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Reduction Formulae for Kampé de Fériet Functions $F^{p:2;1}_{q:1;0}$

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Several reduction formulae for Kampé de Fériet functions $F^{p:2;1}_{q:1;0}$ are given in terms of a finite number of generalized hypergeometric functions.

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REDUCTION FORMULAE FOR KAMPÉ DE FÉRIET FUNCTIONS $F^{p:2;1}_{q:1;0}$

INTRODUCTION

In 1921, Kampé de Fériet defined and studied the functions named after him. Using the notation introduced in 1976 by Srivastava and Panda, the Kampé de Fériet functions in two variables are defined:

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k);\\(\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\ x,y\right]:=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}.$$

The exact region of convergence for these functions is determined by using Horn's theorem for double series. Many properties and applications of Kampé de Fériet functions are found in Refs. 1-3.

There appears, however, to be a paucity of reduction formulae for Kampé de Fériet functions. Srivastava and Karlsson [1] list only 20 nontrivial instances in which Kampé de Fériet functions can be expressed in terms of generalized hypergeometric functions. In particular, for the functions $F^{p:2;1}_{q:1;0}$, they give the three results:

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p: & \lambda,\mu; & \nu-\lambda-\mu\ ;\\ \beta_1,\ldots,\beta_q: & \nu; & \text{———};\end{matrix}\ x,x\right]$$

$$={}_{p+2}F_{q+1}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\nu-\lambda,\nu-\mu;\\ \beta_1,\ldots,\beta_q,\nu;\end{matrix}\ x\right], \tag{1}$$

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix}\alpha_1,\cdots,\alpha_p: & \lambda,\mu; & -1;\\ \beta_1,\ldots,\beta_q: & \lambda-\mu+1; & —;\end{matrix}\ x,x\right]$$

$$={}_{p+3}F_{q+2}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p,\lambda-1,\mu-1,\frac{1}{2}\lambda+\frac{1}{2};\\ \beta_1,\ldots,\beta_q,\lambda-\mu+1,\frac{1}{2}\lambda-\frac{1}{2};\end{matrix}\ x\right], \tag{2}$$

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$$F^{p:1;1}_{q:0;0}\left[\begin{matrix}\alpha_1,\ldots,\alpha_p: & \mu; & \nu; \\ \beta_1,\ldots,\beta_q: & —; & —;\end{matrix}\ x, x\right]$$

$$= {}_{p+1}F_q\,[\alpha_1,\ldots,\alpha_p,\mu+\nu;\beta_1,\ldots,\beta_q;x], \tag{3}$$

the latter being viewed as a special case of $F^{p:2;1}_{q:1;0}$.

In a series of recent papers [4-7], the author has shown that the function $F^{0:2;1}_{2:1;0}$ is intimately connected with representations for incomplete Lipschitz-Hankel integrals of cylindrical functions (see Eqs. (14-16)), where the class of cylindrical functions C includes Bessel functions of the first kind J, modified Bessel functions I, Bessel functions of the second kind or Neumann functions Y (or N), Bessel functions of imaginary argument or MacDonald functions K, and Bessel functions of the third kind that include Hankel functions of the first and second kind, $H^{(1)}$ and $H^{(2)}$.

As a byproduct of these investigations, several reduction formulae for $F^{0:2;1}_{2:1;0}$ not included in Eqs. (1) and (2) were derived. It is the purpose of this report, in addition to collecting these results, to generalize them somewhat and to derive four reduction formulae for $F^{0:2;1}_{2:1;0}$ that were not previously given.

REDUCTION FORMULAE FOR $F^{p:2;1}_{q:1;0}\left[\begin{matrix}\mu_1,\ldots,\mu_p: & \alpha, 1; & 1; \\ \nu_1,\ldots,\nu_q: & \beta; & -;\end{matrix}\ x, x\right]$

The following is derived in [4]:

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}——: & \alpha,\ 1; & 1; \\ \gamma,\delta: & \beta; & -;\end{matrix}\ x, x\right]$$

$$= \frac{1-\beta}{\alpha-\beta+1}\,{}_1F_2[1;\gamma,\delta;x] + \frac{\alpha}{\alpha-\beta+1}\,{}_2F_3[1,\alpha+1;\gamma,\delta,\beta;x]. \tag{4}$$

In particular, the following are special cases of Eq. (4) given essentially in terms of modified Bessel functions [5]:

