Exact Wave Functions and Coherent States of a Damped Driven
Harmonic Oscillator

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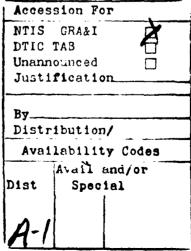# Exact wave functions and coherent states of a damped driven harmonic oscillator

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1. Introduction

In the past few decades there has been extensive effort[1] to obtain exact solutions to the Schödinger equation for oscillator systems with time-dependent Hamiltonians. The path integral formalism of Feynman[2] provides a general approach to quantum systems. In this theory one must obtain the exact propagator associated with the classical action of a given system. Provided the exact propagator is obtained, the wave function of the system can be readily calculated. Even though the solution of the Schrödinger equation is possible through this method, there remains the problem of second-quantization, which is important in connection with construction of the explicitly time-dependent invariant (dynamical invariant) and finding the coherent states of the system.

Since Lewis and Riesenfeld[3] derived a simple relation between the eigenstates of the dynamical invariant and the solution of the Schrödinger equation, several authors have employed the dynamical invariant method to investigate systems with a time-dependent Hamiltonian, including the time-dependent harmonic oscillator. The so-called Ermakov-Lewis problem and its generalization have been investigated by the following four methods:[4] (i) exact adiabatic invariants, (ii) time-dependent canonical transformation, (iii) Noether's theorem and (iv) Lie theory of extended groups. The dynamical invariant involves an auxiliary function which is related to the amplitude of the classical harmonic oscillator and satisfies a nonlinear second-order differential equation (auxiliary equation). Each particular solution to the auxiliary equation determines the dynamical invariant.

Coherent states were used by Glauber[5] to discuss the photon statistics of radiation fields. After that, they have been widely used in various fields of physics.[6] Hartley and Ray[7] constructed coherent states for a time-dependent harmonic oscillator on the basis of the Lewis and Riesenfeld theory.[3] Recently, Yeon, Um and George[8] obtained the exact coherent states for a damped harmonic oscillator with constant frequency. Quesne[9] examined the unitary-operator coherent states.[10] The ordinary coherent states may be defined in alternate, but essentially equivalent, ways. For example, these states are defined as the eigenstates of a destruction operator and are also obtained by applying a unitary operator, consisting of destruction and creation operators, to the ground state of the system. The coherent states have several novel properties,[11] including the minimum uncertainty product in position and momentum.

In previous work[12] we considered a molecular system adsorbed on a dielectric solid surface, modeled as a damped harmonic ascillator driven by an external electric field. The induced dipole moment of the adsorbed molecule obeys the equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_m^2 x = f(t)/m_0 \quad , \tag{1-1}$$

where m_0, γ and ω_m are, respectively, the mass of the adsorbed molecule, modified damping constant and modified frequency due to the presence of the solid surface. The dots denote the derivative with respect to time t, and f(t) is an external driving force given by

$$f(t) = g(\omega',\theta) \cos(\omega't - \phi_0) \tag{1-2}$$

where $g(\omega',\theta)$ is the amplitude of the driving force, which depends on the frequency ω' of the incident field and on the incident angle defined with respect to the normal direction to the solid surface, and ϕ_0 is the phase determined by incident and reflected fields. The Lagrangian L and classical Hamiltonian H corresponding to the equation of motion (Eq. (1-1)) are

$$L(\dot{x},x.t) = \left[\frac{1}{2}m_0\dot{x}^2 - \frac{1}{2}m_0\omega_m^2 x^2 + xf(t)\right]e^{\gamma t} \quad , \tag{1-3}$$

$$H(x,p,t) = \left[\frac{p^2}{2m_0}e^{-2\gamma t} + \frac{1}{2}m_0\omega_m^2 x^2 - xf(t)\right]e^{\gamma t} \quad , \tag{1-4}$$

where p is the classical linear momentum. Hereafter we shall use the units $\hbar$ (Planck's constant) $= m_0 = 1$.

In Sec. 2.A we review the results of our previous work for finding the wave function of the damped driven harmonic oscillator within the path integral formalism.[12,13] The second-quantization formalism is presented in Sec. 2.B. The appropriate destruction and creation operators are defined, and by using these operators we obtain the wave functions for this system. In Sec. 2.C we use the dynamical invariant method, constructing the exact invariant by solving the auxiliary equation. The coherent states are obtained, and their properties are investigated in Sec. 3. In Sec. 4 we briefly discuss the results and present conclusions.

2. Three Methods

A. Path Integral Method

The Hamiltonian operator $\hat{H}(t)$ corresponding to the classical Hamiltonian (Eq. 1-4) is obtained by making the replacement $p \rightarrow \frac{1}{i}\frac{\partial}{\partial x}$,

$$\hat{H}(t) = \left[-\frac{1}{2}\frac{\partial^2}{\partial x^2}e^{-2\gamma t} + \frac{1}{2}\omega_m^2 x^2 - xf(t)\right]e^{\gamma t} \quad , \tag{2-1}$$

where x and p satisfy the commutation relation $[x,p] = i$. In order to solve the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(x,t) = \hat{H}(t)\,\psi(x,t) \quad , \tag{2-2}$$

we follow Feynman's path integral method[2] and adopt a Gaussian type propagator K,

$$K(x,t;\,x_0,0) = A_0 \exp[-a_1(t)x^2 e^{\gamma t} - a_2(t)xe^{\gamma t/2} - a_3(t)] \quad , \tag{2-3}$$

where A_0 is a normalization constant and we have used the notation $x_0 = x(0)$ for simplicity. We assume the external driving force to be turned on at time zero ($t = 0$), and for $t \leq 0$ the system is described by the damped oscillator wave function.[8] The propagator contains all information about the system in the time interval $[0,t]$ and satisfies the wave equation

$$i\frac{\partial K}{\partial t} = \hat{H}(t)\,K \, . \tag{2-4}$$

From Eqs. (2-1), (2-3) and (2-4) we obtain three first-order differential equations for the coefficients in the propagator (2-3). The solutions of those differential equations yield the explicit form of the propagator,[12,13]

$$K(x,t; x_0,0) = \left[\frac{\omega\, e^{\gamma t/2}}{2\pi i\, \sin(\omega t)}\right]^{\frac{1}{2}} \exp\left[A_1(t)x^2 + A_2(t)x - A_3(t)x_0^2 + A_4(t)x_0\right],$$

