TESTS FOR THE GAMMA DISTRIBUTION WITH ESTIMATED SHAPE PARAMETERS

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R. A. LOCKHART and M. A. STEPHENS

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Summary.

It is well known that the distributions of the goodness-of-fit statistics W^2, U^2, and A^2 do not depend on location and scale parameters even when these parameters must be estimated. It has generally been assumed, however, that when shape parameters must be estimated the null distributions of these statistics would depend too severely on the unknown parameters to permit their use in practice. Exact asymptotic distributions of these statistics are presented for the two parameter gamma family of distributions. Critical points are given in tables constructed so as to avoid any need for extrapolation. The results presented show that the tests can be expected to be useful in practice.

1. Introduction.

Goodness-of-fit tests such as the Cramer-Von Mises statistic W^2 are distribution free when the distribution being tested is completely specified. When parameters must be estimated, as when testing for the normal family, the situation is not so nice. If the parameters are those of location and scale the null distribution of W^2 will depend on the particular family being tested but not on the exact values of the parameters being estimated. When a shape parameter must be estimated, as in the gamma family, the distribution of the test statistic will generally depend, even asymptotically, on the unknown true value of the shape parameter. In this paper we explore the extent of the dependence of critical points on the shape parameter in the two-parameter gamma family. Our major conclusion is that the dependence is small enough to make the tests useful in practice.

In this paper we discuss the behavior of the goodness-of-fit statistics W^2, U^2, and A^2 for testing the hypothesis H_o: that a random sample of size n comes from the gamma distribution with density

$$(1.1) \qquad f(x;\alpha,\beta) = \Gamma(\alpha)^{-1}\beta^{-\alpha}x^{\alpha-1}\exp(-x/\beta) \qquad 0 < x < \infty .$$

Following Stephens (1976) we distinguish four situations: Case 0, where both α and β are known; Case 1, where β is known but α must be estimated; Case 2, where α is known but β must be estimated and Case 3, where both parameters are unknown and must be estimated. Case 2 has been studied by Pettitt and Stephens (1975). In this paper we concentrate on Case 3 but include the critical points for cases 1 and 2 for the sake of completeness. Those for Case 2 have been recomputed.

In section 2 we review briefly the relevant theory for Case 3 in order to fix notation and to record the relevant formulae for the gamma distribution. In section 3 we discuss the problem of computing the asymptotic critical points of the statistics and tabulate the results. An important feature of these tables is that there is never any need for extrapolation--only interpolation. Section 3 contains the relevant theory as well as some useful new identities for computing asymptotic moments of the statistics. Some general conclusions and speculations are in section 4.

2. Asymptotic Distributions of W^2, U^2, and A^2.

Suppose $X_1, X_2, \ldots, X_n$ are a sample from the gamma distribution (1.1) with distribution function $F(x,\theta)$ where $\theta = (\alpha,\beta)$ is in $(0,\infty)\times(0,\infty)$. Let F_n be the empirical distribution function. Let $\hat{\theta} = (\hat{\alpha},\hat{\beta})$ be the maximum likelihood estimates of (α,β) -- see Johnson and Kotz (1970, page 187). The Cramer-Von Mises statistic is defined by

$$W^2 = n \int_{-\infty}^{\infty} \{F_n(x) - F(x,\hat{\theta})\}^2 F(dx,\hat{\theta}) \quad (2.1)$$

(See Stephens (1977) for computing formulae.) The asymptotic theory reproduced here is now standard--see for example Stephens (1976).

Asymptotically (2.1) is equivalent to

$$n \int_{-\infty}^{\infty} \{F_n(x)-F(x,\hat{\theta})\}^2 F(dx,\theta) = \int_0^1 Y_n^2(t)\,dt \tag{2.2}$$

where

$$Y_n(t) = n^{\frac{1}{2}}[F_n\{F_\theta^{-1}(t)\}-F\{F_\theta^{-1}(t),\hat{\theta}\}] \ . \tag{2.3}$$

Here $F_\theta^{-1}(t)$ is the inverse of the map $x \to F(x,\theta)$.

