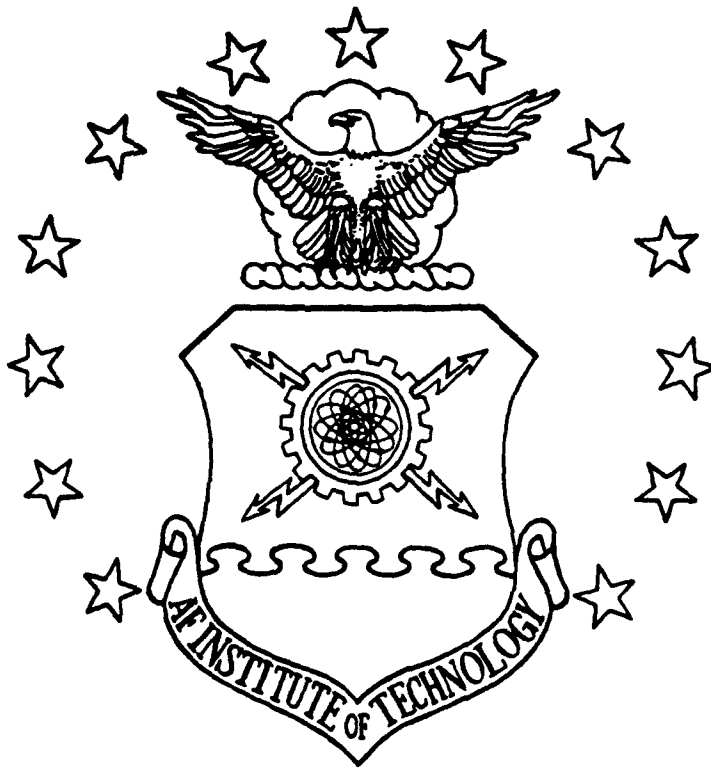


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FRACTIONAL CALCULUS FORMULATION OF THE  
 QUASI-STATIC VISCOELASTIC PROBLEM  
 THESIS

Joseph B. McCullough III  
 First Lieutenant, USAF

AFIT/GA/AA/89M-02

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THESIS

Presented to the Faculty of the School of Engineering  
of the Air Force Institute of Technology  
Air University

In Partial Fulfillment of the  
Requirements for the Degree of  
Master of Science in Astronautical Engineering



Joseph B. McCullough III, B.S.  
First Lieutenant, USAF

March 1989

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## Preface

The purpose of this study was to incorporate temperature and frequency dependency of viscoelastic material behavior in the complex modulus with the use of fractional calculus. The use of fractional derivatives and a temperature shift function provided a complex modulus model that could be used in a finite element formulation to analyze the system response of a structure that contained both elastic and viscoelastic members. While the model is limited to low frequencies and use of the complex modulus in the transition region, the system response is a relatively simple expression with only a few parameters.

I am deeply indebted to the invaluable assistance I have received from my advisor, Lt Col Ronald L. Bagley. He has shown me there is more than one way to skin a cat, and I am grateful to him for the confidence he has had in my ability to work this project through to the end. I would also like to thank Dr Peter Torvik and Major Paul Copp for their guidance and assistance throughout this research effort. Finally, I would like to express my thanks to my mother, Sylvia Ann Cobeaga, for the sacrifices, support, and encouragement she has given me over the years.

Joseph B. McCullough III

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Abstract

The purpose of this study was to demonstrate the use of fractional derivatives to capture the frequency and temperature dependency of viscoelastic material behavior. To model the frequency dependency of viscoelastic material, fractional derivatives were included in the complex modulus. Solution techniques were performed in the Laplace domain to allow for easy manipulation of the fractional derivative terms. To incorporate the temperature dependency of viscoelastic material in the complex modulus model, the method of reduced variables was employed with the use of the WLF equation. With the frequency and temperature dependency built into the complex modulus, a finite element formulation was devised that incorporated elastic and viscoelastic response of a truss structure. The formulation was limited to the use of the complex modulus in the transition region, the region where the damping ability of viscoelastic material is maximized. Quasi-static motion was also assumed, which limited the response to low frequencies. *theses. (m-jm)* ←

The solution technique was demonstrated on a nine degree of freedom truss composed of aluminum and neoprene rubber rods subject to a temperature variation. The results of the example problem show that temperature and frequency dependency of viscoelastic material can be incorporated with

the use of fractional derivatives. The simple solution format and model robustness, along with the theoretical basis, provides encouragement for additional work on the complex modulus model.

# FRACTIONAL CALCULUS FORMULATION OF THE QUASI-STATIC VISCOELASTIC PROBLEM

## I. Introduction

### 1.1 Overview

The purpose of this research effort is to demonstrate the use of fractional derivatives to capture the frequency and temperature dependency of viscoelastic material behavior. This chapter provides a basic introduction to the development of the use of fractional derivatives. In chapter two, the fractional derivative is incorporated into a complex modulus model. The effects of strain, frequency, and temperature are investigated as they affect the viscoelastic behavior. With the fractional derivative complex modulus model established, a two-dimensional finite element formulation is developed in chapter three. This solution technique imposes the condition of quasi-static motion and incorporates temperature and frequency dependence in the transition region. The use of the finite element formulation is demonstrated in chapter four. This example problem provides a solution technique for the system response of a typical space truss configuration composed of elastic and viscoelastic members. The final chapter is devoted to conclusions of the solution technique.

## 1.2 Background

Recently, there has been an increasing demand to model the frequency-dependent behavior of viscoelastic materials. The driving force behind this resurgence of viscoelastic behavior models has been the increasing use of viscoelastic material in engineering structures to take advantage of their energy dissipation characteristics. One common approach to modeling the viscoelastic behavior is to include a large number of derivative terms to model the frequency-dependent stiffness and damping characteristics of the material (1:918). The result is a complicated model that produces even more complex equations of motion to be solved. Another variation is to store the real and imaginary components of the complex modulus in a computer (1:918). When the data for a particular frequency is required, the data can be accessed. This works well for a single frequency, but this process still results in a complicated solution for transient analysis. The use of fractional derivatives to model the complex modulus has shown great accuracy and simplicity.

The concept of using fractional calculus is not new. An article by Ross (2) provides an interesting historical summary of the development of using derivatives of fractional order. Ross states that, in 1695, L'Hospital posed a question to Leibniz of the use of a derivative of fractional order. Leibniz's response was that it would lead to a paradox, but he also concluded that "some day it would lead

to useful consequences." Even back in 1819, derivatives of fractional orders were discussed in a text by Lacroix. While the idea of fractional calculus has been around for nearly three hundred years, the use of fractional derivatives seems non-existent as compared to integer order derivatives. In a series of papers by Bagley and Torvik (1), (3), (4), and (5), the development of the use of fractional calculus to describe viscoelastic behavior is well documented. The initial observations by Nutting, and the subsequent findings by Gemant, Scott-Blair, Caputo, and others are traced. The fractional order derivatives were used by Bagley and Torvik (1), (3), (4), and (5), to model the frequency-dependent behavior of viscoelastic materials. The advantage of the fractional derivative model lies in its simplicity. By being able to model the complex modulus with as few as three or four parameters, a very accurate least-squares curve fit can be obtained to model measured frequency-dependent mechanical properties. The mathematics involved lends itself to the use of either Laplace or Fourier transformations. Results obtained have shown accurate descriptions of complex modulus data up to eight decades of frequency with a four parameter model (5:133).

The use of the fractional derivative in the finite element formulation for determining transient response is in agreement with the continuum model (1). The ability to include the fractional derivative in the finite element model

provides greater flexibility to analyze structures containing both elastic and viscoelastic components. In a more recent paper by Bagley and Calico (6), the fractional derivative was utilized to produce equations of motion capable of describing the system in terms of its initial states. The construction of the state equations with fractional order derivatives provide additional forms of feedback, thus enabling engineers to improve feedback control systems.

While the fractional derivative model provides an accurate empirical model of viscoelastic material behavior, a theoretical basis would increase the confidence in its use. The fact that the fractional derivative formulation created a simple, yet quite accurate, representation of the complex modulus leads one to question its fundamental nature. In a paper by Bagley and Torvik (4), this theoretical basis was established. The result of the molecular theory by Rouse "provides us a non-empirical basis for including fractional derivatives." (4:208) Ferry, Landel, and Williams adapted Rouse's theory to concentrated polymer solutions and polymer solids with no crosslinking (4:208). The result was that the viscoelastic shear stress was a function of the fractional derivative of the shear strain history.

The successful ability to model frequency-dependent viscoelastic behavior, along with its theoretical basis, provides a foundation for further uses of the model. Demonstrations by Bagley, Torvik and others have shown that

the use of fractional derivatives can be used to improve feedback control systems (6), conduct transient analysis of elastic and viscoelastic configurations (1), predict material response outside of experimental data (5), and various other applications.

## II. Fractional Derivative Viscoelastic Model

The basis of this research is to model viscoelastic material with fractional derivatives. In this chapter, the use of fractional derivatives is developed and incorporated into the complex modulus model. The effects of strain, frequency, and temperature are also investigated since they affect the viscoelastic behavior.

### 2.1 Fractional Derivative Formulation

In an attempt to model viscoelastic materials with fractional powers, the use of the fractional derivative is required. In a paper by Torvik and Bagley (5:126), the fractional derivative is in the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = D^\alpha[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(t-\tau)}{\tau^\alpha} d\tau \quad (1)$$

where

$$0 \leq \alpha < 1$$

$\Gamma$  is the gamma function

Using Leibniz's Rule (7:282-84) to expand the integral yields:

$$D^\alpha[x(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(t-\tau)}{\tau^\alpha} d\tau + \frac{1}{\Gamma(1-\alpha)} \frac{x(0)}{t^\alpha} \quad (2)$$

The goal of this process is to have a usable form of the fractional derivative for use in the viscoelastic model formulation. By working in the Laplace domain, this is achieved. The Laplace transformation is defined as

$$\mathcal{L}[x(t)] = \int_0^{\infty} e^{-st} x(t) dt \quad (3)$$

Taking the Laplace transformation of the integral in equation (2) leads to

$$\mathcal{L} \left[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(t-\tau)}{\tau^\alpha} d\tau \right] = \frac{s\mathcal{L}[x(t)]}{\Gamma(1-\alpha)} \int_0^t \frac{e^{-s\tau}}{\tau^\alpha} d\tau \quad (4)$$

Using the relationship (8:22)

$$\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \quad (5)$$

and letting  $n = -\alpha$ , equation (4) reduces to

$$\mathcal{L} \left[ \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(t-\tau)}{\tau^\alpha} d\tau \right] = s^{\alpha-1} [s \mathcal{L}[x(t)] - x(0)] \quad (6)$$

Similarly,

$$\mathcal{L} \left[ \frac{1}{\Gamma(1-\alpha)} \frac{x(0)}{t^\alpha} \right] = s^{\alpha-1} x(0) \quad (7)$$

The final result is

$$\mathcal{L} \{ D^\alpha [x(t)] \} = s^\alpha \mathcal{L} [x(t)] \quad (8)$$

The result of equation (8) is that it is possible to solve problems when the material is modeled by fractional derivatives.

## 2.2 Complex Modulus Model

For analysis of linear viscoelastic behavior, a popular frequency-dependent complex modulus form often used is

$$E^* = E' + iE'' \quad (9)$$

where

$E^*$  = complex modulus

$E'$  = storage modulus

$E''$  = loss modulus

The storage modulus,  $E'(\omega)$ , is the real component of the complex modulus and represents the stress in phase with the strain divided by the strain. The storage modulus is a measure of the energy stored elastically and recovered per cycle in the material (9:41). The loss modulus,  $E''(\omega)$ , is

the imaginary component of the complex modulus and represents the stress  $90^\circ$  out of phase with the strain divided by the strain. The loss modulus is a measure of the energy lost per cycle due to viscous dissipation (9:42).

A common relationship between the imaginary and real components of the complex modulus is the loss factor, which is defined as

$$\tan \delta = \frac{E''}{E'} \quad (10)$$

The loss factor,  $\tan \delta$ , represents the energy loss of the viscoelastic material. This energy loss, described in Payne and Scott (10:18-22), can be used to calculate the damping factor of the material. To maximize the use of the damping characteristics of viscoelastic material, a high loss factor is desired. This occurs in the transition region of viscoelastic materials.