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix}———: & 2+\nu,\ 1; & 1; \\ 3+\nu,\ 3: & 5/2; & -;\end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{2+\nu}{1+2\nu}\,\frac{48}{x^4}\left\{\cosh x - 2\,\Gamma(2+\nu)\left(\frac{2}{x}\right)^{\nu} I_\nu(x) + 1 + 2\nu\right\}, \tag{5}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———}: 1+\nu,\ 1; & 1; \\ 2+\nu,\ 2: \quad 3/2; & -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{1+\nu}{1+2\nu}\,\frac{4}{x^2}\left\{\cosh x - \left(\frac{2}{x}\right)^{\nu}\Gamma(1+\nu) I_\nu(x)\right\}, \tag{6}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———}: 1+\nu,\ 1; & 1; \\ 1+\nu,\ 2: \quad 3/2; & -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{2}{1+2\nu}\,\frac{1}{x}\left\{\frac{2\nu\cosh x}{x} + \sinh x - \left(\frac{2}{x}\right)^{\nu}\Gamma(1+\nu) I_{\nu-1}(x)\right\}, \tag{7}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———}: 2+\nu,\ 1; & 1; \\ 2+\nu,\ 3: \quad 5/2; & -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{1}{1+2\nu}\,\frac{24}{x^4}\left\{x\sinh x + 2\nu\cosh x - 4\Gamma(2+\nu)\left(\frac{2}{x}\right)^{\nu-1} I_{\nu-1}(x) + 2\nu(1+2\nu)\right\}. \tag{8}$$

Equation (4) is, in fact, included in the reduction formula

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix} \mu_1,\ldots,\mu_p: \alpha,\ 1; & 1; \\ \nu_1,\ldots,\nu_q: \quad \beta; & -; \end{matrix}\ x, x\right]$$

$$= \frac{1-\beta}{1-\beta+\alpha}\ {}_{p+1}F_q\,[\mu_1,\ldots,\mu_p, 1; \nu_1,\ldots,\nu_q; x]$$

$$+ \frac{\alpha}{1-\beta+\alpha}\ {}_{p+2}F_{q+1}\,[\mu_1,\ldots,\mu_p, 1, \alpha+1; \nu_1,\ldots,\nu_q, \beta; x], \tag{9}$$

which we now show.

By definition

$$F^{p:2;1}_{q:1;0}\left[\begin{matrix} \mu_1,\ldots,\mu_p: \alpha,\ 1; & 1; \\ \nu_1,\ldots,\nu_q: \quad \beta; & -; \end{matrix}\ x, x\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\mu_1)_{m+n}\cdots(\mu_p)_{m+n}}{(\nu_1)_{m+n}\cdots(\nu_q)_{m+n}}\ \frac{(\alpha)_m\,(1)_m\,(1)_n}{(\beta)_m}\ \frac{x^m x^n}{m!\,n!}$$

$$= \sum_{r=0}^{\infty}\left[\sum_{m=0}^{r}\frac{(\alpha)_m}{(\beta)_m}\right]\frac{(\mu_1)_r\cdots(\mu_p)_r}{(\nu_1)_r\cdots(\nu_q)_r}\,x^r.$$

Since [8, Eq. (7.1.1), p. 151]

$$\sum_{m=0}^{r} \frac{(\alpha)_m}{(\beta)_m} = \frac{1}{1-\beta+\alpha} \left[1 - \beta + \alpha \frac{(\alpha+1)_r}{(\beta)_r} \right],$$

we arrive at

$$F^{p:2;1}_{q:1;0} \left[\begin{matrix} \mu_1, \ldots, \mu_p : \alpha, 1; & 1; \\ \nu_1, \ldots, \nu_q : \quad \beta; & -; \end{matrix} x, x \right]$$

$$= \frac{1-\beta}{1-\beta+\alpha} \sum_{r=0}^{\infty} \frac{(\mu_1)_r \ldots (\mu_p)_r \, (1)_r}{(\nu_1)_r \ldots (\nu_q)_r} \frac{x^r}{r!}$$

$$+ \frac{\alpha}{1-\beta+\alpha} \sum_{r=0}^{\infty} \frac{(\mu_1)_r \ldots (\mu_p)_r}{(\nu_1)_r \ldots (\nu_q)_r} \frac{(1)_r \, (\alpha+1)_r}{(\beta)_r} \frac{x^r}{r!},$$

and Eq. (9) follows.