(2-5)

where

$$A_1(t) = \frac{\omega}{2i}e^{\gamma t}\left\{\frac{\gamma}{\omega} - \xi(t)\right\}$$

$$A_2(t) = i\omega A(\omega',\theta)e^{\gamma t/2}\left[\left\{\frac{\gamma}{\omega} - \xi(t)\right\} \cos(\omega' t - \phi_0 - \phi_1) - \frac{\omega'}{\omega} \sin(\omega' t - \theta_0 - \theta_1)\right]$$

$$A_3(t) = \frac{\omega}{2i}\xi(t)$$

$$A_4(t) = \frac{\omega\, e^{\gamma t}}{i\, \sin(\omega t)}\left\{x - A(\omega',\theta) \cos(\omega' t - \phi_0 - \phi_1)\right\}$$

$$\xi(t) = \frac{\gamma}{2\omega} + \cot(\omega t) \tag{2-6}$$

$$\omega = \left\{\omega_m^2 - \frac{\gamma^2}{4}\right\}^{\frac{1}{2}}$$

$$A(\omega',\theta) = g(\omega',\theta)\Big/\left[(\gamma\omega')^2 + \left\{\omega_m^2 - \omega'^2\right\}^2\right]^{\frac{1}{2}}$$

$$\phi_1 = \tan^{-1}\left[\gamma\omega'/(\omega_m^2 - \omega'^2)\right].$$

ω represents the reduced frequency, which is assumed to be real throughout this paper. This implies that we are concerned with the underdamped case, although the results can be taken over to the overdamped or critical damping case.

The wave functions of the system are readily calculated using the formula

$$\psi_n(x,t) = \int_{-\infty}^{\infty} dx_0 \, K[x,t; \, x_0,0] \, \psi_n(x_0,0), \tag{2-7}$$

where $\psi_n(x_0,0)$ is the wave function for a simple harmonic oscillator at $t = 0$,

$$\psi_n(x_0,0) = \left[\frac{\sqrt{\omega}}{2^n n! \sqrt{\pi}}\right]^{\frac{1}{2}} H_n(\sqrt{\omega}x_0) \, \exp[-\tfrac{1}{2}\omega x_0^2] \tag{2-8}$$

and H_n is the usual Hermite polynomial. Substituting Eqs. (2-5) and (2-8) into Eq. (2-7) and performing the integration, we obtain

$$\psi_n(x,t) = \left[\frac{D(t)}{2^n n! \sqrt{\pi}}\right]^{\frac{1}{2}} \exp\left[-B_1(t)x^2 + B_2(t)x - B_3(t)\right.$$

$$\left. - i(n+\tfrac{1}{2})\cot^{-1}(\xi(t))\right] H_n\left[D(t)\left\{x-E(t)\right\}\right] \,, \tag{2-9}$$

where

$$D(t) = \left\{\omega e^{\gamma t}/\zeta_1(t)\right\}^{\frac{1}{2}}$$

$$E(t) = A(\omega',\theta) \cos(\omega' t - \phi_0 - \phi_1)$$

$$B_1(t) = \tfrac{1}{2} D^2(t) \left\{1 + i\zeta_2(t)\right\}$$

$$B_2(t) = D^2(t)E(t)\left\{1 + i\varsigma_3(t)\right\}$$

(2-10)

$$B_3(t) = \frac{1}{2}D^2(t)E^2(t)\left\{1 + i\xi(t)\right\}$$

$$\varsigma_1(t) = \sin^2(\omega t)\left\{1 + \xi^2(t)\right\}$$

$$\varsigma_2(t) = \xi(t) + \varsigma_1(t)\left\{\frac{\gamma}{\omega} - \xi(t)\right\}$$

$$\varsigma_3(t) = \varsigma_2(t) - \varsigma_1(t)\frac{\omega'}{\omega}\tan(\omega't - \phi_0 - \phi_1) \quad .$$

Hereafter, we shall use the notation $D = D(t)$, $E = E(t)$, ..., $\varsigma_3 = \varsigma_3(t)$ and $\xi = \xi(t)$ whenever there is no ambiguity. We note that there exist useful relations between coefficients in the exponential terms in the wave function,

$$B_1 + B_1^* = D^2, \quad B_2 + B_2^* = 2D^2E, \quad B_3 + B_3^* = D^2E^2 \quad ,$$

(2-11)

which will be used in later calculations.

B. Second-quantization method

To implement the second-quantization formalism for the damped driven harmonic oscillator, we introduce the time-dependent annihilation operator $\hat{a}(t)$ and creation operator $\hat{a}^\dagger(t)$

$$\hat{a}(t) = \frac{\sqrt{2}}{\omega_m}u(t)\left\{2B_1x + ip - B_2\right\}$$

(2-12)

$$\hat{a}^\dagger(t) = \frac{\sqrt{2}}{\omega_m} u^*(t) \left\{ 2B_1^* x - ip - B_2^* \right\} ,$$

where

$$u(t) = u_0(t) \, e^{iy(t)}$$

$$u_0(t) = \frac{\omega_m}{2D(t)} \tag{2-13}$$

$$y(t) = \cot^{-1}(\xi(t)) ,$$

x and p are the canonically conjugate coordinate and momentum such that $[x,p] = i$, and B_1, B_2 and D are all given in Eq. (2-10). It is obvious through Eq. (2-11) that the non-Hermitian operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfy the commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 . \tag{2-14}$$

