Technical Report ARAED-TR-89006

ANALYTICAL TRANSFER FUNCTIONS IN PARALLEL CONFIGURATIONS

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March 1989

US ARMY
ARMAMENT MUNITIONS
& CHEMICAL COMMAND
ARMAMENT RDE CENTER

U.S. ARMY ARMAMENT RESEARCH, DEVELOPMENT AND ENGINEERING CENTER

Armament Engineering Directorate

Picatinny Arsenal, New Jersey

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER Technical Report ARAED-TR-89006	5. MONITORING ORGANIZATION REPORT NUMBER)
6a. NAME OF PERFORMING ORGANIZATION ARDEC, AED 6b. OFFICE SYMBOL SMCAR-AET-A	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (CITY, STATE, AND ZIP CODE) Armaments Technology Div Picatinny Arsenal, NJ 07806-5000	7b. ADDRESS (CITY, STATE, AND ZIP CODE)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION ARDEC, IMD STINFO Br 8b. OFFICE SYMBOL SMCAR-IMI-I	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (CITY, STATE, AND ZIP CODE) Picatinny Arsenal, NJ 07806-5000	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (INCLUDE SECURITY CLASSIFICATION)

ANALYTICAL TRANSFER FUNCTIONS IN PARALLEL CONFIGURATIONS

12. PERSONAL AUTHOR(S)

Michael J. Amoruso, Romel Campbell, and Eugene M. Friedman

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (YEAR, MONTH, DAY)	15. PAGE COUNT
Final	FROM ______ TO ______	March 1989	

16. SUPPLEMENTARY NOTATION

This report supercedes a similarly titled report, Campbell, R.; Amoruso, M. J.; and Friedman, E. M., "Analytical Transfer Functions in Parallel Configurations," Technical Report ARAED-TR-86041, ARDEC, Dover, NJ.

17. COSATI CODES			18. SUBJECT TERMS (CONTINUE ON REVERSE IF NECESSARY AND IDENTIFY BY BLOCK NUMBER)
FIELD	GROUP	SUB-GROUP	Simulation, Guided projectile, ~~Six degrees of freedom~~, Guided munition, Integration, Transfer function.

19. ABSTRACT (CONTINUE ON REVERSE IF NECESSARY AND IDENTIFY BY BLOCK NUMBER)

Piecewise analytical solutions for typical autopilot transfer functions were developed to increase speed of digital simulations of guided munitions over standard numerical integration techniques. The replacement of serial autopilot configurations by equivalent parallel configurations avoids difficulties arising from propagation of signals through a sequential network of widely varying natural frequencies and always leads to a decomposition with readily integrated terms.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION	
☐ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL I. HAZNEDARI	22b. TELEPHONE (INCLUDE AREA CODE) (201) 724-3316	22c. OFFICE SYMBOL SMCAR-IMI-I

DD FORM 1473, 84 MAR

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

ACKNOWLEDGMENT

The authors wish to express their appreciation to Herbert Cohen of the U.S. Army Materiel Systems Analysis Activity, Aberdeen Proving Ground, MD, whose efforts contributed immeasurably to the success of this method.

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INTRODUCTION

A substantial need has been demonstrated within the Armament Research, Development and Engineering Center (ARDEC) for a convenient-to-use and computationally efficient flight simulation for smart munitions. To meet this need, ARDEC is formulating methods and writing modular software for rapidly developing autopilot simulations for guided projectiles and other munitions.

Time constants associated with autopilot components are often small compared with their driving terms. The integration time step is consequently driven to very small values to achieve stable numerical integration, which results in increased computer run time. An innovative technique was developed in which exact analytic solutions to differential equations and transfer functions are applied in a piecewise manner within a larger but lower frequency problem that is solved numerically.[1]

Closed form analytical solutions were previously obtained for the following: the first-order lag, the first-order lag with differentiation, the first-order lead/lag, the first-order lag with integrator, the second-order lag/oscillator,[1] a two-axis gimbaled gyro,[2] and an impulse thruster.[3] The typical autopilot can be described as a concatenation of these functions and rate sensors, switches, limiters, and dead zones. When small integration time steps are used, as in the numerical integration of transfer functions, it is straightforward to concatenate solutions in serial order since the sampling rate of the signal is high because of the small integration time step. However, if analytical solutions to transfer functions are to be used to make possible the use of relatively large time steps, the sampling rate may no longer be adequate to sample the frequency content of the signal as it moves from one transfer function to another.

This report discusses the replacement of serial configurations of transfer functions by equivalent parallel configurations. This approach avoids difficulties arising from propagation of the signal through a sequential network of widely varying natural frequencies when using a relatively large piecewise integration time step, and produces a decomposition that always leads to terms that can be readily integrated analytically.

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[1] Eugene M. Friedman and Michael J. Amoruso, "An Analytical Modularized Treatment of Autopilots for Guided Projectile Simulations," US Army Armament Research and Development Center, Technical Report ARLCD-TR-85025, Picatinny, NJ, August 1985.

[2] M. J. Amoruso, "An Analytical Technique for Modeling Gyroscopes in Guided Projectile Simulations," US Army Armament Research and Development Center, Technical Report ARAED-TR-86010, Picatinny, NJ, June 1986.

[3] Eugene M. Friedman, Michael J. Amoruso, and Romel Campbell, "An Analytical Model of an Impulsive Thruster," Technical Report ARAED-TR-86034, Picatinny Arsenal, NJ, November 1986.

DISCUSSION

When modeling guided projectiles in 6 DOF (six degree of freedom) simulations, differential equations are obtained that describe the various component subsystems. These are then typically integrated numerically within the framework of the 6 DOF simulation. The largest allowable time step to perform the integration is bounded by two constraints. The driving term or input must not vary appreciably during a time step and the time step must be sufficiently small to insure a stable integration.

Since the driving term rates are commensurate with the airframe motion rates, inherently slower processes than those associated with the autopilot, stable integration is a lower bound to integration step size than the requirement for driving terms that remain essentially constant during the integration time step. By using analytic closed-form solutions for the differential equations for the autopilot, the second constraint appears to be eliminated. An innovative technique was developed[1,4,5] by which exact analytical solutions to the required transfer functions are applied in a piecewise manner within a larger but lower frequency problem which must be solved numerically. The use of these piecewise analytical solutions to the transfer functions guarantees valid integration of the autopilot transfer functions regardless of the integration time step (app A).

The overall system is usually analyzed into simpler terms in sequential order that are separately solved by numerical integration techniques, assuming that the input or driving term is essentially constant during the integration time step. These factors are concatenated with the output of one factor or block becoming the input to the next block. Since the integration time step is generally quite small to achieve stable numerical integration, negligible errors are introduced as the signal propagates through the usually modest number of transfer function blocks down to the output.

This approach was modified by introducing analytical closed-form solutions for the transfer function factors that were previously treated numerically. This approach permitted large time steps across the individual transfer functions and yet guaranteed stable intergration contingent only upon the input remaining essentially constant during the integration time step.

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[4] M. J. Amoruso, T. F. DeYoung, D. F. Ladd, and R. D. Schulz, "A Comprehensive Digital Flight Simulation of the Cannon-Launched Guided Projectile," R-TR-77-007, U.S. Army Armament Command, Rock Island, January 1977.

[5] M. J. Amoruso and D. D. Ladd, "TELUM, A Comprehensive Digital Flight Simulation of the COPPERHEAD Projectile," ARLCD-SP-82003, Picatinny Arsenal, NJ, June 1982.

However, if several factors are concatenated to represent a more complex transfer function, the final output can be found to depend on the order of the transfer function factors. This difficulty arises because, although the input of a block might be essentially constant during an integration time step, the output might not be if the frequency response of the block is relatively high. This output becomes the input to the next transfer function block. The requirement that the input to this next block be essentially constant can break down unless the sampling rate is high. This requires a smaller integration time step and longer computer execution time. This dilemma does not arise in the former numerical integration approach because the very fine time step required avoided difficulties associated with incompatibilities of bandwidth and frequency content as the signal propagated from block to block.

The approach finally used was the conversion of a complex transfer function to a parallel representation instead of a serial representation. This technique not only saves computer run time but can make the task of programming simpler and nearly automatic.