The process Y_n converges weakly in $D[0,1]$ to a mean zero Gaussian process $Y(t)$. The covariance of Y is given by

$$\rho(s,t) = \rho_0(s,t) - \phi(s)^T \ddagger \phi(t) \tag{2.4}$$

where $\rho_0(s,t) = \min(s,t)-st$ is the covariance of the Brownian bridge appropriate for W^2 in Case 0, $\ddagger$ is the inverse of the Fisher information matrix and $\phi(s) = \frac{\partial}{\partial\theta} F(x,\theta)$ evaluated at $x = F_\theta^{-1}(s)$. For the gamma distribution in Case 3 we have

$$\phi(s) = \begin{bmatrix} \Gamma(\alpha)^{-1} \int_0^{x/\beta} e^{-u} u^{\alpha-1} \ln u \, du - s\psi(\alpha) \\ \\ -\Gamma(\alpha)^{-1} x^{\alpha} \beta^{-(\alpha+1)} \qquad\qquad e^{-x/\beta} \end{bmatrix} \tag{2.5}$$

and

$$(2.6) \qquad \ddagger = \begin{bmatrix} \alpha c(\alpha) & -c(\alpha) \\ -c(\alpha) & c(\alpha)\psi'(\alpha) \end{bmatrix}$$

where $\psi(a) = \frac{d}{da}\ln(\Gamma(a))$ is the digamma function, $\psi'(a)$ is its derivative, the trigamma function, and $c(a) = (a\psi'(a)-1)^{-1}$. (See Abramowitz and Stegen (1965) for a discussion of these functions and useful recurrence relations.)

For A^2 the situation is similar except that Y_n must be divided by $w(s) = \sqrt{s(1-s)}$ and ρ must be divided by $w(s)w(t)$. For U^2, Y_n is replaced by $Y_n(t) - \int_0^1 Y_n(s)ds$ and $\rho(s,t)$ is replaced by

$$\rho_0(s,t)-(\phi(s) - \int_0^1 \phi(u)du)^T \ddagger (\phi(t) - \int_0^1 \phi(u)du)$$

where now $\rho_0(s,t) = \min(s,t) - st + 1/12 - (s-s^2+t-t^2)/2$.

For all three statistics then the asymptotic distribution is that of $\int_0^1 Y^2(t)dt$ where Y is a mean zero Gaussian process with covariance ρ of the form $\rho_0(s,t)-\phi(s)^T\ddagger\phi(t)$ where ρ_0 is independent of θ. This distribution is that of $\Sigma_{i=1}^{\infty} V_i/\lambda_i$ where the V_i are independent and identically distributed chi-square random variables on one degree of freedom and $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of the integral equation

$$(2.7) \qquad h(s) = \lambda \int_0^1 h(t)\rho(s,t)dt \ .$$

In Case 0 the eigenvalues and eigenfunctions can be found explicitly for the three statistics studied here (again the reader is referred to Stephens (1976) for details and history). We will denote the Case 0 eigenvalues by λ_i^* and call them the standard weights. We use f_i^* to denote the corresponding orthonormal eigenfuntions.

Now expand (2.4) in terms of f_i^*. Call

$$a_i = \int_0^1 \phi(s) f_i^*(s) ds \ . \tag{2.8}$$

The eigenvalues of (2.7) are then the roots of

$$D(\lambda) = 0 \tag{2.9}$$

where D is the Fredholm determinant of (2.7) given by

$$D(\lambda) = D_0(\lambda) \mathrm{Det}(I + \sum_{i=1}^{\infty} a_i a_i^T \ddagger / (1-\lambda/\lambda_i^*)) \tag{2.10}$$

with D_0 the Case 0 Fredholm determinant and I the 2x2 identity matrix.

3. Computation of Asymptotic Critical Points.

The process of calculating the asymptotic critical points of these statistics is long and will be described step-by-step. Since all the statistics are scale invariant the results do not depend on β. Accordingly, we use $\beta = 1$ for the remainder of the paper.

Step 1: Fourier coefficients.

The first step is to calculate the Fourier coefficients a_i with (2.8).