Thus we set

$$\hat{a}(t)|n,t> = \sqrt{n}|n-1,t>$$

$$\hat{a}^\dagger(t)|n,t> = \sqrt{n+1}|n+1,t>$$

$$\hat{a}(t)|0,t> = 0 . \qquad (n = 0,1,2,3...) \tag{2-15}$$

The state vector $|n,t>$ implies that the quantum state depends on the quantum number n and time t. Even though the definitions of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in Eq. (2-12) are correct, it is more convenient to introduce a new function for later calculations,

$$\eta(t) = \frac{2i}{\omega_m} u(t) B_1 e^{-\gamma t} \quad . \tag{2-16}$$

Then, the time-dependent operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ are rewritten as

$$\hat{a}(t) = -\sqrt{2}i \ e^{\gamma t} \left[\eta(t)(x-E) - \frac{u(t)}{\omega_m} e^{-\gamma t} \left\{ p + \varsigma_4(t) \right\} \right]$$

$$\tag{2-17}$$

$$\hat{a}^\dagger(t) = -\sqrt{2}i \ e^{\gamma t} \left[\frac{u^*(t)}{\omega_m} e^{-\gamma t} \left\{ p + \varsigma_4(t) \right\} - \eta^*(t)(x-E) \right]$$

where

$$\varsigma_4(t) = D^2 E \ (\varsigma_2 - \varsigma_3) \quad . \tag{2-18}$$

From now on we use the notation $\hat{a} = \hat{a}(t)$, $\hat{a}^\dagger = \hat{a}^\dagger(t)$, $u = u(t)$, etc. x and p are expressed in terms of $\hat{a}$ and $\hat{a}^\dagger$ as

$$x = E + \frac{\sqrt{2}}{\omega_m} (u^*\hat{a} + u\hat{a}^\dagger)$$

$$\tag{2-19}$$

$$p = -\varsigma_4 + \sqrt{2} \ e^{\gamma t} (\eta^*\hat{a} + \eta\hat{a}^\dagger) \quad .$$

Substitution of Eq. (2-20) into (2-1) gives the second-quantized expression of the Hamiltonian operator $\hat{H}(t)$,

$$\hat{H}(t) = e^{\gamma t}\left[(|u|^2 + |\eta|^2)(2\hat{a}^{\dagger}\hat{a} + 1) + \left\{(u^2 + \eta^2)\hat{a}^{\dagger}\hat{a}^{\dagger}\right.\right.$$

$$\left.\left. + \left[\frac{u}{\omega_m}[\omega_m^2 E - f(t)] - \eta\varsigma_4 e^{-\gamma t}\right]\hat{a}^{\dagger} + h.c.\right\}\right] + G(t)$$

$$G(t) = e^{\gamma t}\left[\frac{1}{2}\omega_m^2 E^2 + \frac{1}{2}\varsigma_4^2 e^{-2\gamma t} - Ef(t)\right] \quad , \tag{2-20}$$

where h.c. implies the Hermitian conjugate.

The ground-state wave function in the coordinate representation, $\psi_0(x,t) = \langle x|0,t\rangle$, is readily calculated using the definition of $\hat{a}$ in Eq. (2-12) and $\hat{a}|0,t\rangle = 0$. The normalized form is

$$\psi_0(x,t) = \left[\frac{D}{\sqrt{\pi}}\right]^{\frac{1}{2}} \exp\left[-B_1 x^2 + B_2 x - B_3 - \frac{i}{2} y(t)\right] \quad , \tag{2-21}$$

which corresponds to Eq. (2-9) when $n = 0$. The n-th state wave function $\psi_n(x,t)$ is obtained through

$$\psi_n(x,t) = \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^n \; \psi_0(x,t) \quad . \tag{2-22}$$

With $\hat{a}^{\dagger}$ as defined in Eq. (2-12), we readily obtain

$$(\hat{a}^{\dagger})^n = \left\{\frac{\sqrt{2}u^*}{\omega_m}\right\}^n \left\{2B_1^* x - \frac{\partial}{\partial x} - B_2^*\right\}^n$$

$$= \left\{ \frac{e^{-iy(t)}}{\sqrt{2}D} \right\}^n \exp\left[b_1^* x^2 - B_2^* x \right] \left(\frac{\hat{\partial}}{\partial x} \right)^n \exp\left[-B_1^* x^2 + B_2^* x \right] \quad . \qquad (2\text{-}23)$$

Substituting Eqs. (2-21) and (2-23) into (2-22) and performing some calculations using Eq. (2-11), we obtain

$$\psi_n(x,t) = (-1)^n \exp\left[-B_1 x^2 + B_2 x - B_3 - i(n+\tfrac{1}{2})y(t) \right] e^{z^2} (\frac{\partial}{\partial z})^n e^{-z^2} \quad , \qquad (2\text{-}24)$$

where $z = D(x-E)$, and the Hermite polynomial function is expressed as

$$H_n(z) = (-1)^n e^{z^2} (\frac{\partial}{\partial z})^n e^{-z^2} \quad . \qquad (2\text{-}25)$$

Then the wave function in Eq. (2.24) is exactly the same as Eq. (2-9). It is easy to show that $\psi_n(x,t)$ has the orthonormal property,

$$\langle \psi_m(x,t) | \psi_n(x,t,) \rangle = \langle m,t | n,t \rangle = \delta_{m,n} \quad . \qquad (2\text{-}26)$$

C. Time-dependent operator method

For the system characterized by the time-dependent Hamiltonian $\hat{H}(t)$ [Eq. (2-1)], we assume that there exists a Hermitian operator $\hat{I}(x,p,t)$ which is explicitly time-dependent and satisfies the invariant condition

$$\frac{d}{dt} \hat{I} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i} [\hat{I},\hat{H}] = 0 \quad . \qquad (2\text{-}27)$$