The obvious advantage to an equivalent parallel representation is that each block receives the same input at the same time and each block produces its output at the end of the same single time step. These outputs do not become inputs to other autopilot transfer function blocks, but are instead summed to produce the overall output for the overall transfer function. Generally, this technique not only avoids simulation errors arising from time step size incompatibilities with bandwidth, internal lag, and frequency content of the signal propagating through the sequence of transfer function blocks but also has additional benefits. Noise should be treated in a manner that is consistent with the analytical treatment of the signals. The treatment of noise can be considerably simplified by using a parallel representation of the transfer function, for reasons analogous to those adduced above. In addition to producing an algorithm that is easy to apply and considerably faster than previous numerical integration approaches, parallel decomposition generally leads to a combination of elementary expressions, whose Laplace inverse and analytical integration are well known.

A specific example of this approach is given below to clarify this approach. A general treatment is developed in the next section and computer software to implement this technique is to be found in the appendixes.

This approach consists of the following steps:

(1) Writing down the Laplace operator expression *including non-zero initial conditions* from the differential equation description or from the block diagram (which usually will not show the initial conditions)

(2) Factoring

(3) Making a formal partial fraction expansion

(4) Finding the expansion coefficients

(5) Writing down the expanded Laplace transform.

The latter can then be inverted from standard tables of inverse Laplace transforms to obtain the analytic solutions in parallel decomposition or calculated using the residue theorem of complex variables.

It is worthwhile to emphasize the first of the steps enumerated above. Autopilots, seekers, control actuators, and other components of guided projectiles are conventionally described in block diagrams in terms of the Laplace operator s. Typically, these block diagrams represent the underlying differential equations only if the initial conditions vanish. This is a convenient shorthand notation but can also be a source of confusion. Recall that for initial conditions $y_o = y(t=0)$, $\dot{y}_o = \frac{dy}{dt}(t=0)$, etc., the Laplace transform of the derivative of a time domain function is given by [6,7]

$$L[\dot{y}(t)] = sL[y(t)] - y_0 \qquad (1)$$

$$L[\ddot{y}(t)] = s^2 L[y(t)] - s\,y_0 - \dot{y}_0 \qquad (2)$$

or, in general, where $y^{(n)}$ represents the n^{th} derivative of y(t) with respect to t,

$$L[y^{(n)}] = s^n L[y(t)] - s^{n-1}\,y_0 - s^{n-2}\,\dot{y}_0 - \ldots - y_0^{(n-1)} \qquad (3)$$

This will be illustrated in the example given below. There are two methods for implementing this procedure. The first involves the taking of limits and derivatives and requires the factoring of the transfer functions into first-order systems. The second requires the solution of sets of simultaneous equations and does not involve limits or derivatives. Factoring the transfer function into first-order terms is optional in the second method. These two approaches are illustrated below by a concrete example.

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6 Erwin Kreysig, Advanced Engineering Mathematics , Fourth Edition, John Wiley and Sons, New York, NY, 1979.

7 Equation (1) can be derived as follows. Let $\zeta = y - y_0$. Thus $\zeta(t=0) = \zeta_0 = 0$ and $\dot{\zeta} = \dot{y}$. Then $L(\dot{y}) = L(\dot{\zeta}) = sL(\zeta) = sL(y) - sL(y_0) = sL(y) - y_0$. Recall L(A) = A/s if A is a constant. A similar derivation holds for higher orders.

First Method: Limit Approach

Step 1 - Obtain Laplace Transform: Consider an autopilot component represented by the following block diagram in figure 1. The differential equation corresponding to this block diagram is

$$\ddot{y}(t) + \dot{y}(t) - 2y(t) = \textit{Driving term} \tag{4}$$

$$= K(t-t_0) + K_0 = Kt + L$$

$$\longrightarrow \boxed{\frac{1}{s^2 + s - 2}} \longrightarrow$$

Figure 1. Example of block diagram

The driving term will be taken to be $K(t-t_o)+K_0 = Kt+L$ in this example, where K and L are constants and t is time. This has the Laplace transform $K/s^2 + L/s$. Let $\mathbf{L}\{y(t)\} = Y(s)$. The Laplace transform of the differential equation is **not**

$$s^2Y(s) + sY(s) - 2Y(s) = \frac{K}{s^2} + \frac{L}{s} \tag{5}$$

but is rather

$$s^2Y(s) - sy_0 - \dot{y}_0 + sY(s) - y_0 - 2Y(s) = \frac{K}{s^2} + \frac{L}{s} \tag{6}$$

or

$$(s^2 + s - 2)Y(s) - y_0(1+s) - \dot{y}_0 = \frac{K + sL}{s^2} \tag{7}$$

where the additional terms are due to the initial conditions defined by $y_0 = y_{|t=t_0}$ and $\dot{y}_0 = \frac{dy}{dt}\Big|_{t=t_0}$. This may be written

$$Y(s) = \frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{s^2(s^2 + s - 2)} \tag{8}$$

Step 2 - Factor: This may be factored

$$Y(s) = \frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{s^2(s+2)(s-1)} \tag{9}$$

Step 3 - Expansion: This may be formally expanded as follows:

$$Y(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} + \frac{D}{s-1} \tag{10}$$

Step 4 - Find Expansion Coefficients: The partial fraction expansion coefficients A through D are now evaluated.

Find A:

$$\lim_{s \to 0} s^2 Y(s) = \lim_{s \to 0} \left[\frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{(s+2)(s-1)} \right] = -\frac{K}{2}$$

$$= A + \lim_{s \to 0} \left[\frac{B}{s} + \frac{C}{s+2} + \frac{D}{s-1} \right]$$

$$= A = -\frac{K}{2} \tag{11}$$

Find B:

$$\lim_{s \to 0} \frac{d}{ds}\left[s^2 Y(s)\right] = \lim_{s \to 0} \left[\frac{d}{ds} \frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{(s+2)(s-1)} \right] \tag{12}$$

$$= \lim_{s \to 0} \left[\frac{3y_0 s^2 + 2s(y_0 + \dot{y}_0) + L}{(s+2)(s-1)} - \frac{[K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0](2s+1)}{(s+2)^2(s-1)^2} \right]$$

$$= -\frac{(K + 2L)}{4} = B$$

Find C:

$$\lim_{s \to -2} (s+2)Y(s) = \lim_{s \to -2} \frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{s^2(s-1)}$$

$$= \frac{4y_0 - 4\dot{y}_0 - K + 2L}{12} = C \qquad (13)$$

Find D:

$$\lim_{s \to 1} (s-1)Y(s) = \lim_{s \to 1} \frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{s^2(s+2)}$$

$$= \frac{K + L + 2y_0 + \dot{y}_0}{3} = D \qquad (14)$$

Thus

$$Y(s) = -\frac{K}{2s^2} - \frac{(K+2L)}{4s} + \frac{(4y_0 - 4\dot{y}_0 - K + 2L)}{12(s+2)} + \frac{(K + L + 2y_0 + \dot{y}_0)}{3(s-1)} \qquad (15)$$

This transition from parallel decomposition or expansion is shown in figure 2 for this simple case. It is a simple matter to invert this expression into the time domain. Tables of inverse Laplace transforms can be found in standard math tables.[8] Here, the inversion of (15) is given by

$$y(t) = -\frac{Kt}{2} - \frac{(K+2L)}{4} + \left[\frac{4y_0 - 4\dot{y}_0 - K + 2L}{12}\right] e^{-2t} + \left[\frac{K + L + 2y_0 + \dot{y}_0}{3}\right] e^{t} \qquad (16)$$

[8] Charles D. Hodgman, ed., C.R.C. Standard Mathematical Tables, Chemical Rubber Publishing Co., Cleveland, OH, 1952.

