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ANALYSIS OF LAMINATED COMPOSITE PLATES USING HYBRID-STRESS ELEMENTS

by

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# ANALYSIS OF LAMINATED COMPOSITE PLATES USING HYBRID-STRESS ELEMENTS

Deng Liangbo

Summary

Based on the modified complementary energy principle, the paper describes a hybrid-stress rectangular flexural element suitable for characteristics of laminated fiber composite plates. This element can take into account the effects of transverse shear deformation and local warping. In the dynamic analysis, the interior element displacements are assumed to be compatible with the boundary displacements. The element has the characteristics of fewer degrees of freedom, convenience in application, and high accuracy in numerical calculations.

## I. Introduction

When applying the method of finite elements to analyze laminated plates made of composites, not only should transverse shear deformations be considered, but also the local warping effect. In this respect, the hybrid-stress finite elements method has unique advantages. When constituting elements in [1], the interlaminar displacement of nodal points was taken as a degree of freedom. Hence, the number of degrees of freedom of an element will rise with an increase in the number of layers, thus incurring difficulties in numerical calculations, especially in multilaminated plates. An element is proposed in [2]; only a set of stress parameters is assumed in the element interior, and a hypothesis of YNS theory (hypothesis of rectilinear normal line) is applied on the element boundary, therefore the local warping effect can not be taken into account. With the foregoing in mind, the author conceptualizes the title element; the hypothesis of YNS theory is adopted for the element boundary, and stress parameters are assumed layer by layer in the element interior. Hence, the element can take into account both the effect of transverse shearing deformation and the local warping effect with fewer degrees of freedom.

## II. Theory

In the analysis of laminated plates, the modified arbitrary function of the rest energy is:

$$\pi_{mc} = \sum_{n} \Big[ \sum_{i=1}^{m} \Big( \int_{v_{ni}} \tfrac{1}{2} \{\sigma^{(i)}\}^T [s^{(i)}] \{\sigma^{(i)}\} dv - \int_{\partial v_{ni}} \{P^{(i)}\}^T \{u^{(i)}\} ds + \int_{S\sigma_{ni}} \tfrac{1}{2} \{\overline{p}^{(i)}\} \{u^{(i)}\} ds \Big) \Big] \tag{1}$$

In the equation,

$\{\sigma^{(i)}\}$ is the stress vector of the i-th subzone (i-th layer);
$\{s^{(i)}\}$ is the elasticity matrix of the i-th subzone;
$\{u^{(i)}\}$ is the boundary displacement vector of the i-th subzone;
$\{p^{(i)}\}$ is the boundary force vector of the i-th subzone;
$\{\overline{p}^{(i)}$ is the assigned boundary force vector of the i-th subzone;
$\partial v_{ni}$ is all boundary $\partial v_{ni} = Su_{ni} + S\sigma_{ni} + S_{ni}$ of the i-th subzone of the n-th element;
$v_{ni}$ is the volume of the i-th subzone of the n-th element;
$S\sigma_{ni}$ is the delimited boundary of the i-th subzone boundary force of the n-th element;
$Su_{ni}$ is the delimited boundary in terms of displacements;
and $S_{ni}$ is the boundary adjacent to other elements.

When one applies the arbitrary function (1), the assumed stress vector should satisfy the homogeneous equilibrium equation in each subzone. At the boundary $Su_{ni}$, $\{u^{(i)}\}=\{\bar{u}^{(i)}\}$ should be satisfied; in the equation, $\{\bar{u}^{(i)}\}$ is the known displacement vector at the boundary.

We assume that the stress field of each layer is:

$$\{\sigma^{(i)}\}=[A^{(i)}]\{\beta^{(i)}\} \tag{2}$$

In the equation, $[A^{(i)}]$ is the coefficient matrix of stress parameters; this is a function of x, y and z. $\{\beta^{(i)}\}$ is the stress parameter of the i-th layer.

In terms of the stress parameter, the unit boundary force vector can be expressed as:

$$\{P^{(i)}\}=[R^{(i)}]\{\beta^{(i)}\} \tag{3}$$

In the equation, $[R^{(i)}]$ is constituted with the boundary value of the matrix $[A^{(i)}]$ in satisfying the boundary force equation $P_i = =\sigma_{ij}n_j$.

We assume that the boundary displacement of the i-th layer is:

$$\{u^{(i)}\}=[L^{(i)}]\{\delta^{(i)}\} \tag{4}$$

In the equation, $\{\delta^{(i)}$ is the nodal point displacement (in the broad sense) of the i-th layer. The boundary displacement field (4) should satisfy the displacement compatibility conditions among elements.