In order to analyze the behavior of viscoelastic material, the complex modulus needs to be modeled over the entire frequency range. In a series of papers by Bagley and Torvik (1), (3), (4), and (5), the frequency-dependent complex modulus is represented in the following form:

$$E(i\omega) = E_0 + E_1(i\omega)^\alpha \quad (11)$$

Expanding equation (11) yields:

$$\bar{E}(i\omega) = E_0 + \omega^\alpha E_1 (i)^\alpha \quad (12)$$

The following relationships are used to expand equation (12):

$$(i)^\alpha = \left[ e^{i\pi/2} \right]^\alpha \quad (13)$$

$$\left[ e^{i\pi/2} \right]^\alpha = \cos \frac{\pi\alpha}{2} + i \sin \frac{\pi\alpha}{2} \quad (14)$$

Substituting equations (13) and (14) into equation (12) results in the following complex modulus:

$$E(i\omega) = E_0 + \omega^\alpha E_1 \left[ \cos \frac{\pi\alpha}{2} + i \sin \frac{\pi\alpha}{2} \right] \quad (15)$$

Separating equation (15) into its real and imaginary components results in the fractional derivative model for the storage and loss moduli. The storage modulus is the real component of the complex modulus and is defined as

$$E'(\omega) = E_0 + \omega^\alpha E_1 \cos \frac{\pi\alpha}{2} \quad (16)$$

Similarly, the loss modulus is the imaginary component of the complex modulus and is defined as:

$$E''(\omega) = \omega^\alpha E_1 \sin \frac{\pi\alpha}{2} \quad (17)$$

The result of equations (16) and (17) is that the complex modulus for viscoelastic material can be modeled by the use of fractional derivatives. Since fractional derivatives are easier to manipulate in the Laplace domain, equation (11) is transformed to the form

$$E(s) = E_0 + E_1 s^\alpha \quad (18)$$

Equation (18) is limited to frequencies below the glass transition region. While this model predicts an exponential increase in the modulus for increasing frequencies as expected, the model is unbounded and does not level off in the glassy region. To extend this model into the higher frequency ranges, another parameter is included in the model. The complex modulus now has the form

$$E(s) = \frac{E_0 + E_1 s^\alpha}{1 + bs^\alpha} \quad (19)$$

The denominator,  $1 + bs^\alpha$ , causes the storage modulus to plateau and the loss modulus to diminish at high frequencies in the glassy region. The result is that the complex modulus can be modeled by the use of fractional derivatives and the selection of parameters  $E_0$ ,  $E_1$ ,  $b$  and  $\alpha$ .

### 2.3 Viscoelastic Material Properties

While equation (19) encompasses a broad frequency range, viscoelastic behavior is also affected by strain and

temperature. The effects of strain, frequency, and temperature are investigated and incorporated into the complex modulus model.

2.3.1 Strain. Since the use of the complex modulus is limited to linear viscoelastic behavior, the boundary of the linear region is required. In Figure 1, the relationship between strain level and modulus is shown.

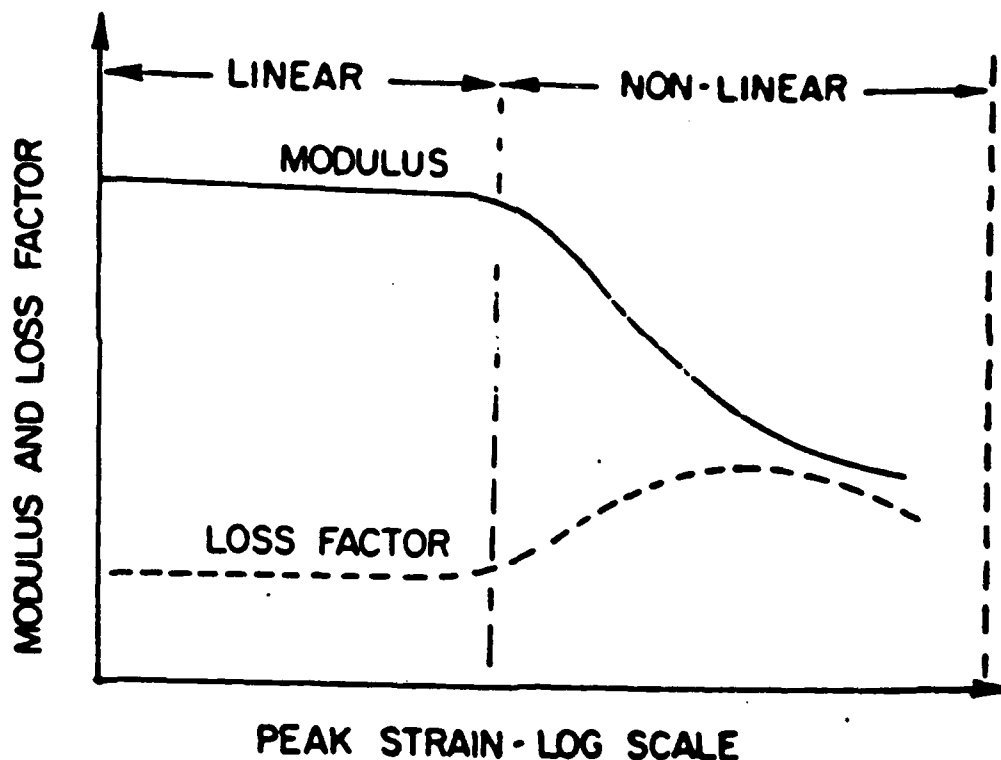


Figure 1. Variation of Complex Modulus with Strain Level  
(11:40)

For strains levels less than 1 or 2%, the complex modulus remains linear (12:261). By remaining below this strain level, use of the complex modulus is applicable.

2.3.2 Temperature. In the presence of significant temperature gradients, the variation of the complex modulus with temperature must be taken into account. The effect of temperature variations is shown in Figure 2.

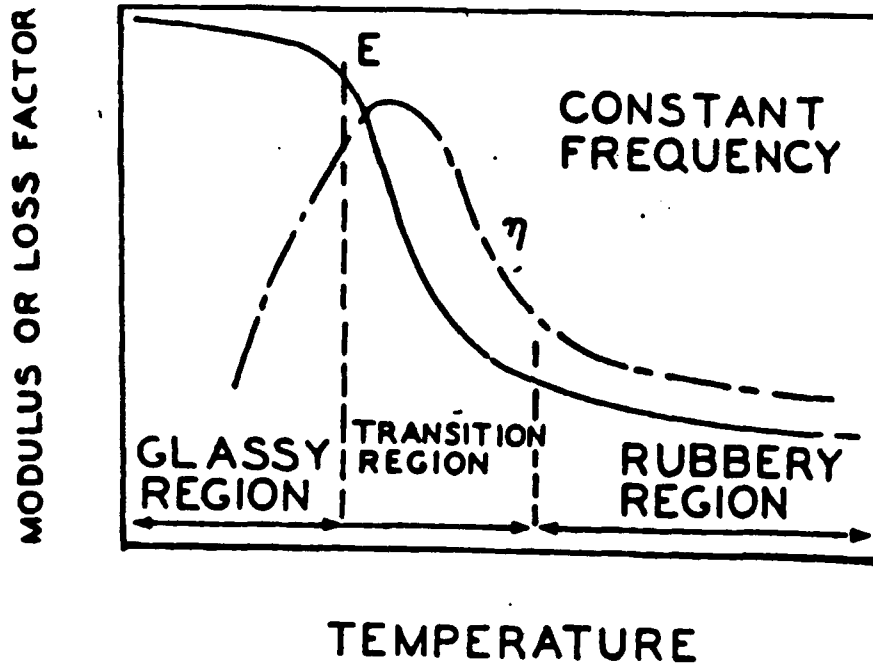


Figure 2. Variation of Complex Modulus with Temperature  
(11:38)

In Figure 2, the temperature scale has been divided into three regions: glassy, transition, and rubbery. The transition from one modulus magnitude to another is an inherent property of viscoelastic material. While the modulus is in the glassy region, the physical temperature is relatively low. At this low temperature, the material is "frozen" and the only movement occurs by straining the inter-molecular bonds. This requires extremely high forces

and thus accounts for the high modulus. At these low temperatures, the material is somewhat brittle and the behavior resembles that of glass; hence, the glassy region. As the temperature increases, it becomes easier and easier for the molecular chains to slide over one another. This process occurs over a narrow temperature range in the transition region and is characterized by a rapid change in the coefficient of thermal expansion. At higher temperatures, the modulus enters the rubbery region where an equilibrium is reached (12:230-31).

The effects of temperature on the molecular chains is also described in Payne and Scott (10:14). They describe a "potential barrier" that must be overcome for the molecules to slide over one another. The probability of overcoming this barrier is proportional to  $\exp(-E/kT)$ , where  $E$  is an activation energy,  $k$  is Boltzmann's constant, and  $T$  is the absolute temperature. As the temperature is reduced, the probability of exceeding the potential barrier is reduced. This probability is exponential and corresponds to a rapid change in the modulus with changing temperature. As the temperature is further reduced, the probability of movement is greatly reduced and the material becomes glass-like. The opposite effect occurs as the temperature is increased. The end result is that in the glassy region, the modulus is high and the loss factor is low. In the transition region, the modulus is changing rapidly and the loss factor is high. The rubbery region is characterized by a low modulus and a loss

factor that is lower than the transition region.

2.3.3 Frequency. The effect of frequency on the complex modulus is quite similar to temperature effects. In the glassy region, the frequency is high. As the frequency is reduced, the modulus transitions to the rubbery region. The result of changing frequencies is that increasing frequency increases modulus while decreasing the frequency decreases the modulus.

2.3.4 Combined Temperature-Frequency Effect. In order to account for both frequency and temperature changes in the complex modulus model, a reduced frequency principle is used. Using the method of reduced variables (9:266-73), the effects of temperature variations can be incorporated into a frequency-dependent complex modulus model. The effect of changing the temperature results in a horizontal shift of the complex modulus along the frequency axis. Thus, a modulus at frequency  $\omega$  and temperature  $T$  would be equivalent to the complex modulus at a reference temperature  $T_0$  and using a shifted frequency  $\omega a_T$ , where  $a_T$  is the shift factor. The complex modulus developed in section 2.2, equation (11), would now have the following form:

$$E(i\omega) = E_0 + E_1(i\omega a_T)^\alpha \quad (20)$$

The shift factor,  $a_T$ , is temperature dependent and is determined empirically by superimposing modulus data at various temperatures and frequencies. The empirical

expression used to incorporate temperature effects was developed by Williams, Landel, and Ferry, and is generally referred to as the WLF equation. The WLF equation provides an expression for the general curve of the shift factor. The equation for the temperature shift factor is described in Ferry (9:273-80) and has the form

$$\log a_T = \frac{c_1(T - T_o)}{c_2 + T - T_o} \quad (21)$$

where

$c_i$  = coefficients dependent upon material

$T$  = temperature

$T_o$  = complex modulus reference temperature

The WLF equation allows for the use of complex moduli data for temperatures other than the reference temperature of the viscoelastic material. The result is that frequency and temperature dependency can be incorporated into the fractional derivative complex modulus model.

### III. Rod Element Formulation

In this chapter, the formulation of the equations of motion for elastic and viscoelastic rods are developed using the complex modulus that is modeled with the use of fractional derivatives. The solution technique incorporates the temperature dependency of the complex modulus, and a two dimensional matrix formulation is developed.

#### 3.1 Rod Element Formulation for Constant Temperature

In the development of the equations of motion for a slender rod, a simple rod element shown in Figure 3 is used.

