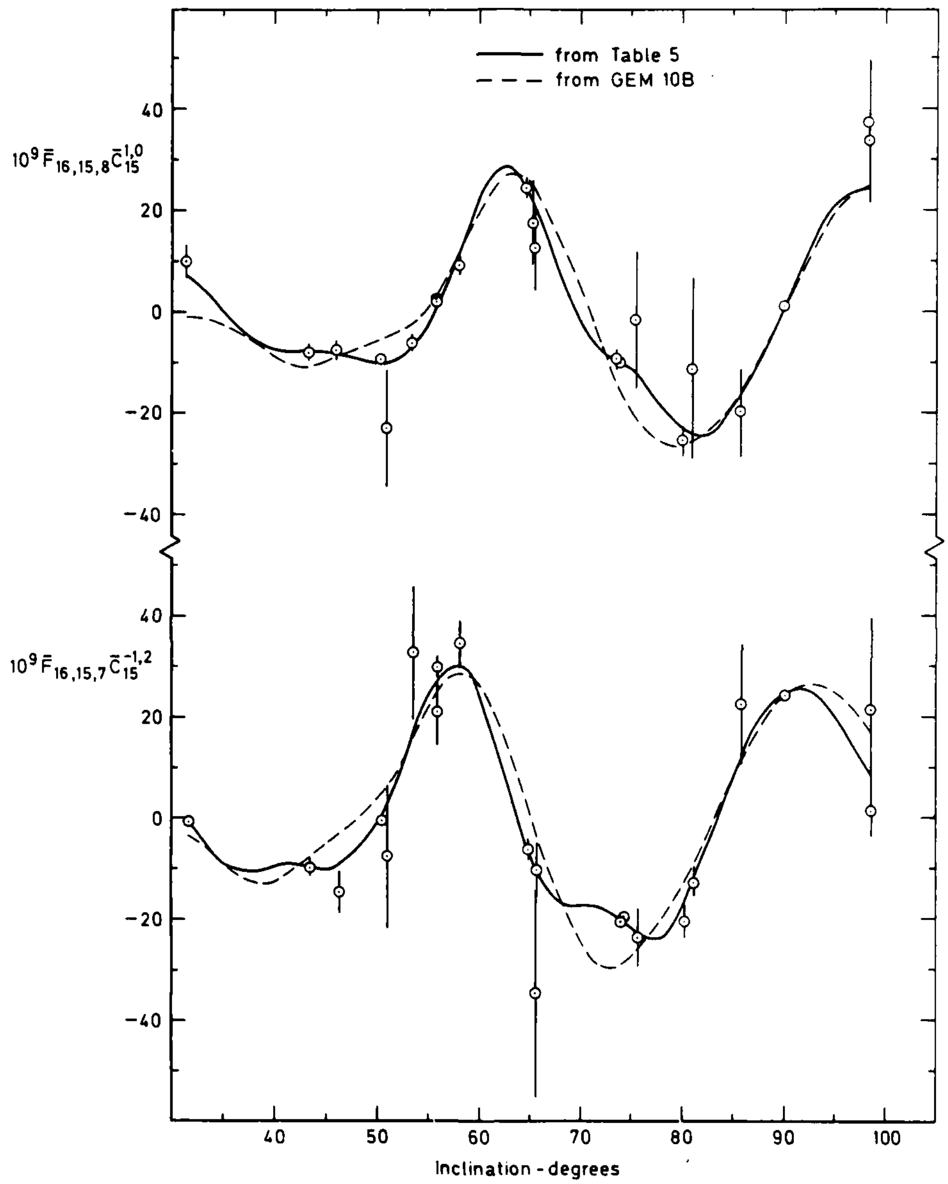

Fig 2 Values of $\bar{F}_{16,15,8}\bar{C}_{15}^{0,1}$ and $\bar{F}_{16,15,7}\bar{C}_{15}^{-1,2}$ from Table 2, with the curves given by the coefficients in Table 5 and by GEM 10B