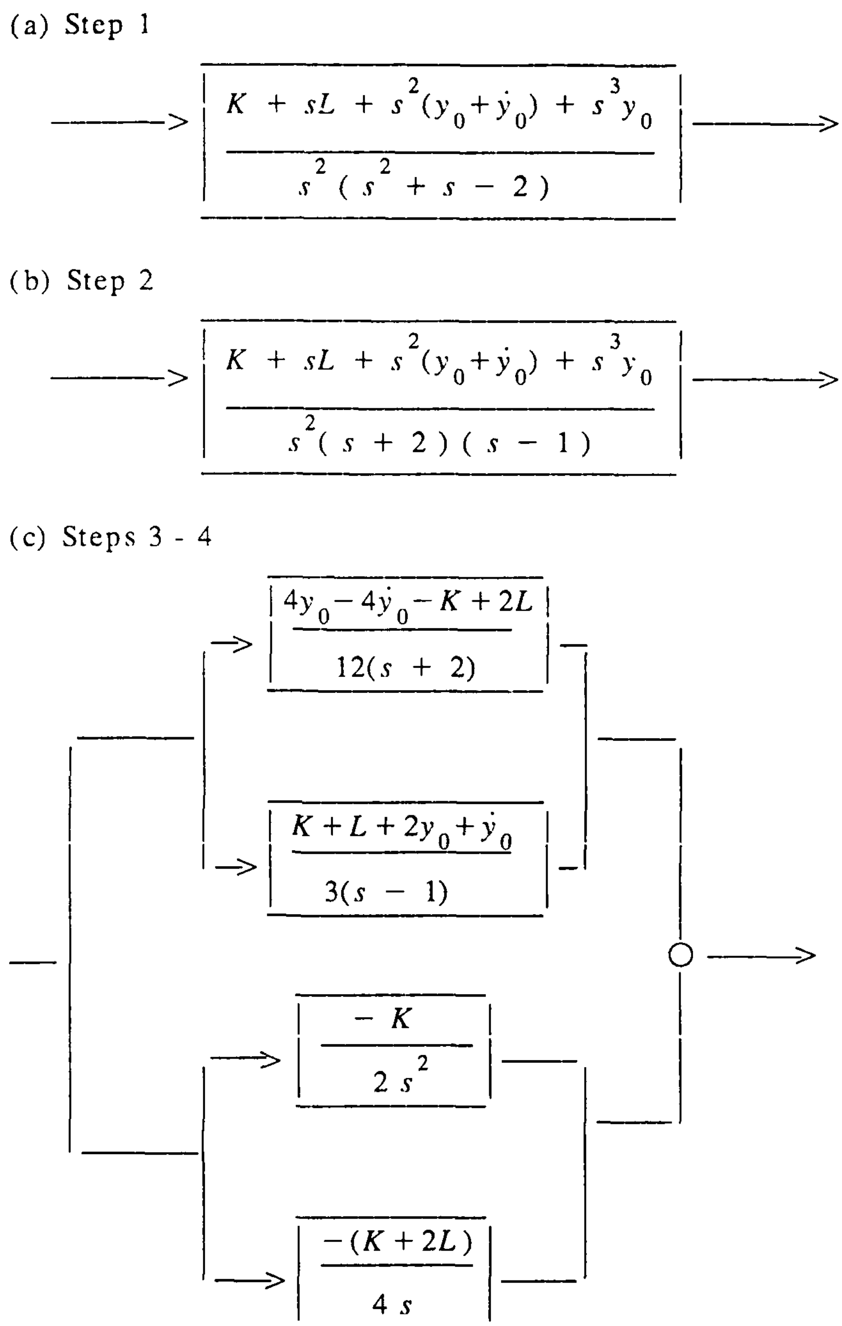

Figure 2. Example of transition to parallel representation

Second Method: Algebraic Approach

The first step in the algebraic approach is always the same as above. The second and third steps may be done as before, i.e., factoring the denominator of the transfer function into monomials and expanding in terms of these monomials. Later higher order terms will be retained.

Step 4 - Find Expansion Coefficients: Write the right side of (10) with a least common denominator and equate to (9).

$$
\begin{aligned}
Y(s) &= \frac{K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0}{s^2(s^2 + s - 2)} \\
&= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+2)} + \frac{D}{s-1} \\
&= \frac{A(s^2+s-2) + Bs(s^2+s-2) + Cs^2(s-1) + Ds^2(s+2)}{s^2(s^2+s-2)}
\end{aligned}
\qquad (17)
$$

The denominators are equal. Therefore the numerators are also equal. Collecting powers of s yields

$$K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0 = s^3[B + C + D] + s^2[A + B - C + 2D] + s[A - 2B] - 2A \qquad (18)$$

Since the powers of s are linearly independent, this is equivalent to the following set of equations which must be solved simultaneously.

$$y_0 = B + C + D \qquad (19)$$

$$y_0 + \dot{y}_0 = A + B - C + 2D \qquad (20)$$

$$L = A - 2B \qquad (21)$$

$$K = -2A \qquad (22)$$

The solution of these equations for the coefficients A through D is elementary. The values of the coefficients are identical to the previous result, equations (11) through (14).

Alternatively, the investigator may wish to retain some higher order terms rather than reduce all the denominators to monomials, perhaps to retain a physical interpretation of terms. Note that step 1 in figure 2 contains a second order term, viz., $(s^2 + s - 2)$. It is possible to make a partial fraction expansion without going to a decomposition that only includes first order terms. This can be illustrated by the following example. Replace (10) by a more general expression, viz.

$$Y(s) = \frac{V(s)}{G(s)} \tag{23}$$

$$= \sum_{i=1}^{l} \frac{C_i}{D_i} + \sum_{j=l-1}^{n} \frac{P_j + sQ_j}{E_j}$$

where the D_j are first order terms of the form $s - r_i$ and the E_j are second order terms of the form $(s^2 + \alpha s + \beta)$. This may be expanded

$$Y(s) = \frac{V(s)}{G(s)} \quad (24)$$

$$= \frac{C_1 D_2 \ldots D_l E_{l+1} \ldots E_n + C_2 D_1 D_3 \ldots E_n + \ldots [P_{l+1} + sQ_{l+1}] D_1 \ldots D_l E_{l+2} \ldots E_n + \ldots [P_n + sQ_n] D_1 \ldots D_l E_{l+1} \ldots E_{n-1}}{D_1 D_2 D_3 \cdots D_l E_{l+1} \cdots E_n}$$

The denominators are equal. Therefore this expression can be true if and only if the numerators are equal. For the example given in this report

$$Y(s) = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{P_3 + sQ_3}{s^2 + s - 2} \tag{25}$$

$$= \frac{C_1(s^2 + s - 2) + C_2 s(s^2 + s - 2) + P_3 s^2 + s^3 Q_3}{s^2(s^2 + s - 2)}$$

equating the numerator of (25) to the numerator of (9) and collecting similar powers of s gives

$$K + sL + s^2(y_0 + \dot{y}_0) + s^3 y_0 = s^3(Q_3 + C_2) + s^2(C_1 + C_2 + P_3) + s(C_1 - 2C_2) - 2C_1 \tag{26}$$

Equating similar powers of s gives the following simultaneous equations:

$$y_0 = Q_3 + C_2 \tag{27}$$

$$y_0 + \dot{y}_0 = C_1 + C_2 + P_3 \tag{28}$$

$$L = C_1 - 2C_2 \tag{29}$$

$$K = -2C_1 \tag{30}$$

When these are solved

$$C_1 = -\frac{K}{2} \tag{31}$$

$$C_2 = -\frac{(K+2L)}{4} \tag{32}$$

$$P_3 = \frac{(3K+2L)}{4} + y_0 + \dot{y}_0 \tag{33}$$

$$Q_3 = \frac{(4y_0 + K + 2L)}{4} \tag{34}$$

Thus

$$Y(s) = -\frac{K}{2s^2} - \frac{(K+2L)}{4s} + \frac{3K + 2L + 4(y_0 + \dot{y}_0)}{4(s^2+s-2)} + \frac{(4y_0 + K + 2L)s}{4(s^2+s-2)} \tag{35}$$

When Laplace inverted, (35) becomes

$$y(t) = -\frac{Kt}{2} - \frac{(K+2L)}{4} + \frac{3K+2L+4(y_0+\dot{y}_0)}{12}[e^t - e^{-2t}] + \frac{(4y_0+K+2L)}{12}[e^t + 2e^{-2t}] \tag{36}$$

This expression is equivalent to equation (16) even though equation (35) is not identical to equation (15). The decomposition characterized by (35) can be found in figure 3.

