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A Dilogarithmic Extension of Liouville's Theorem on Integration in Finite Terms

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Abstract

The result obtained generalizes Liouville's Theorem by allowing, in addition to the elementary functions, dilogarithms to appear in the integral of an elementary function. The basic conclusion is that an associated function to the dilogarithm, if dilogarithms appear in the integral, appears linearly, with logarithms appearing in a non-linear way.

Let k be a differential field in one variable of characteristic zero. The derivation operator of k into itself will be denoted by ', that is, the derivative of x is x'. Elements of k of derivative equal to zero are called constants; they form a subfield of k called the subfield of constants and are denoted by C(k).

Let K be a differential extension field of k such that K = k(t) for some t ε K. t is called elementary over k if the field of constants C(K) is equal to C(k) and t is such that:

(1") $t' = a'/a$, for some $a \varepsilon k^*$. In this case, we write t = log a and call t logarithmic over k.

(2") $t' = a't$ for some $a \varepsilon k$. In this case, we write t = exp a and call t exponential over k.

For a differential extension field K of k such that $K = k(\psi, t)$ for two elements t and $\psi \varepsilon K$, t is called dilogarithmic over k. If C(K) = C(k), and t, ψ are such that:

$$\exists\, a \,\varepsilon\, k\,\{0,1\}, \quad t' = -\frac{a'}{a}\,\psi, \quad \text{where } \psi' = \frac{(1-a)'}{(1-a)},$$

we write $t = \ell_2(a)$ and call t dilogarithmic over k. Also, we assume that C(K) = C(k).

Remark: We observe that the above definition does not imply that $t' \varepsilon k$ since ψ is not assumed to be an element of k. A differential extension field of a differential field is said to be dilogarithmic-elementary if there exists a finite tower of intermediate fields starting with the given small field and ending with the given extention field, such that each field in the tower after the first is obtained from its predecessor by the adjunction of a single element that is elementary over the preceding field or by the adjunction of two elements t and t', where $t = \ell_2(a)$, and a is an element of the preceding field.

If K is a differential extension of k such that $K = k(t)$ for some $t \in K$, and $t' = a \in k$, we call t primitive over k and write $t = \int a$.

And, finally, if k is a differential field of characteristic zero, K is a differential field extension of k such that $K = k(t,\phi,\psi)$, and:

$$t' = -\frac{1}{2}\frac{a'}{a}\psi + \frac{1}{2}\frac{(1-a)'}{(1-a)}\phi, \quad a \in k\ \{0,1\}$$

$$\text{where:} \quad \psi' = \frac{(1-a)'}{(1-a)} \quad \text{and} \quad \phi' = \frac{a'}{a}$$

we write in this case:

$$t = \ell_2(a) + \frac{1}{2}\log a \log(1-a) = D(a)$$

and we assume also that $C(K) = C(k)$.

We recall now a version of Ostrowski's theorem which is useful in all the upcoming proofs.

<u>Theorem</u>: (Ostrowski [__]. See also Kolchin [__].) Let k be a differential field of characteristic zero and let $K = k(\log v_1,\ldots,\log v_n)$, where $v_i \in k$ $(1 \le i \le n)$ and $C(K) = C(k)$. Assume that $\log v_1, \ldots, \log v_r$ $(0 \le r \le n)$ are algebraically independent over k and that K and $k(\log v_1,\ldots,\log v_r)$ have the same transcendence degree v over k. Then, there exist $c_{ij} \in C(k)$ $(1 \le i \le r,\ r < j \le n)$, $s_j \in k$ $(r < j \le n)$ such that:

$$\log v_j = \sum_{i=1}^{r} c_{ij} \log v_i + s_j, \quad \text{for } j \in \{r+1,\ldots,n\}$$

and if $r = 0$, $\log v_s \in k$ for all $s \in \{1,\ldots,n\}$.

<u>Definition</u>: Given two differential fields k K, we say that K is a Liouville extension of k if there exist $t_1,\ldots,t_n \in K$ such that $K = k(t_1,\ldots,tn)$ and each t_i is either algebraic, elementary, or primitive over $k(t_1,\ldots,t_{i-1})$.

We recall now a useful lemma due to Singer [__].

Lemma: Let $k \subset K$ be differential fields of characteristic zero with the same field of constants C supposed to be algebraically closed. Assume that k is a Liouville extension of C and that K is algebraic over k. Suppose that $c_1, \ldots, c_n \in C$ are linearly independent over Q, that $u_1, \ldots u_n \in K^*$, $v \in K$, and that we have:

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + V' \in k$$

Then, $v \in k$ and there is a non-zero integer N such that $u_i^N \in k$, $i = 1, \ldots, n$.

Definition: Let k be a differential field of characteristic 0. We call an expression S a simple elementary-dilogarithmic expression over k if:

$$S = g + \sum_{i \in I} c_i \log w_i + \sum_{j} [s_j \log(1 - h_j) + t_j \log h_j + d_j D(h_j)]$$

where I and J are finite sets, $g, w_i, s_j, t_j, h_j \in k$, and c_i, d_j are constants.

Lemma 1: Let k be a differential field of characteristic 0, which is a Liouville extension of its subfield of constants assumed algebraically closed. Suppose that we have an expression of the form:

$$\int f = g + \sum_{i \in I} c_i \log w_i + \sum_{j \in J} [s_j \log(1 - h_j) + t_j \log h_j + d_j D(h_j)] \tag{1.1}$$

where I and J are finite sets, $f \in k$, s_j, t_j, g, and w_i are algebraic over k, $h_j \in k$, and c_i, d_j are constants.

Then, we can write $\int f = S$, where S is a simple elementary-dilogarithmic expression over k. (So, we get g, w_i, s_j, t_j in k instead of being algebraics.)

<u>Proof</u>: Let K be a finite normal algebraic extension field of k that contains g, w_i ($i \in I$), s_j, t_j ($j \in J$) (the smallest normal extension $k(g, w_1, \ldots, w_i, \ldots, s_1, \ldots, s_j, \ldots, t_1, \ldots, t_j)$). Consider the vector space E over k spanned by the vectors 1, $\log h_1$, ... , $\log h_j$, ... , $\log(1 - h_1)$, ... , $\log(1 - h_j)$, ... Then, we choose among these vectors a k-basis $(1, e_1, \ldots, e_N)$ for E. By Ostrowski's theorem, we can write:

$$\left.\begin{aligned} \log h_j &= \sum_{m=1}^{N} a_{jm} e_m + p_j, \qquad a_{jm} \in C,\ p_j \in k \\ \log(1 - h_j) &= \sum_{m=1}^{N} b_{jm} e_m + q_j, \qquad b_{jm} \in C,\ q_j \in k \end{aligned}\right] \qquad (*)$$

We claim that $1, e_1, \ldots, e_N$ are still linearly independent over K. Otherwise, and by Ostrowski's theorem, there exist constants α_m ($2 \le m \le N$) and $Q_0 \in K$ such that:

$$e_1 = \sum_{m=2}^{N} \alpha_m e_m + Q_0 \implies e_1' = \sum_{m=2}^{N} \alpha_m e_m' + Q_0' \qquad (1.2)$$

By assumption, $e_m = \log H_m$ ($1 \le m \le N$), where $H_m \in \{(1 - h_1), \ldots, h_1, \ldots\}$.

Let $\gamma_0 = 1, \gamma_1, \ldots, \gamma_r$ be a vector space basis for the Q-span of $1, \alpha_2, \ldots, \alpha_n$, and write:

$$\alpha_m = \sum_{i=0}^{r} n_{mi} \gamma_i$$

with each $n_{mi} \in Q$. Replacing each γ_i by $\gamma_i / LCD(n_{mi})$ if necessary, we can assume $n_{mi} \in Z$ (where LCD means least common denominator).