For W^2 the eigenfunctions are $f_j^*(s) = \sqrt{2}\,\sin(\pi js)$. With $\phi(s)$ as recorded in (2.5) the resulting formulae involve double integrals but integration by parts produces the following simpler formula:

$$(3.1) \qquad a_j = \begin{bmatrix} \sqrt{2}\ (\pi j\Gamma(\alpha))^{-1} \displaystyle\int_0^\infty \cos(\pi jF(x,\alpha,1))e^{-x}x^{\alpha-1}\ \ln x\,dx \\ -\sqrt{2}\ (\pi j\Gamma(\alpha))^{-1} \displaystyle\int_0^\infty \cos(\pi jF(x,\alpha,1))e^{-x}x^{\alpha}dx \end{bmatrix} .$$

For U^2 the eigenfunctions are

$$f_{2j}^*(s) = \sqrt{2}\ \sin(2\pi js)$$

and

$$f_{2j-1}^*(s) = \sqrt{2}\ \cos(2\pi js) .$$

Hence a_{2j} for U^2 is given by a_{2j} for W^2 while after some integration by parts we obtain

$$(3.2) \qquad a_{2j-1} = \begin{bmatrix} -2^{-1/2}(\pi j\Gamma(\alpha))^{-1} \displaystyle\int_0^\infty \sin(2\pi jF(x,\alpha,1))e^{-x}x^{\alpha-1}\ln x\,dx \\ \sqrt{2}\ \Gamma(\alpha)^{-2} \displaystyle\int_0^\infty \cos(2\pi jF(x,\alpha,1))e^{-2x}x^{2\alpha-1}dx \end{bmatrix} .$$

For A^2 the i^{th} eigenfunction is the i^{th} Ferrar associated Legendre function (see Anderson and Darling, 1952). After some algebra and integration by parts we obtain

$$(3.3) \qquad a_j = c_j \begin{bmatrix} \Gamma(\alpha)^{-1} \int_0^\infty P_j(2s-1)e^{-x}x^{\alpha-1}\ln x\,dx \\ \\ \Gamma(\alpha)^{-1} \int_0^\infty P_j(2s-1)e^{-x}x^{\alpha-1}(\alpha-x)dx \end{bmatrix}$$

where $c_j = [(2j+1)/(j(j+1))]^{1/2}$ and P_j is the j^{th} Legendre polynomial (see Abramowitz and Stegun 1965, page 332).

It was judged necessary for the purposes of Step 2 to obtain numerical values of the integrals (3.1-3) for j from 1 to 50 and for a variety of shapes α. Thus for each value of the shape there were 300 numerical integrals to be done. Many of the integrands involve similar functions. Moreover, the eigenfunctions for large j may (and for A^2 must) be evaulated by recursion from those for small j. In order to take advantage of these properties we wrote a routine to do Romberg extrapolation (see Ralston (1965, page 121)) of the trapezoidal rule for an array of functions. For integer values of α the gamma cumulative $F(x,\alpha,1)$ can be computed explicitly. We therefore computed the six integrals for α from 1 to 20 and for j from 1 to 50 simultaneously. The result was an important saving in computer time--the integrals still required around 10 hours of CPU time on an IBM 4341.

Step 2: Roots of the Fredholm Determinant.

The next step is to find the roots of $D(\lambda) = 0$. In all cases the poles in the determinant term in D cancel the zeros in D_0 so it suffices to search for the roots of the determinant term. The series $\Sigma_{j=1}^{\infty} a_j a_j^{r} \ddagger/(1-\lambda/\lambda_j^*)$ must be truncated somewhere. We found by experiment that 400 terms are adequate. The difficulty is that we were able to compute only 50 a_j's. It proved possible, however, to extrapolate the a_j's with confidence since the a_j's were seen to fall away smoothly with j. We therefore carried out the extrapolation from 50 to 400 by fitting $a_j = k\,j^{-r}$. We then searched each interval $\lambda_j^* < \lambda < \lambda_{j+1}^*$ exhaustively stopping when about 200 roots were found.

Step 3: Critical Values.

The final step is to compute the critical points of $Q = \Sigma_{j=1}^{\infty} V_j/\lambda_j$. We used two different methods. First, Pearson curves were fitted using the first four cumulants of Q. Second, the Imhof technique (see Imhof, 1961) of numerically inverting the characteristic function of Q was used.