By operating on the left-hand side of Eq. (2-27) with the time-dependent Schrödinger state vector $|\rangle_s$ which satisfies the wave equation

$$i \frac{\partial}{\partial t} |>_s = \hat{H}(t) |>_s \quad , \tag{2-28}$$

we obtain

$$i \frac{\partial}{\partial t} (\hat{I} |>_s) = \hat{H}(t) (\hat{I} |>_s) \quad . \tag{2-29}$$

This means that the operation of the invariant on the Schrödinger state vector yields another solution of the Schrödinger wave equation.[14] Now, we assume that the time-dependent invariant operator has the form[2]

$$\hat{I}(x,p,t) = \frac{1}{2} \Big[\delta_1(t) x^2 + \delta_2(t)(xp+px) + \delta_3(t) p^2 + \delta_4(t) x$$

$$+ \delta_5(t) p + \delta_6(t) \Big] \quad , \tag{2-30}$$

where the δ_i's are all real functions of time and do not involve time-differential terms. Applying Eq. (2-30) to (2-27), we have six differential equations for the δ_i's,

$$\dot{\delta}_1 - 2\delta_2 \, \omega_m^2 \, e^{\gamma t} = 0 \tag{2-30-1}$$

$$\dot{\delta}_2 + \delta_1 \, e^{-\gamma t} - \delta_3 \, \omega_m^2 \, e^{\gamma t} = 0 \tag{2-31-2}$$

$$\dot{\delta}_3 + 2\delta_2 \, e^{-\gamma t} = 0 \tag{2-31-3}$$

$$\dot{\delta}_4 + (2\delta_2 f - \delta_5 \omega_m^2) \, e^{\gamma t} = 0 \tag{2-31-4}$$

$$\dot{\delta}_5 + \delta_4 \, e^{-\gamma t} + 2\delta_3 f \, e^{\gamma t} = 0 \tag{2-31-5}$$

$$\dot{\delta}_6 + \delta_5 f \, e^{\gamma t} = 0 \quad , \tag{2-31-6}$$

where the dots represent partial derivative with respect to time.

To solve the above equations, we write

$$\delta_3(t) = \rho^2(t) \, e^{-\gamma t} \quad , \tag{2-32}$$

where $\rho(t)$ is a real function of time which will be determined later. Substituting Eq. (2-32) into Eqs. (2-31-3), (2-32-2) and (2-31-1), we get

$$\delta_1(t) = \left[\dot{\rho}^2 + \rho \left\{ \rho - \gamma\dot{\rho} + \omega_m^2 \right\} \right] e^{\gamma t} \tag{2-33}$$

$$\delta_2(t) = -\rho \left[\dot{\rho} - \frac{1}{2}\gamma\rho \right] \quad .$$

where $\rho(t)$ obeys the nonlinear differential equation

$$\ddot{\rho} + \omega^2\rho = \rho^{-3} \quad , \tag{2-34}$$

and ω is the reduced frequency defined by Eq. (2.6). Any particular solution of Eq. (2-34) can be used to construct the invariant $\hat{I}(x,p,t)$. It is straightforward to show that

$$\rho(t) = e^{\gamma/2}/D(t) = \left[\frac{\zeta_1(t)}{\omega} \right]^{\frac{1}{2}} \tag{2-35}$$

is a particular solution of Eq. (2-34), where $D(t)$ and $\zeta_1(t)$ are defined in Eqs. (2-10). Inserting this solution into Eqs. (2-31), we obtain

$$\delta_1(t) = D^2[1 + \zeta_2^2]$$

$$(2-36)$$

$$\delta_2(t) = \zeta_2 = \frac{1}{2\omega} [\gamma\zeta_1 - \dot{\zeta}_1] \quad .$$

Substitution of Eq. (2-36) into Eqs. (2-31-4) and (2-31-5) yields a nonhomogeneous differential equation with constant coefficients,

$$\ddot{\delta}_5 + \gamma\dot{\delta}_5 + \omega_m^2\delta_5 = \left[2\zeta_1\dot{E} + \left\{\gamma\zeta_1 + 3\dot{\zeta}_1\right\}E\right]/\omega \qquad (2-37)$$

the solution of which is

$$\delta_5(t) = -2E \zeta_3 \quad , \qquad\qquad (2-38)$$

where $E(t)$ and $\zeta_3(t)$ have been defined in Eqs. (2-10). Again, substituting Eq. (2-38) into (2-31-4), (2-31-5) and (2-31-6), we obtain

$$\delta_4(t) = -2D^2 E (1 + \zeta_2\zeta_3)$$

$$(2-39)$$

$$\delta_6(t) = D^2 E^2 (1 + \zeta_3^2) \quad .$$

We have determined all coefficients in Eq. (2-30). Thus the invariant operator $\hat{I}$ can be explicitly written as

$$\hat{I}(x,p,t) = \frac{1}{2}\left[D^2(x-E)^2 + \left\{D(t)\zeta_2 x + \frac{p}{D} - DE\zeta_3\right\}^2\right] \quad . \qquad (2-40)$$