This implies that $1 = n_1 \gamma_0$, so we can write (1.2) as:

$$\frac{(H_1^{n_1})'}{H_1^{n_1}} \gamma_0 = \sum_{i=0}^{r} \gamma_i \frac{(H_2^{n2i} H_3^{n3i} \ldots H_N^{nNi})'}{H_2^{n2i} H_3^{n3i} \ldots H_N^{nNi}} + Q_0'$$

which can also be written as:

$$\gamma_0 \frac{(H_1^{-n_1} H_2^{n_{20}} \cdots H_N^{n_{NO}})'}{H_1^{-n_1} H_2^{n_{20}} \cdots H_N^{n_{NO}}} + \sum_{i=1}^{r} \gamma_i \frac{(H_2^{n_{2i}} H_3^{n_{3i}} \cdots H_N^{n_{Ni}})'}{H_2^{n_{2i}} H_3^{n_{3i}} \cdots H_N^{n_{Ni}}} + Q_0' = 0 \qquad (1.3)$$

Using Singer's lemma, we deduce that $Q_0 \in k$. Investigating (1.1) again, we get:

$$e_1 = \sum_{m=2}^{N} \alpha_m e_m + Q_0$$

with $Q_0 \in k$, $\alpha_m \in k \quad C = C$. This is a contradiction, since the e_m $(1 \leq m \leq N)$ and 1 were assumed to be linearly independent over k.

So, $1, e_1, \ldots, e_N$ are linearly dependent over K.

Now, we write (1.1) in terms of the relations (*):

$$\int f = g_0 + \sum_{m=1}^{N} r_m e_m + \sum_{j \in J} d_j D(h_j) + \sum_{i \in I} c_i \log w_i \qquad (1.4)$$

(where $g_0 \in K_1$, $r_m \in K$).

Taking the derivative of the previous relation, we obtain:

$$f = g_0' + \sum_{i \in I} c_i \frac{w_i'}{w_i} + \sum_{m=1}^{N} r_m e_m' + \sum_{m=1}^{N} r_m' e_m - \frac{1}{2} \sum_{j \in J} d_j \frac{h_j'}{h_j} \log(1 - h_j)$$

$$+ \frac{1}{2} \sum_{j \in J} d_j \frac{(1 - h_j)'}{(1 - h_j)} \log h_j \qquad (1.5)$$

Using again the relation (*) for $\log(1 - h_j)$ and $\log h_j$, and assembling coefficients of (1.5) according to the K-basis $(1, e_1, \ldots, e_N)$, we obtain:

$$f = g_0' + \sum_{i \in I} c_i \frac{w_i'}{w_i} + \sum_{m=1}^{N} r_m e_m' - \frac{1}{2} \sum_{j \in J} d_j \frac{h_j'}{h_j} q_j$$

$$+ \frac{1}{2} \sum_{j \in J} d_j \frac{(1 - h_j)'}{(1 - h_j)} p_j \qquad (1.6)$$

(the above is the coefficient of the vector 1), and:

$$r_m' - \sum_{j \varepsilon J} \frac{1}{2} d_j b_{jm} \frac{h_j'}{h_j} + \sum_{j \varepsilon J} \frac{1}{2} d_j a_{jm} \frac{(1 - h_j)'}{(1 - h_j)} = 0, \quad 1 \le m \le N \quad (1.7)$$

(the above is the coefficient of the vector e_m).

From (1.7), we deduce that $r_m \varepsilon k$ (using Singer's lemma and exactly the same argument used in the above proof).

Assume that $M = [K:k]$. For any $\sigma \varepsilon Awt(K/k)$, we have (using 1.6):

$$f = \sigma(f) = \sigma(g_o') + \sum_{r \varepsilon I} c_i \frac{\sigma(w_i)'}{\sigma(w_i)} + \sum_{m=1}^{N} r_m e_m' - \frac{1}{2} \sum_{j \varepsilon J} d_j \frac{h_j'}{h_j} q_j + \frac{1}{2} \sum_{j \varepsilon J} d_j \frac{(1 - h_j)'}{(1 - h_j)} p_j$$

Taking the sum over all the σ's in Awt(K/k), we obtain:

$$Mf = \sum_{\sigma} \sigma(g_o) + \sum_{i \varepsilon I} c_i \sum_{\sigma} \frac{\sigma(w_i)'}{\sigma(w_i)} + M \left[\sum_{m=1}^{N} r_m e_m' - \frac{1}{2} \sum_{j \varepsilon J} d_j q_j \frac{h_j'}{h_j} + \frac{1}{2} \sum_{j \varepsilon J} d_j p_j \frac{(1 - h_j)'}{(1 - h_j)} \right]$$

which implies that:

$$- f + \left(\frac{Tr(g_o)}{M} \right) + \sum_{i \varepsilon I} \frac{c_i}{M} \frac{N(w_i)'}{N(w_i)} + \left[\sum_{m=1}^{N} r_m e_m' - \frac{1}{2} \sum_{j \varepsilon J} d_j q_j \frac{h_j'}{h_j} + \frac{1}{2} \sum_{j \varepsilon J} d_j p_j \frac{(1 - h_j)'}{(1 - h_j)} \right] = 0 \quad (1.8)$$

where Tr( ) and N( ) are the trace and norm maps, respectively, from K to k.

Now, multiplying (1.8) by 1 and each (1.7) by c_m, adding them using again relations (*), and integrating, we get:

$$\int f = \frac{Tr(g_o)}{M} + \sum_{i \varepsilon I} \frac{c_i}{M} \log N(w_i) + \sum_{m=1}^{n} r_m e_m + \sum_{j \varepsilon J} d_j D(h_j) \quad (1.9)$$

Note that $Tr(g_0)/M \in k$ and $N(w_i) \in k$, and also $e_m = \log H_m$, where $H_m \in \{h_1, h_2, \ldots, 1-h_1, 1-h_2, \ldots\}$. So, the right-hand side of (1.9) is a simple elementary-dilogarithmic expression over k, which is what we wanted to prove.

Definition: Let k be a differential field of characteristic zero. K is a finite algebraic extension of k_1, and $\log h_1, \ldots, \log h_m$ are logarithmics over k (that is, $h_1, \ldots, h_m \in k$). Assume that $C(k) = C(K(\log h_1, \ldots, \log h_m))$. We call L a linear logarithmic expression over $K(\log h_1, \ldots, \log h_m)$ if:

$$L = \sum_{i=1}^{m} c_i \log h_i + r$$

where the c_i are constants and $r \in K$. L is said to be dependent on $\log h_j$ $(1 \leq j \leq m)$ if $c_j \neq 0$.

Proposition 1: Let k be a differential field of characteristic zero which is a Liouville extension of its field of constants C assumed algebraically closed. Suppose that $f \in k$; $h_1, \ldots, h_n \in k$; K a finite algebraic extension of k; $a_1, \ldots, a_m \in C$; $d_1, \ldots, d_n \in C$; and $L_1, \ldots, L_m$ are linear logarithmic expressions over $K\left(\log(1-h_1), \ldots, \log(1-h_n)\right)$. Then, if:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i=1}^{m} a_i \log L_i \in K\left(\log(1-h_1), \ldots, \log(1-h_n)\right) \tag{1.10}$$

$\int f$ is a simple elementary-dilogarithmic expression over k.