The cumulants needed for use with Pearson curves were found several ways. First, the r^{th} cumulant κ_r is given by (Stephens, 1976):

$$\kappa_r = 2^{r-1}(r-1)!\sum_{j=1}^{\infty} \lambda_j^{-r} . \tag{3.4}$$

This sum converges quickly for $r \geq 2$ but for $r = 1$ is unsatisfactory. The sum of those terms truncated away in a finite approximation may, however, be calculated approximately using the fact that $j^2\lambda_j$ has a computable limit.

A second formula for κ_r is (Stephens, 1976)

$$\kappa_r = \int_0^1 \rho_r(s,s)ds \tag{3.5}$$

where $\rho_1 = \rho$ and $\rho_{r+1}(s,t) = \int_0^1 \rho_r(s,u)\rho(u,t)du$. For $r = 1$ and integral values of α these integrals are tractable but very messy. Lengthy recurrence relations were developed (details may be obtained from the authors) for these calculations, reducing the calculations to gamma, digamma and trigamma function evaluations--at least for W^2 and U^2. For non-integral α and A^2 even numerical integration seemed formidable.

A third, and seemingly new, approach to the calculation of κ_r is as follows. Expand the integrals in (3.5) in terms of the eigenfunctions f_j^* of ρ_0. The result is the following:

(3.6)
$$\int_0^1 \rho(s,s)ds = \int_0^1 \rho_0(s,s)ds - \sum_{j=1}^{\infty} a_j^T \ddagger a_j$$

$$\int_0^1 \rho_2(s,s)ds = \int_0^1 \rho_{0,2}(s,s)ds - 2\sum_j a_j^T \ddagger a_j/\lambda_j^* + \sum_{j,k} a_j^T \ddagger a_k a_k^T \ddagger a_j$$

$$\int_0^1 \rho_3(s,s)ds = \int_0^1 \rho_{0,3}(s,s)ds - 3\sum_j a_j^T \ddagger a_j/(\lambda_j^*)^2 + 3\sum_{j,k} a_j^T \ddagger a_k a_k^T \ddagger a_j/\lambda_j^*$$
$$- \sum_{j,k,\ell} a_j^T \ddagger a_k a_k^T \ddagger a_\ell a_\ell^T \ddagger a_j$$

$$\int_0^1 \rho_4(s,s)ds = \int_0^1 \rho_{0,4}(s,s)ds - 4\sum_j a_j^T \ddagger a_j/(\lambda_j^*)^3$$
$$+ 2\sum_{j,k} a_j^T \ddagger a_k a_k^T \ddagger a_j/\lambda_j^* \lambda_k^* + 4\sum_{j,k} a_j^T \ddagger a_k a_k^T \ddagger a_j/(\lambda_k^*)^2$$
$$- 4\sum_{j,k,\ell} a_j^T \ddagger a_k a_k^T \ddagger a_\ell a_\ell^T \ddagger a_j/\lambda_j^*$$
$$+ \sum_{j,k,\ell,m} a_j^T \ddagger a_k a_k^T \ddagger a_\ell a_\ell^T \ddagger a_m a_m^T \ddagger a_j \ .$$

These series have the advantage of converging very quickly even for the mean--that is $r=1$. For non-integral shapes and for A^2 these calculations were the only ones available for the mean. For integral α and Case 2 where the integrals can be done explicitly for $r = 1$ (see Pettitt and Stephens, 1975), the series (3.6) agreed very well with the results of direct integration (differing only in the seventh or eighth significant digit). These series also can be used to check the accuracy of the computation of the roots of (2.9) since they do not use the computed values of the eigenvalues λ_j. Comparison of (3.6) and (3.4) suggests that Step 2, the calculation of the roots of the Fredholm determinant, has been performed very accurately. Finally it should be remarked that the double, triple and even quadruple sums in (3.6) can all be written as products of singly infinite sums.

The critical points were also found using the technique of Imhof (1961) of numerical inversion of the characteristic function of a weighted sum of chi-squares. The series $\Sigma_{j=1}^{\infty} V_j/\lambda_j$ was truncated at 40 terms and the tail of the sum replaced by its expected value. Since the standard deviation of the tail is small compared to its mean, this approximation was expected to perform well. The approximation was checked in several cases by using 150 terms of the sum and was found to be excellent.