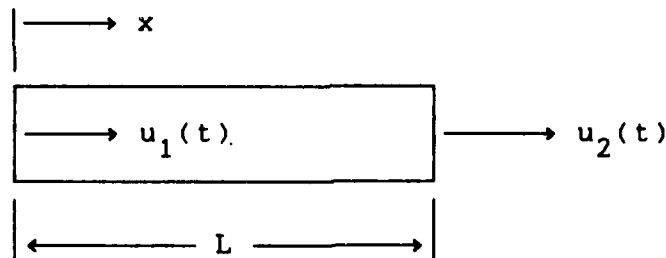


Figure 3. Rod Element with Constant Properties

Using constant properties and a constant temperature, the following differential equation of motion for a slender rod is used (13:45-46):

$$EA \frac{\partial^2 u}{\partial x^2} = \rho A \frac{\partial^2 u}{\partial t^2} \quad (22)$$

where

$u$  = longitudinal displacement

If quasi-static equilibrium is assumed, the equation of motion for the rod reduces to

$$EA \frac{\partial^2 u}{\partial x^2} = 0 \quad (23)$$

The result of equation (23) is that the displacement field is a linear function of the form

$$u(x,t) = u_1 + (u_2 - u_1)(x/L) \quad (24)$$

The displacement field shown in equation (24) is then used in the strain energy formulation for a rod (13:34-37). The strain energy takes the form of

$$U(t) = \int_0^L \frac{EA}{2} \left( \frac{\partial u}{\partial x} \right)^2 dx \quad (25)$$

where

$$U(t) = \text{strain energy}$$

By using the linear displacement field established in equation (24), the strain energy for the rod element is

$$U(t) = \int_0^L \frac{EA}{2L^2} (u_2^2 - 2u_1u_2 + u_1^2) dx \quad (26)$$

The final step in the development of a matrix form for a rod element is to apply Castigliano's First Theorem to the strain energy. Castigliano's theorem, described in Saada (14:453), requires that

$$\frac{\partial U(t)}{\partial u_i} = F_i \quad (27)$$

where

$$F_i = \text{external force}$$

Applying Castigliano's theorem to the strain energy in equation (26) results in the following matrix formulation for a slender rod:

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \quad (28)$$

Equation (28) is the equation of motion for elastic and viscoelastic rods with constant temperature. The formulation of the stiffness matrix agrees with the development in the text by Cook (15:6-9).

### 3.2 Temperature Effects

While the displacement field in equation (28) is limited to constant temperature, temperature changes can greatly affect the complex modulus. To increase the usefulness of the viscoelastic formulation, the viscoelastic stiffness matrix needs to be altered to incorporate temperature

variations. The strain energy for a rod element used in equation (25) is used as a starting point to construct a new stiffness matrix. The differential equation of motion is developed by applying Hamilton's principle to the strain energy for a rod (neglecting the kinetic energy due to the quasi-static assumption). This method is described in Meirovitch (13:42-46). When Hamilton's principle is applied to equation (25), the result is

$$\delta \int_{t_1}^{t_2} \int_0^L \frac{AE(s)}{2} \left( \frac{\partial u}{\partial x} \right)^2 dx dt = 0 \quad (29)$$

Applying the variation to the integral yields the following equation of motion:

$$\frac{\partial}{\partial x} \left[ AE(s) \left( \frac{\partial u}{\partial x} \right) \right] = 0 \quad (30)$$

Young's modulus,  $E(s)$ , is a function of frequency and temperature. For this model, the temperature variation is a function of position,  $x$ , along the rod. Solving equation (30) for the longitudinal displacement leads to

$$u(x) = \int \frac{c_1}{AE(s)} dx + c_2 \quad (31)$$

where

$c_1$  = integration constants

For equation (31) to be solved, the complex modulus must be integrable. In this case, the temperature variation is a function of position,  $x$ , and this temperature change can be incorporated by the use of reduced variables described in section 2.3.4. An integrable complex modulus that includes the temperature dependency has the form

$$E(s) = E_1(s)^\alpha (a_T)^\alpha \quad (32)$$

Substituting equation (32) into equation (31) results in

$$u(x) = \int \frac{c_1}{AE_1(s)^\alpha (a_T)^\alpha} dx + c_2 \quad (33)$$

The modified complex modulus in equation (32) is limited to use in the transition region. While this may appear to place narrow constraints on the model, it must be understood that the transition region is where the damping capability of the viscoelastic material is maximized. Since the viscoelastic material is used for its damping ability, it is beneficial to remain in the transition region. This transition region can be manipulated by use of frequency loading or temperature applications. Thus, careful selection of proper materials, temperature gradients, and frequency loadings can shift the transition region to the operating region.

To integrate equation (33), an integrable temperature shift function needs to be used. An exponential function is

used to model the temperature shift factor,  $(a_T)^\alpha$ . Examples in Payne and Scott (10:24-33) and Jones (11:40-43) show that the temperature shift resembles an exponential function. The temperature shift is modeled as

$$(a_T)^\alpha = a \exp[-b(T-T_0)] \quad (34)$$

where

$a, b$  = curve fit parameters

$T_0$  = reference temperature for  $E(s)$  data

$T$  = actual temperature

To determine the temperature distribution, the heat equation for a rod is used along with the assumption of quasi-static equilibrium. The heat equation reduces to the form:

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad (35)$$

Equation (35) results in a linear temperature distribution as a function of position,  $x$ , along the rod. For a basic rod in Figure 3 with  $T(x=0) = T_1$  and  $T(x=L) = T_2$ , the linear temperature function is

$$T(x) = T_1 + (T_2 - T_1)(x/L) \quad (36)$$

Substituting equation (36) into equation (34) results in

$$(a_T)^\alpha = a \exp\left\{-b\left[T_1 - T_0 + (T_2 - T_1)(x/L)\right]\right\} \quad (37)$$

This temperature shift function is substituted into equation (33) and then integrated. The result is

$$u(x) = c_1(L) \exp\left\{-b\left[T_1 - T_0 + (T_2 - T_1)(x/L)\right]\right\} \\ + \left[abAE(s)(T_2 - T_1)\right] + c_2 \quad (38)$$

To solve for the integration constants, the boundary conditions  $u(x=0) = u_1$  and  $u(x=L) = u_2$  are used. Once the displacement field,  $u(x)$ , is solved, it is then substituted into the strain energy equation for a viscoelastic rod. The strain energy has the form

$$U(s) = \int_0^L \frac{AE(s)(a_T)^\alpha}{2} \left(\frac{\partial u}{\partial x}\right)^2 dx \quad (39)$$

Substituting the results of equation (38) into equation (39) and then integrating yields the strain energy of the form:

$$U(s) = abAE(s)(T_2 - T_1)(u_1 - u_2)^2 \\ + (2L) \left\{ \exp[b(T_2 - T_0)] - \exp[b(T_1 - T_0)] \right\} \quad (40)$$

The final step is to apply Castigliano's theorem to the strain energy in equation (40). The result is the following equation of motion for a viscoelastic rod that incorporates temperature effects:

$$\frac{CAE(s)}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1(s) \\ u_2(s) \end{Bmatrix} = \begin{Bmatrix} f_1(s) \\ f_2(s) \end{Bmatrix} \quad (41)$$

where

$$C = ab(T_2 - T_1) + \left\{ \exp[b(T_2 - T_0)] - \exp[b(T_1 - T_0)] \right\}$$

Equation (41) represents the viscoelastic formulation incorporating a temperature variation along the rod into the complex modulus. A special case exists when the temperature along the rod is constant but different from the reference temperature,  $T_0$ , for the complex modulus data. If a constant temperature other than the reference temperature exists, the temperature shift function in equation (37) is reduced to

$$(a_T)^\alpha = a \exp[-b(T - T_0)] \quad (42)$$

Since the temperature shift function in equation (42) is not a function of position,  $x$ , along the rod, the displacement field is linear in the form of equation (24). The end result is that, for a constant temperature other than the reference

temperature, the rod formulation of equation (28) is modified by substituting the complex modulus of equation (32) and using the temperature shift function of equation (42).

### 3.3 Matrix Assembly

The final part of the formulation involves the incorporation of elastic and viscoelastic elements into a matrix structure. Figure 4 shows the details of a long slender rod composed of elastic element 1 and viscoelastic elements 2 and 3.

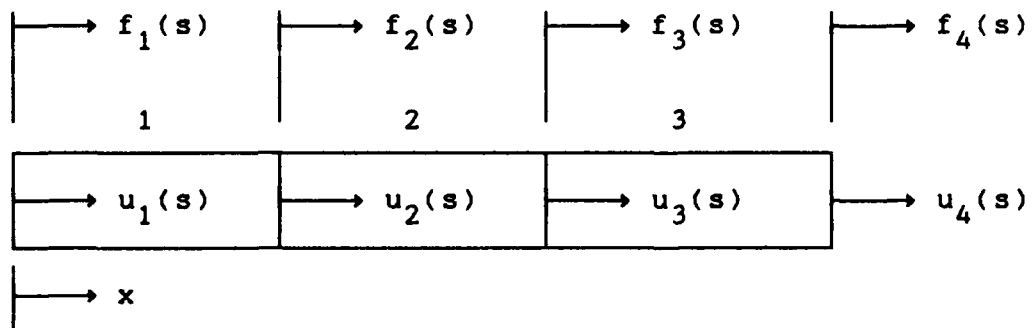


Figure 4. Rod of Elastic and Viscoelastic Elements

At this point, the displacement field and the forces will be placed in the Laplace domain. Work in the Laplace domain will allow easier manipulation of the mathematical operations, especially in the use of fractional derivatives that are used to model the stiffness of the viscoelastic elements. Using the format established in equation (28), the matrix form for the equation of motion for Figure 4 can be

constructed as follows:

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{Bmatrix} = \begin{Bmatrix} f_1(s) \\ f_2(s) \\ f_3(s) \\ f_4(s) \end{Bmatrix} \quad (43)$$

where

$k_i$  = stiffness of rod element  $i$

At this point, the stiffness matrix is split up into elastic and viscoelastic stiffness matrices of the form

$$E(s) \left[ K_v \right] \{u(s)\} + \left[ K_e \right] \{u(s)\} = \{f(s)\} \quad (44)$$

The complex modulus has been factored out of the viscoelastic stiffness matrix. This formulation can be extended to two-dimensional form. The rod in Figure 5 shows a general configuration.

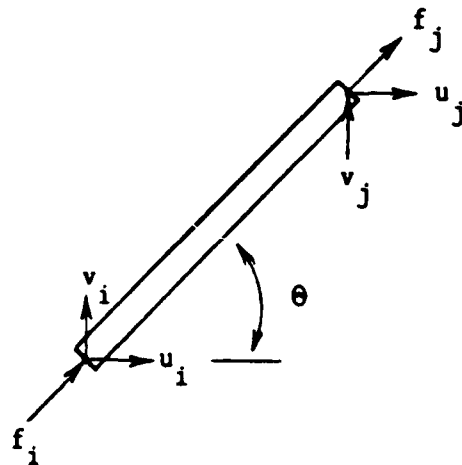


Figure 5. General Rod Configuration

Using the transformation matrix described in Cook (15:25-26), the two dimensional rod formulation can be represented by the following form:

$$\frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix} \quad (45)$$

where

$$c = \cos \theta$$

$$s = \sin \theta$$

#### IV. Example Problem

The purpose of this example problem is to demonstrate the solution technique based upon the use of fractional derivatives to model viscoelastic behavior. The solution is formulated with the assumption of quasi-static equilibrium. This requires that the operating frequency is low enough that inertia effects are considered negligible. A typical space structure in Figure 6 is the basis for the example.