For brevity, we define below two functions that occur often in what follows:

$${}_pS_q\,[(a_p); (b_q); z]$$

$$:= \frac{1}{2} \{e^z \, {}_pF_q[(a_p); (b_q); -2z] - e^{-z} \, {}_pF_q\,[(a_p); (b_q); 2z]\},$$

$${}_pC_q\,[(a_p); (b_q); z]$$

$$:= \frac{1}{2} \{e^z \, {}_pF_q\,[(a_p); (b_q); -2z] + e^{-z} \, {}_pF_q\,[(a_p); (b_q); 2z]\}.$$

We observe that

$${}_pS_q\,[(a_p); (b_q); z] = \left\{ 1 - 2 \frac{\prod_{k=1}^{p} a_k}{\prod_{k=1}^{q} b_k} \right\} z + O(z^3) \qquad z \to 0,$$

$${}_pC_q\,[(a_p); (b_q); z] = 1 + O(z^2) \qquad z \to 0.$$

Hence,

$$\lim_{z \to 0} \frac{{}_pS_q\ [(a_p);\ (b_q);\ z]}{z} = 1 - 2\,\frac{\prod_{k=1}^{p} a_k}{\prod_{k=1}^{q} b_k},$$

$${}_pC_q\ [(a_p);\ (b_q);\ 0] = 1.$$

REDUCTION FORMULAE FOR $F^{0:2;1}_{2:1;0}$

In addition, using the functions ${}_2C_2$ and ${}_2S_2$, we have from results in [6]

$$F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} & : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} & ; & 1; \\ \frac{2+\mu+\nu}{2}, \frac{3+\mu+\nu}{2} & : \quad 1+\nu & ; & -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= {}_2C_2\left[\begin{matrix} 1+\mu+\nu,\ 1/2+\nu; \\ 2+\mu+\nu,\ \ 1+2\nu; \end{matrix}\ x\right], \qquad (10)$$

$$F^{0:2;1}_{2:1:0}\left[\begin{matrix} \text{———} & : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} & ; & 1; \\ \frac{3+\mu+\nu}{2}, \frac{4+\mu+\nu}{2} & : \quad 1+\nu & ; & -; \end{matrix}\ \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \frac{2+\mu+\nu}{2x}\ {}_2S_2\left[\begin{matrix} 1+\mu+\nu,\ 1/2+\nu; \\ 2+\mu+\nu,\ \ 1+2\nu; \end{matrix}\ x\right]. \qquad (11)$$

Equations (10) and (11) may also be written [6]

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} ; & 1; \\ \frac{2+\mu+\nu}{2}, \frac{3+\mu+\nu}{2} : \quad 1+\nu ; & -; \end{matrix} \; \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= \cosh x \; {}_3F_4\left[\begin{matrix} \frac{1+\mu+\nu}{2}, \frac{1/2+\nu}{2}, \frac{3/2+\nu}{2} ; \\ \frac{3+\mu+\nu}{2}, 1/2+\nu, 1+\nu, 1/2 ; \end{matrix} \; x^2\right]$$

$$- \frac{1+\mu+\nu}{2+\mu+\nu} x \sinh x \; {}_3F_4\left[\begin{matrix} \frac{2+\mu+\nu}{2}, \frac{3/2+\nu}{2}, \frac{5/2+\nu}{2} ; \\ \frac{4+\mu+\nu}{2}, 3/2+\nu, 1+\nu, 3/2 ; \end{matrix} \; x^2\right], \tag{12}$$

$$F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} ; & 1; \\ \frac{3+\mu+\nu}{2}, \frac{4+\mu+\nu}{2} : \quad 1+\nu ; & -; \end{matrix} \; \frac{x^2}{4}, \frac{x^2}{4}\right]$$

$$= (2+\mu+\nu) \frac{\sinh x}{x} \; {}_3F_4\left[\begin{matrix} \frac{1+\mu+\nu}{2}, \frac{1/2+\nu}{2}, \frac{3/2+\nu}{2} ; \\ \frac{3+\mu+\nu}{2}, 1/2+\nu, 1+\nu, 1/2 ; \end{matrix} \; x^2\right]$$

$$- (1+\mu+\nu) \cosh x \; {}_3F_4\left[\begin{matrix} \frac{2+\mu+\nu}{2}, \frac{3/2+\nu}{2}, \frac{5/2+\nu}{2} ; \\ \frac{4+\mu+\nu}{2}, 3/2+\nu, 1+\nu, 3/2 ; \end{matrix} \; x^2\right] \tag{13}$$