Proof: Let r = trans-degree $K\left(\log(1-h_1), \ldots, \log(1-h_n)\right)$ over k. If $r = 0$, then, by Ostrowski's theorem, $\log(1-h_j) \in k$ $(1 \leq j \leq n)$ ==> $K\left(\log(1-h_1), \ldots, \log(1-h_n)\right) = K$, and $L_i \in K$ $(1 \leq i \leq m)$. So, (1.10) implies that:

$$\int f = \sum_{j=1}^{n} d_j \ell_2(h_j) + \sum_{i=1}^{m} a_i \log L_i + g, \qquad g \in K,\ L_i \in K$$

$$\Longrightarrow \int f = \sum_{j=1}^{n} d_j D(h_j) + g + \sum_{i=1}^{m} a_i \log L_i - \frac{1}{2} \sum d_j \log(1 - h_j) \log h_j$$

So, if:

$s_j = -\frac{1}{2} d_j \log(1 - h_j) \in k$, we get:

$$\int f = \sum_{j=1}^{k} d_j D(h_j) + g + \sum_{i=1}^{m} a_i \log L_i + \sum_{j=1}^{n} s_j \log h_j,$$

$$s_j \in k,\ g \in K,\ L_i \in k$$

So, by lemma 1, $\int f$ is a simple elementary-dilogarithmic expression over k and the proposition is proved for $r = 0$. Let r be greater than 0 and assume without loss of generality that $\log(1 - h_1), \ldots, \log(1 - h_r)$ are algebracially independent over K so that by Ostrowski's theorem again we find constants c_{jp} such that:

$$\log(1 - h_j) = \sum_{p=1}^{r} c_{jp} \log(1 - h_p) + R_j,$$

where: $R_j \in k,\ r < j \leq n$ (**)

So, $K\big(\log(1 - h_1), \log(1 - h_2), \ldots, \log(1 - h_n)\big) = K\big(\log(1 - h_1), \log(1 - h_2), \ldots, \log(1 - h_r)\big)$.

Let $K_{i_0} = K\big(\log(1 - h_1), \ldots, \log(1 - h_{i_{0-1}}), \log(1 - h_{i_{0+1}}), \ldots, \log(1 - h_r)\big)$ $(1 \leq i_0 \leq r)$. Clearly, $t_{i_0} = \log(1 - h_{i_0})$ is transcendental over K_{i_0} since we have assumed that $\log(1 - h_j)$ $(1 \leq j \leq r)$ are algebraically independent over K.

For each $i_0 \in \{1,2,\ldots,r\}$, let I_0 be the subset of $\{1,\ldots,m\}$ such that, for all $i \in I_0$, Li is dependent on $t_{i_0} = \log(1 - h_{i_0})$. Then, (1.10) implies that:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i=1}^{m} a_i \log L_i \in K_{i_0}(t_{i_0}) \tag{1.11}$$

We want to prove that:

$$\left[\sum_{i \in I_0} a_i \log L_i \right]' = 0$$

and that:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \notin I_0} a_i \log L_i \in K_{i_0}[t_{i_0}] \tag{1.12}$$

Once (1.12) is proved for each index $i_0 \in \{1,\dots,r\}$, we deduce that:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{00}} a_i \log L_i \in \bigcap_{i_0 \in \{1,\dots,r\}} K_i[t_{i_0}]$$

$$= K[t_1, t_2, \dots, t_r]$$

where I_{00} is such that, for all $i \in I_{00}$, L_i is not dependent on any $t_j = \log(1 - h_j)$, for all $j \in \{1,\dots,r\}$. So, $L_i \in K$ for all $i \in I_{00}$, and:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{00}} a_i \log L_i = P(t_1,\dots,t_r)$$

where P is a polynomial.

So, let $K_0 = K_{i_0} = K(t_1,\dots,t_{i_0-1},t_{i_0+1},\dots,t_r)$ and $t = t_{i_0}$. Then, if L_i depends on t, $L_i = b_i t + r_i$, where $r_i \in K_0$, and b_i is a constant, $b_i \neq 0$. By assumption, we had:

$$\int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i=1}^{m} a_i \log L_i = g(t) \in K_0(t) \tag{1.13}$$

If K° is a finite normal algebraic extension of K_0 where g(t) splits into linear factors, we write:

$$g(t) = g_0(t) + \sum_{\alpha,\beta} \frac{r_{\alpha,\beta}}{(t - T_\alpha)^\beta}, \quad r_{\alpha,\beta} \in K^\circ,\ T_\alpha \in K^\circ,\ \beta \in N^*$$

α and β range over a finite set of integers, and $g_0(t) \in K^0[t]$.

(1.13) yields:

$$f - \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} \log(1 - h_j) - \sum_{i=1}^{m} a_i \frac{L_i'}{L_i} - g_0'(t) - \sum_{\alpha,\beta} \frac{r_{\alpha,\beta}'}{(t - T_\alpha)^\beta}$$

$$+ \sum \frac{\beta r_{\alpha,\beta}(t' - T_\alpha')}{(t - T_\alpha)^{\beta+1}} = 0 \qquad (1.14)$$

The key idea in the on-going proof is that, when we use the relations (**) on page (8), the expression:

$$f - \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} \log(1 - h_j)$$

is a linear polynomial in t over K_0. Also, $g_0'(t)$ is a polynomial in t since $t' = (1 - h_{i_0})'/(1 - h_{i_0}) \in k$. So:

$$\sum_{\substack{L_i \text{ depends} \\ \text{on } t}} a_i \frac{L_i'}{L_i} - \sum_{\alpha,\beta} \frac{r_{\alpha,\beta}'}{(t - T_\alpha)^\beta} + \sum \frac{\beta r_{\alpha,\beta}(t' - T_\alpha')}{(t - T_\alpha)^{\beta+1}}$$

must cancel.

Let $I_t = \{i \text{ such that } L_i = b_i t + r_i,\ b_i \neq 0\}$ and $I_t^0 = \{1,\dots,m\} - I_t$.

(1.14) then becomes:

$$f - \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} \log(1 - h_j) - \sum_{i \in I_t^0} \frac{r_i'}{r_i} - \sum_{i \in I_t} a_i \frac{(b_i t' + r_i')}{(b_i t + r_i)} - g_0'(t)$$

$$- \sum_{\alpha,\beta} \frac{r_{\alpha,\beta}'}{(t - T_\alpha)^\beta} + \sum \frac{\beta r_{\alpha,\beta}(t' - T_\alpha')}{(t - T_\alpha)^{\beta+1}} = 0, \quad \text{where: } r_i \in K_0 \qquad (1.15)$$

First, $t' - T_\alpha' \neq 0$, otherwise we would have $t' = T_\alpha'$; and for each $\sigma \in \mathrm{Awt}(K^0/K_0)$ we have $t' = \sigma(T_\alpha)' \Longrightarrow [K^0:K_0]t' = \mathrm{Tr}(T_\alpha)' \Longrightarrow$ $t = 1/[K^0:K_0]\,\mathrm{Tr}(T_\alpha) + C$, where C is a constant and Tr is the trace map from K^0 to K_0. But this gives a contradiction since t was supposed to be transcendental over K_0.