The Pearson curve and Imhof calculations were found to agree well, rarely differing by more than 1 in the third significant digit. In tables 1, 2, and 3 we report the Imhof calculations as these are based on the exact asymptotic distribution. For the sake of reference we record the cumulants of Q in tables 4, 5, and 6. Case 1 is in tables 1 and 4, while Case 2 is in tables 2 and 5 and Case 3 is in tables 3 and 6.

The test is now carried out as follows:

1) Estimate whichever parameters are unknown by maximum likelihood.

2) Order the sample data points $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ and calculate $Z_{(i)} = F(X_{(i)}, \alpha, \beta)$ where any parameter which has been estimated is replaced by that estimate.

3) Calculate W^2, U^2, or A^2 using the computing formulae of Stephens (1977).

4) Enter the appropriate table using α, or when this is estimated its estimate $\hat{\alpha}$, interpolating when necessary to find the appropriate critical point. Linear interpolation in α is adequate except between $\alpha = 20$ and $\alpha = \infty$ where we recommend linear interpolation in $1/\alpha$.

The tests are asymptotically exact because $\hat{\alpha}$ is consistent. In finite samples, however, entering the tables using $\hat{\alpha}$ seems risky since one will normally be taking the critical point from the wrong row. In the present case, however, the critical points are seen to depend very little on α, particularly for α bigger than 2 or so. In the case of U^2 the dependence is only one digit in the third significant digit. For α less than 1 the critical points change rapidly with α but it should be noted that the standard error of $\hat{\alpha}$ is small for α near 0--partially compensating for this defect.

The second source of error in using the tables is, of course, the use of asymptotic theory in finite samples. A Monte Carlo study by Schneider and Clickner (1977) shows the critical points even for samples of size 20 are in excellent agreement with the asymptotic points at least for shapes over 0.5. For samples of size 10 their Monte Carlo points are still generally within 2 or 3 percent of our asymptotic points. Schneider and

Clickner have studied the problem of entering the tables using estimated shapes and report levels of 4 to 6 percent when testing at a nominal 5 percent level with the true levels rising to 7 or 8 percent for a shape near 0 and sample sizes of 20. Schneider and Clickner had no points for a shape of 0. When they encountered estimated shapes smaller than the least value of α for which they had Monte Carlo points they were forced to use some sort of extrapolation. This may have led to erroneous readings of the true level of the test. We plan our own Monte Carlo studies on this point.

Our tables list two shape parameter values outside the parameter space--namely, $\alpha = 0$, $\alpha = \infty$. These points are provided to eliminate the need for extrapolation beyond the ends of the tables. They were computed using the following theorem:

Theorem 3.1: Suppose $\rho_n(s,t)$ is a sequence of non-negative definite covariance functions of mean zero Gaussian process $\{Y_n(t);\ 0 \le t \le 1\}$. Suppose that, as n tends to infinity, ρ_n converges, uniformly in s and t, to $\rho(s,t)$, the covariance function of a mean zero Gaussian process $\{Y(t);\ 0 \le t \le 1\}$. Then Y_n converges weakly to Y in $C[0,1]$.

Hence, under the conditions of the theorem, $\int_0^1 Y_n^2(t)dt$ converges in distribution to $\int_0^1 Y^2(t)dt$.

As $\alpha \to 0$ we find, for W^2, that $\rho(s,t)$ converges to

$$\rho_0(s,t)-s\ln(s)t\ln(t)\ . \tag{3.7}$$

For A^2 the limit is that for W^2 divided by $w(s)w(t)$. For U^2 the limit is $\rho_0(s,t)-(s\ln(s)+1/4)(t\ln(t)+1/4)$ where now ρ_0 is the Case 0 covariance for U^2. We remark that these covariances are those encountered in the one-parameter exponential distribution--see Stephens (1976).

As $\alpha \to \infty$ the limit is somewhat harder to compute, but a Cornish-Fisher expansion shows that the covariances converge to those of Case 3 for the normal distribution as defined in Stephens (1976). It should be noted that as $\alpha \to \infty$ the gamma density becomes more and more normal. This suggests, but does not prove, the conclusion just reached concerning the critical points.