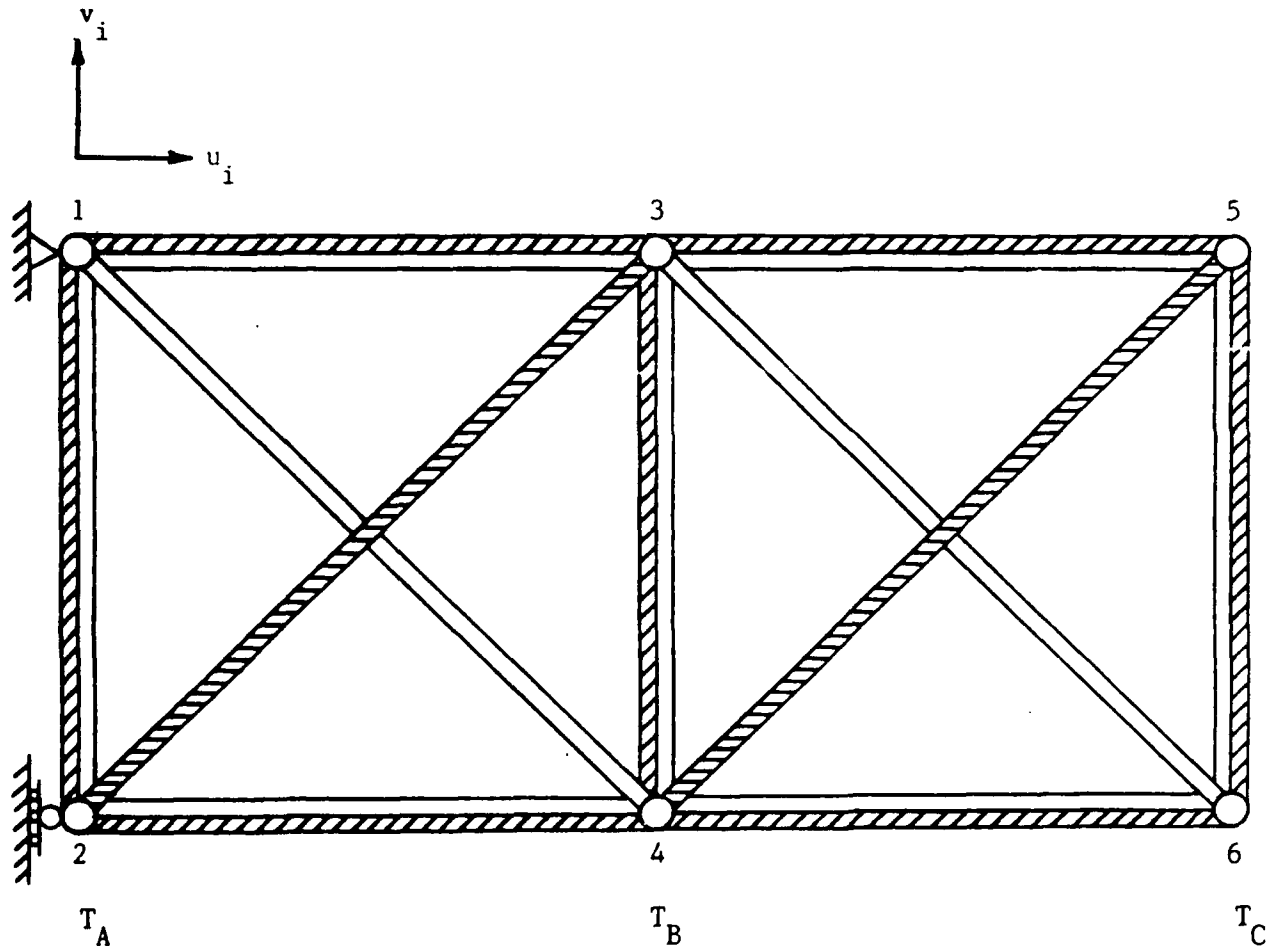


Figure 6. Elastic and Viscoelastic Member Truss

The truss is composed of elastic rods made of aluminum and viscoelastic rods made of neoprene (polychloroprene) rubber. The viscoelastic rods are distinguished by their cross-hatching. The example will also incorporate a temperature gradient across the truss. The node temperatures are marked such that nodes 1 and 2 operate at temperature  $T_A$ ; nodes 3 and 4 at temperature  $T_B$ ; and nodes 5 and 6 at temperature  $T_C$ .

In order to analyze the system structural response, the complex modulus for neoprene rubber needs to be modeled. The actual data for neoprene rubber is taken from a paper by Madigosky and Lee (16). One note of caution is that the example problem uses approximate numbers for the complex modulus. The main purpose is to demonstrate the solution technique. Full use of the model can be employed with actual data for a specified operating environment. The master curve for neoprene rubber that was available only contained complex modulus data in the transition region; therefore, this problem will be limited to that region. To limit the complex modulus to the transition region, the complex modulus is modeled with the form of equation (15) and setting the parameter  $E_0$  equal to zero. Using the format established in equations (16) and (17) with  $E_0$  equal to zero produces the storage and loss moduli models to estimate the fractional derivative complex modulus for neoprene rubber. The master curve referenced has data for the storage modulus and the

loss factor (16:348). To develop the data for the loss modulus, the relationship in equation (10) is used. With the data calculated for storage and loss moduli, the fractional derivative model parameters can be estimated. Examination of the storage and loss modulus curves revealed that their magnitudes were nearly equal in the transition region. For this to occur, the arguments of sine and cosine in equations (16) and (17) need to be equal. The arguments are equal at  $45^\circ$ ; thus, the fractional power  $\alpha$  must be equal to  $1/2$ . The final parameter,  $E_1$ , is then estimated to provide a best fit fractional derivative model. The master curve for storage modulus and the fractional derivative model is shown in Figure 7; and the master curve for loss modulus and its fractional derivative model is displayed in Figure 8. Figures (7) and (8) were created using the GRAPHER software program (17), and the data points used are contained in Appendix A. The fractional derivative model uses the parameters

$$\alpha = 0.5$$

$$E_1 = 1.4 \times 10^6 \text{ N/m}^2 (\text{sec})^{0.5}$$

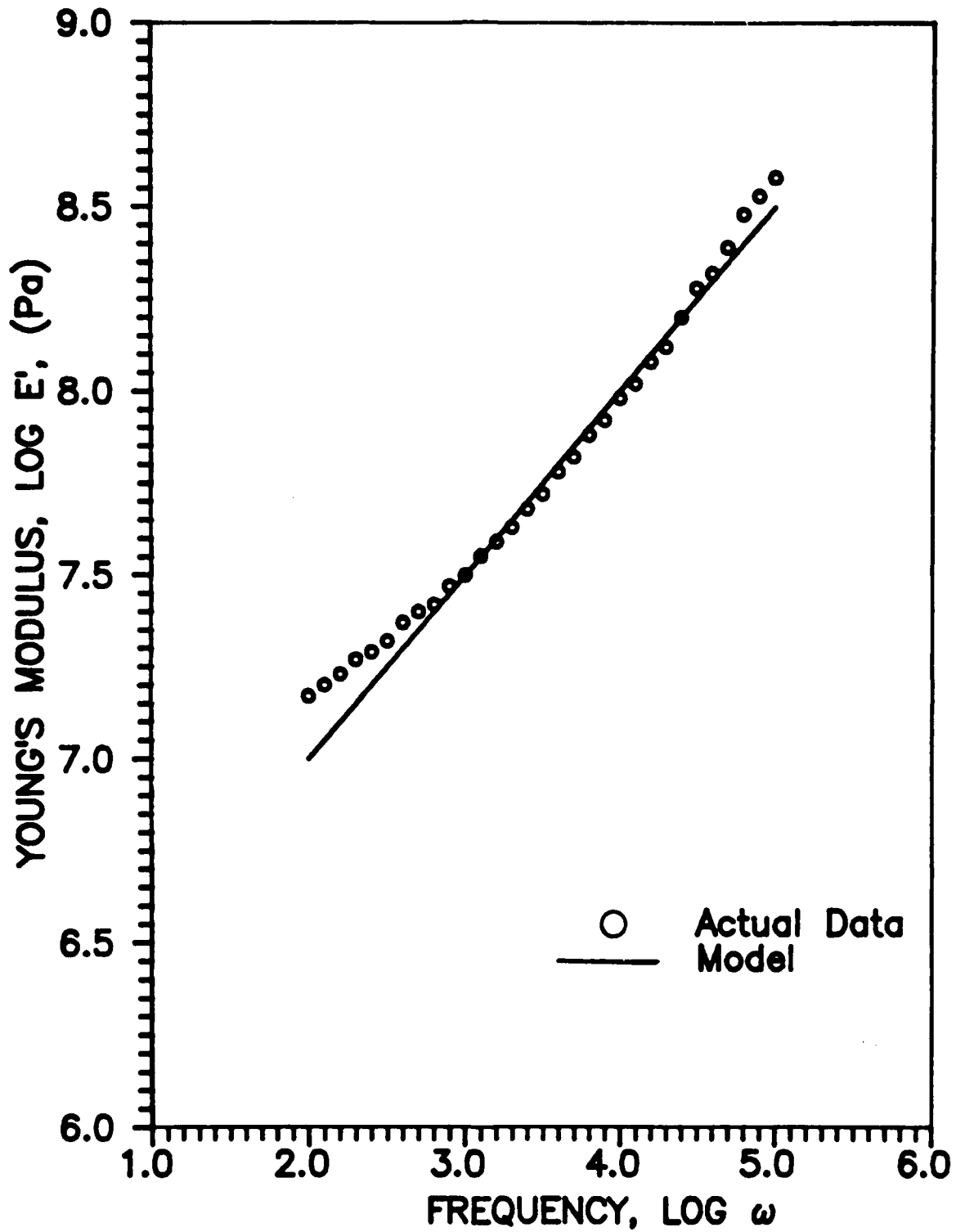


Figure 7. Young's Modulus,  $E'$ , and Parameter Curve Fit at  $T_0 = 20^\circ\text{C}$  for Neoprene Rubber

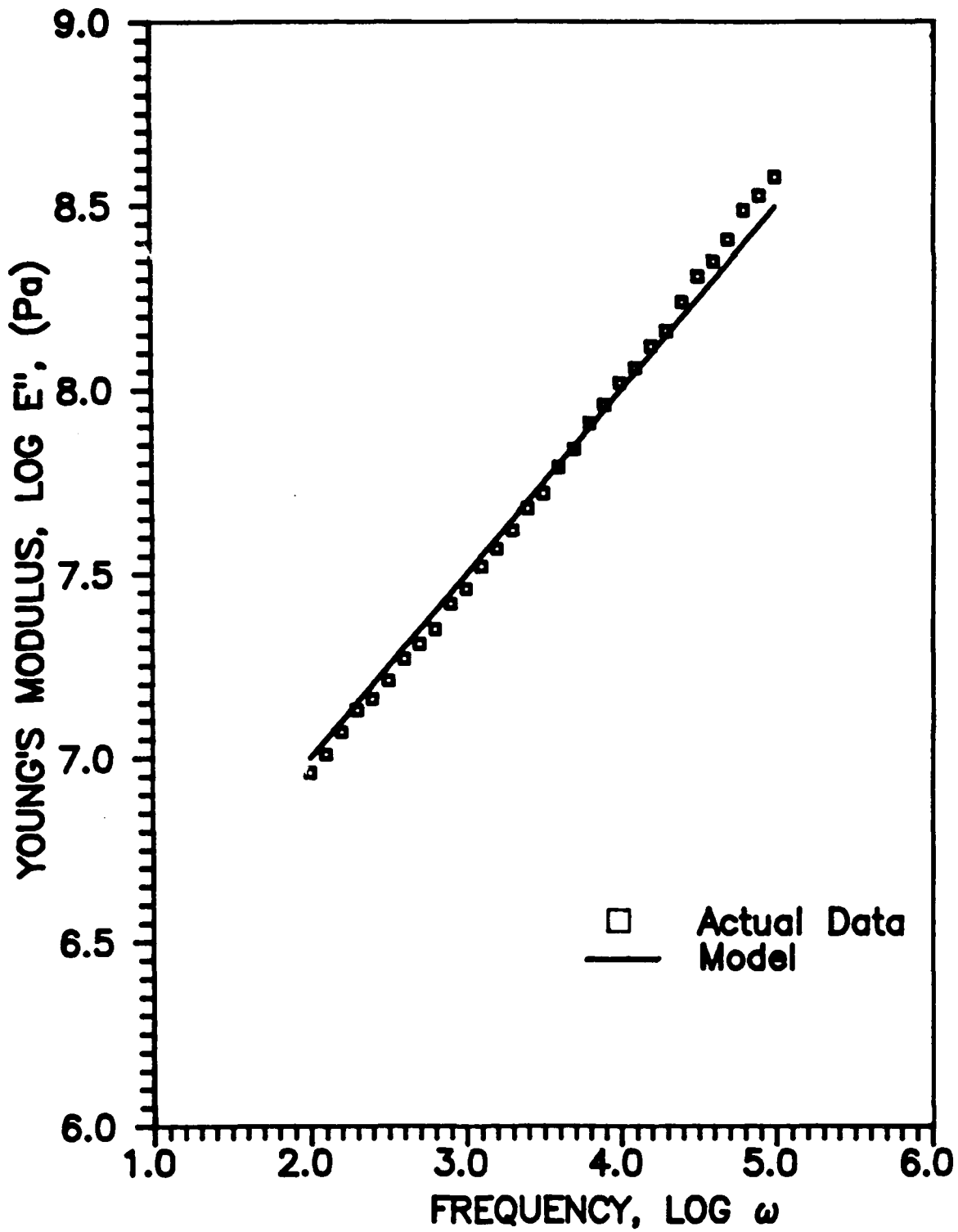


Figure 8. Young's Modulus,  $E'$ , and Parameter Curve Fit at  $T_0 = 20^\circ\text{C}$  for Neoprene Rubber

To incorporate temperature effects into the viscoelastic response, the complex modulus established in equation (32) is used. The temperature shift function,  $(a_T)^\alpha$ , is calculated using equation (21). Using a reference temperature,  $T_0$ , of 20°C and material coefficients  $c_1 = 4.09$  and  $c_2 = 67$ , a shift function is determined using the data tabulated in Table 1.