So, if we look at the partial fraction decomposition we have in (1.15), we deduce that $r_{\alpha,\beta} = 0$ for all α, β, and we get:

$$f - \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} \log(1 - h_j) - \sum_{i \in I_t^0} a_i \frac{r_i'}{r_i}$$

$$- \sum_{i \in I_t} a_i \frac{(b_i t' + r_i')}{(b_i t + r_i)} - g_0'(t) = 0$$

which also implies that:

$$\sum_{i \in I_t} a_i \frac{(b_i t' + r_i')}{(b_i t + r_i)} = 0$$

(by looking at partial fraction decomposition). Also, $g_0 \in K^0[t] \quad K_0(t)$ $\Longrightarrow g_0 \in K_0[t]$, and:

$$\sum_{i \in I_t} a_i \log L_i = \text{constant}$$

By induction on $t = t_{i_0} \in \{\log(1 - h_1), \ldots, \log(1 - h_r)\}$, we deduce that:

$$g(t_1, \ldots, t_r) = \int f - \sum_{j=1}^{n} d_j \ell_2(h_j) - \sum_{i \in I_{00}} a_i \log L_i \in K[t_1, \ldots, t_r]$$

$$\text{and } L_i \in K \qquad (1.16)$$

We claim that g is a polynomial of degree 2 with constant coefficients, for all the terms in $t_1, \ldots, t_r$ of degree 2. In fact, let $A_{\alpha_1 \alpha_2 \ldots \alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r}$ be one monomial in the leading homogenous term of g, with $A_{\alpha_1 \alpha_2 \ldots \alpha_r} \neq 0$. Then:

$$(A_{\alpha_1 \ldots \alpha_r} t^{\alpha_1} \ldots t^{\alpha_r})' = A'_{\alpha_1 \ldots \alpha_r} t_1^{\alpha_1} \ldots t_r^{\alpha_r}$$

$$+ \sum_{j=1}^{r} A_{\alpha_1 \ldots \alpha_r} \alpha_j t_j' t_1^{\alpha_1} \ldots t_j^{\alpha_j - 1} \ldots t_r^{\alpha_r}$$

Assuming:

$$\sum_{j=1}^{r} \alpha_j \geq 2$$

and noticing that the derivative of the right-hand side of (1.16) is of degree 1 in $t_1,\ldots,t_r$, we deduce that:

$$A'_{\alpha_1\ldots\alpha_r} = 0 \quad \Longrightarrow \quad A_{\alpha_1\ldots\alpha_r} \text{ is a constant}$$

If:

$$\sum_{j=1}^{r} \alpha_j > 2$$

then there exists i_0 such that $\alpha_{i_0} \neq 0$. The coefficient of $t_1^{\alpha_1}\ldots t_{i_0}^{\alpha_{i_0}-1}\ldots t_r^{\alpha_r}$ must be zero in the derivative of $g(t_1,\ldots,t_r)$. So:

$$A_{\alpha_1\ldots\alpha_r}\alpha_{i_0}t'_{i_0} + A'_{\alpha_1\alpha_2\ldots\alpha_{i_0}-1\ldots\alpha_r} = 0$$

$$\Longrightarrow \quad t_{i_0} = -\frac{1}{A_{\alpha_1\ldots\alpha_r}}\frac{1}{\alpha_0}A_{\alpha_1\alpha_2\ldots\alpha_{i_0}-1\ldots\alpha_r} + C$$

where C is a constant. But this is a contradiction since t is transcendental over K.

So, we deduce that g is a polynomial of degree 2 with constant coefficients, for all the terms in $t_1,\ldots,t_r$ of degree 2. That is:

$$g(t_1,\ldots,t_r) = A_0 + \sum_{p=1}^{r} A_p t_p + \sum_{\substack{\alpha,\beta \\ \beta\geq\alpha}} A_{\alpha,\beta}t_\alpha t_\beta$$

where t_α, $t_\beta \in \{t_1,\ldots,t_r\}$, and $A_{\alpha,\beta}$ are constants.

$$g'(t_1,\ldots,t_r) = A'_0 + \sum_{p=1}^{r} A_p t'_p + \sum_{p=1}^{r} A'_p t_p + \sum_{\substack{\alpha,\beta \\ \beta\geq\alpha}} A_{\alpha,\beta}t'_\alpha t_\beta + \sum_{\substack{\alpha,\beta \\ \beta\geq\alpha}} A_{\alpha,\beta}t_a t'_\beta \qquad (1.17)$$

and:

$$g'(t_1,\ldots,t_r) = f + \sum_{j=1}^{n} d_j \frac{h_j'}{h_j} \log(1 - h_j) - \sum_{i=1}^{m} a_i \frac{L_i'}{L_i} \tag{1.18}$$

Using the dependency relations (**) (page 8), we obtain from (1.17) and (1.18):

$$f - A_0' - \sum_{p=1}^{r} A_p t_p' - \sum_{j=r+1}^{n} d_j \frac{h_j'}{h_j} R_j$$

$$= \sum_{p=1}^{r} \left[d_p \frac{h_p'}{h_p} - \sum_{j=r+1}^{n} c_{jp} d_j \frac{h_j'}{h_j} + 2A_{pp} t_p' + \sum_{\alpha \neq p} A_{\alpha p} t_\alpha' + A_p' \right] t_p$$

(where $A_{\alpha p} = A_{p\alpha}$ if $\alpha > p$).

From the above, we deduce that:

$$2A_{pp} t_p' + \sum_{\alpha \neq p} A_{\alpha p} t_\alpha' = - d_p \frac{h_p'}{h_p} - \sum_{j=r+1}^{n} c_{jp} d_j \frac{h_j'}{h_j} - A_p'$$

and, by integration, we get:

$$A_{pp} t_p + \sum_{\alpha \neq p} \frac{1}{2} A_{\alpha p} t_\alpha$$

$$= \frac{1}{2} \left[- d_p \log h_p - \sum_{j=r+1}^{n} c_{jp} d_j \log h_j - A_p \right] + C_p \tag{1.19}$$

where C_p is a constant.

Notice that we can write:

$$g(t_1,\ldots,t_r) = A_0 + \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} \left[A_{pp} t_p + \sum_{\alpha \neq p} \frac{1}{2} A_{\alpha p} t_\alpha \right] t_p$$

and, using (1.19) and (1.16), we get:

$$\int f = \sum_{j=1}^{n} d_j \ell_2(h_j) + A_0 + \sum_{p=1}^{r} A_p t_p - \frac{1}{2} \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} c_p t_p$$

$$+ \frac{1}{2} \sum_{p=1}^{r} \left[d_p \log h_p + \sum_{j=r+1}^{n} c_{jp} d_j \log h_j \right] t_p + \sum_{i=1}^{m} a_i \log L_i$$

which gives:

$$\int f = \sum_{p=1}^{r} d_j \left[\ell_2(h_j) + \frac{1}{2} (\log h_p)\, t_p \right]$$

$$+ \sum_{j=r+1}^{n} d_j \left[\ell_2(h_j) + \left[\sum_{p=1}^{r} c_{jp} t_p \right] \log h_j \right]$$

$$+ A_0 + \frac{1}{2} \sum_{p=1}^{r} A_p t_p + \sum_{p=1}^{r} c_p t_p + \sum_{i=1}^{m} a_i \log L_i$$

But we had:

$$\sum_{p=1}^{r} c_{jp} t_p = \log(1 - h_j) - R_j$$

for $j \in \{r+1,\ldots,n\}$ and $t_p = \log(1 - h_p)$. So:

$$\int f = \sum_{j=1}^{n} d_j D(h_j) - \sum_{j=r+1}^{n} d_j R_j \log h_j + A_0 + \frac{1}{2} \sum_{p=1}^{r} A_p \log(1 - h_p)$$

$$+ \sum_{p=1}^{r} c_p \log(1 - h_p) + \sum_{i=1}^{m} a_i \log L_i,$$

$$R_j \in k,\ A_0, A_p \in K,\ L_i \in K$$

and, by lemma 1, $\int f = S$, where S is a simple elementary-dilogarithmic expression. This completes the proof of proposition 1.