If we substitute Eqs. (2), (3) and (4) into Eq. (1), then we obtain:

$$\pi_{mc}=\sum_{n}\left(\frac{1}{2}\{\beta^e\}^T[H^e]\{\beta^e\}-\{\beta^e\}[G^e]\{\delta^e\}+\{Q_0\}^T\{\delta^e\}\right) \tag{5}$$

where $[H^e]$, $[G^e]$ and $\{Q_0\}$ are combined, respectively, according to the specified sites by:

$$[H^{(i)}] = \int_{V_{ni}} [A^{(i)}]^T [S^{(i)}][A^{(i)}] dv$$

$$[G^{(i)}] = \int_{\partial V_{ni}} [R^{(i)}]^T [L^{(i)}] ds$$

$$\{Q^{(i)}\}^T = \int_{S\sigma_{ni}} \{\overline{P}^{(i)}\}^T [L^{(i)}] ds$$

$\{\beta^e\}$ and $\{\delta^e\}$ are stress parameters and nodal point displacement vector (in the broad sense) of the entire element, and combined, respectively, with $\{\beta^{(i)}\}$ and $\{\delta^{(i)}\}$ (i=1, 2, ..., m).

Stresses $\sigma_{xz}$, $\sigma_{yz}$ and $\sigma_z$ should continue among layers; this continuation condition can be expressed as:

$$[\sigma_{xz}^{(1)}, \sigma_{yz}^{(1)}, \sigma_z^{(1)}] = [0, 0, 0]$$

$$[\sigma_{xz}^{(i+1)}, \sigma_{yz}^{(i+1)}, \sigma_z^{(i+1)}] - [\sigma_{xz}^{(i)}, \sigma_{yz}^{(i)}, \sigma_z^{(i)}] = [0, 0, 0] \tag{6}$$

$$[\sigma_{xz}^{(m)}, \sigma_{yz}^{(m)}, \sigma_z^{(m)}] = [0, 0, 0]$$

We substitute Eq. (2) into (6); this continuation condition can be expressed with the stress parameter as:

$$[B]\{\beta^e\} = \{0\} \tag{7}$$

Using Lagrange's operator on Eq. (7) and introducing into arbitrary function (5), then the following equations are obtained through differentiation and rearrangement:

$$\pi_{mc} = \sum_n \left(\frac{1}{2}\{\delta^e\}^T [K_0]\{\delta^e\} - \{Q_0\}^T\{\delta^e\}\right) \tag{8}$$

$$\{\beta^e\} = ([H^e]^{-1}[G^e] - [H^e]^{-1}[B]^T([B][H^e]^{-1}[B]^T)^{-1}[B][H^e]^{-1}[G^e])\{\delta^e\} \tag{9}$$

In the equations,

$$[K_0] = [G^e][H^e]^{-1}[G^e] - [G^e][H^e]^{-1}[B]^T([B][H^e]^{-1}[B]^{T-1}[B][H^e]^{-1}[G^e]$$

This is apparent from the above-mentioned deduction process. Since stress $\{\sigma^{(i)}\}$ is different for each layer, and the elasticity matrix $[s^{(i)}]$ of each layer is generally different, therefore the stress field $\{\varepsilon^{(i)}\} = [s^{(i)}]\{\sigma^{(i)}\}$ of each layer differs layer by layer. Based on geometric equations, it is known that there are different variation (in terms of coordinates) equations for the displacement of each layer with the exception of transverse displacement w; this is the different function of the coordinates. Hence, the post-deformation normal line is partitioned on different segments according to the layer. This element can be used to take into account the local warping effect.

On the element boundary, according to the hypothesis of YNS theory, we have:

$$\{\delta^e\} = [w]\{q^e\} \tag{10}$$

where $\{q^e\}$ is the nodal point displacement (in the broad sense) at the midplane of the laminated plate.

Substituting Eq. (10) into (8), we obtain:

$$\pi_{mc} = \sum_n \left(\frac{1}{2}\{q^e\}^T[K^e]\{q^e\} - \{Q^e\}^T\{q^e\}\right) \tag{11}$$

In the equation, $[K^e] = [W]^T[K_0][W]$ is a single-element rigidity matrix: $\{Q^e\} = \{Q_0\}^T[W]$ is the single-element nodal point as vector.