PRELIMINARY RESULTS AND DEFINITIONS

The general incomplete Lipschitz-Hankel integral of cylindrical functions $C_\nu(z)$ may be defined as the following function of two complex variables:

$$Ce_{\mu,\nu}(a, z) := \int_0^z e^{at} t^\mu C_\nu(t)\, dt . \tag{14}$$

Here the symbol e denotes the presence of the exponential function, and μ, ν may be complex. Analogously, we may define integrals that contain the functions $\sin(at)$ and $\cos(at)$ in place of $\exp(at)$:

$$C_{s_{\mu,\nu}}(a,z) := \int_0^z \sin(at)\, t^{\mu} C_{\nu}(t)\, dt, \tag{15}$$

$$C_{c_{\mu,\nu}}(a,z) := \int_0^z \cos(at)\, t^{\mu} C_{\nu}(t)\, dt. \tag{16}$$

To assure convergence of $C_{e_{\mu,\nu}}(a,z)$ and $C_{c_{\mu,\nu}}(a,z)$, it is necessary that $\mathrm{Re}\,(\mu+1) > |\mathrm{Re}\,\nu|$ when $C = K, Y, H^{(1)}, H^{(2)}$; $\mathrm{Re}\,(1+\mu+\nu) > 0$ when $C = I, J$. For convergence of $C_{s_{\mu,\nu}}(a,z)$, replace μ by $\mu+1$ in the latter two inequalities.

Defining

$$\xi := \begin{cases} 1: C = I, K \\ -1: C = H, J, Y \end{cases} \qquad \eta := \begin{cases} 1: C = K \\ -1: C = H, I, J, Y \end{cases}$$

$$Q\,[A_1(\mu,\nu); x, y] := F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+1}{2} ; 1; \\ \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+3}{2} : \quad 1/2 \quad ; -; \end{matrix}\; x, y\right]$$

$$Q\,[A_2(\mu,\nu); x, y] := F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+1}{2}, \dfrac{\mu-\nu+1}{2} ; 1; \\ \dfrac{\mu+\nu+3}{2}, \dfrac{\mu-\nu+3}{2} : \quad 1/2 \quad ; -; \end{matrix}\; x, y\right]$$

$$Q\,[B_1(\mu,\nu); x, y] := F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+2}{2}, \dfrac{\mu-\nu+2}{2} ; 1; \\ \dfrac{\mu+\nu+2}{2}, \dfrac{\mu-\nu+4}{2} : \quad 3/2 \quad ; -; \end{matrix}\; x, y\right]$$

$$Q\,[B_2(\mu,\nu); x, y] := F^{0:2;1}_{2:1;0}\left[\begin{matrix} \text{———} : \dfrac{\mu+\nu+2}{2}, \dfrac{\mu-\nu+2}{2} ; 1; \\ \dfrac{\mu+\nu+4}{2}, \dfrac{\mu-\nu+4}{2} : \quad 3/2 \quad : -; \end{matrix}\; x, y\right],$$

it is shown in [7] that

$$C_{s_{\mu,\nu}}(a,z) = \frac{az^{2+\mu}}{\mu-\nu+2}\left\{C_{\nu}(z)Q\left[B_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{\eta z}{\mu+\nu+2} C_{\nu-1}(z)Q\left[B_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}, \tag{17}$$

$$C_{c_{\mu,\nu}}(a,z) = \frac{z^{1+\mu}}{\mu-\nu+1}\left\{C_{\nu}(z)Q\left[A_1; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right] + \frac{\eta z}{\mu+\nu+1} C_{\nu-1}(z)Q\left[A_2; \frac{-a^2z^2}{4}, \frac{\xi z^2}{4}\right]\right\}. \tag{18}$$

ADDITIONAL RESULTS FOR $F^{0:2;1}_{2:1;0}$

In Eq (17), set firstly $\xi = 1$, $\eta = -1$, $a = i$ and secondly $\xi = 1$, $\eta = 1$, $a = i$. This gives the system of linear equations in two unknowns $Q[B_1; z^2/4, z^2/4]$ and $Q[B_2; z^2/4, z^2/4]$:

$$I_\nu(z)\, Q\left[B_1; \frac{z^2}{4}, \frac{z^2}{4}\right] - \frac{z}{\mu+\nu+2}\, I_{\nu-1}(z)\, Q\left[B_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+2}{z^{2+\mu}} \int_0^z \sinh t\; t^\mu I_\nu(t)\, dt, \tag{19}$$

$$K_\nu(z)\, Q\left[B_1; \frac{z^2}{4}, \frac{z^2}{4}\right] + \frac{z}{\mu+\nu+2}\, K_{\nu-1}(z)\, Q\left[B_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+2}{z^{2+\mu}} \int_0^z \sinh t\; t^\mu K_\nu(t)\, dt. \tag{20}$$