If we go back and look at the definition of t dilogarithmic over k we observe the following fact: t is defined up to a constant multiple of a log, or, more precisely: if $t' = - a'/a\, \psi$, where $\psi' = (1 - a)'/(1 - a)$, ψ is defined up to a constant. So, $\psi_1' = (1 - a)'/(1 - a) \Longrightarrow \psi_1 = c + \psi$, where c is a constant, and this implies that $t' = - a'/a\, (\psi_1 - c) = - a'/a\, \psi_1 + c\, a'/a \Longrightarrow$ t is defined up to c log a.

This motivates considering the dilog as defined mod the vector space generated by constant multiples of logarithms over k. We denote from now on this vector space by M_k for any differential field k of characteristic zero.

<u>Lemma 2</u>: If k is a differential field of characteristic zero, then $D(1/f) \equiv - D(f) \bmod M_k$.

<u>Proof</u>:

$$D'\left(\frac{1}{f}\right) \equiv + \frac{1}{2}\frac{f'}{f}\,\phi + \frac{1}{2}\,\frac{n\;1 - \frac{1}{f})'}{1 - \frac{1}{f}}\,\theta$$

$$\text{where:}\quad \phi' = \frac{\left(1 - \frac{1}{f}\right)'}{1 - \frac{1}{f}} \quad \text{and} \quad \theta' = \frac{\left(\frac{1}{f}\right)'}{\frac{1}{f}} = - \frac{f'}{f}$$

So:

$$\phi' = \frac{(1 - f)'}{(1 - f)} - \frac{f'}{f}$$

$$\Longrightarrow D'\left(\frac{1}{f}\right) \equiv \frac{1}{2}\frac{f'}{f}\left(\log(1 - f) - \log f\right)$$

$$- \frac{1}{2}\left(\frac{(1 - f)'}{(1 - f)} - \frac{f'}{f}\right)\log f \quad \bmod M_k'$$

$$\Longrightarrow D'\left(\frac{1}{f}\right) \equiv \frac{1}{2}\frac{f'}{f}\log(1 - f) - \frac{1}{2}\frac{(1 - f)'}{(1 - f)}\log f \equiv - D'(f) \quad \bmod M_k'$$

$$\Longrightarrow D\left(\frac{1}{f}\right) \equiv - D(f) \quad \bmod M_k$$

(M_k' is the space of derivatives of M_k).

<u>Proposition 2</u>: Let k be a differential field of characteristic zero, and let θ be transcendental over k with $k(\theta)$ being a differential field. Let $f(\theta) \in k(\theta)$ and K be the splitting field of $f(\theta)$ and $1 - f(\theta)$. We define, if a is a zero or a pole of $f(\theta)$, $\mathrm{ord}_a f(\theta)$ to be the multiplicity of $(\theta - a)$; this is positive if a is a zero of $f(\theta)$ and negative if a is a pole. Then, there exists $f_1 \in k$ such that:

$$D(f(\theta)) \equiv D(f_1) + \sum_{a,b} \mathrm{ord}_b(1 - f)\,\mathrm{ord}_a f\; D\left(\frac{\theta - b}{\theta - a}\right) \quad \bmod M_{k(\theta)} \qquad \text{(A)}$$

where a and b are the zeros and poles of f and $(1 - f)$, respectively.

<u>Proof</u>: Let $f(\theta) = f_0\, P(\theta)/Q(\theta)$, where $f_0 \in k$, and $P(\theta)$, $Q(\theta)$ are relatively prime polynomials over k which are monic. We can also assume that $\deg P(\theta) \geq \deg Q(\theta)$, otherwise, using lemma 2, we replace f by 1/f.

$$1 - f(\theta) = \frac{Q(\theta) - f_0 P(\theta)}{Q(\theta)} = g_0 \frac{R(\theta)}{Q(\theta)}$$

where $g_0 \in k$, and $R(\theta)$ is a monic polynomial relatively prime with both P and Q.

<u>First step</u>:

$$D'(f) = -\frac{1}{2}\frac{f'}{f}\log(1 - f) + \frac{1}{2}\frac{(1 - f)'}{(1 - f)}\log f$$

is well-defined mod $M'_{K(\theta)}$. We can check easily that, if $a \neq b$ and $a,b \in K$, then:

$$D'\left(\frac{\theta - b}{\theta - a}\right) = \frac{1}{2}\left(\frac{\theta' - b'}{\theta - b} - \frac{b' - a'}{b - a}\right)\log(\theta - a)$$

$$+ \frac{1}{2}\left(\frac{b' - a'}{b - a} - \frac{\theta' - a'}{\theta - a}\right)\log(\theta - b)$$

$$+ \frac{1}{2}\left(\frac{\theta' - a'}{\theta - a} - \frac{\theta' - b'}{\theta - b}\right)\log(b - a) \quad \mathrm{mod}\ M'_{K(\theta)}$$

(This is because log gh = log g + log h + constant, and log 1/g = - log g + constant.)

<u>Second step</u>: Consider the set $I_1 = \{ (a,b) \mid$ a is a zero of P or of Q, b is a zero of R or of Q, but whenever one of a and b is a zero of Q the other is not $\}$. (So the set (a,b), a zero of Q and b zero of Q, is excluded.)

We have:

$$f_0 \frac{P(\theta)}{Q(\theta)} + g_0 \frac{R(\theta)}{Q(\theta)} = 1 \tag{B}$$

$$\Longleftrightarrow \quad f_0 P(\theta) + g_0 R(\theta) = Q(\theta) \tag{C}$$

Let us compute:

$$- \frac{1}{2} \left[\sum_{(a,b)\epsilon I_1} + \text{ord}_a f \; \text{ord}_b(1 - f) \frac{b' - a'}{b - a} \log(\theta - a) \right]$$

$$+ \frac{1}{2} \sum_{(a,b)\epsilon I_1} \text{ord}_a f \; \text{ord}_b(1 - f) \frac{b' - a'}{b - a} \log(\theta - b) \quad \text{mod } M_K'$$

We call the above quantity or sum S_1:

$$S_1 = - \frac{1}{2} \sum_{\substack{a \text{ zero} \\ \text{of } P}} \text{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ (1 - f)}} \text{ord}_b(1 - f) \frac{b' - a'}{b - a} \right] \log(\theta - a)$$

$$- \frac{1}{2} \sum_{\substack{a \text{ zero} \\ \text{of } Q}} \text{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ R}} \text{ord}_b(1 - f) \frac{b' - a'}{b - a} \right] \log(\theta - a)$$

$$+ \frac{1}{2} \sum_{\substack{b \text{ zero} \\ \text{of } R}} \text{ord}_b(1 - f) \left[\sum_{\substack{a \text{ zero of} \\ f}} \text{ord}_a f \frac{b' - a'}{b - a} \right] \log(\theta - b)$$

$$+ \frac{1}{2} \sum_{\substack{b \text{ zero} \\ \text{of } Q}} \text{ord}_b(1 - f) \left[\sum_{\substack{a \text{ zero of} \\ P}} \text{ord}_a f \frac{b' - a'}{b - a} \right] \log(\theta - b)$$

since $(a,b) \epsilon I_1$.

Now, (B) above implies, if a is a zero of P, that:

$$g_0 \frac{R(a)}{Q(a)} = 1 \quad \Longrightarrow \quad \frac{g_0'}{g_0} + \frac{R'(a)}{R(a)} - \frac{Q'(a)}{Q(a)} = 0$$

but, as we can easily check:

$$\sum_{\substack{b \text{ zero of} \\ (1 - f)}} \text{ord}_b(1 - f) \frac{b' - a'}{b - a} = \frac{R'(a)}{R(a)} - \frac{Q'(a)}{Q(a)} = - \frac{g_0'}{g_0} \qquad (2.1)$$

(where a is a zero of P).