To obtain the operator form of $\hat{I}$, we introduce the two operators

$$\hat{a}_0(t) = \frac{\sqrt{2}}{\omega_m} u_0(t) \ (2B_1 x + ip - B_2)$$

$$(2\text{-}41)$$

$$\hat{a}_0^\dagger(t) = \frac{\sqrt{2}}{\omega_m} u_0(t) \ (2B_1^* x - ip - B_2^*) \quad ,$$

where $u_0(t)$ is defined in Eq. (2-13). We note that $\hat{a}_0(t) = \hat{a}(t)e^{-iy(t)}$ and $\hat{a}_0^\dagger(t) = \hat{a}^\dagger(t)e^{iy(t)}$. Equation (2-40) is expressed in terms of $\hat{a}_0(t)$ and $\hat{a}_0^\dagger(t)$ as

$$\hat{I}(t) = \hat{a}_0^\dagger(t)\hat{a}_0(t) + \frac{1}{2} \quad .$$

$$(2\text{-}42)$$

Obviously, $\hat{a}_0(t)$ and $\hat{a}_0^\dagger(t)$ satisfy the same commutation rule as $\hat{a}(t)$ and $\hat{a}^\dagger(t)$, i.e.,

$$[\hat{a}_0(t), \ \hat{a}_0^\dagger(t)] = 1 \quad .$$

$$(2\text{-}43)$$

Therefore, we can write

$$\hat{a}_0(t)|n,t\rangle_I = \sqrt{n}|n\text{-}1,t\rangle_I$$

$$\hat{a}_0^\dagger(t)|n,t\rangle_I = \sqrt{n+1}|n+1,t\rangle_I$$

$$(2\text{-}44)$$

$$\hat{a}_0(t)|n,t\rangle_I = 0 \qquad (n = 0,1,2,3,\ldots)$$

Here we assume that the eigenfunctions $\phi_n(x,t) = \langle x|n,t\rangle_I$ of $\hat{I}$ form a complete orthonormal set corresponding to the eigenvalues λ_n. Then we have

$$\hat{I}\phi_n(x,t) = \lambda_n\phi_n(x,t)$$

$$\lambda_n = n + \frac{1}{2} \quad . \tag{2-45}$$

Even though the eigenfunctions of $\hat{I}$ are time-dependent, the eigenvalues are time-independent.[3]

In order to obtain $\phi_n(x,t)$, we first obtain the normalized form of $\phi_0(x,t)$ using the explicit expression of $\hat{a}_0(t)$ in Eq. (2-44),

$$\phi_0(x,t) = \left[\frac{D}{\sqrt{\pi}}\right]^{\frac{1}{2}} \exp\left[-B_1 x^2 + B_2 x - B_3\right] \quad , \tag{2-46}$$

which is the same as $\psi_0(x,t)$ except for the $y(t)$ term in Eq. (2-21). The eigenstate $\phi_n(x,t)$ can be obtained by applying $\hat{a}_0^\dagger(t)$ n times successively to $\phi_0(x,t)$. Through the same procedure for obtaining Eq. (2-24), we get

$$\phi_n(x,t) = \frac{1}{\sqrt{n!}} (\hat{a}_0^\dagger)^n \phi_0(x,t)$$

$$= \left[\frac{D}{2^n n!\sqrt{\pi}}\right]^{\frac{1}{2}} \exp\left[B_1 x^2 + B_2 x - B_3\right] H_n\left[D(x-E)\right] \quad . \tag{2-47}$$

Now the solution, $\psi_n(x,t)$, of the Schrödinger wave equation can be obtained from[3]

$$\psi_n(x,t) = e^{i\alpha_n(t)} \phi_n(x,t) \quad , \tag{2-48}$$

where the phase function $\alpha_n(t)$ is the solution of the equation

$$\frac{d\alpha_n(t)}{dt} = <\phi_n|\, i\, \frac{\partial}{\partial t} - \hat{H}|\phi_n> \quad . \tag{2.49}$$

The diagonal matrix element of $\frac{\partial}{\partial t}$ with respect to the eigenvector $|n,t>_I$ of $\hat{I}$ is obtained by taking the partial time derivative of $\hat{a}_0^\dagger(t)$ in Eq. (2-44) and operating taking a scalar product:

$$_I<n,t|\frac{\partial}{\partial t}|n,t>_I = {}_I<n-1,t|\frac{\partial}{\partial t}|n-1,t>_I + \frac{1}{\sqrt{n}}\, {}_I<n,t|\frac{\partial \hat{a}_0^\dagger}{\partial t}|n-1,t>_I$$

$$= {}_I<0,t|\frac{\partial}{\partial t}|0,t>_I + \sqrt{n}\, {}_I<n,t|\frac{\partial \hat{a}_0^\dagger}{\partial t}|n-1,t>_I \quad . \tag{2-50}$$

In obtaining the second line in Eq. (2.50), we have used a recurrence relation. $\frac{\partial \hat{a}_0^\dagger}{\partial t}$ is easily calculated from Eq. (2-41) as

$$\frac{\partial \hat{a}_0^\dagger}{\partial t} = \frac{2i}{\omega_m} e^{\gamma t} \left\{ \eta_0^*(\gamma u_0 - \dot{u}_0) + u_0 \dot{\eta}_0^* \right\} \hat{a}_0 + \left[\gamma + \frac{2i}{\omega_m} e^{\gamma t} \left\{ u_0 \dot{\eta}_0^* + \eta_0(\gamma u_0 - \dot{u}_0) \right\} \right] \hat{a}_0^\dagger$$

$$- \frac{\sqrt{2}i}{\omega_m} \left[u_0 \dot{\zeta}_4 + \omega_m \eta_0^* \dot{E} e^{\gamma t} \right] \quad , \tag{2-51}$$

where $\eta_0(t)$ is a function which is obtained from Eq. (2-16) by making the replacement $u(t) \to u_0(t)$. Thus, Eq. (2-50) becomes

$$_I<n,t|\frac{\partial}{\partial t}|n,t>_I = {}_I<0,t|\frac{\partial}{\partial t}|0,t>_I + n\left[\gamma + \frac{2i}{\omega_m} e^{\gamma t} \left\{ u_0 \dot{\eta}_0^* + \eta_0(\gamma u_0 - \dot{u}_0) \right\} \right] \quad . \tag{2-52}$$