TABLE 1: Temperature Shift Data for Neoprene Rubber

T (°C)	T - T <sub>0</sub> (°C)	(a <sub>T</sub> ) <sup>α</sup>
4.0	-16.0	4.38
7.8	-12.2	2.85
12.5	-7.5	1.81
17.0	-3.0	1.25
25.8	5.8	0.69
31.0	11.0	0.51
41.8	21.8	0.31
47.0	27.0	0.26

A temperature shift function is estimated by plotting  $(a_T)^\alpha$  versus  $(T - T_0)$ . An exponential function in the form of equation (34) is used to obtain a best fit. The software program GRAPHER (17) is used to calculate a best fit exponential curve and the model and actual data are plotted in Figure 9. To obtain an accurate temperature shift model, data points at  $T - T_0 = 21.8$  and 27°C were not used in the curve fitting routine. The reason for this is that to force the curve through these points created larger errors at the lower temperature region, where the shift factor becomes exponentially large. Deleting these two points allowed for a more representative curve fit, and the resulting temperature

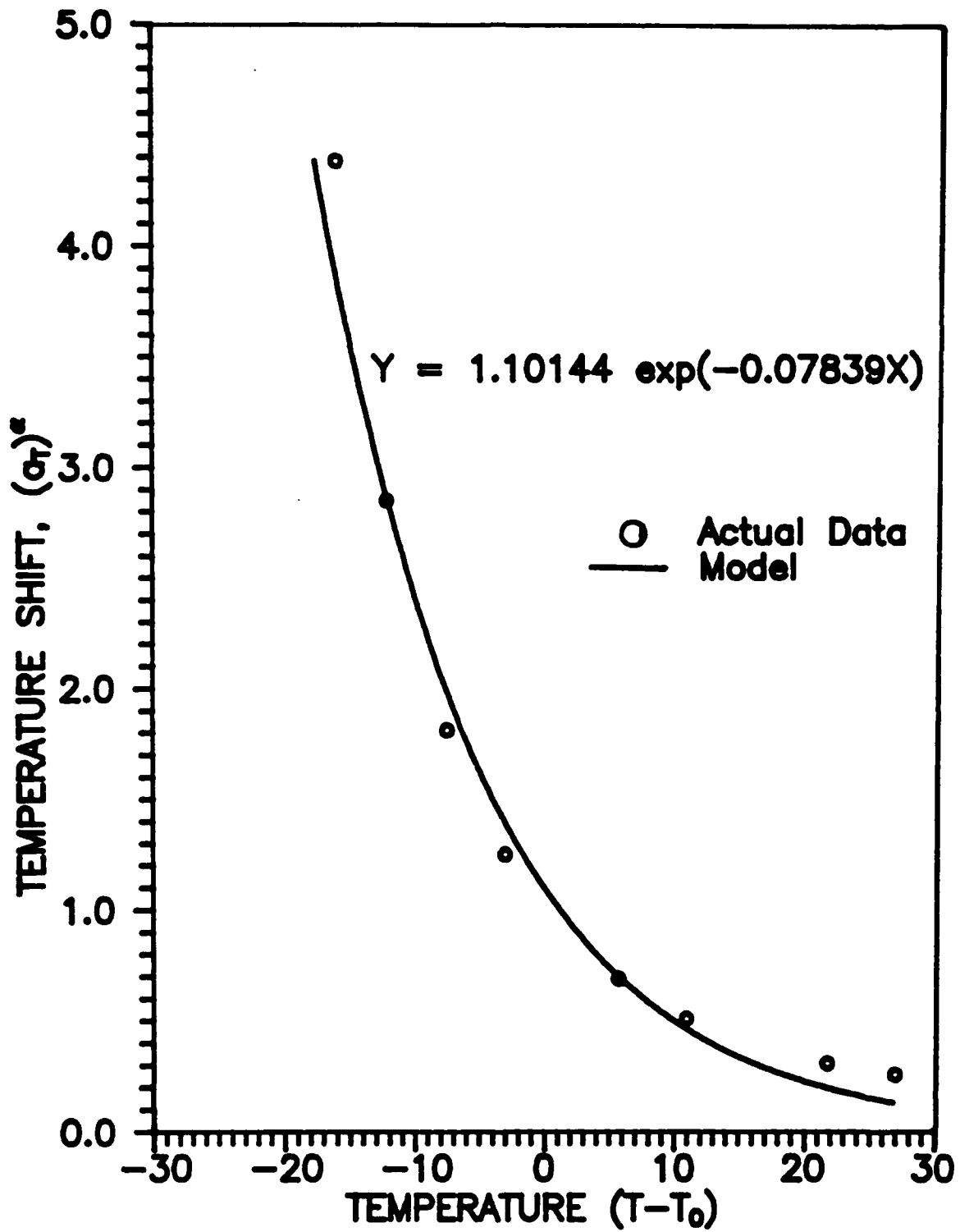


Figure 9. Temperature Shift,  $(a_T)^\alpha$ , at  $T_0 = 20^\circ\text{C}$   
for Neoprene Rubber (Actual Data: Table 1)

shift model still approximated the deleted points quite well. Since the model agrees well with the actual data, the following temperature shift function is used:

$$(a_T)^\alpha = 1.10144 \exp[-0.07839 (T - T_0)] \quad (46)$$

The complex modulus model including frequency and temperature dependency is the basis for analyzing the truss in Figure 6. The model assumes quasi-static equilibrium. The complex modulus data for neoprene rubber in Figures (7) and (8) is in the frequency range of 100 to 100,000 Hz at a reference temperature of 20°C. For the inertia effects to be considered negligible, the transition region needs to be in the region of 1.0 Hz. This can be accomplished by using the temperature shift property described in section 2.3.4. Since lowering the temperature shifts the transition region to a lower frequency domain, selecting an appropriate temperature variation has the effect of shifting the transition region to a frequency range where the inertia effects can be considered negligible. The data in Table 2 represents the results of applying the temperature shift function to a range of temperatures at a reference frequency,  $\omega$ , of 1.0 Hz.

TABLE 2: Frequency Shift Due to Temperature Changes for Neoprene Rubber

T (°C)	T - T <sub>o</sub> (°C)	a <sub>T</sub>	log (ω a <sub>T</sub> )
0	-20	27.9	1.44
-10	-30	133.8	2.13
-20	-40	641.9	2.81
-30	-50	3078.6	3.49
-35	-55	6742.2	3.83
-40	-60	14765.3	4.17
-45	-65	32336.0	4.51

The result of Table 2 is that choosing temperatures in the range of -20 to -40°C at a frequency of 1.0 Hz is equivalent to operating at the reference temperature, T<sub>o</sub> = 20°C, in the transition frequency region (10<sup>2</sup> to 10<sup>5</sup> Hz). The lower limit on the temperature range is approximately -50°C, the glass transition temperature for neoprene rubber (18:1564). The truss will be modeled with a temperature variation horizontally along the truss.

To determine the system response, the equation of motion developed in equation (44) is used with the following format:

$$E(s) \begin{bmatrix} K_v \\ \end{bmatrix} \begin{Bmatrix} u(s) \\ v(s) \end{Bmatrix} + \begin{bmatrix} K_e \\ \end{bmatrix} \begin{Bmatrix} u(s) \\ v(s) \end{Bmatrix} = \begin{Bmatrix} f(s) \end{Bmatrix} \quad (47)$$

To solve for the system response, the eigenvalues and eigenvectors need to be determined from the homogeneous solution. The complex modulus format in equation (32) is substituted into equation (47). Consolidating the constants into the viscoelastic stiffness matrix results in the

following homogeneous equation of motion:

$$\left[ s^\alpha \begin{bmatrix} K_v \end{bmatrix} + \begin{bmatrix} K_e \end{bmatrix} \right] \begin{Bmatrix} u(s) \\ v(s) \end{Bmatrix} = \{0\} \quad (48)$$

The eigenvalues and eigenvectors are determined by setting the determinant of the stiffness matrix equal to zero. The system eigenvectors represent a set of orthogonal vectors that span the system response. Any general motion can be represented by a superposition of the normal modes of the system. Because of this orthogonality property, the system eigenvectors can be used to uncouple the equations of motion (13:76-81). This is done by the transformation

$$\begin{Bmatrix} u(s) \\ v(s) \end{Bmatrix} = [\Phi] \{w(s)\} \quad (49)$$

where

$\{w(s)\} \equiv$  orthogonal coordinates of system

Substitution of this transformation into equation (47) yields

$$\left[ s^\alpha \begin{bmatrix} K_v \end{bmatrix} [\Phi] + \begin{bmatrix} K_e \end{bmatrix} [\Phi] \right] \{w(s)\} = \{f(s)\} \quad (50)$$

Premultiplying by  $[\Phi]^T$  transforms equation (50) into

$$\left[ s^\alpha \begin{bmatrix} K_v \end{bmatrix} + \begin{bmatrix} K_e \end{bmatrix} \right] \{w(s)\} = [\Phi]^T \{f(s)\} \quad (51)$$

The orthogonal coordinate vector  $\{w(s)\}$  is normalized and equation (51) is transformed to

$$\left[ s^\alpha [K_v] + [K_e] \right] \left[ [K_v]^{-1} \right]^{1/2} \{\tilde{w}(s)\} = [\Phi]^T \{f(s)\} \quad (52)$$

where

$$\{\tilde{w}(s)\} \equiv \text{orthonormal coordinates of system}$$

Equation (52) is pre-multiplied by the matrix  $\left[ [K_v]^{-1} \right]^{1/2}$ .

This process results in the diagonal viscoelastic stiffness matrix to reduce to the identity matrix while the diagonal elastic stiffness matrix reduces to the system eigenvalue matrix. The reduced equation has the form

$$\left[ s^\alpha [I] + [\lambda] \right] \{\tilde{w}(s)\} = [\phi] \{f(s)\} \quad (53)$$

where

$$[\phi] = \left[ [K_v]^{-1} \right]^{1/2} [\Phi]^T$$

Equation (53) is the general form of the equation of motion for the viscoelastic formulation developed in a paper by Bagley and Calico (6). The notation is modified to parallel the solution technique developed by Bagley and Calico.