Also, if b is a zero of R, we have, using (B) above:

$$f_0 \frac{R(b)}{Q(b)} = 1 \quad \Longrightarrow \quad \frac{f_0'}{f_0} = - \frac{P'(b)}{P(b)} + \frac{Q'(b)}{Q(b)}$$

So we get:

$$\sum_{\substack{a \text{ zero of} \\ f}} \mathrm{ord}_a f \, \frac{b' - a'}{b - a} = \frac{P'(b)}{P(b)} - \frac{Q'(b)}{Q(b)} = - \frac{f_0'}{f_0} \tag{2.2}$$

(where, in the above, b is a zero of R).

Now, we look at the sum:

$$S_2 = -\frac{1}{2} \sum_{\substack{a \text{ zero} \\ \text{of } Q}} \mathrm{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ R}} \mathrm{ord}_b(1 - f) \frac{b' - a'}{b - a} \right] \log(\theta - a)$$

$$+ \frac{1}{2} \sum_{\substack{b \text{ zero} \\ \text{of } Q}} \mathrm{ord}_b(1 - f) \left[\sum_{\substack{a \text{ zero of} \\ P}} \mathrm{ord}_a f \frac{b' - a'}{b - a} \right] \log(\theta - b)$$

$$\Longrightarrow \quad S_2 = \frac{1}{2} \sum_{\substack{a \text{ zero} \\ \text{of } Q}} \mathrm{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ R}} - \mathrm{ord}_b(1 - f) \frac{b' - a'}{b - a} + \sum_{\substack{b \text{ zero of} \\ P}} \mathrm{ord}_b f \frac{b' - a'}{b - a} \right] \log(\theta - a)$$

But the relation $f_0 P(\theta) + g_0 R(\theta) = Q(\theta)$ implies, if a is a zero of Q, that:

$$f_0 P(a) + g_0 R(a) = 0$$

$$\Longrightarrow \quad \frac{f_0'}{f_0} + \frac{P'(a)}{P(a)} = \frac{g_0'}{g_0} + \frac{R'(a)}{R(a)}$$

$$\Longrightarrow \quad \frac{P'(a)}{P(a)} - \frac{R'(a)}{R(a)} = \frac{g_0'}{g_0} - \frac{f_0'}{f_0} \tag{2.3}$$

and:

$$\left[\begin{array}{l} - \sum_{\substack{b \text{ zero of} \\ R}} \text{ord}_b(1 - f) \dfrac{b' - a'}{b - a} = - \dfrac{R'(a)}{R(a)} \\ \\ + \sum_{\substack{b \text{ zero of} \\ P}} \text{ord}_b f \dfrac{b' - a'}{b - a} = \dfrac{P'(a)}{P(a)} \end{array}\right. \qquad \text{if } a \text{ is a zero of } Q$$

(2.3) and the above imply that:

$$S_2 = \frac{1}{2} \sum_{\substack{a \text{ zero of} \\ Q}} \text{ord}_a f \left[\frac{g_0'}{g_0} - \frac{f_0'}{f_0} \right] \log(\theta - a)$$

which is exactly:

$$S_2 = - \frac{1}{2} \sum_{\substack{b \text{ zero of} \\ Q}} \text{ord}_b(1 - f) \frac{f_0'}{f_0} \log(\theta - b)$$

$$+ \frac{1}{2} \sum_{\substack{a \text{ zero of} \\ Q}} \text{ord}_a f \frac{g_0'}{g_0} \log(\theta - a)$$

(2.1) and (2.2) imply, respectively, that:

$$- \frac{1}{2} \sum_{\substack{a \text{ zero of} \\ P}} \text{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ (1 - f)}} \text{ord}_b(1 - f) \frac{b' - a'}{b - a} \right] \log(\theta - a)$$

$$= \frac{1}{2} \sum_{\substack{a \text{ zero of} \\ P}} \text{ord}_a f \frac{g_0'}{g_0} \log(\theta - a)$$

This sum will be denoted by S_3.

$$+ \frac{1}{2} \sum_{\substack{a \text{ zero of} \\ R}} \text{ord}_b(1 - f) \left[\sum_{\substack{a \text{ zero of} \\ f}} \text{ord}_a f \frac{b' - a'}{b - a} \right] \log(\theta - b)$$

$$= - \frac{1}{2} \sum_{\substack{b \text{ zero of} \\ P}} \text{ord}_b(1 - f) \frac{f_0'}{f_0} \log(\theta - b)$$

This sum will be denoted by S_4.

Now, $S_1 = S_2 + S_3 + S_4$, and by regrouping the terms in S_2, S_3, and S_4 we deduce that:

$$S_1 = +\frac{1}{2} \sum_{\substack{a \text{ zero or pole} \\ \text{of } f}} \operatorname{ord}_a f \, \frac{g_0'}{g_0} \log(\theta - a)$$

$$-\frac{1}{2} \sum_{\substack{b \text{ zero or pole} \\ \text{of } (1 - f)}} \operatorname{ord}_b (1 - f) \, \frac{f_0'}{f_0} \log(\theta - b)$$

Now, consider the four following sums:

$$\Sigma_3 = +\frac{1}{2} \sum_{\substack{a \text{ zero of} \\ P}} \operatorname{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ (1 - f)}} \operatorname{ord}_b (1 - f) \log(b - a) \right] \frac{\theta' - a'}{\theta - a}$$

$$\Sigma_4 = -\frac{1}{2} \sum_{\substack{b \text{ zero of} \\ R}} \operatorname{ord}_b (1 - f) \left[\sum_{\substack{a \text{ zero of} \\ f}} \operatorname{ord}_a f \log(b - a) \right] \frac{\theta' - b'}{\theta - b}$$

$$\Sigma_2 = +\frac{1}{2} \sum_{\substack{a \text{ zero of} \\ Q}} \operatorname{ord}_a f \left[\sum_{\substack{b \text{ zero of} \\ R}} \operatorname{ord}_b (1 - f) \log(b - a) \right] \frac{\theta' - a'}{\theta - a}$$

$$-\frac{1}{2} \sum_{\substack{b \text{ zero of} \\ Q}} \operatorname{ord}_b (1 - f) \left[\sum_{\substack{a \text{ zero of} \\ P}} \operatorname{ord}_a f \log(b - a) \right] \frac{\theta' - b'}{\theta - b}$$

and: $\Sigma_1 = \Sigma_2 + \Sigma_3 + \Sigma_4$

It follows immediately that:

$$\Sigma_1 = \sum_{(a,b) \in I_1} \frac{1}{2} \operatorname{ord}_a f \operatorname{ord}_b (1 - f) \left(\frac{\theta' - a'}{\theta - a} - \frac{\theta' - b'}{\theta - b} \right) \log (b - a)$$

Now, and as before, integrating (2.1), (2.2), and (2.3), we deduce:

$$\sum_{\substack{b \text{ zero of} \\ (1 - f)}} \operatorname{ord}_b (1 - f) \log(b - a) = \log R(a) - \log Q(a) + \text{etc.}$$

$$= -\log g_0 + \text{etc.}, \quad \text{where } a \text{ is a zero of } P \qquad (2.1')$$

$$\sum_{\substack{a \text{ zero of} \\ f}} \operatorname{ord}_a f \log(b - a) = \log P(b) - \log Q(b) + \text{etc.}$$

$$= -\log f_0 + \text{etc.}, \quad \text{where b is a zero of R} \qquad (2.2')$$

$$\sum_{\substack{b \text{ zero of} \\ R}} -\operatorname{ord}_b(1 - f) \log(b - a) + \sum_{\substack{b \text{ zero of} \\ P}} \operatorname{ord}_b(f) \log(b - a)$$

$$= \log g_0 - \log f_0 + \text{etc.}, \quad \text{where a is a zero of Q} \qquad (2.3')$$