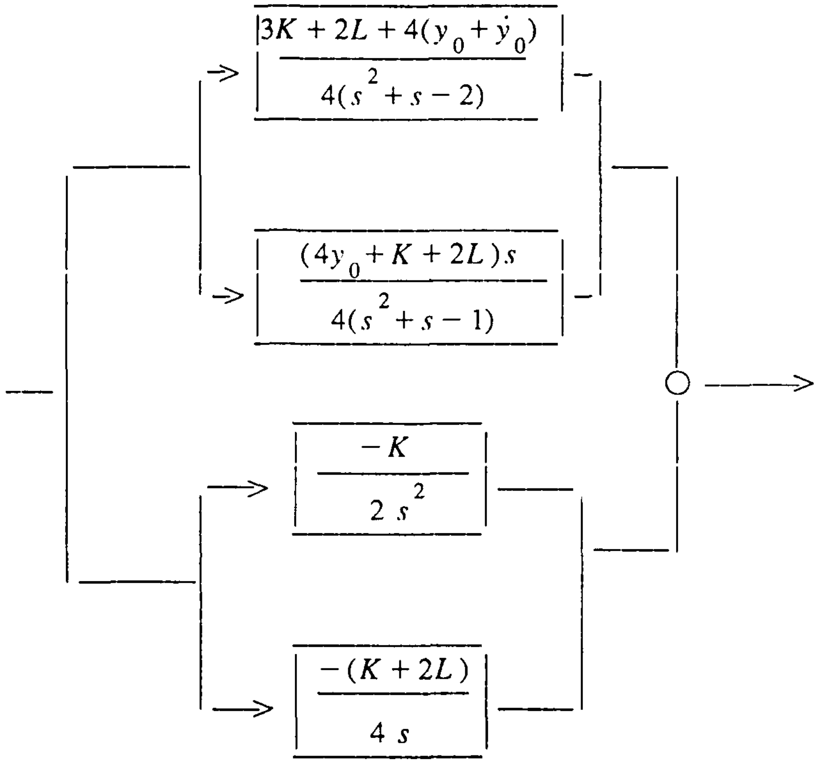

Figure 3. Alternate example of transition to parallel representation

The application of this result in a six degree of freedom simulation is illustrated in figure 4. At any time step t_n, the above analytic result can propagate the output of the transfer function to t_{n+1}. The initial conditions come from the results of the previous time step, the driving terms come from the aerodynamic motions of the projectile (inherently slower processes) which are integrated numerically during the time step interval, and possibly other transfer functions.

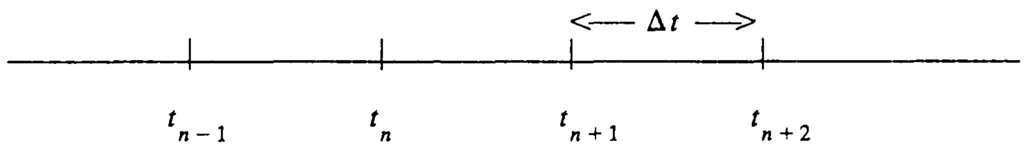

Figure 4. Iterative propagation of the solution

When a result is obtained for y at $t = t_{n+1}$, it is saved to become the new initial condition to be used for once again propagating the piecewise analytic solution for y to the next time step t_{n-2}. Note that not only must $y(t_n)$ be saved to serve as an initial condition for the next time step, but higher order derivatives as well. For the example used in this report, only

$$\dot{y}(t = t_n) = \frac{dy}{dt}\Big|_{t = t_n} \tag{37}$$

is required. This is easily obtained by differentiating equation (16) or (36).

For this example, equations (4), (5) and (16) are used iteratively as follows:

Step A: $L = K_0 - Kt_0$ (38)
K is the slope of the driving term at t_0.
K_0 is the value of the driving term at t_0.
Initially, take

$$y_0 = 0 \tag{39}$$

$$\dot{y}_0 = 0 \tag{40}$$

$$n = 0 \tag{41}$$

Step B: Let

$$n = n + 1 \tag{42}$$

$$t = \Delta t \tag{43}$$

Find y and $\dot{y}$ at the end of the time step, $t = \Delta t$.

$$y(\Delta t) = -\frac{K\Delta t}{2} - \frac{K+2L}{4} + \frac{e^{-2\Delta t}}{12}\left[4y_0 - 4\dot{y}_0 - K + 2L\right] + \frac{e^{\Delta t}}{3}\left[K + L + 2y_0 + \dot{y}_0\right] \tag{44}$$

$$\dot{y}(\Delta t) = -\frac{K}{2} - \frac{e^{-2\Delta t}}{6}\left[4y_0 - 4\dot{y}_0 - K + 2L\right] + \frac{e^{\Delta t}}{3}\left[K + L + 2y_0 + \dot{y}_0\right] \tag{45}$$

Step C: Let

$$\begin{aligned} y_0 &= y(\Delta t) \\ \dot{y}_0 &= \dot{y}(\Delta t) \end{aligned} \tag{46}$$

Evaluate K and L at t_n (these depend ultimately on air frame motion which is integrated numerically elsewhere in the six degree of freedom simulation.)

Step D: Repeat from step B to advance to the end of the next time step.

In summary, if the product of several sequential transfer functions can be recast into an equivalent network of transfer functions in parallel, difficulties arising from propagation of the signal through a sequential network can be eliminated even when relatively large integration time steps are used. This can be done by making a partial fraction decomposition of the Laplace operator representation. This technique produces a decomposition that always leads to terms that can be readily integrated. The use of a partial fraction decomposition also facilitates the treatment of noise. The next section contains a more general and formal treatment of some aspects of these techniques.

GENERAL ANALYSIS

Partial Fraction Expansion

In this section, a formal treatment is given for carrying out the partial fraction decomposition using the limit method. The function Y(s) for which an inverse transform is desired typically appears as the ratio of two polynomials.

$$Y(s) = \frac{V(s)}{G(s)} = \frac{\sum_{j=0}^{i} a_j s^j}{\sum_{k=0}^{n} b_k s^k}$$

$$= \frac{a_i s^i + a_{i-1} s^{i-1} + \dots + a_1 s + a_0}{s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0} \tag{47}$$

with n > i and $b_n = 1$.

Note that Y(s) must be a proper fraction (n > i). If this is not the case, an improper fraction can always be reduced to a proper fraction by long division.

It is desired to decompose this single complex transfer function into a group of simpler parallel transfer functions that may be more readily handled, as explained in the previous section. The technique to be used to achieve parallel decomposition is called partial fraction expansion. Suppose that the denominator of equation (27) can be factored to yield the form

$$Y(s) = \frac{V(s)}{G(s)} = \frac{V(s)}{(s-r_1)(s-r_2)(s-r_3) \dots (s-r_n)} \tag{48}$$

where the r_j , $j = 1$ to n are distinct roots of G(s) = 0. (A discussion of obtaining the roots r_k will be treated subsequently.) Then the partial fraction expansion *for distinct roots* is given by

$$Y(s) = \frac{C_1}{s - r_1} + \frac{C_2}{s - r_2} + \ldots. + \frac{C_n}{s - r_n} \tag{49}$$

The expansion coefficients C_1 to C_n are to be determined. Equating the right sides of equations (48) and (49) gives

$$\frac{C_1}{s - r_1} + \frac{C_2}{s - r_2} + \ldots. + \frac{C_n}{s - r_n}$$

$$= \frac{V(s)}{(s - r_1)(s - r_2)(s - r_3) \ldots. (s - r_n)}$$

$$= \frac{V(s)}{G(s)} \tag{50}$$

The i^{th} constant C_i can be determined by multiplying both sides of equation (50) by the denominator of the ith fraction in equation (50), viz. $(s - r_i)$. Therefore, the i^{th} factor will be canceled on the left side of equation (50). Finally, if r_i is not a repeated root, take the limit as s goes to r_i and solve for C_i . This process is expressed mathematically by the following relationship.

$$C_i = \lim_{s \to r_i} \frac{V(s)\,(s - r_i)}{G(s)} \tag{51}$$

for $1 = 1,2,3,\ldots..,n$.