Equation (53) now has the form

$$\tilde{w}(s) = \frac{\tilde{f}(s)}{s^\alpha + a^\alpha} \quad (54)$$

where

$$\begin{aligned} \tilde{f}(s) &= [\phi] \{f(s)\} \\ a^\alpha &= \lambda, \text{ system eigenvalues} \end{aligned}$$

Factoring out  $s^{-\alpha}$  from equation (54) yields

$$\tilde{w}(s) = \tilde{f}(s) s^{-\alpha} \left[ 1 + \left(\frac{a}{s}\right)^\alpha \right]^{-1} \quad (55)$$

Using the binomial expansion (19:347), equation (55) has the form

$$\tilde{w}(s) = \tilde{f}(s) s^{-\alpha} \left[ 1 - \left(\frac{a}{s}\right)^\alpha + \left(\frac{a}{s}\right)^{2\alpha} - \left(\frac{a}{s}\right)^{3\alpha} + \dots \right] \quad (56)$$

To obtain the system response in the time domain, several steps need to be taken. First, the parameter,  $s$ , is factored out of the expansion. To bring equation (56) into the time domain, the convolution integral (13:534-35) is used. This example problem will use a forcing function of a unit step for demonstration purposes. Using the Laplace transformation property in equation (5), equation (56) has the following

form in the time domain:

$$\tilde{w}(t) = \int_0^t D^{1-\alpha} \left[ 1 - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(1+2\alpha)} - \dots \right] [\tilde{\phi}] dt \quad (57)$$

The expansion term in equation (57) is the alpha order Mittag-Leffler function. Equation (57) can be represented in the following reduced form:

$$\tilde{w}(t) = \int_0^t D^{1-\alpha} \left\{ E_\alpha \left[ -(\lambda t^\alpha) \right] \right\} [\tilde{\phi}] dt \quad (58)$$

Equation (58) is integrated and has the form

$$\tilde{w}(t) = -\lambda^{-1} \left\{ E_\alpha \left[ -(\lambda t^\alpha) \right] - 1 \right\} [\tilde{\phi}] \quad (59)$$

where

$$E_\alpha \left[ -(\lambda t^\alpha) \right] = \sum_{p=0}^{\infty} \frac{[-(\lambda t^\alpha)]^p}{\Gamma(1 + \alpha p)}$$

This solution can then be substituted back into the general system response equation:

$$\begin{Bmatrix} u(t) \\ v(t) \end{Bmatrix} = [\Phi]^T \{w(t)\} \quad (60)$$

#### 4.1 Example Setup

The system response formulation, equation (60), is demonstrated by comparing three variations to the truss in Figure 6. The first example, case A, is a truss composed only of elastic members (aluminum). This case will serve as a baseline to determine the effects of temperature on the damping characteristics of the viscoelastic material. Case B will use the truss layout in Figure 6 with elastic (aluminum) and viscoelastic (neoprene rubber) members at a constant temperature. Case C will also use the same layout as case B but will have a temperature variation horizontally along the truss. To solve for the system response, the following parameters are used:

$$A = 0.001 \text{ m}^2$$

$$L = 1.0 \text{ m for horizontal and vertical rods}$$

$$F = 1000 \text{ N } \hat{v}_6 \text{ (unit step forcing function)}$$

$$E = 7.0 \times 10^{10} \text{ N/m}^2 \quad (18:2211)$$

$$E_1 = 1.4 \times 10^6 \text{ N/m}^2 \text{ (sec)}^{0.5}$$

$$\alpha = 0.5$$

$$a = 1.10144$$

$$b = 0.07839$$

The truss in Figure 6 has six nodes, labeled 1-6, and has nine degrees of freedom. The system response will be calculated for a unit step function applied in the vertical direction at node 6, ( $v_6$ ), and the response at nodes  $v_3$ ,  $v_4$ ,  $v_5$ , and  $v_6$  will be compared for all three cases.

4.1.1 Case A. The response for the elastic truss can be found by using the format established in equation (28) and inverting the stiffness matrix.

4.1.2 Case B. This case was originally intended to be a constant temperature case. Using an operating temperature of  $-20^{\circ}\text{C}$  at a frequency of 1.0 Hz, the complex modulus fell in the beginning of the transition region (see Table 2 and Figures 7 and 8). The eigenvalues and eigenvectors were determined by the use of equation (48). Of the nine eigenvalues calculated, five were repeated with a value of 1973.0. The existence of repeated eigenvalues also resulted in complex eigenvectors. To achieve real eigenvectors, a small temperature was imposed as follows:  $T_A = -20.00^{\circ}\text{C}$ ;  $T_B = -20.01^{\circ}\text{C}$ ; and  $T_C = -20.02^{\circ}\text{C}$ . While the temperature variation seems negligible, the slight change was enough to produce nine distinct eigenvalues and real eigenvectors. The eigenvalues and eigenvectors for case B are included in Appendix C.

4.1.3 Case C. The temperature variation for this case is chosen to capture the entire transition region of the complex modulus for neoprene rubber. Referencing Table 2, the node temperatures were chosen as follows:  $T_A = -20^{\circ}\text{C}$ ;  $T_B = -30^{\circ}\text{C}$ ; and  $T_C = -40^{\circ}\text{C}$ . The same procedure as case B is used to determine the system response. The eigenvalues and eigenvectors for case C are included in Appendix D.

## 4.2 Results

To determine the system response for cases B and C, the Mittag-Leffler series needs to be evaluated. A FORTRAN program was created to evaluate the Mittag-Leffler series for each eigenvalue. The program can be found in Appendix E. After the Mittag-Leffler series is calculated for the system eigenvalues at selected time increments, the system response can be determined. The matrix operations to determine the system eigenvalues, eigenvectors, and response were accomplished by use of the software program MATLAB (20).

The system response for cases A, B, and C are plotted for selected nodes. The vertical nodes,  $v_3$  (Figure 10),  $v_4$  (Figure 11),  $v_5$  (Figure 12), and  $v_6$  (Figure 13), were chosen since their displacements were significantly affected by the vertical unit step forcing function applied at node 6. The time increments displayed in Figures 10-13 are defined as follows:  $T_0 = 0.0$  sec;  $T_1 = 10^{-9}$  sec;  $T_2 = 10^{-8}$  sec;  $T_3 = 10^{-7}$  sec;  $T_4 = 10^{-6}$  sec;  $T_5 = 10^{-5}$  sec;  $T_6 = 10^{-4}$  sec; and  $T_7 = 10^{-3}$  sec. The system response for each node at each time increment are included in Appendix B for case A; in Appendix C for case B; and in Appendix D for case C.

Examination of Figures 10-13 reveals the effects of temperature on the system response. Since the complex modulus for case C encompasses the entire transition region due to temperature variations, its damping ability is increased. This is evident by noting the slower response