From dynamic analysis, the modified variation principle is:

$$\begin{aligned}\pi_{mD} = \int_{t1}^{t2} \sum_n \Big\{ \sum_{i=1}^{m} \Big[ \int_{v_{ni}} \frac{1}{2}\{\dot{u}_I^{(i)}\}^T \rho^{(i)} \{\dot{u}_I^{(i)}\} dv + \int_{v_{ni}} \frac{1}{2}\{a^{(i)}\}^T[s^{(i)}]\{\sigma^{(i)}\} dv - \\ - \int_{\partial v_{ni}} \{p^{(i)}\}^T\{u^{(i)}\} ds + \int_{S\sigma_{ni}} \{\bar{p}^{(i)}\}\{u^{(i)}\} ds \Big] \Big\} dt\end{aligned} \tag{12}$$

where $\sigma^{(i)}$ is the mass density of the i-th layer; $\{u_I\}$ is the derivative with respect to time of the internal displacement vector of the i-th layer.

In Eq. (12), in principle the internal displacement field $\{u_I\}$ and single-element boundary displacement field can be independently assumed. However, for convenience the paper adopts the internal displacement field compatible with the boundary displacement field. We assume that the internal displacement field is:

$$\{u_I^{(i)}\} = [N^{(i)}]\{\delta^{(i)}\} \tag{13}$$

Substituting Eqs. (2), (3), (4) and (13) into Eq. (12), we obtain the following after rearrangement:

$$\pi_{mD} = \int_{t1}^{t2} \sum_{n} \left(\frac{1}{2}\{\dot{q}\}^T[M^e]\{\dot{q}\} - \frac{1}{2}\{q^e\}[K^e]\{q^e\}\right)dt \tag{14}$$

In the equation, $[M^e]$ is the mass matrix of a single element and is composed of the mass matrices of all the layers. When the free vibration problem is considered, no load item appears in Eq. (14).

$$[M^{(i)}] = \int_{v_{ni}} \rho^{(i)} [N^{(i)}]^T [N^{(i)}] dv$$

From Eq. (14) a characteristic equation with the intrinsic property of power can be derived:

$$([K] - \omega^2[M])\{\bar{q}\} = 0 \tag{15}$$

In the equation, [K] is the overall rigidity matrix; [M] is the overall mass matrix.

## III. Hypotheses of Field Variates

(1) In a single element, the stress field hypothesis of each layer is:

$$\begin{aligned}
\sigma_x^{(i)} &= \beta_1^{(i)} + \beta_4^{(i)} x + \beta_7^{(i)} y + \beta_{10}^{(i)} z + \beta_{13}^{(i)} xz + \beta_{16}^{(i)} yz \\
\sigma_y^{(i)} &= \beta_2^{(i)} + \beta_5^{(i)} x + \beta_8^{(i)} y + \beta_{11}^{(i)} z + \beta_{14}^{(i)} xz + \beta_{17}^{(i)} yz \\
\sigma_{xy}^{(i)} &= \beta_3^{(i)} + \beta_6^{(i)} x + \beta_9^{(i)} y + \beta_{12}^{(i)} z + \beta_{15}^{(i)} xz + \beta_{18}^{(i)} yz \\
\sigma_{yz}^{(i)} &= -\beta_6^{(i)} z - \beta_8^{(i)} z - \frac{z^2}{2}(\beta_{15}^{(i)} + \beta_{17}^{(i)}) + \beta_{20}^{(i)} \\
\sigma_{xz}^{(i)} &= -\beta_4^{(i)} z - \beta_9^{(i)} z - \frac{z^2}{2}(\beta_{13}^{(i)} + \beta_{18}^{(i)}) + \beta_{19}^{(i)} \\
\sigma_z^{(i)} &= 0
\end{aligned} \tag{16}$$

In the equation, z is the distance from a point in each layer to the midplane of that layer. Equation (16) satisfies the stress homogeneous equilibrium equation.

(2) The boundary displacement field of each layer in a single element:

We take displacements u, v and w along the interlayer directions x, y and z, considered as the nodal point displacements (as in Fig. 1) in the broad sense. Here, the author and his colleagues assume that w is a constant along the entire thickness direction of the single element in order to be consistent with $\sigma_z=0$. For boundary AB, the equation expressing boundary displacement is:

$$\begin{aligned}
u_{AB}^{(i)} &= \frac{1}{2}(u_1^{(i)} + u_5^{(i)})(1-\frac{x}{a}) + \frac{1}{2}(u_2^{(i)} + u_6^{(i)})\frac{x}{a} + \\
&+ \frac{z}{h^{(i)}}[(u_5^{(i)} - u_1^{(i)})(1-\frac{x}{a}) + (u_6^{(i)} - u_2^{(i)})\frac{x}{a}] \\
v_{AB}^{(i)} &= \frac{1}{2}(v_1^{(i)} + v_6^{(i)})(1-\frac{x}{a}) + \frac{1}{2}(v_2^{(i)} + v_6^{(i)})\frac{x}{a} + \frac{z}{h^{(i)}}[(v_6^{(i)} \qquad (17) \\
&- u_1^{(i)})(1-\frac{x}{a}) + (v_6^{(i)} - v_2^{(i)})\frac{x}{a}] \\
w_{AB}^{(i)} &= w_1^{(i)}(1-\frac{x}{a}) + w_2^{(i)}\frac{x}{a} = w_1(1-\frac{x}{a}) + w_2\frac{x}{a}
\end{aligned}$$

In the equation, $h^{(i)}$ is the thickness of the i-th layer. Displacements of other boundaries can be written out following the previous line of reasoning.