Note that G(s) always contains a term of the form $(s - r_i)$ which must be removed by division before going to the limit. (For this reason, that root cannot be a repeated root. See below if this is not the case.) Since the inverse Laplace transform of $A / (s - r_i)$ is $A\,e^{r_i}$, it is always a trivial matter to invert Y(s) in equation (27) to find y(t),

$$y(t) = \mathbf{L}^{-1}\left[Y(s)\right]$$

$$= \mathbf{L}^{-1}\left[\frac{C_1}{(s-r_1)} + \frac{C_2}{(s-r_2)} + \ldots\ldots + \frac{C_n}{(s-r_n)}\right] \tag{52}$$

$$= \sum_{i=1}^{n} y_i(t)$$

where

$$y_i(t) = C_i\,\mathbf{L}^{-1}\left[\frac{1}{(s-r_i)}\right] = C_i A e^{r_i t} \tag{53}$$

If Y(s) contains *repeated roots* in the denominator, the procedure for finding the C_i for the partial fraction expansion must be modified. Consider,

$$Y(s) = \frac{V(s)}{G(s)} = \frac{V(s)}{(s-r_1)^m (s-r_{m-1})(s-r_{m-2}) \ldots\ldots (s-r_{m+n})} \tag{54}$$

The root r_1 is repeated here m times, and there are n nonrepeated (distinct) roots making a total of (m + n) factors of G(s). Expanding in partial fraction form, equation (54) becomes

$$Y(s) = \frac{V(s)}{G(s)} = \frac{C_1}{(s-r_1)^m} + \frac{C_2}{(s-r_1)^{m-1}} +$$

$$\frac{C_3}{(s-r_1)^{m-2}} + \cdots + \frac{C_j}{(s-r_1)^{m-j+1}} + \ldots + \frac{C_m}{(s-r_1)} +$$

$$\frac{C_{m+1}}{(s-r_{m+1})} + \cdots + \frac{C_{m+n}}{(s-r_{m+n})} \tag{55}$$

The constants $C_1 \ldots C_m$ for the repeated roots are determined from the following relationships:

$$C_1 = \lim_{s \to r_1} \left[\frac{V(s)}{G(s)} \left(s - r_1\right)^m \right] \tag{56}$$

$$C_2 = \lim_{s \to r_1} \frac{d}{ds} \left[\frac{V(s)}{G(s)} \left(s - r_1\right)^m \right] \tag{57}$$

$$C_k = \lim_{s \to r_1} \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left[\frac{V(s)}{G(s)} \left(s - r_1\right)^m \right] \tag{58}$$

Expression (58) may be derived as follows: Consider equation (55), which is rewritten here in more compact form for convenience.

$$Y(s) = \sum_{l=1}^{j} C_l (s - r_1)^{l-(m-1)} + \sum_{l=m-1}^{m-n} C_l (s - r_l)^{-1} \tag{59}$$

The terms in braces and in the second summation represent the distinct roots of G(s). It is desired to obtain an expression for C_j. To obtain this, multiply both sides by $(s - r_1)^m$ or

$$\frac{V(s)}{G(s)} (s - r_1)^m = \sum_{l=1}^{j} C_l (s - r_1)^{l-1} + (s - r_1)^m \sum_{l=m-1}^{m-n} C_l (s - r_l)^{-1} \tag{60}$$

If this is differentiated, the leading term C_1 vanishes. The C_2 vanishes when the second derivative is taken, and so forth. It is clear that taking k-1 derivatives reduces the typical (j^{th}) term to a constant (all prior terms vanishing), with each differentiation bringing down an exponent of $(s - r_1)$ that leads to a factorial coefficient, viz.

$$\frac{d^{k-1}}{ds^{k-1}} \left[\frac{V(s)}{G(s)} (s - r_1)^m \right] = \sum_{l=1}^{m} C_l \frac{d^{k-1}}{ds^{k-1}} (s - r_1)^{l-1} + \sum_{l=m+1}^{m+n} C_l \frac{d^{k-1}}{ds^{k-1}} (s - r_1)^m (s - r_l)^{-1}$$

$$= (k-1)!\, C_k + \sum_{l=k+1}^{m} C_l \frac{(l-1)!}{(l-k)!} (s - r_1)^{l-k} + \sum_{l=m+1}^{m+n} C_l \frac{m!}{(m-k+1)!} (s - r_1)^{m-k+1} (s - r_1)^{-1}$$

$$+ \text{ similar terms in } (s - r_2) \tag{61}$$

This is almost what is required, viz. an expression for C_k. If at this point the limit is taken as s goes to r_1, all extra terms vanish leaving the expression in equation (58).

Finding the roots

One major obstacle that may arise in applying this approach is the task factoring the denominator G(s), i.e., finding the roots $r_1, r_2, \ldots, r_n$ for the n^{th} order polynomial G(s) in equation (27). In most practical cases, the transfer function is already in the form of factors of no higher than second order. These roots can be found by using well-known formulae such as the quadratic formula, viz.

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{62}$$

where

$$as^2 + bs + c = 0 \tag{63}$$

Solving low order polynomial for roots is not difficult. However, if higher orders of n are encountered, techniques of higher sophistication must be used. There are many published techniques which solve n^{th} order polynomials. The "Laguerre's Method" for polynomial expansion[9] was adapted here.

The n roots $r_1, r_2, \ldots, r_n$ of the polynomial G(s) are sought where

$$\begin{aligned} G(s) &= (s - r_1)(s - r_2) \ldots\ldots (s - r_n) \\ &= s^n + C_{n-1}s^{n-1} + \ldots\ldots + C_2 s^2 + C_1 s + C_0 \end{aligned} \tag{64}$$

C_n does not explicitly appear since all the coefficients of s in the factors $(s - r_j)$ are unity. Two expressions will be needed in the following development. The first is obtained by differentiating G(s) with respect to s.

$$G(s)' = G(s) \sum_{j=1}^{n} \frac{1}{s - r_j} \tag{65}$$

rearranging gives

$$J(s) \equiv \frac{G(s)'}{G(s)}$$

$$= \frac{1}{s - r_1} + \frac{1}{s - r_2} + \ldots\ldots + \frac{1}{s - r_n} \qquad (66)$$

$$= \frac{1}{a} + \sum_{i=1}^{n-1} \frac{1}{s - r_i}$$

where $a = s - r_n$ (the interval separating the variable s from the n^{th} root). The second expression is obtained by differentiating $G(s)'$ with respect to s again. From (65),

$$G(s)'' = G(s)' \sum_{j=1}^{n} \frac{1}{s - r_j} - G(s) \sum_{j=1}^{n} \frac{1}{(s - r_j)^2} \qquad (67)$$

dividing through by G(s) and rearranging gives

$$\frac{G(s)''}{G(s)} = J(s)J(s) - \sum_{j=1}^{n} \frac{1}{(s - r_j)^2} \qquad (68)$$

or

---

[9] Forman S. Acton, Numerical Methods That Work, Harper and Row, New York, NY, 1970. Acton asserts that Laguerre's method always converges to some root from any starting approximation, even to a complex root from a real first approximation if the implementation of the algorithm permits complex arithmetic.

$$H(s) \equiv J(s)^2 - \frac{G(s)''}{G(s)} = \sum_{j=1}^{n} \frac{1}{(s-r_j)^2}$$

$$= \frac{1}{(s-r_1)^2} + \frac{1}{(s-r_2)^2} + \ldots\ldots + \frac{1}{(s-r_n)^2} \tag{69}$$

$$= \frac{1}{a^2} + \sum_{j=1}^{n-1} \frac{1}{\left[s - r_j\right]^2}$$

While seeking the n^{th} root r_n, consider the standard deviation σ_{n-1} of all the other roots.

$$\sigma_{n-1}^2 = \frac{\displaystyle\sum_{i=1}^{n-1} \frac{1}{(s-r_i)^2} - \frac{1}{n-1}\left[\sum_{i=1}^{n-1} \frac{1}{(s-r_i)}\right]^2}{n-2} \tag{70}$$