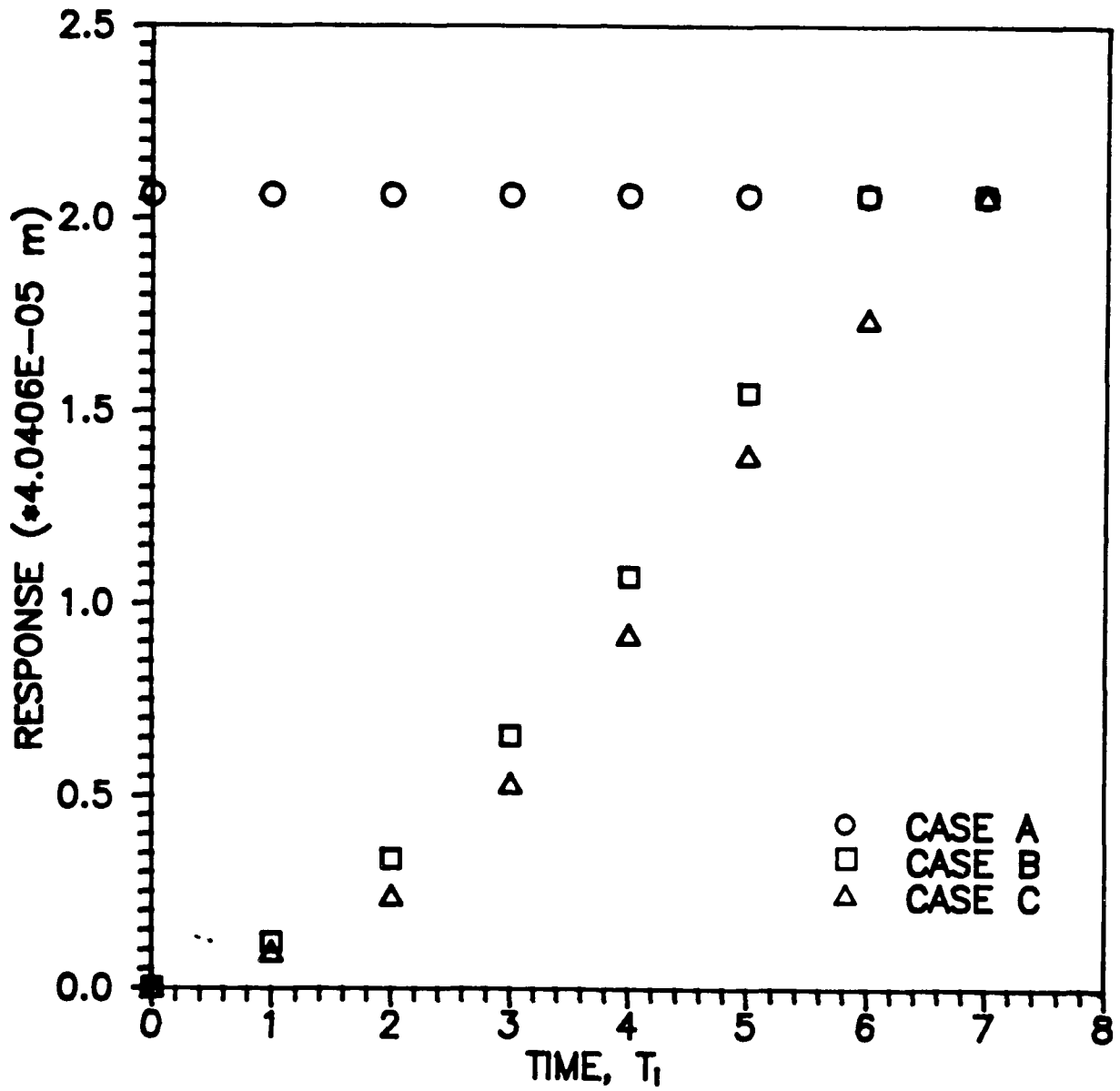


Figure 10. System Response for Node  $V_3$

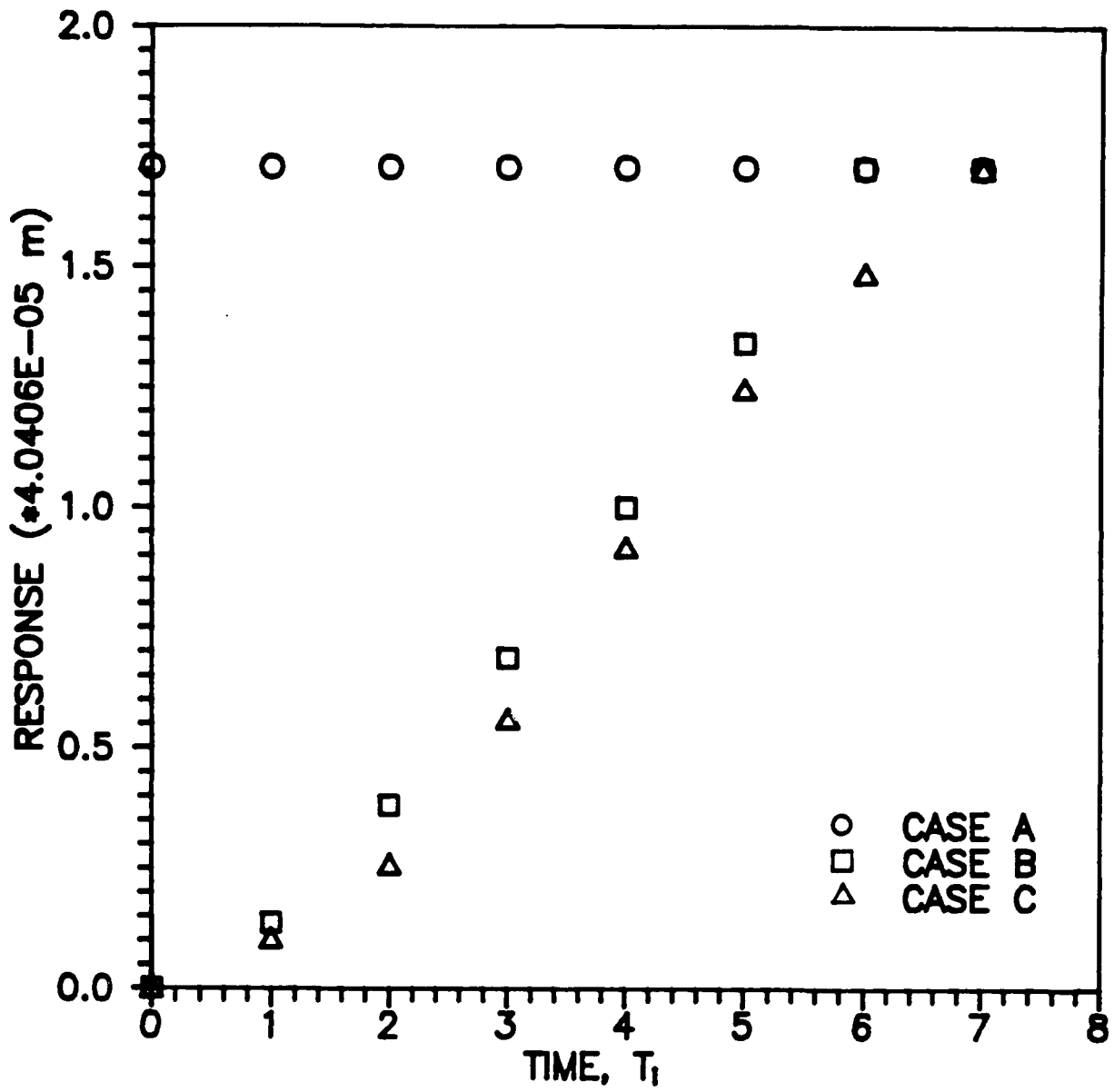


Figure 11. System Response for Node  $V_4$

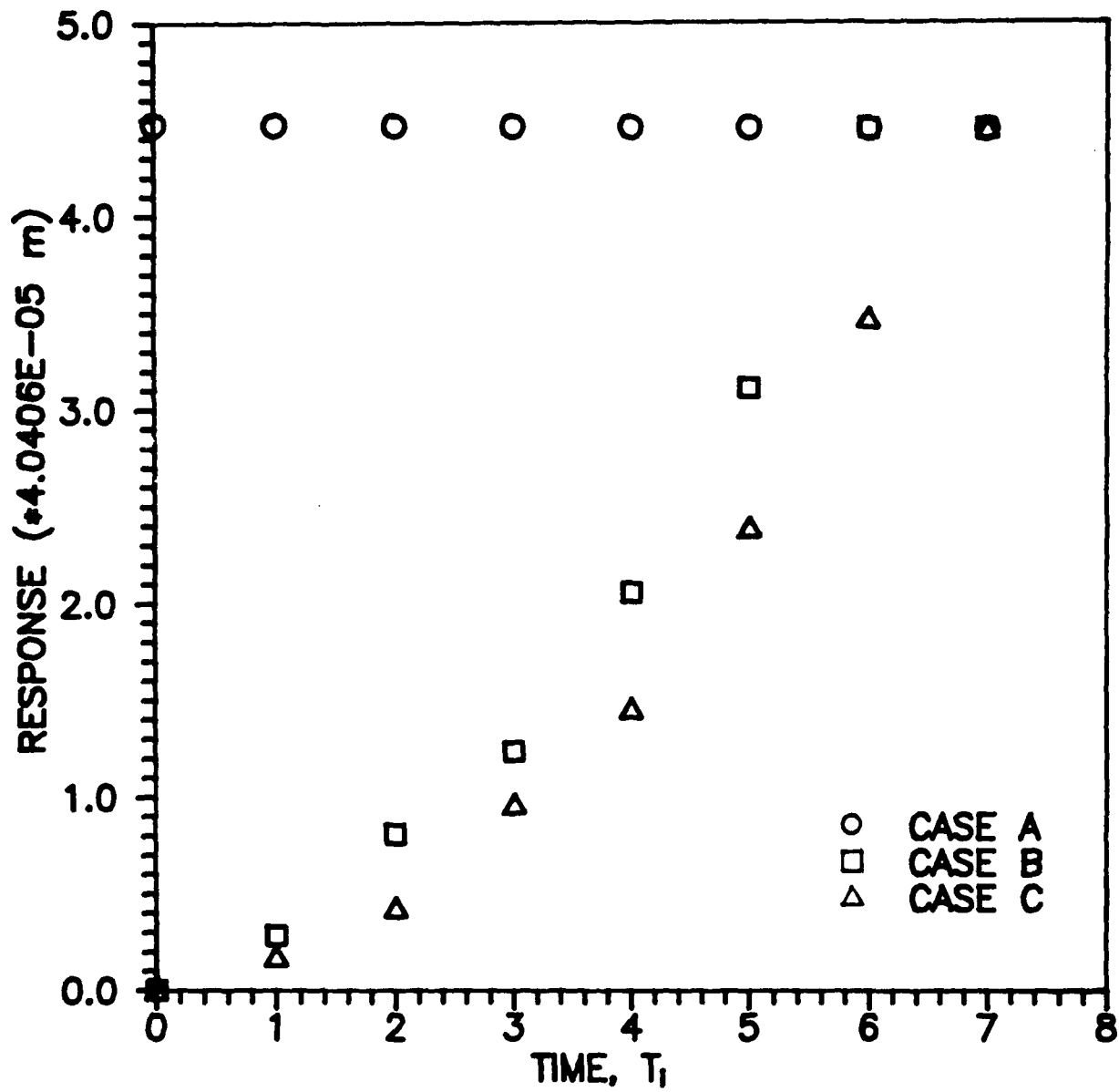


Figure 12. System Response for Node  $V_5$

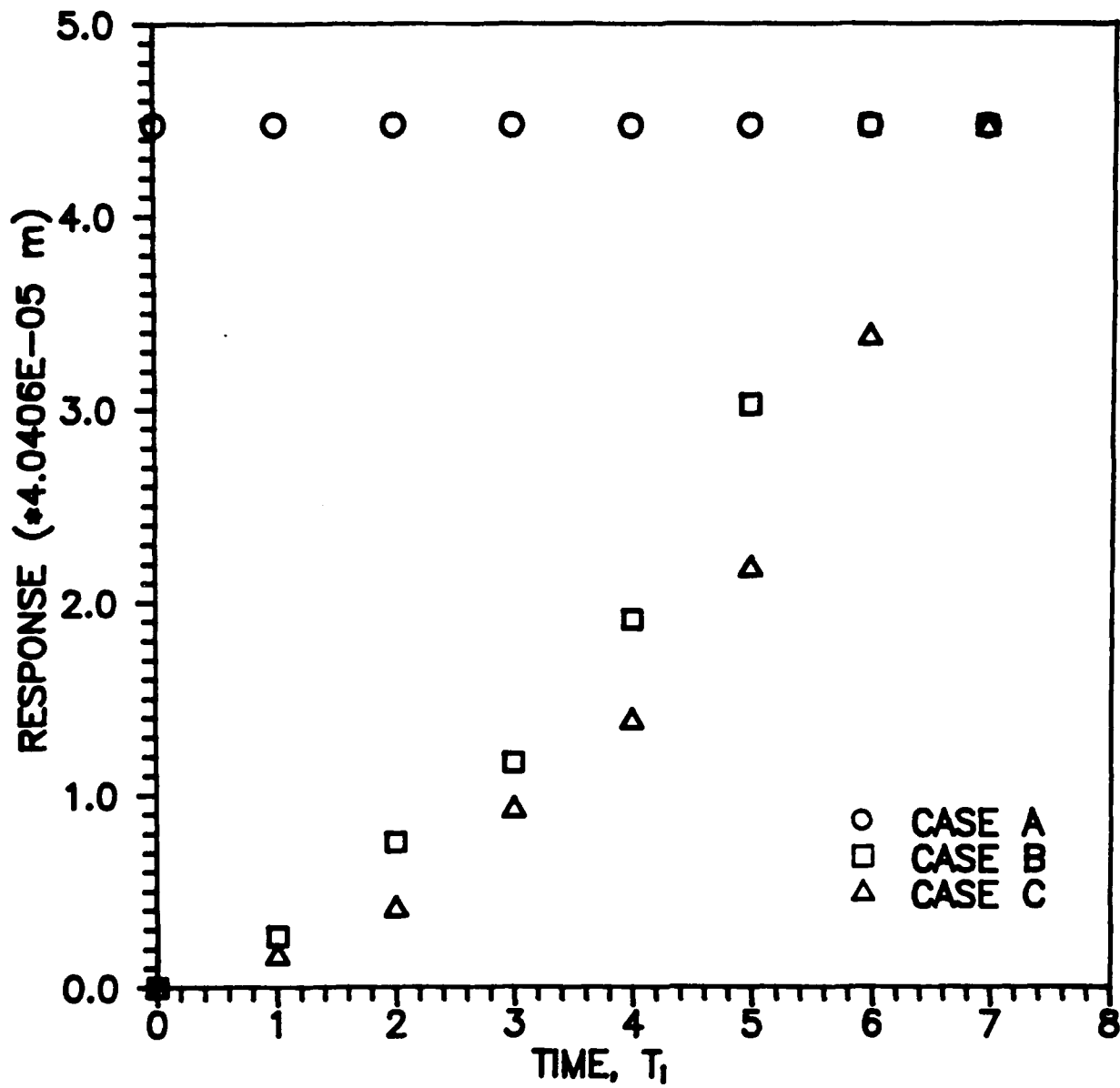


Figure 13. System Response for Node  $V_6$

of case C. While the response for cases B and C appear to reach the static response of case A, they actually only asymptotically approach case A. The responses that were plotted contained data only to four decimal places. Actual differences between the cases take place past this accuracy.

The results of Figures 10-13 demonstrate the ability to determine the system response of a structure containing both elastic and viscoelastic elements. This is accomplished by incorporating temperature and frequency dependency in the complex modulus model with fractional derivatives and temperature shift functions.

## V. Conclusions/Recommendations

For nearly three hundred years, the fractional derivative has gone virtually unnoticed. With the increased emphasis on using viscoelastic materials to capture their damping and energy dissipation characteristics, more accurate and manageable techniques are needed to model the material behavior. The fractional derivative has shown great promise by its ability to model the complex modulus data of numerous viscoelastic materials for up to eight decades of frequency with the determination of three or four parameters. The confidence in using the fractional derivative model has been enhanced by the theoretical basis at the molecular level.

The cornerstone of this research was the derivation of a fractional derivative model for the complex modulus that incorporated temperature as well as frequency dependency. To formulate the solution, the following requirements were established:

1. Formulation limited to longitudinal motion.
2. Quasi-static equilibrium was specifically assumed to produce low frequency responses.
3. Linear relationships allowed the use of Laplace transformations.
4. Strain levels below 1 - 2% required.
5. Temperatures must be below the glass transition temperature for the viscoelastic material.

The temperature and frequency dependency was built into the model due to their inter-relationship. By using the principle of reduced variables, temperature variations can be accounted for by applying a temperature shift function to the frequency. The inclusion of temperature dependency limited the model to the transition region of the complex modulus. While this removed the rubbery and glassy plateau regions from the model, a very versatile model was established for the transition region. Since the transition region is characterized by a high value for the loss factor, maintaining the operating environment in this region maximizes the damping ability of the viscoelastic material.

The construction of the temperature shift function by use of the WLF equation provided the necessary tools to solve the example problem. This example problem was chosen to demonstrate the ability to use fractional derivatives to analyze the structural response of a truss composed of elastic and viscoelastic members subject to a temperature variation along the truss. The viscoelastic rods were chosen to be made of a commercially available neoprene rubber while the elastic rods were composed of aluminum. In order to maintain the quasi-static equilibrium assumption, the complex modulus data needed to be shifted to a lower frequency range. This was accomplished by selecting a temperature range lower than the reference temperature through the use of the

temperature shift function. With the appropriate temperature and materials selected, the fractional derivative complex model was incorporated into the finite element formulation of the equations of motion for the truss. By working in the Laplace domain, the solution was simplified, another advantage of the fractional derivative.

The example problem demonstrated the ability to incorporate the temperature and frequency dependency of the viscoelastic material for operating conditions in the transition region. The results of this research and others has shown great potential for the continued investigation of fractional derivatives to model viscoelastic material behavior. The benefits such as increased feedback states, simple but accurate models, robustness, and the ability to predict material behavior outside of the measured data, are incentives to continue expanding the model.

While the fractional derivative model for the complex modulus incorporated temperature effects in the transition region, future work is needed to extend the model to the rubbery and glassy regions. Follow-on work is also suggested to evaluate case B at constant temperature. Solution techniques do exist for repeated eigenvalues, although somewhat complicated, and the results in this example problem can serve as a basis for comparison.