Plugging (2.1'), (2.2'), and (2.3') in Σ_3, Σ_4, and Σ_2, respectively, and regrouping, as we have done for computing S_1, we obtain:

$$\Sigma_1 = -\frac{1}{2} \sum_{\substack{a \text{ zero or pole} \\ \text{of } f}} \operatorname{ord}_a f \, \frac{\theta' - a'}{\theta - a} \log g_0$$

$$+ \frac{1}{2} \sum_{\substack{b \text{ zero or pole} \\ \text{of } (1 - f)}} \operatorname{ord}_b(1 - f) \, \frac{\theta' - b'}{\theta - b} \log f_0 \mod M'_{K(\theta)}$$

(This is because we had constants in relations (2.1'), (2.2'), and (2.3').)

<u>Third Step</u>: We compute $D'(f(\theta)) \mod M'_{K(\theta)}$, which can be immediately verifed to be:

$$D'(f(\theta)) = \frac{1}{2} \Bigg[-\sum_{a,b} \operatorname{ord}_a f \operatorname{ord}_b(1 - f) \frac{(\theta - a)'}{(\theta - a)} \log(\theta - b)$$

$$+ \sum_{a,b} \operatorname{ord}_a f \operatorname{ord}_b(1 - f) \frac{(\theta - b)'}{(\theta - b)} \log(\theta - a) \Bigg]$$

$$-\frac{1}{2} \sum_a \operatorname{ord}_a f \frac{(\theta - a)'}{(\theta - a)} \log g_0 + \frac{1}{2} \sum_b \operatorname{ord}b(1 - f) \frac{(\theta - b)'}{(\theta - b)} \log f_0$$

$$-\frac{1}{2} \sum_b \operatorname{ord}_b(1 - f) \frac{f_0'}{f_0} \log(\theta - b) + \frac{1}{2} \sum_a \operatorname{ord}_a f \frac{g_0'}{g_0} \log(\theta - a)$$

$$-\frac{1}{2} \frac{f_0'}{f_0} \log g_0 + \frac{1}{2} \frac{g_0'}{g_0} \log f_0 \mod M'_{K(\theta)}$$

(where $\sum_{a,b}$ runs over all zeros and poles of f and $1 - f$, respectively, $\sum_a$ runs over the zeros and poles of f, and $\sum_b$ runs over the zeros and poles of $(1 - f)$).

The term:

$$- \sum_{(a,b)\in I_1} \left[\mathrm{ord}_a f \ \mathrm{ord}_b(1 - f) \frac{(\theta - a)'}{(\theta - a)} \log(\theta - b) + \mathrm{ord}_a f \ \mathrm{ord}_b(1 - f) \frac{(\theta - b)'}{(\theta - b)} \log(\theta - a) \right]$$

is zero since a and b run over the roots of Q.

So:

$$D'(f(\theta)) \equiv - \frac{1}{2} \frac{f_0'}{f_0} \log g_0 + \frac{1}{2} \frac{g_0'}{g_0} \log f_0 - \sum_{(a,b)\in I_1} \frac{1}{2} \mathrm{ord}_a f \ \mathrm{ord}_b(1 - f) \left(\frac{(\theta - b)'}{(\theta - b)} \log(\theta - a) - \frac{(\theta - a)'}{(\theta - b)} \log(\theta - b) \right) + \Sigma_1 + S_1 \mod M'_{K(\theta)} \quad \text{(by (2.4) and (2.4'))}$$

(2.5) ==>

$$D'(f(\theta)) = \left[\sum_{(a,b)\in I_1} \mathrm{ord}_a f \ \mathrm{ord}_b(1 - f) \ D\left(\frac{\theta - b}{\theta - a} \right) \right]' - \frac{1}{2} \frac{f_0'}{f_0} \log g_0 + \frac{1}{2} \frac{g_0'}{g_0} \log f_0 \mod M'_{K(\theta)}$$

Now, we distinguish 3 cases:

<u>Case 1</u>:

deg P > deg Q (strict inequality)

==> deg $(Q(\theta) - f_0 P(\theta))$ = deg $P(\theta)$ ==> $g_0 = -f_0$

And:

$$\log -f_0 = \log g_0 = \log f_0 + \text{etc.}$$

So:

$$- \frac{1}{2} \frac{f_0'}{f_0} \log g_0 + \frac{1}{2} \frac{g_0'}{g_0} \log f_0 \equiv 0 \mod M'_{K(\theta)}$$

and we take f_1 in proposition 2 to be a constant. So $D'(f_1) = 0$.

Case 2: If deg P = deg Q (and $f_0 \neq 1$), then the leading coefficient of $Q(\theta) - f_0 p(\theta)$ is $1 - f_0 \Longrightarrow g_0 = 1 - f_0$

$$\Longrightarrow \quad -\frac{1}{2}\frac{f_0'}{f_0}\log g_0 + \frac{1}{2}\frac{g_0'}{g_0}\log f_0 = -\frac{1}{2}\frac{f_0'}{f_0}\log(1 - f_0) + \frac{1}{2}\frac{(1 - f_0)'}{(1 - f_0)}\log f_0$$

and we take f_1 in proposition 2 to be f_0.

Case 3: deg P = deg Q and $f_0 = 1$.

Let $I = \{ (a,b) \mid a \text{ pole or zero of } f,\ b \text{ pole or zero of } (1 - f) \}$.

Then, $I - I_1 = \{ (a,b) \mid a \text{ zero of } Q,\ b \text{ zero of } Q \}$. But:

$$D\left(\frac{z - b}{z - a}\right) \equiv -D\left(\frac{z - a}{z - b}\right) \mod M_{K(\theta)}$$

$$\Longrightarrow \sum_{(a,b)\in I-I_1} \text{ord}_a f\ \text{ord}_b(1 - f)\ D\left(\frac{z - b}{z - a}\right) \equiv 0 \mod M_{K(\theta)}$$

So:

$$\sum_{(a,b)\in I_1} \text{ord}_a f\ \text{ord}_b(1-f)\ D\left(\frac{z - b}{z - a}\right)$$

$$\equiv \sum_{(a,b)\in I} \text{ord}_a f\ \text{ord}_b(1 - f)\ D\left(\frac{z - b}{z - a}\right) \mod M_{K(\theta)} \qquad (2.6)$$

Now, deg P = deg Q, and $f_0 = 1 \Longrightarrow$

$$1 - f = \frac{Q(\theta) - P(\theta)}{Q(\theta)} \quad \Longrightarrow \quad \deg\left(Q(\theta) - P(\theta) \right) < \deg Q(\theta)$$

But, since:

$$D(f) \equiv -D(1 - f) \equiv D\left(\frac{1}{1 - f}\right) \mod M_{K(\theta)}$$

and:

$$\sum_{(a,b)\in I_1} \text{ord}_a f\ \text{ord}_b(1 - f)\ D\left(\frac{z - b}{z - a}\right)$$

is unchanged if we replace f by $1/(1 - f)$, we are again in case 1.

But, by the results of case 1 and case 2, and relations (2.5) and (2.6), proposition 2 is proved.