(3) Transformation matrix [W]:

We take four corner points at the geometric midplane (of the laminated plate) as nodal points; there are five degrees of freedom, three displacement degrees of freedom (u, v and w), and two degrees of freedom of the angle of rotation (namely the angle of rotation $\theta_x$ around the x-axis and the angle of rotation $\theta_y$ around the y-axis) for each nodal point, as:

$$u_i^{(i)} = u_1 + l_i\theta_{y}$$

$$v_i^{(i)} = v_1 + l_i\theta_{x1} \tag{18}$$

$$w_i^{(i)} = w_1$$

In the equation, $l_i$ is the distance from the point (to be solved) of nodal point displacement to the geometric midplane. The other nodal point displacements of each layer in the broad sense can be similarly expressed in terms of the nodal point displacement on the geometric midplane. From Eq. (18), the transformation matrix [W] can be formed.

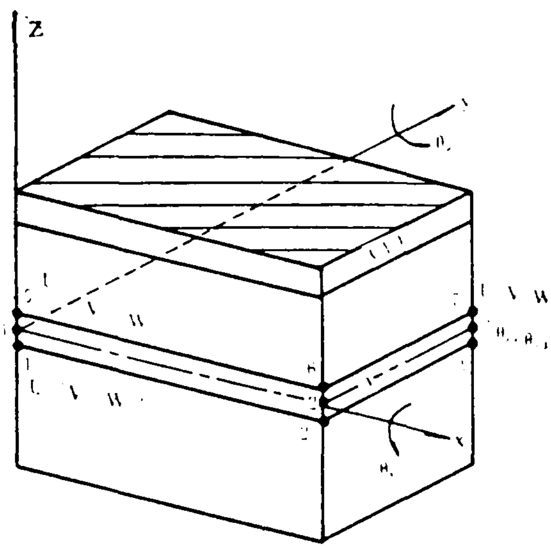


Fig. 1. Nodal point displacement.

(4) Hypothesis of internal displacement field:

$$\begin{aligned}
u_i^{(i)} &= (1-\frac{x}{a}-\frac{y}{b}+\frac{xy}{ab})u_1 + (1-\frac{x}{a}-\frac{y}{b}+\frac{xy}{ab})(z+h_i)\theta_{y1} + \\
&\quad + (\frac{x}{a}-\frac{xy}{ab})u_2 + (\frac{x}{a}-\frac{xy}{ab})(z+h_i)\theta_{y2} + \frac{xy}{ab}u_3 + \frac{xy}{ab}(z+h_i)\theta_{y3} + \\
&(\frac{y}{b}-\frac{xy}{ab})u_4 + (\frac{y}{b}-\frac{xy}{ab})(z+h_i)\theta_{y4} \\
v_i^{(i)} &= (1-\frac{x}{a}-\frac{y}{b}+\frac{xy}{ab})v_1 + (1-\frac{x}{a}-\frac{y}{b}+\frac{xy}{ab})(z+h_i)\theta_{x1} \\
&\quad + (\frac{x}{a}-\frac{xy}{ab})v_2 + (\frac{x}{a}-\frac{xy}{ab})(z+h_i)\theta_{x2} + \frac{xy}{ab}v_3 + \frac{xy}{ab}(z+h_i)\theta_{x3} +
\end{aligned} \tag{19}$$

$$+\left(\frac{y}{b}-\frac{xy}{ab}\right)v_4+\left(\frac{y}{b}-\frac{xy}{ab}\right)(z+h_i)\theta_{x4} \qquad (19)$$

$$w_i^{(i)}=\left(1-\frac{x}{a}-\frac{y}{b}+\frac{xy}{ab}\right)w_1+\left(\frac{x}{a}-\frac{xy}{ab}\right)w_2+\frac{xy}{ab}w_3+\left(\frac{y}{b}-\frac{xy}{ab}\right)w_4$$

where $h_i$ is the distance from the midplane of the i-th layer to the geometric midplane of the plate. It can be proved that Eq. (19) is compatible with the displacement field (of each boundary) at the boundary.