Using the definitions given in (66) and (69) to eliminate the summations in (70) yields

$$(n-2)\sigma_{n-1}^2 = \left[H(s) - \frac{1}{a^2}\right] - \frac{1}{n-1}\left[J(s) - \frac{1}{a}\right]^2 \tag{71}$$

Expressed as a quadratic equation in the reciprocal for a, this is

$$-\frac{n}{a^2} + 2J(s)\frac{1}{a} - J(s)^2 + (n-1)H(s) - (n-1)(n-2)\,\sigma_{n-1}^2 = 0 \tag{72}$$

Using the quadratic formula to solve for 1/a and then inverting gives

$$a = \frac{n}{J \pm \left\{(n-1)(nH(s) - J^2) - n(n-1)(n-2)\sigma_{n-1}^2\right\}^{1/2}} \tag{73}$$

where the sign of the denominator of this expression for a is chosen to minimize the absolute value of a. To use (73) to solve for the roots iteratively, the standard deviation σ_{n-1} is neglected. The justification for this step is discussed at the end of this section. Dropping σ_{n-1} gives

$$a = \frac{n}{J \pm \left\{(n-1)(nH(s) - J^2)\right\}^{1/2}} \tag{74}$$

where, from (66) and (69)

$$J(s) = \frac{G(s)'}{G(s)} \tag{75}$$

and

$$H(s) = J(s) - \frac{G(s)''}{G(s)} \tag{76}$$

The iterative algorithm for applying Laguerre's method is as follows:

1. Select an initial trial root, s_0, for equation (64).

2. Use the trial root to evaluate the expression for a, using (75) and (76) in (74).

3. Generate an improved trial root by replacing s_0 by s_0 - a.

4. Repeat step 2 to obtain an improved value for a.

Continue the iteration of steps 2 and 3 until convergence of a has occurred ("a" sufficiently small.)

Laguerre's method can converge to a pair of complex conjugate roots, since the radicand in equation (74) can be negative. Rapid convergence depends on a good initial selection for a trial root. A rule of thumb is to select a root that is much larger than the coefficients in the polynomial G(s).

At this point, note that neglecting σ_{n-1} in equation (73) can be justified as follows. If the roots are equal, then σ_{n-1} is identically zero. This can be seen from equation (70) if the roots become equal as a limiting process. In the limit, all the expressions $s - r_i$ tend toward the same small value, which will be denoted by b. Then equation (70) becomes

$$\sigma_{n-1}^{2} = \frac{\displaystyle\sum_{i=1}^{n-1} \frac{1}{b^{2}} - \frac{1}{n-1}\left[\sum_{i=1}^{n-1} \frac{1}{b}\right]^{2}}{n-2} \tag{77}$$

or

$$\sigma_{n-1}^{2} = \frac{\displaystyle (n-1)\frac{1}{b^{2}} - \frac{1}{(n-1)}\left[\frac{n-1}{b}\right]^{2}}{n-2} = 0 \tag{78}$$

On the other hand, if at least some of the roots are distinct, neglecting σ_{n-1} is intuitively equivalent to saying that the root r_n is far removed from the other roots, which are bunched. Relatively speaking, this is always true as the solution begins to converge.

A FORTRAN program implementing this technique is to be found in appendix A.

CONCLUSIONS

The use of piecewise closed integration techniques can result in considerable relaxation in integration time step size and significant savings in computer run time. However, the user must use an equivalent parallel rather than a sequential representation to avoid unphysical propagation delay effects. The use of an equivalent parallel network is also convenient when treating autopilot noise sources. The partial fraction expansion techniques described here are a convenient method for transforming a product of transfer functions into an equivalent parallel network.

SYMBOLS

VARIABLE	DESCRIPTION
$C_{1,2,3,..,n}$	Expansion Coefficients of Partial Fraction Decomposition
D	Coefficient of the Laplace operator s in the second-order lag/ oscillator with differentiation
G(s)	Denominator of transfer function Y(s)
H(s)	Transfer function $J(s) - \dfrac{G(s)''}{G(s)}$
I	Coefficient of the square of the Laplace operator s in the second-order lag/ oscillator with differentiation
J(s)	Transfer function $\dfrac{G(s)''}{G(s)}$
K	Constant term in the transfer function for the second-order lag/ oscillator with differentiation
L	Laplace transform
L^{-1}	Laplace transform
$r_{1,2,3,..,n}$	Roots of G(s)
s	Laplace transform operator
T	Driving term in the transfer function for the second-order lag/ oscillator with differentiation
t	Time
V(s)	Numerator of transfer function Y(s)
Y(s)	Transfer function, written as $\dfrac{V(s)}{G(s)}$; Laplace transform of y(t)
y(t)	Inverse Laplace Transform of Y(s)

APPENDIX A

FINDING ROOTS BY LAGUERRE'S METHOD

```
      PROGRAM ROOTS
                  IMPLICIT COMPLEX*16 (A-H,O-Z)
      COMPLEX*8 J
      REAL*8 ACC,RR,RI
      DIMENSION C(21),ROOT(20),CD(21),CDD(21),D(21)
C
C     THIS ROUTINE FINDS THE ROOTS OF A NTH ORDER POLYNOMIAL USING
C     LAGUERRE'S METHOD. POINT OF CONTACT: MICHAEL J. AMORUSO OR
C     ROMEL CAMPBELL, ARDEC,DOVER, NJ 201-724-4523.
C
C *** DATA MUST APPEAR IN A FILE CALLED RDATA IN FIELDS 20 COLUMNS
C *** WIDE. SEE EXAMPLE OF FORMAT BELOW.
C
      OPEN (3,FILE='RDATA')
      REWIND 3
C
C     INITIALIZING VARIABLES AND CONSTANTS FOR POLYNOMIAL EQUATION.
C
      ZERO = DCMPLX(0.,0.)
      DO 5 I = 1,21
      C(I)   = ZERO
      CD(I)   = ZERO
      CDD(I)  = ZERO
    5 ROOT(I) = ZERO
C
C     TRIAL ROOT
C
      S = DCMPLX(100.0D0,0.D0)
C
C     CONVERGENCE-TEST TOLLERANCE
C
      ACC  = 1.0D-12

C     EVALUATING EQUATIONS FOR N'TH ORDER POLYNOMIAL
C     AND ITS FIRST AND SECOND DERIVATIVES.
C
C *** THE ORDER OF THE POLYNOMIAL "N" AND ITS N+1 COEFFICIENTS ARE
C *** ARE READ IN HERE. COFFICIENTS ARE READ FROM HIGHEST POWER OF
C *** VARIABLE TO CONSTANT TERM.
C
C *** EXAMPLE: FIND THE ROOTS OF 4TH ORDER POLYNOMIAL:
C
C         4    3
C        S  + S  - 2 S  =  0
C
C *** THE INPUT DATA WOULD BE IN THE FOLLOWING FORM:
C
```

```
C
C                 4
C               1.0
C               1.0
C               0.0
C               1.0
C               0.0
C
      READ(3,101) N
      WRITE(6,103) N
      DO 9 I=N+1,1,-1
      READ(3,102) RR,RI
      C(I) = DCMPLX(RR,RI)
      IF (I.NE.1) WRITE(6,104) C(I),I-1
      IF (I.EQ.1) WRITE(6,107) C(I)
    9 CONTINUE
      WRITE(6,105)
C
C     NUMBER OF ROOTS TO BE FOUND
C
      K = N
C
C     COEFFICIENTS OF 1ST DERIVATIVE OF POLYNOMIAL
C
    1 DO 6 I=1,20
      RR=DFLOAT(I)
      BUF=DCMPLX(RR,ZERO)
      CD(I) = C(I+1)*BUF
    6 CONTINUE
C
C     COEFFICIENTS OF 2ND DERIVATIVE OF POLYNOMIAL
C
      DO 7 I=1,20
      RR=DFLOAT(I)
      BUF=DCMPLX(RR,ZERO)
      CDD(I) = CD(I+1)*BUF
    7 CONTINUE
C
C     EVALUATING POLYNOMIAL AND ITS FIRST TWO DERIVATIVES
C
      G = C(1)
      GD = CD(1)
      GDD = CDD(1)
      DO 10 I = 1,K
      G   =  (C(I+1)*(S**I)) + G
      GD  =  (CD(I+1)*(S**I)) + GD
   10 GDD = (CDD(I+1)*(S**I)) + GDD
C
C     POLYNOMIAL ROOT SOLVING ROUTINE USING LAGUERRE'S METHOD
```

```
C
      CK  = DCMPLX ( DFLOAT(K) , 0.D0)
      CK1 = CK - DCMPLX ( 1.0D0 , 0.D0)
      J   = GD / G
      H   = J**2 - ( GDD/G )
      DEN = CK1*(CK*H-J*J)
      DEN = CDSQRT(DEN)
      IF (CDABS(J+DEN).GE.CDABS(J-DEN)) THEN
      A = CK / (J+DEN)
      ELSE
      A = CK / (J-DEN)
      END IF
      S = S - A
      IF (CDABS(A/S).LT.ACC) GOTO 20
      GO TO 1
C
C     REDUCTION OF POLYNOMIAL BY SYNTHETIC DIVISION
C
C     THE FACTOR (X-S) WILL BE DIVIDED OUT OF THE POLYNOMIAL
C     TO REMOVE THE ROOT JUST OBTAINED. THEN THE REDUCED POLY-
C     NOMIAL CAN BE SOLVED FOR THE NEXT ROOT.
C     SEE FORMAN S. ACTON, "NUMERICAL METHODS THAT WORK",
C     HARPER AND ROW, NEW YORK, 1970, PP 181-183.
C
   20 CONTINUE
C     RR=DREAL(S)
      RR=S
      RI=DIMAG(S)
      ROOT(K) = S
      WRITE(6,100) ROOT(K)
C
C     THE ORDER OF THE REDUCED POLYNOMIAL IS OBTAINED.
C
      K = K-1
      IF (K.GE.1) THEN
      DO 30 I=K+1,1,-1
      D(I) = (D(I+1)*S) + C(I+1)
   30    CONTINUE
C
C     THE D(I) ARE NOW THE COEFFICIENTS OF THE OLD POLYNOMIAL
C     DIVIDED BY THE FACTOR (X-S), WHERE S IS THE ROOT JUST FOUND.
C     THE C(I) WILL BE SET TO THESE D(I), WHICH WILL BE NORMALIZED
C     SO THAT THE HIGHEST ORDER TERM HAS A COEFFICIENT OF UNITY.
C
C     SET NORMALIZATION FACTOR.
C
      Q = D(K+1)
      DO 40 I =1,K+1
      C(I) = D(I)/Q
```

```
 40 D(I) = 0.
C
    GO TO 1
    END IF
C
    WRITE(6,106)
    STOP
100 FORMAT(' (',G20.8,',',G20.8,')')
101 FORMAT(10X,I10)
102 FORMAT(2D20.0)
103 FORMAT(///,' LAUGERRE METHOD FOR SOLVING FOR POLYNOMIAL ROOTS.',
   * //,' THE ORDER OF THE POLYNOMIAL IS ',I3,//,' THE TERMS ARE:')
104 FORMAT(' (',G20.9,',',G20.9,') * X**',I2)
105 FORMAT(/,' THE ROOTS ARE:')
106 FORMAT(//////)
107 FORMAT(' (',G20.9,',',G20.9,')')
    END
```