$_I<0,t|\frac{\partial}{\partial t}|0,t>_I$ gives the zero-point contribution to Eq. (2-52), which is

$$_I\langle 0,t|\frac{\partial}{\partial t}|0,t\rangle_I = \frac{1}{2}\left[\gamma + \frac{2i}{\omega_m}\left\{u_0\dot{\eta}_0^* + \eta_0(\gamma u_0 - \dot{u}_0)\right\}\right] \quad . \tag{2-53}$$

Using the definition of $u_0(t)$ and $\eta_0(t)$ and Eqs. (2-10), we obtain

$$u_0\dot{\eta}_0^* + \eta_0(\gamma u_0 - \dot{u}_0) = -\frac{1}{2}\omega_m \, e^{-\gamma t}\left[\frac{1}{2}\dot{\zeta}_2 + \frac{\zeta_1}{\omega}\zeta_2^2 - i\gamma\right] \quad . \tag{2-54}$$

Substitution of Eqs. (2-53) and (2-54) into (2-52) yields the diagonal matrix element of $i\frac{\partial}{\partial t}$,

$$_I\langle n,t|i\frac{\partial}{\partial t}|n,t\rangle_I = \left(n+\frac{1}{2}\right)\left(\frac{1}{2}\dot{\zeta}_2 + \frac{\omega}{\zeta_1}\zeta_2^2\right) \quad . \tag{2-55}$$

Now we can express the Hamiltonian $\hat{H}$ in terms of $\hat{a}_0(t)$ and $\hat{a}_0^\dagger(t)$ by making the replacements $u(t) \rightarrow u_0(t)$, $\eta(t) \rightarrow \eta_0(t)$, $\hat{a}(t) \rightarrow \hat{a}_0(t)$ and $\hat{a}^\dagger(t) \rightarrow \hat{a}_0^\dagger(t)$ in Eq. (2-20). By this manner we obtain the diagonal matrix element of $\hat{H}$ as

$$_I\langle n,t|\hat{H}|n,t\rangle_I = 2\left(n+\frac{1}{2}\right)\left[u_0^2 + |\eta_0|^2\right]e^{\gamma t} + G(t) \quad . \tag{2-56}$$

Here, we note that the $G(t)$ term in Eqs. (2-20) and (2-56) does not have any effect on the dynamics of the system. Therefore it is always possible to remove this term. In later calculations, we neglect this term.

Since $\rho = (\zeta_1/\omega)^{\frac{1}{2}}$ is the solution of the auxiliary equation (2-34), this equation can be written in the equivalent form

$$\omega + \frac{\gamma^2}{8\omega} - \frac{1}{2}\omega\zeta_1 - \frac{\dot{\zeta}_1^2}{8\omega\zeta_1} - \frac{\omega}{2\zeta_1} \quad . \tag{2-57}$$

In obtaining the above equation we have used the identity

$$\ddot{\zeta}_1 = 4\omega^2 + \frac{\gamma^2}{2} - 4\omega^2\zeta_1 \quad , \tag{2-58}$$

which is easily verified by using Eqs. (2-10). With use of Eq. (2-57) we obtain

$$\frac{1}{2}\dot{\zeta}_2 + \frac{\omega}{\zeta_1}\zeta_2^2 - 2\left(u_0^2 + |\eta_0|^2\right) = -\frac{\omega}{\zeta_1} \quad . \tag{2-59}$$

Using these results, we finally obtain

$$\frac{d\alpha_n(t)}{dt} = -\left(n + \frac{1}{2}\right)\frac{\omega}{\zeta_1} \quad . \tag{2-60}$$

To integrate the above equation, we use

$$\frac{\zeta_1}{\omega}\dot{y}(t) = 1 \quad , \tag{2-61}$$

which is verified by using the definitions of $\zeta_1(t)$ and $y(t)$. Thus we obtain

$$\alpha_n(t) = (n + \frac{1}{2}) \cot^{-1}[\xi(t)] + constant \quad , \tag{2-62}$$

where the constant gives a constant phase in the wave function $\psi_n(x,t)$, and can thus be neglected. Substitution of Eq. (6-62) into (2-48) yields the

Schrödinger wave function, which is exactly the same as Eqs. (2-9) and (2-24). Thus, we have shown that the three methods described in this section yield the same wave function.

3. Coherent states

In this section we obtain the coherent-state wave function by means of three equivalent definitions and show that the coherent states correspond to minimum uncertainty states. We define a coherent state $|\alpha,t>$ as the eigenstate of $a(t)$ corresponding to the eigenvalue α,

$$\hat{a}(t)|\alpha,t> = \alpha|\alpha,t> \quad , \tag{3-1}$$

where α is a complex number and $\hat{a}(t)$ has been defined in Eq. (2-12) or (2-17). Equation (3-1) can thus be rewritten in the coordinate representation as

$$\frac{\sqrt{2}}{\omega_m} u(t)\left\{2B_1(t)x - B_2(t) + \frac{\partial}{\partial x}\right\} <x|\alpha,t> = \alpha <x|\alpha,t> \tag{3-2}$$

$$\psi_\alpha(x,t) \equiv <x|\alpha,t>$$

$$= N(t) \exp\left[-B_1 x^2 + \left\{B_2 + \frac{\alpha\omega_m}{\sqrt{2}u}\right\} x\right] \quad , \tag{3-3}$$

where $N(t)$ is a normalization factor. Through the normalization procedure we obtain

$$\psi_\alpha(x,t) = \left[\frac{D^2 u^*}{\pi u}\right]^{\frac{1}{2}} \exp\left[-B_1 x^2 + \left\{B_2 + \alpha D \sqrt{\frac{2u^*}{u}}\right\}x - B_3 - \frac{1}{2}|\alpha|^2 - \frac{\alpha^2 u^*}{2u}\right.$$