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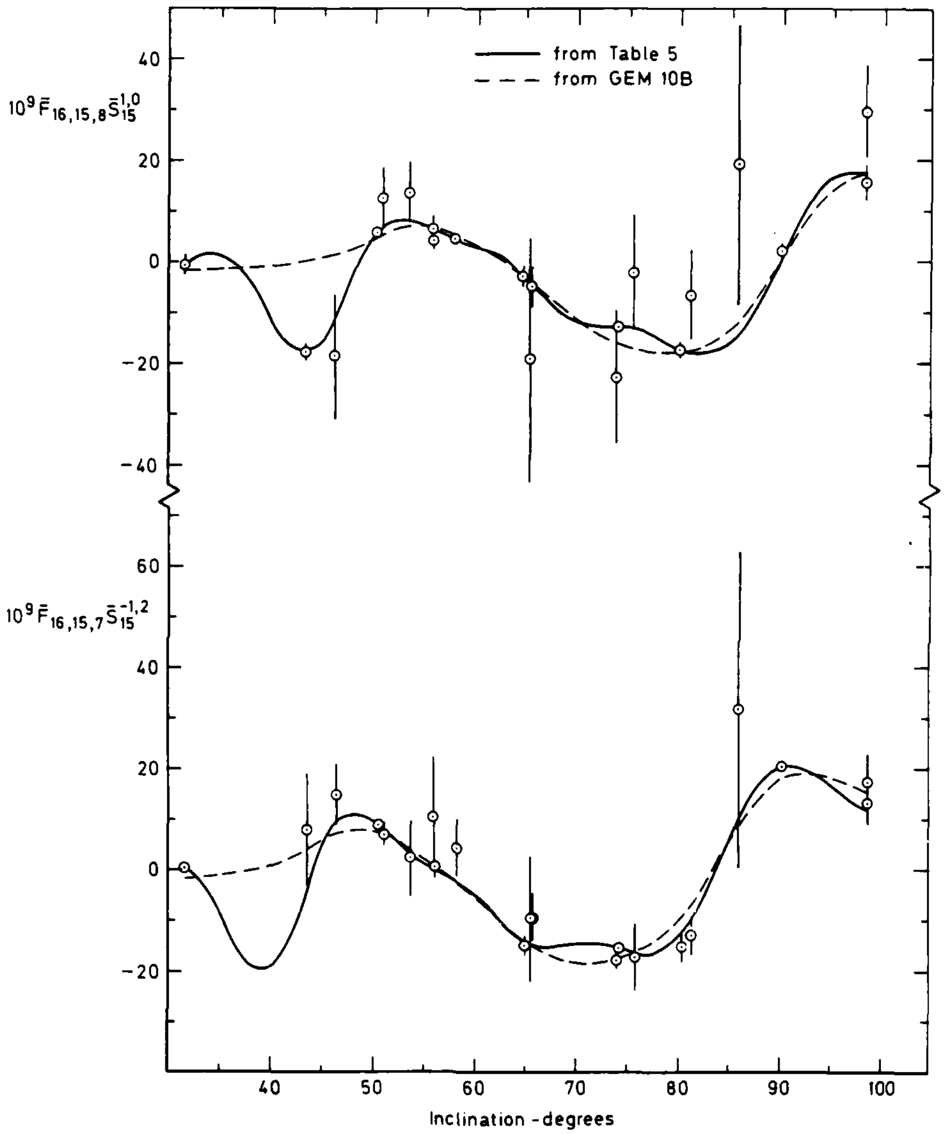

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Fig 3 Values of $\bar{F}_{16,15,8}\bar{S}_{15}^{1,0}$ and $\bar{F}_{16,15,7}\bar{S}_{15}^{-1,2}$ from Table 2, with the curves given by the coefficients in Table 5 and by GEM 10B