APPENDIX B

SECOND ORDER LAG/OSCILLATOR WITH DIFFERENTIATOR

In a previous report,[1] transfer functions were treated using this analytic approach. Solutions were provided for lead/lag, lag, lag with differentiator, and second order lag/oscillator. Sometimes these are sufficient and there is no need for parallel decomposition. For completeness, the second order lag/oscillator with differentiator is developed here. Consider the differential equation

$$I \frac{d^2 y}{dt^2} + D \frac{dy}{dt} + Ky = \frac{dT}{dt} \tag{B-1}$$

where both I and K are nonzero. The derivative of the driving term appears on the right side as an input, which is undesirable. A standard technique introduces an auxiliary variable, say Z, and substitutes a pair of coupled differential equations in the variables Z and y that do not contain any derivatives of the driving term T. These equivalent differential equations are

$$\frac{dZ}{dt} = Ky \tag{B-2}$$

and

$$I \frac{dy}{dt} + Dy = T - Z \tag{B-3}$$

The equivalence of (B -2) and (B -3) to (B -1) can be readily verified by differentiating (B -3) with respect to t and substituting (B -2) into the result.

To implement the solution of the coupled equations (B -2) and (B -3), an equation for Z in which y has been eliminated is sought. This is done by using (B -3) to eliminate y in the equation (B -2). This yields

$$\frac{dZ}{dt} = Ky \tag{B-4}$$

$$= \left[\frac{K}{D}\right] \left[T - Z - I \frac{dy}{dt}\right]$$

---

[1] Eugene M. Friedman and Michael J. Amoruso, "An Analytical Modularized Treatment of Autopilots for Guided Projectile Simulations," US Army Armament Research and Development Center, Technical Report ARLCD-TR-85025, Picatinny Arsenal, NJ, August 1985.

$$= \left[\frac{K}{D}\right]\left[T - Z - \frac{I}{K}\frac{d^2Z}{dt^2}\right]$$

or

$$I\frac{d^2Z}{dt^2} + D\frac{dZ}{dt} + KZ = KT \tag{B -5}$$

Once Z(t) is found, y(t) is obtained from (B -2), i.e. $y(t) = \frac{1}{K}\frac{dZ}{dt}$.

This is the equation of a forced harmonic oscillator, with driving term KT. The general solution to (B -5) is obtained from the sum of a particular integral of (B -5) and the general solution to the inhomogeneous form of (B -5). Since (B -5) is second order, the general solution to (B -5) must contain two linearly independent particular solutions to (B -5). It is natural to try

$$Z(t) = e^{Lt} \tag{B -6}$$

Substituting into the homogeneous form (right side of equality sign set to zero) of equation (B -5) obtains the characteristic equation

$$IL^2 + DL + K = 0 \tag{B -7}$$

which has the following roots

$$M = \frac{-D + \sqrt{D^2 - 4IK}}{2I} \equiv \lambda + i\omega \tag{B -8a}$$

$$N = \frac{-D - \sqrt{D^2 - 4IK}}{2I} \equiv \lambda - i\omega \tag{B -8b}$$

The radicand $D^2 - 4IK$ in (B -8) divides the solution into classes. If the radicand is nonzero, the homogeneous solution may be written

$$Z_H(t) = A\,e^{Mt} + B\,e^{Nt} \tag{B -9a}$$

When the radicand doesn't vanish, M and N are distinct and the two exponentials are linearly independent and (B -9a) is a complete general solution to the homogeneous form of (B -5). When the radicand is positive, (B -9a) represents a pure damped solution. If the radicand is negative, then M and N are complex conjugates of one another. In this case, the solution is a damped oscillation. It may be desirable to show this behavior explicitly. The solution (B -9a) may be rewritten

$$Z_H(t) = A\ e^{\lambda t} \sin\left[\omega t + \phi\right] \tag{B -9b}$$

where λ and ω are the real and imaginary parts of the complex conjugate pair M and N, and C and ϕ are the constants of integration instead of A and B.