$$\left. - \alpha DE \sqrt{\frac{2u^*}{u}}\right] \quad , \tag{3-4}$$

where we have used the relation $u/u^* = e^{2iy(t)} = \exp[\cot^{-1}\zeta(t)]$. The wave function $\psi_\alpha(x,t)$ can be obtained through the second definition of the coherent states such that

$$\psi_\alpha(x,t) = \exp[\alpha\hat{a}^\dagger(t) - \alpha^*\hat{a}(t)] \ \psi_0(x,t) \tag{3-5}$$

where $\psi_0(x,t)$ is the ground-state wave function given by Eq. (2-21). Thus $\exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]$ plays the role of a displacement operator. If we use the identity

$$e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \ e^{-|\alpha|^2/2} \quad , \tag{3-6}$$

equation (3-5) can be rewritten as

$$\psi_\alpha(x,t) = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\hat{a}^\dagger)^n \ \psi_0(x,t)$$

$$\tag{3-7}$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \ \psi_n(x,t) \quad ,$$

where we have used Eqs. (2-15) and (2-22). Eq. (3-7) can also be used for the definition of the coherent states. Since $\psi_n(x,t)$ is given by Eq. (2-9) or (2-24), Eq. (3-7) becomes

$$\psi_\alpha(x,t) = \left[\frac{D^2 u^*}{\pi u}\right]^{\frac{1}{4}} \exp\left[-B_1 x^2 + B_2 x - B_3 - \frac{1}{2}|\alpha|^2\right] \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\alpha^2 u^*}{2u}\right)^{n/2}$$

$$\times H_n[D(x-E)] \quad , \tag{3-8}$$

The summation in Eq. (3-8) can be easily carried out by use of the expansion of the Hermite polynomial H_n:

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\alpha^2 u^*}{2u}\right]^{\frac{n}{2}} H_n[D(x-E)] = \exp\left[\alpha D(x-E)\overline{\sqrt{2u^*/u}} - \frac{\alpha^2 u^*}{2u}\right] \quad . \tag{3-9}$$

Substitution of Eq.(3-9) into (3-8) yields Eq. (3-4). Thus we have shown that three definitions of coherent states [Eqs. (3-1), (3-5) and (3-7)] are equivalent. Here we note that when there is no driving force (i.e., $f = 0$, $E = 0$), the coherent state wave function [Eq. (3-4)] is reduced to that of the damped (not driven) harmonic oscillator,[15]

$$\psi_\alpha(x,t) = \left[\frac{D^2 u^*}{\pi u}\right]^{\frac{1}{4}} \exp\left[-B_1 x^2 + \alpha D\overline{\sqrt{\frac{2u^*}{u}}}x - \frac{1}{2}|\alpha|^2 - \frac{\alpha^2 u^*}{2u}\right] \quad . \tag{3-10}$$

We now show that the coherent state vectors form a nonorthogonal complete set:

$$\langle \alpha, t | \beta, t \rangle = \int_{-\infty}^{\infty} dx \; \langle \alpha, t | x \rangle \langle x | \beta, t \rangle$$

$$= e^{-[(|\alpha| + |\beta|^2)/2]} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha^*)^n \beta^m}{\sqrt{n! m!}} \int_{-\infty}^{\infty} dx \; \psi_n^*(x, t) \; \psi_m(x, t)$$

$$= \exp\left[-\frac{1}{2} \left\{ |\alpha - \beta|^2 + (\alpha\beta^* - \alpha^*\beta) \right\} \right] \; . \tag{3-11}$$

Here we have used the orthonormality of $\psi_n(x,t)$ given by Eq.(2-26). Since Eq. (3.11) has nonzero values for $\alpha \neq \beta$, the states are not orthogonal, but as $|\alpha - \beta|^2 \to \infty$ the states become orthogonal. However, the completeness of the coherent states is easily proved:

$$\frac{1}{\pi} \int d^2\alpha \; | \alpha, t \rangle \langle \alpha, t |$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{| n, t \rangle \langle m, t |}{\sqrt{n! m!}} \int d^2\alpha \; (\alpha^*)^n (\alpha)^m \; e^{-|\alpha|^2}$$

$$= \sum_{n=0}^{\infty} | n, t \rangle \langle n, t |$$

$$= \hat{1} \qquad\qquad [d^2\alpha = d(\mathrm{Re}\alpha) \cdot d(\mathrm{Im}\alpha)] \tag{3-12}$$

Here a polar coordinate $\alpha = |\alpha| e^{i\theta}$ has been used at the intermediate stage of the calculation, and $\hat{1}$ means the unit operator. From Eq. (3-11) we easily obtain

$$|\beta,t> - \frac{1}{\pi} \int d^2\alpha \ |\alpha,t> \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^*\beta\right] \quad . \tag{3-13}$$

This means that the coherent states are not linearly independent of one another. Even though these states form a complete set, there are more states than are necessary for expanding any given state in terms of the coherent states. In this sense, these states are said to be "overcomplete".

The coherent states correspond to the minimum uncertainty states, and thus we can show easily the following relation:

$$[(\Delta x)(\Delta p)]_\alpha - \frac{1}{2}(1 + \varsigma_2^2)^{\frac{1}{2}} \quad . \tag{3.14}$$

Equation (3.14) is the minimum uncertainty corresponding to the ground state.