## IV. Numerical Results

### (1) Flexural stress analysis of simply supported rectangular laminated plate [0°/90°/0°]

The plate dimensions are 3a×a; the bearing load is $q=q_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{3a}$. The origin of the corresponding coordinate system is taken at a corner point of the plate; the y-axis extends along the long side; the x-axis along the short side; and the ratio between plate width and plate thickness is (a/h)=10. In [3], Pagano gives an exact solution for this problem. In [1] a finite element analysis is made for this problem; the numerical results and the calculated results in the paper are listed in Table 1.

Table 1. Stress and deflection of laminated plate.

| (a) 方法 | (b) 网格 | (c) 自由度 | $\bar{\sigma}_x$ | $\bar{\sigma}_y$ | $\bar{\sigma}_{xz}$ | $\bar{\sigma}_{yz}$ | $\bar{\sigma}_{xy}$ | $\bar{w}$ |
|---|---|---|---|---|---|---|---|---|
| | | | $(\frac{a}{2},\frac{3a}{2},\pm\frac{h}{2})$ | $(\frac{a}{2},\frac{3a}{2},\pm\frac{h}{6})$ | $(0,\frac{3a}{2},0)$ | $(\frac{a}{2},0,0)$ | $(0,0,\pm\frac{h}{2})$ | $(\frac{a}{2},\frac{3a}{2},0)$ |
| (d) 〔1〕单元 | 2×1 | 54 | ±0.828 | ±0.0462 | 0.287 | 0.0104 | ±0.0136 | 1.0131 |
| (e) 本文单元 | 2×1 | 30 | ±0.834 | ±0.0467 | 0.283 | 0.0101 | ±0.0139 | 1.0133 |
| (d) 〔1〕单元 | 4×2 | 135 | ±0.741 | ±0.0433 | 0.386 | 0.0139 | ±0.0126 | 0.9427 |
| (e) 本文单元 | 4×2 | 75 | ±0.744 | ±0.0440 | 0.383 | 0.0136 | ±0.0128 | 0.9430 |
| (d) 〔1〕单元 | 8×4 | 405 | ±0.714 | ±0.0426 | 0.415 | 0.0150 | ±0.0122 | 0.9255 |
| (e) 本文单元 | 8×4 | 225 | ±0.737 | ±0.0428 | 0.413 | 0.0149 | ±0.0123 | 0.9260 |
| (f) 精确解 | | | 0.726<br>-0.725 | 0.0413<br>-0.0435 | 0.420 | 0.0152 | -0.0120<br>0.0123 | 0.919 |

(Key on following page.)

Key to Table 1: (a) Method; (b) network; (c) degree of freedom; (d) single element from [1]; (e) single element in the paper; (f) exact solution

Table 2. Effect ($\frac{a}{h}=10$) of dimensionless frequency $\lambda=\omega a^2(\rho/E_2h^2)^{\frac{1}{2}}$ of square plate by layer number.

| (a) 层 数 | (b) [4] 单元 | (c) 文 [5] | (d) 本 文 |
|---|---|---|---|
| 2 | 15.714 | 13.04 | 13.581 |
| 4 | 18.069 | 18.46 | 18.385 |
| 6 | 18.295 | 19.09 | 19.003 |
| 8 | 19.028 | 19.29 | 19.230 |
| 10 | 19.074 | 19.38 | 19.351 |
| 12 | 13.038 | — | 19.429 |
| 14 | 13.113 | — | 19.484 |
| 16 | 19.122 | 19.48 | 19.500 |

Key: (a) Layer number; (b) single element from [4]; (c) reference [5]; (d) this paper.

(2) Intrinsic property analysis of bias-laid laminated plate

The fiber-laying angle of the plate is [-45°/45°/-45°/45°/...]; the elastic constant of the material is

$$\frac{E_1}{E_2}=40,\quad \frac{G_{12}}{E_2}=0.6,\quad \frac{G_{13}}{E_2}=\frac{G_{23}}{E_2}=0.5,\quad \nu=0.25.$$

The boundary condition of the plate is u=w=0, $\theta_x=0$. At a, where x=0, v=w=0, and $\theta_y=0$. At b, the dimensionless frequency values of a plate with different layer numbers are listed in Table 2.

## V。Conclusion

From the above analyses and numerical calculations, the following conclusion can be obtained: to conform to the properties of a laminated plate, stress parameters should be assumed layer by layer. However, at the boundary of a single element, the displacement hypothesis in the YNS theory can be adopted. A single element thus constituted can not only take into account the effect of shear deformation and the local warping effect, but also involves fewer degrees of freedom of a single element.

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