Although critical points are reported in Stephens (1976) for Case 3 of the normal distribution using Pearson curves we have recomputed the points using Imhof's technique. We report these new computations here.

4. Conclusions.

The tests studied will be useful in practice. The convergence of the finite sample points to the asymptotic points is rapid. In addition the asymptotic points depend little on the shape for shapes larger than 1 or 2. For small samples and estimated shapes near 0 the asymptotic points are slightly larger than the finite sample points. Since the effect of entering the tables with an estimated shape appears to be an increase in the frequency of rejection use of the asymptotic points even in small samples is to be recommended.

TABLE 1.

PERCENTAGE POINTS FOR THE NULL DISTRIBUTIONS OF THE TEST STATISTICS W^2, U^2, AND A^2 IN CASE 1
KNOWN SCALE

Statistic	Shape Parameter	Significance Level 0.25	0.10	0.05	0.025	0.01	0.005
W^2	0	.115	.174	.221	.269	.341	.395
	0.2	.113	.169	.214	.260	.322	.368
	0.5	.108	.160	.200	.243	.299	.346
	1	.103	.150	.186	.223	.273	.311
	2	.0988	.143	.176	.210	.256	.291
	5	.0958	.138	.169	.202	.245	.278
	10	.0948	.136	.167	.199	.241	.274
	20	.0943	.135	.166	.198	.240	.272
	∞	.0938	.134	.165	.196	.238	.270
U^2	0	.0903	.129	.158	.191	.234	.256
	0.2	.0908	.130	.159	.192	.235	.257
	0.5	.0908	.130	.159	.192	.235	.257
	1	.0899	.129	.159	.189	.230	.261
	2	.0894	.128	.158	.189	.229	.261
	5	.0891	.128	.158	.188	.229	.260
	10	.0889	.128	.157	.188	.228	.260
	20	.0888	.127	.157	.187	.228	.260
	∞	.0888	.127	.157	.187	.228	.259
A^2	0	.735	1.06	1.32	1.59	1.96	2.24
	0.2	.725	1.04	1.29	1.55	1.90	2.17
	0.5	.703	.997	1.22	1.46	1.78	2.03
	1	.680	.956	1.17	1.39	1.69	1.92
	2	.661	.926	1.13	1.34	1.62	1.84
	5	.649	.906	1.10	1.30	1.58	1.79
	10	.645	.899	1.09	1.29	1.56	1.77
	20	.643	.896	1.09	1.29	1.56	1.76
	∞	.641	.893	1.09	1.28	1.55	1.75

Remarks: 1) Points for shape 0 are from the one parameter exponential family--Case 4 of Stephens (1976).

2) Points for shape ∞ are from Case 1 of the normal family--see Stephens (1976).

TABLE 2.

PERCENTAGE POINTS FOR THE NULL DISTRIBUTIONS OF THE TEST STATISTICS W^2, U^2, AND A^2 IN CASE 2
KNOWN SHAPE

Statistic	Shape Parameter	Significance Level 0.25	0.10	0.05	0.025	0.01	0.005
W^2	0	.209	.348	.461	.580	.745	.967
	0.2	.158	.257	.339	.423	.542	.629
	0.5	.132	.205	.265	.328	.412	.484
	1	.116	.174	.222	.270	.337	.389
	2	.106	.155	.194	.235	.289	.332
	5	.0987	.143	.177	.212	.259	.294
	10	.0963	.139	.171	.204	.248	.282
	20	.0951	.137	.168	.200	.243	.276
	∞	.0938	.134	.165	.196	.238	.270
U^2	0	.105	.151	.188	.221	.265	.303
	0.2	.0950	.137	.166	.199	.244	.266
	0.5	.0916	.131	.160	.193	.237	.258
	1	.0897	.129	.159	.189	.230	.261
	2	.0892	.128	.158	.188	.229	.260
	5	.0889	.128	.157	.188	.228	.260
	10	.0889	.127	.157	.187	.228	.260
	20	.0888	.127	.157	.187	.228	.259
	∞	.0888	.127	.157	.187	.228	.259
A^2	0	1.25	1.93	2.49	3.08	3.88	4.50
	0.2	0.923	1.41	1.80	2.22	2.78	3.22
	0.5	0.803	1.19	1.50	1.82	2.27	2.61
	1	0.735	1.06	1.32	1.59	1.96	2.24
	2	0.692	0.982	1.21	1.44	1.76	2.01
	5	0.662	0.929	1.14	1.35	1.64	1.86
	10	0.652	0.911	1.11	1.32	1.59	1.81
	20	0.646	0.902	1.10	1.30	1.57	1.78
	∞	0.641	0.893	1.09	1.28	1.55	1.75

Remarks: 1) Points for shape 0 are the Case 0 points.