Now making the previous substitutions in Eq. (18) gives the system of linear equations in unknowns $Q[A_1; z^2/4, z^2/4]$ and $Q[A_2; z^2/4, z^2/4]$:

$$I_\nu(z)\, Q\left[A_1; \frac{z^2}{4}, \frac{z^2}{4}\right] - \frac{z}{\mu+\nu+1}\, I_{\nu-1}(z)\, Q\left[A_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+1}{z^{1+\mu}} \int_0^z \cosh t\; t^\mu I_\nu(t)\, dt, \tag{21}$$

$$K_\nu(z)\, Q\left[A_1; \frac{z^2}{4}, \frac{z^2}{4}\right] + \frac{z}{\mu+\nu+1}\, K_{\nu-1}(z)\, Q\left[A_2; \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{\mu-\nu+1}{z^{1+\mu}} \int_0^z \cosh t\; t^\mu K_\nu(t)\, dt. \tag{22}$$

Solving the systems Eqs. (19) and (20), and Eqs. (21) and (22), respectively, using Cramer's rule and noting the Wronskian

$$I_\nu(z)\, K_{\nu-1}(z) + I_{\nu-1}(z)\, K_\nu(z) = 1/z, \tag{23}$$

we obtain

$$Q\left[B_1; \frac{z^2}{4}, \frac{z^2}{4}\right] = \frac{\mu-\nu+2}{z^{1+\mu}} \begin{vmatrix} \int_0^z \sinh t\; t^\mu I_\nu(t)dt & -I_{\nu-1}(z) \\ \int_0^z \sinh t\; t^\mu K_\nu(t)dt & K_{\nu-1}(z) \end{vmatrix}, \tag{24}$$

$$Q\left[B_2;\ \frac{z^2}{4},\ \frac{z^2}{4}\right] = \frac{(\mu-\nu+2)(\mu+\nu+2)}{z^{2+\mu}}\begin{vmatrix} I_\nu(z) & \int_0^z \sinh t\ t^\mu I_\nu(t)dt \\ K_\nu(z) & \int_0^z \sinh t\ t^\mu K_\nu(t)dt \end{vmatrix}, \tag{25}$$

$$Q\left[A_1;\ \frac{z^2}{4},\ \frac{z^2}{4}\right] = \frac{(\mu-\nu+1)}{z^{\mu}}\begin{vmatrix} \int_0^z \cosh t\ t^\mu I_\nu(t)dt & -I_{\nu-1}(z) \\ \int_0^z \cosh t\ t^\mu K_\nu(t)dt & K_{\nu-1}(z) \end{vmatrix}, \tag{26}$$

$$Q\left[A_2;\ \frac{z^2}{4},\ \frac{z^2}{4}\right] = \frac{(\mu-\nu+1)(\mu+\nu+1)}{z^{1+\mu}}\begin{vmatrix} I_\nu(z) & \int_0^z \cosh t\ t^\mu I_\nu(t)dt \\ K_\nu(z) & \int_0^z \cosh t\ t^\mu K_\nu(t)dt \end{vmatrix}. \tag{27}$$

Equations (24) and (25) are valid for $\mathrm{Re}(\mu \pm \nu + 2) > 0$; Eqs. (26) and (27) are valid for $\mathrm{Re}(\mu \pm \nu + 1) > 0$. Further, Eqs. (24-27) provide integral representations for the function Q appearing in each equation.

Using [9, p. 117, Eq. (2)], we evaluate the four integrals in Eqs. (24-27) in terms of the functions ${}_2\boldsymbol{C}_2$ and ${}_2\boldsymbol{S}_2$:

$$\int_0^z \sinh t\ t^\mu\ C_\nu(t)dt = \frac{z^{\mu+1}}{\mu+\nu+1}\left\{\sinh z\ C_\nu(z)\right.$$

$$\left. - \frac{zC_\nu(z)}{\mu+\nu+2}\ {}_2\boldsymbol{C}_2\left[\begin{matrix} 1, & \mu+3/2; \\ \mu-\nu+1, & \mu+\nu+3; \end{matrix} z\right] + \frac{\delta z C_{\nu+1}(z)}{\mu-\nu+1}\ {}_2\boldsymbol{S}_2\left[\begin{matrix} 1, & \mu+3/2; \\ \mu-\nu+2, & \mu+\nu+2; \end{matrix} z\right]\right\},$$