If the radicand vanishes, (B -9a) is not general since it does not contain two distinct, linearly-independent solutions. Substitution verifies that the second independent solution can be obtained from te^{Mt} where $M = -D/2I$. In this case,

$$Z_H(t) = A\ e^{Mt} + B\ t\ e^{Mt} \tag{B -9c}$$

The next step is finding a particular integral of (B -5). The damped positive-radicand solution will be treated first. The method of variation of parameters is applicable.[2] Assume the following form for the solution to the inhomogeneous equation, viz. (B -5).

$$Z_p(t) = u_1(t)\ e^{Mt} + u_2(t)\ e^{Nt} \tag{B -10}$$

where u_1 and u_2 are to be determined. It can be shown that the u_1 and u_2 of (B -10) satisfy

$$u_1'(t) = -e^{-Mt} \frac{K\ T(t)}{I\ (N - M)} \tag{B -11}$$

$$u_2'(t) = +e^{-Nt} \frac{K\ T(t)}{I\ (N - M)} \tag{B -12}$$

where the prime denotes the derivative with respect to t.[2] Integrating u_1' and u_2' yields the particular solution required in order to generate the complete general solution to (B -5). However, this can not be done in closed form in the general case since the form of T(t) is not known analytically *a priori*. In order to carry out these integrations explicitly, some generality can be given up that is in the spirit of this approach. The simplest nontrivial assumption is that the time step is small enough to approximate T(t) linearly, viz.

$$T(t) = At + b = T'(t - t_o) + T_o \tag{B -13}$$

where A, b, T_o, and T' are constants.[3] Integrating u_1' and u_2' with this assumption yields

$$u_1(t) = \frac{K e^{-Mt}}{MI(N-M)} \left[At + b + \frac{A}{M} \right]$$

$$= \frac{K e^{-Mt}}{MI(N-M)} \left[T(t) + \frac{T'}{M} \right] \tag{B -14}$$

and

$$u_2(t) = \frac{-K e^{-Nt}}{NI(N-M)} \left[At + b + \frac{A}{N} \right]$$

$$= \frac{-K e^{-Nt}}{NI(N-M)} \left[T + \frac{T'}{N} \right] \tag{B -15}$$

Putting (B -14) and (B -15) into (B -10) yields

---

[2] C. R. Wylie, Jr., Advanced Engineering Mathematics, McGraw Hill, New York, NY, 1960.

[3] With this assumption, (B-1) can be solved directly without (B-2) or (B-3) since the right hand side of (B-1) is replaced by the constant A = T ' (i.e., $I\ddot{y} + D\dot{y} + Ky = A$).

$$Z_p(t) = \frac{K}{I\,(N-M)}\left[T\left\{\frac{1}{M} - \frac{1}{N}\right\} + T'\left\{\frac{1}{M^2} - \frac{1}{N^2}\right\}\right]$$

$$= \frac{K}{I\,(N-M)}\left[T\,\frac{(N-M)}{MN} + T'\,\frac{(N^2-M^2)}{M^2N^2}\right]$$

$$= \frac{K}{INM}\left[T + T'\frac{(N+M)}{MN}\right] \qquad \text{(B -16)}$$

From (B -8), $MN = K/I$ and $M+N = -D/I$.

$$Z_p(t) = T - \frac{D}{K}T' \qquad \text{(B -17)}$$

This solution can be verified by substituting (B -17) into (B -5), keeping in mind that T' is constant during the time step. Since this constancy is the only assumption used in verifying (B -17) by substitution into (B -5), this particular solution Z_p holds for all three cases even though it was formally derived for only one case. This result can be combined with the results for the three homogeneous solutions [viz., (B -9a) through (B -9c)] and the boundary conditions applied to determine the constants of integration.

POSITIVE RADICAND

Combining (B -9a) and (B -17) gives the complete solution for the purely damped case

$$Z(t) = Ae^{Mt} + Be^{Nt} + T - \frac{D}{K}T' \qquad \text{(B -18)}$$

$$\frac{dZ(t)}{dt} = Z' = AMe^{Mt} + BNe^{Nt} + T' \tag{B-19}$$

The initial conditions

$$Z_o = Z \text{ at } t_o \tag{B-20}$$

$$Z'_o = \frac{dZ}{dt} \text{ at } t_o$$

$$T_o = T \text{ at } t_o$$

$$T'_o = T' \text{ or } \frac{dT}{dt} \text{ at } t_o$$

determine the constants of integration A and B. Substituting into (B-18) and (B-19) yields

$$Z_o = A\,e^{Mt_o} + B\,e^{Nt_o} + T_o - \frac{D}{K}T'_o \tag{B-21}$$

$$Z'_o = AMe^{Mt_o} + B\,N\,e^{Nt_o} + T'_o \tag{B-22}$$

solving simultaneously yields

$$A = \left[Z_o - \frac{Z'_o}{N} - T_o + T'_o \left\{ \frac{1}{N} + \frac{D}{K} \right\} \right] \frac{e^{-Mt_o}}{1 - \frac{M}{N}}$$

$$B = \left[Z_o - \frac{Z'_o}{M} - T_o + T'_o \left\{ \frac{1}{M} + \frac{D}{K} \right\} \right] \frac{e^{-Nt_o}}{1 - \frac{N}{M}} \qquad \text{(B -23)}$$

Finally, y(t) can be obtained by substituting (B -19) into (B -2), viz.,

$$y(t) = \frac{1}{K} \left[AMe^{Mt} + BNe^{Nt} + T' \right] \qquad \text{(B -24)}$$

NEGATIVE RADICAND

Combining (B -9b) and (B -17) give the complete solution for the damped oscillatory case

$$Z(t) = Ce^{\lambda t} \sin[\omega t + \phi] + T - \frac{D}{K} T' \qquad \text{(B -25)}$$

$$\frac{dZ(t)}{dt} = Z' = Ce^{\lambda t} \left\{ \lambda \sin[\omega t + \phi] + \omega \cos[\omega t + \phi] \right\} + T' \qquad \text{(B -26)}$$

where $\lambda = -\frac{D}{2I}$ and $\omega = \frac{\sqrt{4IK - D^2}}{2I}$ [See (B -8)]. See (B -20) for the initial conditions. The constants of integration C and ϕ can be determined now from

$$Z_o = Ce^{\lambda t_o} \sin[\omega t_o + \phi] + T_o - \frac{D}{K} T'_o \qquad \text{(B -27)}$$

$$Z'_o = Ce^{\lambda t_o}\left\{\lambda \sin[\omega t_o + \phi] + \omega \cos[\omega t_o + \phi]\right\} + T'_o \tag{B-28}$$

From (B -28),

$$C = \frac{[Z'_o - T'_o]\, e^{-\lambda t_o}}{\lambda \sin[\omega t_o + \phi] + \omega \cos[\omega t_o + \phi]} \tag{B-29}$$

Substituting (B -29) in (B -27) gives

$$\phi = \tan^{-1}\left[\frac{\omega\left(Z_o - T_o - \frac{D}{K}T'_o\right)}{Z'_o - T'_o - \lambda\left(Z_o - T_o + \frac{D}{K}T'_o\right)}\right] - \omega t_o \tag{B-30}$$

Finally, y(t) can be obtained by substituting (B -26) into (B -2), viz.,

$$y(t) = \frac{1}{K}\left[Ce^{\lambda t}\left\{\lambda \sin[\omega t + \phi] + \omega \cos[\omega t + \phi]\right\} + T'\right] \tag{B-31}$$

ZERO RADICAND

Here $M = -D/2I$ [see equation (B -8a)]. Combining (B -9c) with (B -17) gives the complete solution for this case.

$$Z(t) = Ae^{Mt} + Bte^{Mt} + T - \frac{D}{K}T' \tag{B-32}$$

$$\frac{dZ(t)}{dt} = Z' = AMe^{Mt} + BMte^{Mt} + Be^{Mt} + T' \qquad \text{(B -33)}$$

The constants of integration can be determined now from the initial conditions.

$$Z_o = Ae^{Mt_o} + Bt_o e^{Mt_o} + T_o - \frac{D}{K}T'_o \qquad \text{(B -34)}$$

$$Z'_o = AMe^{Mt_o} + BMt_o e^{Mt_o} + Be^{Mt_o} + T'_o \qquad \text{(B -35)}$$

Solving (B -34) and (B -35) simultaneously yields

$$A = e^{-Mt_o}\left[Z_o - T_o + \frac{D}{K}T'_o + t_o[\, MZ_o - Z'_o - MT_o + \frac{MD}{K}T'_o + T'_o] \right] \text{(B -36)}$$

and

$$B = -e^{-Mt_o}\left[MZ_o - Z'_o - MT_o + \frac{MD}{K}T'_o + T'_o \right] \qquad \text{(B -37)}$$

Finally, y(t) can be obtained by substituting (B -33) into (B -2), viz.,

$$y(t) = \frac{1}{K}\left[AMe^{Mt} + BMte^{Mt} + Be^{Mt} + T' \right] \qquad \text{(B -38)}$$

Initially, Z_o can be taken to be zero on the very first integration time step and Z'_o can be derived from the initial condition $y_o = y$ at t_0 by means of equation (B -2).

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