2) Points for the shape ∞ are the Case 1 points of the normal distribution--see Stephens (1976).

TABLE 3.

PERCENTAGE POINTS FOR THE NULL DISTRIBUTIONS OF THE TEST STATISTICS W^2, U^2, AND A^2 IN CASE 3

Statistic	Shape Parameter	Significance Level					
		0.25	0.10	0.05	0.025	0.01	0.005
W^2	0	.115	.174	.221	.269	.341	.395
	0.2	.0924	.136	.168	.204	.252	.296
	0.5	.0828	.119	.147	.175	.214	.244
	1	.0785	.111	.136	.162	.196	.222
	2	.0762	.107	.131	.155	.187	.211
	5	.0748	.105	.128	.151	.182	.206
	10	.0744	.104	.127	.150	.180	.204
	20	.0741	.104	.126	.149	.179	.203
	∞	.0739	.104	.126	.149	.179	.202
U^2	0	.0903	.129	.158	.191	.234	.256
	0.2	.0896	.128	.156	.189	.228	.254
	0.5	.0719	.101	.122	.144	.174	.196
	1	.0705	.0984	.119	.141	.169	.190
	2	.0699	.0973	.118	.139	.166	.187
	5	.0695	.0966	.117	.138	.165	.186
	10	.0693	.0964	.117	.137	.164	.185
	20	.0692	.0963	.117	.137	.164	.185
	∞	.0692	.0962	.117	.137	.164	.184
A^2	0	.735	1.06	1.32	1.59	1.96	2.24
	0.2	.543	.756	.923	1.09	1.33	1.51
	0.5	.502	.685	.825	.967	1.16	1.31
	1	.486	.657	.786	.917	1.09	1.23
	2	.477	.643	.768	.894	1.06	1.19
	5	.472	.635	.758	.881	1.04	1.17
	10	.471	.633	.755	.877	1.04	1.16
	20	.470	.632	.753	.875	1.04	1.16
	∞	.469	.631	.752	.873	1.03	1.16

Remarks: 1) Points for shape 0 are from the one parameter exponential distribution—see Stephens (1976).

2) Points for shape ∞ are from the normal distribution for Case 3—see Stephens (1976).

TABLE 4.

FIRST FOUR CUMULANTS OF THE STATISTICS
CASE 1--KNOWN SCALE

W^2

Shape	Mean x 100	Variance x 10^3	$k_3 \times 10^3$	$k_4 \times 10^4$
0	9.26	4.36	6.39	1.54
0.2	7.58	4.03	5.54	1.25
0.5	8.66	3.38	4.05	0.796
1	8.22	2.84	2.96	0.507
2	7.87	2.48	2.35	0.362
5	7.64	2.27	2.02	0.291
10	7.56	2.20	1.92	0.271
20	7.52	2.17	1.87	0.262
∞	7.48	2.14	1.83	0.253

U^2

Shape	Mean x 100	Variance x 10^3	$k_4 \times 10^4$	$k_4 \times 10^5$
0	7.18	1.98	1.68	2.32
0.2	7.21	2.01	1.70	2.35
0.5	7.22	2.01	1.70	2.35
1	7.19	1.99	1.69	2.33
2	7.15	1.97	1.67	2.31
5	7.12	1.96	1.65	2.29
10	7.11	1.95	1.65	2.29
20	7.11	1.95	1.65	2.28
∞	7.10	1.95	1.65	2.28

A^2

Shape	Mean x 10	Variance x 10	$k_3 \times 10^2$	$k_4 \times 10^2$
0	5.96	1.39	10.89	14.27
0.2	5.87	1.31	9.65	11.97
0.5	5.69	1.15	7.69	8.62
1	5.50	1.02	6.24	6.40
2	5.36	0.939	5.39	5.20
5	5.26	0.888	4.91	4.56
10	5.23	0.872	4.76	4.37
20	5.21	0.864	4.69	4.28
∞	5.19	0.857	4.62	4.28

TABLE 5.