$$\int_0^z \cosh t\ t^\mu\ C_\nu(t)dt = \frac{z^{1+\mu}}{\mu+\nu+1}\left\{\cosh z\ C_\nu(z)\right.$$

$$\left. - \frac{zC_\nu(z)}{\mu+\nu+2}\ {}_2\boldsymbol{S}_2\left[\begin{matrix} 1, & \mu+3/2; \\ \mu-\nu+1, & \mu+\nu+3; \end{matrix} z\right] + \frac{\delta z C_{\nu+1}(z)}{\mu-\nu+1}\ {}_2\boldsymbol{C}_2\left[\begin{matrix} 1, & \mu+3/2; \\ \mu-\nu+2, & \mu+\nu+2; \end{matrix} z\right]\right\},$$

where $\delta = 1$ if $C = K$, $\delta = -1$ if $C = I$.

Using these results in Eqs. (24-27), the F-symbol for Q and taking note of Eq. (23) together with the easily proved identity

$$K_{\nu+1}(z)\, I_{\nu-1}(z) - I_{\nu+1}(z) K_{\nu-1}(z) = 2\nu/z^2,$$

we deduce the four reduction formulae:

$$F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2}; \ 1; \\ \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+4}{2} : \quad 3/2 \ ; \ -; \end{matrix} \ \frac{z^2}{4}, \frac{z^2}{4}\right] \tag{28}$$

$$= \frac{\mu-\nu+2}{\mu+\nu+1}\left\{\frac{\sinh z}{z} - \frac{1}{\mu+\nu+2}\,{}_2C_2\left[\begin{matrix}1, \ \mu+3/2; \\ \mu-\nu+1, \mu+\nu+3;\end{matrix} z\right] + \frac{2\nu}{\mu-\nu+1}\frac{1}{z}\,{}_2S_2\left[\begin{matrix}1, \ \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{matrix} z\right]\right\},$$

$$F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \frac{\mu+\nu+2}{2}, \frac{\mu-\nu+2}{2}; \ 1; \\ \frac{\mu+\nu+4}{2}, \frac{\mu-\nu+4}{2} : \quad 3/2 \ ; \ -; \end{matrix} \ \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= \frac{(\mu+\nu+2)(\mu-\nu+2)}{(\mu+\nu+1)(\mu-\nu+1)}\,\frac{1}{z}\,{}_2S_2\left[\begin{matrix}1, \ \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{matrix} z\right], \tag{29}$$

$$F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}; \ 1; \\ \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+3}{2} : \quad 1/2 \ ; \ -; \end{matrix} \ \frac{z^2}{4}, \frac{z^2}{4}\right] \tag{30}$$

$$= \frac{\mu-\nu+1}{\mu+\nu+1}\left\{\cosh z - \frac{z}{\mu+\nu+2}\,{}_2S_2\left[\begin{matrix}1, \ \mu+3/2; \\ \mu-\nu+1, \mu+\nu+3;\end{matrix} z\right] + \frac{2\nu}{\mu-\nu+1}\,{}_2C_2\left[\begin{matrix}1, \ \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{matrix} z\right]\right\},$$

$$F^{0:2:1}_{2:1:0}\left[\begin{matrix} \text{———} : \frac{\mu+\nu+1}{2}, \frac{\mu-\nu+1}{2}; \ 1; \\ \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2} : \quad 1/2 \ ; \ -; \end{matrix} \ \frac{z^2}{4}, \frac{z^2}{4}\right]$$

$$= {}_2C_2\left[\begin{matrix}1, \ \mu+3/2; \\ \mu-\nu+2, \mu+\nu+2;\end{matrix} z\right]. \tag{31}$$

If we set $\mu = \nu$, then it is easily shown that Eq. (28) reduces to Eq. (7), and Eq. (29) reduces to Eq. (6). Equations (30) and (31) reduce to $F^{0:1;1}_{2:0;0}$ functions (see Eqs. (22) and (23) of Ref. 5), whose reduction is given by Eq. (3).

We remark that Eqs. (28-31) are valid for $|z| < \infty$ and for all μ, ν such that each function exists. Hence the restrictions on the existence of the integrals in Eqs. (24-27) do not apply here.

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