We now show that the coherent states, which are eigenstates of the destruction operator defined by Eq. (2-12) [or Eq. (2-17)], give the exact classical motion for the damped driven oscillator system. The calculation of the position of the wave packet yields

$$<\alpha,t|x|\alpha,t> - e^{-\gamma t/2}\left[F(\alpha,\alpha^*) \ e^{i\omega t} + F^*(\alpha,\alpha^*) \ e^{-i\omega t}\right]$$

$$+ \frac{g(\omega',\theta)\cos(\omega't-\phi_0-\phi_1)}{\sqrt{(\omega_m^2-\omega'^2)^2 + (\gamma\omega')^2}}$$

$$\tag{3-15}$$

$$F(\alpha,\alpha^*) - \frac{\omega_m}{2\sqrt{\omega}} \left\{(1 - \frac{i\gamma}{4\omega})\alpha^* - \frac{i\gamma\alpha}{4\omega}\right\} \quad ,$$

where we have used the explicit expression for u(t) from Eq. (2-13),

$$u(t) = \frac{\omega_m}{2\sqrt{\omega}} e^{-\gamma t/2} \left\{ \frac{\gamma}{2\omega} \sin(\omega t) + e^{i\omega t} \right\} \quad , \tag{3-16}$$

which is readily obtained by use of the identity

$$\exp\left[\pm in \, \cot^{-1}(\xi(t))\right] = \left[\frac{\xi(t)\pm i}{\xi(t)\mp i}\right]^{\frac{n}{2}} \quad , \tag{3-17}$$

where $\xi(t) = \frac{\gamma}{2\omega} + \cot(\omega t)$ [see Eq. (2-6)]. On the other hand, the classical solution of the equation of motion for the damped driven harmonic oscillator [Eq. (1-1)] is given by

$$x_1(t) = e^{-\gamma t/2}\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right] + \frac{g(\omega',\theta)\cos(\omega' t - \phi_0 - \phi_1)}{\sqrt{(\omega_m^2 - \omega'^2)^2 + (\gamma\omega')^2}} \quad . \tag{3-18}$$

4. Discussion and conclusions

We have considered a molecule adsorbed on a dielectric solid surface, modeled as a damped harmonic oscillator forced by a time-dependent electric field. The exact wave functions of this system have been calculated by three different methods. These methods yield exactly the same results. When there is no external perturbing force, the wave functions naturally reduce to those of a damped harmonic oscillator. Furthermore, when there exists no damping, the wave functions become the time-dependent wave functions for a simple harmonic oscillator. In this sense, we can say that we have obtained the general wave function for a harmonic system.

The Hamiltonian in Eq. (2-20) has off-diagonal terms since the representation defined by Eqs. (2-15) does not diagonalize this operator. When there is no external driving force, Eq. (2-20) becomes

$$\hat{H} = e^{\gamma t}\left[(|u|^2+|\eta|^2)(2\hat{a}^\dagger\hat{a} + 1) + (u^2+\eta^2)\hat{a}^\dagger\hat{a}^\dagger + (u^{*2}+\eta^{*2})\hat{a}\hat{a} \right] \quad . \quad (4\text{-}1)$$

The above equation has the most general form for a Hamiltonian which preserves the coherence of an arbitrary initial SU(1,1) coherent state.[16] Thus this Hamiltonian can be used to describe the dynamics of the SU(1,1) coherent states of a damped oscillator. When there is no damping the Hamiltonian in Eq. (4-1) becomes that of a simple harmonic oscillator. In this case, we easily obtain $u(t) = \frac{1}{2}\sqrt{\omega_0}\, e^{i\omega_0 t}$, $\eta(t) = \frac{1}{2}\sqrt{\omega_0}\, e^{i\omega_0 t}$. Then we have $\hat{H} = \omega_0(\hat{a}^\dagger\hat{a}+\frac{1}{2})$. Therefore, we can say that Eq. (2-20) is the general form for the Hamiltonian of a (damped, driven) harmonic oscillator system.

To obtain the explicitly time-dependent (Hermitian) invariant operator, we have assumed the form of the invariant to be expressed by Eq. (2-30). The same type of invariant was used by Khandekar and Lawande[14] for a damped driven oscillator with a time-dependent frequency. When the frequency of the system depends on time, the auxiliary equation necessary for constructing the invariant should be solved numerically. For the constant frequency case like ours, the auxiliary equation (2-34) is easily solved. The particular solution to Eq. (2-34) is given by Eq. (2-35). With this solution we obtain the explicit form of the dynamical invariant, Eq. (2-40). When there is no driving force, Eq. (2-40) can be written as

$$\hat{I} = \frac{1}{2}\left[(\frac{x}{\rho})^2\, e^{\gamma t} + \left\{ \rho p - x(\dot{\rho} - \frac{\gamma}{2}\rho)e^{\gamma t} \right\}^2 e^{-\gamma t} \right] \quad . \quad (4\text{-}2)$$

This equation agrees with the result of Pedrosa[17] and is somewhat different in form from that of Korsch.[18] Here, we point out that for a given equation of motion there can be various kinds of auxiliary equations.

Hence, a given Hamiltonian can have many different dynamical invariants. Of course, the same physical results must be obtained regardless of which dynamical invariant is chosen. For the case of no damping, Eq. (4-2) becomes the so-called Ermakov-Lewis[19] invariant for a harmonic oscillator,

$$\hat{I} = \frac{1}{2}\left\{ \left(\frac{x}{\rho}\right)^2 + (\rho p - x\dot{\rho})^2 \right\} \quad . \tag{4-3}$$

The operators $\hat{a}_0(t)$ and $\hat{a}_0^+(t)$ in Eq. (2-41), which were used to diagonalize the invariant $\hat{I}$ [Eq. (2-42)], are the same as $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in Eq. (2-12) except for the phase factor. Therefore, the phase function $\alpha_n(t)$ in Eq. (2-49) is readily obtained with use of Eq. (2-20).

The coherent states defined as eigenstates of the operator $\hat{a}(t)$ form a nonorthogonal (over)complete set and correspond to the minimum uncertainty states. The coherent states can be defined in equivalent ways as described in Sec. 3. The product of uncertainty in position and momentum has a periodic characteristic with period π/ω. The uncertainty product in the damping oscillator system is not affected by the external perturbing force. For the $\gamma = 0$ case, the uncertainty product has a minimum value of $\frac{1}{2}$. The coherent states obtained satisfy the Schrödinger equation and give the exact classical motion of the damped driven oscillator system.

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