Appendix A: Complex Modulus and Model Data  
for Neoprene Rubber (16)

Frequency	Actual Data (N/m <sup>2</sup> )		Model (N/m <sup>2</sup> )	
	Log $\omega$	Log E'	Log E''	Log E'
2.0	7.17	6.96	7.00	7.00
2.1	7.20	7.01	7.05	7.05
2.2	7.23	7.07	7.10	7.10
2.3	7.27	7.13	7.15	7.15
2.4	7.29	7.16	7.20	7.20
2.5	7.32	7.21	7.25	7.25
2.6	7.37	7.27	7.30	7.30
2.7	7.40	7.31	7.35	7.35
2.8	7.42	7.35	7.40	7.40
2.9	7.47	7.42	7.45	7.45
3.0	7.50	7.46	7.50	7.50
3.1	7.55	7.52	7.55	7.55
3.2	7.59	7.57	7.60	7.60
3.3	7.63	7.62	7.65	7.65
3.4	7.68	7.68	7.70	7.70
3.5	7.72	7.72	7.75	7.75
3.6	7.78	7.79	7.80	7.80
3.7	7.82	7.84	7.85	7.85
3.8	7.88	7.91	7.90	7.90
3.9	7.92	7.96	7.95	7.95
4.0	7.98	8.02	8.00	8.00
4.1	8.02	8.06	8.05	8.05
4.2	8.08	8.12	8.10	8.10
4.3	8.12	8.16	8.15	8.15
4.4	8.20	8.24	8.20	8.20
4.5	8.28	8.31	8.25	8.25
4.6	8.32	8.35	8.30	8.30
4.7	8.39	8.41	8.35	8.35
4.8	8.48	8.49	8.40	8.40
4.9	8.53	8.53	8.45	8.45
5.0	8.58	8.58	8.50	8.50

Appendix B: Case A Data

$$\begin{Bmatrix} v_2(t) \\ u_3(t) \\ v_3(t) \\ u_4(t) \\ v_4(t) \\ u_5(t) \\ v_5(t) \\ u_6(t) \\ v_6(t) \end{Bmatrix} = \begin{Bmatrix} .0000 \\ -.3536 \\ -2.0607 \\ .7071 \\ -1.7071 \\ -.3536 \\ -4.4749 \\ 1.0607 \\ -4.4749 \end{Bmatrix}$$

### Appendix C: Case B Data

$$\left[ \lambda \right] = \begin{bmatrix}
 13704.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 4112.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 284.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 946.0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1970.6 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1972.5 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1971.9 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1972.7 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1971.2
 \end{bmatrix}$$

$$\left[ \Phi \right] = \begin{bmatrix}
 .0214 & .2891 & .0439 & .5202 & .1149 & -.8275 & .2877 & .0000 & .0000 \\
 -.1104 & -.0111 & -.2265 & -.0203 & .0856 & -.0153 & .5785 & .5000 & .0000 \\
 .1835 & .7252 & -.4679 & -1.0568 & .0288 & -.8110 & -.2914 & -.5000 & .0000 \\
 .1104 & .0111 & .2265 & .0203 & -.0856 & .0153 & -.5785 & .5000 & .0000 \\
 .2494 & .8751 & -.3327 & -.7863 & .0851 & -.0141 & .5778 & -.5000 & .0000 \\
 -.1548 & .1276 & -.3179 & .2298 & .5396 & .3116 & .0002 & .5000 & 1.0000 \\
 .6221 & .5543 & -1.3249 & -.2290 & -.5397 & -.3110 & -.0002 & -.5000 & -1.0000 \\
 .1548 & -.1276 & .3179 & -.2298 & -.5396 & -.3116 & -.0002 & .5000 & 1.0000 \\
 .6666 & .4157 & -1.2336 & -.4793 & .6544 & -.5154 & .2879 & -.5000 & -1.0000
 \end{bmatrix}$$

$$\begin{bmatrix}
 v_2(t) \\
 u_3(t) \\
 v_3(t) \\
 u_4(t) \\
 v_4(t) \\
 u_5(t) \\
 v_5(t) \\
 u_6(t) \\
 v_6(t)
 \end{bmatrix} = \begin{bmatrix}
 T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\
 .0000 & .0207 & .0550 & .1166 & .1286 & .0479 & .0000 & .0000 \\
 .0000 & -.0362 & -.1018 & -.1022 & -.0837 & .1059 & .3536 & .3536 \\
 .0000 & .1189 & .3354 & .6556 & 1.0711 & 1.5490 & 2.0607 & 2.0607 \\
 .0000 & .0126 & .0350 & -.0553 & -.2699 & -.4594 & -.7071 & -.7071 \\
 .0000 & .1355 & .3804 & .6862 & 1.0005 & 1.3433 & 1.7071 & 1.7071 \\
 .0000 & -.0557 & -.1586 & -.1739 & -.2380 & .0060 & .3536 & .3536 \\
 .0000 & .2623 & .7528 & 1.1644 & 1.9082 & 3.0262 & 4.4749 & 4.4749 \\
 .0000 & .0085 & .0251 & -.1413 & -.4691 & -.7131 & -1.0607 & -1.0607 \\
 .0000 & .2818 & .8096 & 1.2360 & 2.0625 & 3.1261 & 4.4749 & 4.4749
 \end{bmatrix}$$

Appendix D: Case C Data

$$\left[ \lambda \right] = \begin{bmatrix}
 8011.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 2973.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 102.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1107.9 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 875.4 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 414.6 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 546.0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1368.0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 624.6
 \end{bmatrix}$$

$$\left[ \Phi \right] = \begin{bmatrix}
 .0486 & .5073 & -.0164 & -.6225 & -.3639 & -.1763 & .3660 & .0000 & .0000 \\
 -.1259 & .0758 & .2062 & .0970 & -.4319 & .2269 & -.1540 & .5000 & .0000 \\
 .2462 & .7659 & .3665 & .2341 & .9630 & .8962 & -1.3559 & -.5000 & .0000 \\
 .1259 & -.0758 & -.2062 & -.0970 & .4319 & -.2269 & .1540 & .5000 & .0000 \\
 .2944 & .8025 & .2502 & .8693 & .1408 & .6208 & -.9027 & -.5000 & .0000 \\
 -.1455 & .1218 & .3109 & .2615 & .0140 & .2550 & .5298 & .5000 & -1.0000 \\
 .6167 & .5022 & 1.2479 & .3077 & .1975 & .1791 & -.9999 & -.5000 & 1.0000 \\
 .1455 & -.1218 & -.3109 & -.2615 & -.0140 & -.2550 & -.5298 & .5000 & -1.0000 \\
 .6293 & .4744 & 1.1314 & .2325 & .0387 & 1.4048 & -.7366 & -.5000 & 1.0000
 \end{bmatrix}$$

$$\begin{bmatrix}
 v_2(t) \\
 u_3(t) \\
 v_3(t) \\
 u_4(t) \\
 v_4(t) \\
 u_5(t) \\
 v_5(t) \\
 u_6(t) \\
 v_6(t)
 \end{bmatrix} = \begin{bmatrix}
 T_0 & T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\
 .0000 & .0214 & .0521 & .1043 & .1180 & .0301 & .0143 & .0000 \\
 .0000 & -.0284 & -.0660 & -.1266 & -.1743 & -.0247 & .1744 & .3536 \\
 .0000 & .0941 & .2386 & .5312 & .9202 & 1.3882 & 1.7422 & 2.0607 \\
 .0000 & .0118 & .0175 & .0034 & -.1792 & -.3288 & -.5279 & -.7071 \\
 .0000 & .1030 & .2575 & .5598 & .9177 & 1.2481 & 1.4897 & 1.7071 \\
 .0000 & -.0354 & -.0847 & -.2180 & -.2598 & -.2169 & .0834 & .3536 \\
 .0000 & .1703 & .4151 & .9331 & 1.3930 & 2.1853 & 3.3904 & 4.4749 \\
 .0000 & .0111 & .0125 & -.0629 & -.2514 & -.4902 & -.7905 & -1.0607 \\
 .0000 & .1749 & .4276 & .9650 & 1.4584 & 2.3991 & 3.4917 & 4.4749
 \end{bmatrix}$$

Appendix E: FORTRAN Program to Calculate  
Mittag-Leffler Series

```

PROGRAM MITLEF
*
REAL LAMDA, DEN1, DEN2, TERM1, TERM2, SUM, TINC,
+ MET(15), TSTART, T
*
INTEGER I, J, TPTS, NMTERM
*
PRINT*, 'ENTER NUM OF TERMS IN M-L SERIES'
READ*, NMTERM
PRINT*, 'ENTER STARTING TIME'
READ*, TSTART
PRINT*, 'ENTER TIME INCREMENT'
READ*, TINC
PRINT*, 'ENTER NUM OF TIME POINTS'
READ*, TPTS
PRINT*, 'ENTER EIGENVALUE'
READ*, LAMDA
*
* **
* ** The following loops read the eigenvalue and calculate
* ** the Mittag-Leffler series. The loops are designed to
* ** calculate the first two terms (TERM1 and TERM2) in
* ** the Mittag-Leffler series and then add further terms
* ** by multiplying the previous terms by appropriate
* ** variables. This method determines the Mittag-Leffler
* ** series in groups of two. The results for each time
* ** increment are printed out at the end of the program
* ** in the MET array.
* **
*
PRINT 20, LAMDA
20 FORMAT (1X, 'LAMDA = ', F8.2)
DO 40 I = 1, TPTS
*
T = TSTART + (I-1)*TINC
T = SQRT(T)
DEN1 = SQRT(4.0*ATAN(1.0)) * 0.5
DEN2 = 1.0
TERM1 = (-LAMDA*T)/DEN1
TERM2 = ((LAMDA*T)**2)/DEN2
SUM = 1.0 + TERM1 + TERM2

```

```

DO 30 J = 1, NMTERM
*
    TERM1 = TERM1 * ((LAMDA*T)**2)/(J + 0.5)
    TERM2 = TERM2 * ((LAMDA*T)**2)/(J*2)
    SUM = SUM + TERM1 + TERM2
    PRINT 28, SUM
28    FORMAT (1X, 'SUM = ', F12.2)
*
30    CONTINUE
*
    MET(I) = SUM
    PRINT 35, I, SUM
35    FORMAT(1X, 'TIME = ', I3, 2X, 'POS = ', F12.2)
*
40    CONTINUE
*
    PRINT 55, ( MET(I), I = 1, TPTS)
55    FORMAT (TPTS(1X, F7.4))
*
    END

```

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Vita

First Lieutenant Joseph B. McCullough III [REDACTED]

[REDACTED]  
[REDACTED] in 1981 and attended the United States Air Force Academy, from which he received the degree of Bachelor of Science in Civil Engineering in May 1985. Upon graduation, he received a commission in the USAF and was assigned to the 47th Civil Engineering Squadron, Laughlin AFB, Texas, where he served as the Environmental Coordinator until entering the School of Engineering, Air Force Institute of Technology, in June 1987.

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Block 19

Abstract

The purpose of this study was to demonstrate the use of fractional derivatives to capture the frequency and temperature dependency of viscoelastic material behavior. To model the frequency dependency of viscoelastic material, fractional derivatives were included in the complex modulus. Solution techniques were performed in the Laplace domain to allow for easy manipulation of the fractional derivative terms. To incorporate the temperature dependency of viscoelastic material in the complex modulus model, the method of reduced variables was employed with the use of the WLF equation. With the frequency and temperature dependency built into the complex modulus, a finite element formulation was devised that incorporated elastic and viscoelastic response of a truss structure. The formulation was limited to the use of the complex modulus in the transition region, the region where the damping ability of viscoelastic material is maximized. Quasi-static motion was also assumed, which limited the response to low frequencies.

The solution technique was demonstrated on a nine degree of freedom truss composed of aluminum and neoprene rubber rods subject to a temperature variation. The results of the example problem show that temperature and frequency dependency of viscoelastic material can be incorporated with the use of fractional derivatives. The simple solution format and model robustness, along with the theoretical basis, provides encouragement for additional work on the complex modulus model.

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