FIRST FOUR CUMULANTS OF THE STATISTICS
CASE 2--KNOWN SHAPE

W^2

Shape	Mean x 100	Variance x 10^3	k_3 x 10^3	k_4 x 10^4
0	16.66	22.22	8.47	50.79
0.2	12.69	11.52	3.11	13.28
0.5	10.54	6.65	1.29	4.06
1	9.26	4.35	0.639	1.54
2	8.44	3.18	0.372	0.717
5	7.88	2.53	0.248	0.397
10	7.68	2.33	0.213	0.319
20	7.58	2.23	0.198	0.284
∞	7.48	2.14	0.183	0.253

U^2

Shape	Mean x 100	Variance x 10^3	k_3 x 10^4	k_4 x 10^5
0	8.33	2.77	2.65	3.97
0.2	7.53	2.20	1.90	2.64
0.5	7.27	2.04	1.73	2.39
1	7.18	1.98	1.68	2.32
2	7.13	1.96	1.66	2.30
5	7.11	1.95	1.65	2.29
10	7.11	1.95	1.65	2.28
20	7.10	1.95	1.65	2.28
∞	7.10	1.95	1.64	2.28

A^2

Shape	Mean x 10	Variance x 10^2	k_3 x 10^2	k_4 x 10^2
0	10.00	57.97	104.32	303.97
0.2	7.45	29.21	37.06	76.48
0.5	6.51	18.93	18.33	29.36
1	5.96	13.92	10.90	14.27
2	5.61	11.20	7.45	8.34
5	5.37	9.59	5.65	5.61
10	5.28	9.07	5.12	4.86
20	5.24	8.82	4.87	4.52
∞	5.19	8.57	4.62	4.19

TABLE 6.

FIRST FOUR CUMULANTS OF THE STATISTICS
CASE 3--NEITHER KNOWN

W^2

Shape	Mean x 100	Variance x 10^3	k_3 x 10^4	k_4 x 10^4
0	9.26	4.36	6.39	1.54
0.2	7.39	2.40	2.50	0.430
0.5	6.66	1.69	1.37	0.185
1	6.32	1.41	0.991	0.115
2	6.13	1.28	0.833	0.0892
5	6.02	1.21	0.754	0.0772
10	5.98	1.19	0.731	0.0738
20	5.96	1.18	0.721	0.0722
∞	5.95	1.16	0.709	0.0705

U^2

Shape	Mean x 100	Variance x 10^2	k_3 x 10^4	k_4 x 10^5
0	7.18	1.98	1.68	2.32
0.2	7.12	1.93	1.60	2.15
0.5	5.78	1.10	0.647	0.623
1	5.68	1.03	0.581	0.531
2	5.63	1.00	0.552	0.492
5	5.59	0.987	0.535	0.470
10	5.58	0.981	0.529	0.463
20	5.58	0.978	0.527	0.459
∞	5.57	0.975	0.525	0.455

A^2

Shape	Mean x 10	Variance x 10	k_3 x 10^2	k_4 x 10^2
0	5.96	1.39	10.89	14.27
0.2	4.44	0.614	2.94	2.39
0.5	4.11	0.460	1.75	1.14
1	3.97	0.406	1.39	0.808
2	3.91	0.382	1.25	0.685
5	3.87	0.370	1.17	0.625
10	3.86	0.366	1.15	0.608
20	3.85	0.364	1.14	0.600
∞	3.84	0.362	1.13	0.591

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20. ABSTRACT

It is well known that the distributions of the goodness-of-fit statistics W^2, U^2, and A^2 do not depend on location and scale parameters even when these parameters must be estimated. It has generally been assumed, however, that when shape parameters must be estimated the null distributions of these statistics would depend too severely on the unknown parameters to permit their use in practice. Exact asymptotic distributions of these statistics are presented for the two parameter gamma family of distributions. Critical points are given in tables constructed so as to avoid any need for extrapolation. The results presented show that the tests can be expected to be useful in practice.