Fig 4

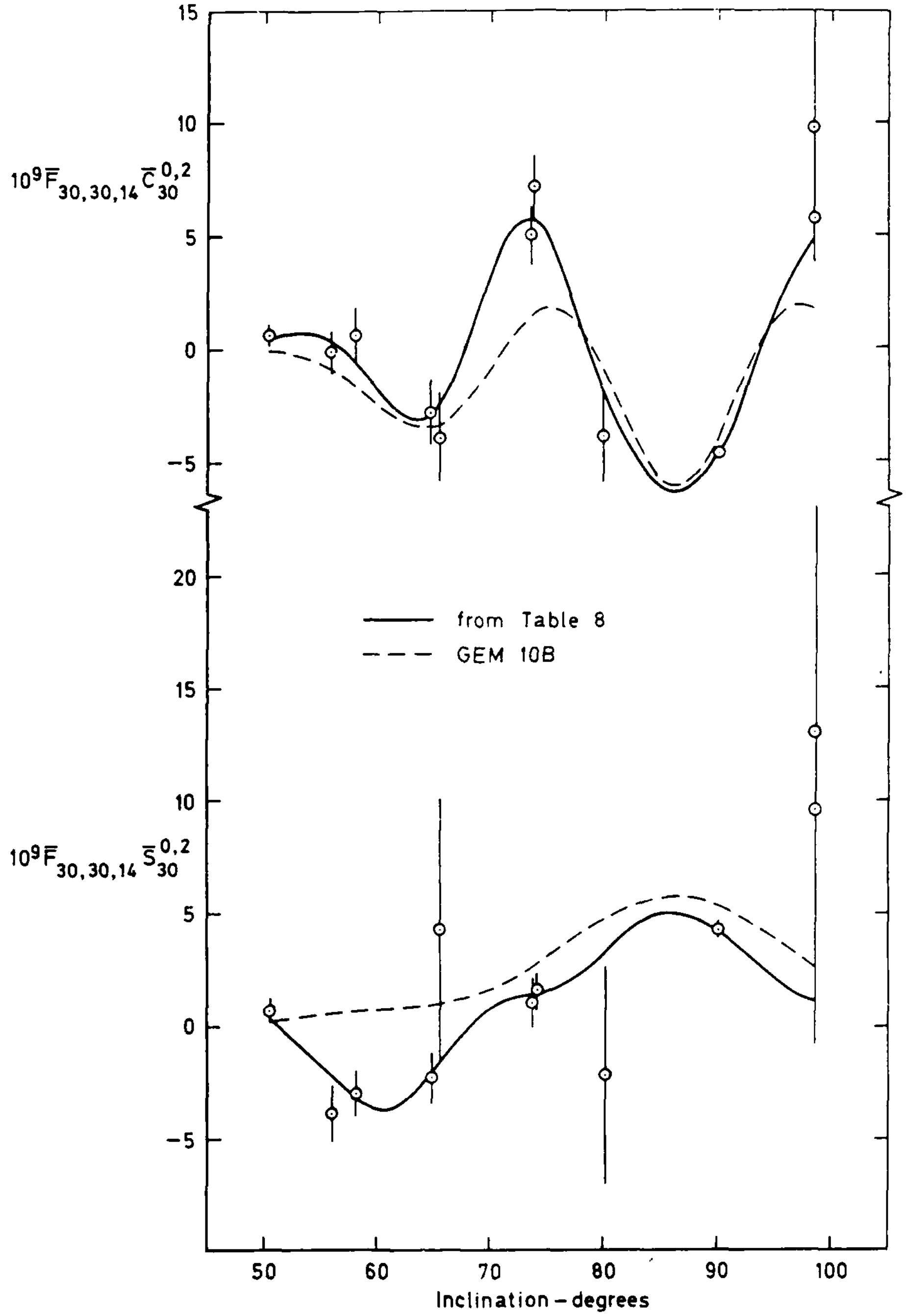

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Fig 4 Values of $\bar{F}_{30,30,14}\bar{C}^{0,2}_{30}$ and $\bar{F}_{30,30,14}\bar{S}^{0,2}_{30}$ from Table 7, with the curves given by the coefficients in Table 8 and by GEM 10B

REPORT DOCUMENTATION PAGE

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7\. Title Evaluation of the 15th- and 30th-order geopotential harmonic coefficients from 26 resonant satellite orbits

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15\. Distribution statement

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16\. Descriptors (Keywords) (Descriptors marked * are selected from TEST)

Satellite orbits. Geopotential. Resonance. Orbit analysis.

17. Abstract The Earth's gravitational potential is usually expressed as an infinite series of harmonics, and the values of harmonic coefficients of order 15 and 30 can be determined most accurately by analysis of satellite orbits which experience 15th-order resonance. The results from two recent resonance analyses, for 1965-09A and 1968-70A, have here been added to those previously available, to produce an improved evaluation of 44 coefficients of order 15 and degree 15-36, and 12 coefficients of order 30 and even degree 30, 32, . . . 40.

Compared with previous results, the new evaluation shows a great improvement in the standard deviations of many of the 15th-order coefficients of even degree, thanks largely to the contribution of 1965-09A at inclination 31.8°: for the coefficients of degree 24, 26, 28 and 30, the standard deviation (sd) has been reduced by a factor of 3.1 on average; and for degree 32, 34 and 36 by a factor of 1.4 on average. For the other coefficients – those of 30th order, and odd-degree 15th order – the changes are relatively small. In the new 15th-order solution, all the 30 coefficients of degree 15-29 have sd $\leqslant 2.0 \times 10^{-9}$, and the average sd of these 30 values is equivalent to an error in geoid height of 0.7 cm.