Lemma 3: (Risch, Kolchin.) Let k be a differential field of characteristic zero. Assume that $u_1,\ldots,u_n$ are logarithmic over k and v exponential over k, and assume also that $k(v,u_1,\ldots,u_n)$ and k have the same field of constants. Then, if $v,u_1,\ldots,u_n$ are algebraically dependent over k, there exists an integer $n \neq 0$ such that $v^n \in k$ (not all c_i are zeros) and constants $c_1,\ldots,c_n$ such that $\Sigma\, c_i u_i \in k$.

Proof: See Kolchin [__].

Corollary 3.1: Consider $k(v,u_1,\ldots,u_n)$, defined as above. Then, if v is transcendental over k, it is transcendental over $k(u_1,\ldots,u_n)$.

Proof: If not, then by lemma 3 there exists $n \neq 0$ such that $v^n \in k$ $\Longrightarrow$ v is algebraic $\Longrightarrow$ contradiction.

Lemma 4: (Ostrowski.) Under the assumptions of lemma 3 concerning k and $u_1,\ldots,u_n$, let t be primitive over k and assume that $t,u_1,\ldots,u_n$ are algebraically dependent over k. Then, there exist $c_1,\ldots,c_n$ constants and $s \in k$ such that $t = \Sigma\, c_i u_i + s$.

Proof: See Ostrowski [__].

Proposition 3: Let k be a differential field of characteristic zero, θ primitive, and transcendental over k. Let $\alpha_1,\ldots,\alpha_n \in k$ ($\alpha_i \neq \alpha_j$ if $i \neq j$), $u_1,\ldots,u_m \in k$, and assume the existence of constants $c_1,\ldots,c_n$, $d_1,\ldots,d_m$ such that:

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j \in k(\theta)$$

(where k and $k(\theta)$ $(\log(\theta - \alpha_1),\ldots,\log u_1,\ldots)$ have the same field of constants). Then, $c_1 = c_2 = \ldots = c_n = 0$.

Proof: As in the proof in proposition 1, by considering partial fraction decomposition (or see Rosenlicht [1]).

Proposition 4: Let k be a differential field of characteristic zero, θ exponential, and transcendental over k. Let $\alpha_1, \ldots, \alpha_n \in k$ ($\alpha_i \neq \alpha_j$ if $i \neq j$, and $\alpha_i \neq 0$ for all i), $u_1, \ldots, u_m \in k$, and assume the existence of constants $c_1, \ldots, c_n$, $d_1, \ldots, d_m$ such that:

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j \in k(\theta)$$

(where k and $k(\theta)$ ($\log u_1, \ldots, \log u_m, \log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n)$) have the same field of constants). Then, $c_1 = c_2 = \ldots = c_n = 0$.

Proof: This is also a classical result. See Risch [1] or Rosenlicht [1].

Corollary 3.2: In the conditions of propositions 3 and 4, $\log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n)$ ($\alpha_i \neq 0$ for all i if θ exponential) are algebraically independent over $k(\theta)$ ($\log u_1, \ldots, \log u_m$).

Proof: If $\log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n)$ were not algebraically independent, and since $\log(\theta - \alpha_1), \ldots, \log(\theta - \alpha_n), \log u_1, \ldots, \log u_m$ are plog's over $k(\theta)$, we deduce by Ostrowski's theorem that there exist $c_1, \ldots, c_n$ not all zero, and constants $d_1, \ldots, d_m$ such that:

$$\sum_{i=1}^{n} c_i \log(\theta - \alpha_i) + \sum_{j=1}^{m} d_j \log u_j \in k(\theta) \implies c_1 = c_2 = \ldots = c_n = 0$$

by propositions 3 and 4 ==> contradiction.

Proposition 5: Let k be a differential field of characteristic zero. Let θ be transcendental over k and $k(\theta)$, k have the same field of constants. Let $s(\theta) \in k(\theta)$ be such that $s'(\theta) \in k$. Then:

(1") If θ is primitive over k, $s(\theta) = c\theta + v$, where c is a constant and $v \in k$.

(2") If θ is exponential over k, $s(\theta) \in k$.

Proof: This is again a well-known fact. See Risch [1] or Rosenlicht [1].

Now we are ready to prove the main theorem in this paper. We recall first the definition of a dilogarithmic-elementary extension of a differential field k, which is a differential field K such that there is a tower of differential fields $k = K_0 \quad K_1 \quad \ldots \quad K_N = K$ all having the same constant field and for each $i = 1,\ldots,N$ we have one of the following three cases:

(1") $K_i = K_{i-1}(\theta_i)$, where θ_i is logarithmic over K_{i-1}.

(2") $K_i = K_{i-1}(\theta_i)$, where θ_i is exponential over K_{i-1}. We also assume θ_i transcendental over K_{i-1} in this case.

(3") $K_i = K_{i-1}(\theta_i,\theta_i')$, where $\theta_i = \ell_2(a)$ for some $a \in K_{i-1}$.

Remark: In (3"), θ_i is defined up to a constant multiple of an element ϕ_i, such that $\phi_i' = (1 - a)'/(1 - a)$, which is always in K_i since $\theta_i' \in K_i$.

The number N will be called the length of K over k.

Theorem: Let k be a differential field of characteristic zero, which is a Liouville extension of its subfield of constants assumed algebraically closed. Let $f \in k$ and suppose that there exists a dilogarithmic-elementary extension K of k such that:

$$\int f \in K$$

Then, $\int f$ is a simple elementary-dilogarithmic expression over k. That is:

$$\int f = g + \sum_{j=1}^{m} s_i \log v_i + \sum_{j=1}^{n} d_j D(h_j) \quad \text{(n,m are positive integers)}$$

where g, s_i, v_i, $h_j \in k$, and the d_j's are constants.

Proof: It is by induction on N, the length of K over k. If $N = 0$, then $\int f = g \in k$ and the theorem is proved. If $N > 0$, we apply the induction hypothesis to $f \in K_\Lambda$ and the tower $K_1 \quad K_2 \quad \ldots \quad K_N = K$, to obtain:

$$\int f = g + \sum_{i=1}^{m} s_i \log v_i + \sum_{j=1}^{n} d_j D(h_j)$$

where g, s_i, v, $h_j \in K_1$, and the d_j's are constants.

We want to modify equation (3.1) in such a way that g, s_i, v_i, and h_j are in $k = K_0$.

For this we consider 3 major cases.

Case 1: $K_1 = k(\theta)$ and θ logarithmic over k. $\theta = \log a$, $a \in k$. If θ is algebraic over k, then, by lemma 4, $\theta \in k$ and there is nothing to prove.

So, we assume θ transcendental, and factor v_i, h_j, $1 - h_j$ over k. So we will be working over k^0, the splitting field of these quantities which we assume normal.

By proposition 2:

$$D(h_j(\theta)) \equiv D(H_j) + \sum \text{ord}_a h_j \, \text{ord}_b(1 - h_j) \, D\left(\frac{\theta - b}{\theta - a}\right) \mod M_{k^0(\theta)}$$

where $H_j \in k$, $b, a \in k^0$, $a \neq b$.

So, (3.1) can be written as:

$$\int f \equiv g(\theta) + \sum_{\alpha} s_\alpha(\theta) \log(\theta - \alpha) + \sum_{p=1}^{r} s_p(\theta) \log f_p$$

$$+ \sum_{j=1}^{m} d_j D(H_j) + \sum_{\beta,\gamma} d_{\beta,\gamma} D\left(\frac{\theta - \gamma}{\theta - \beta}\right) \mod M_{k(\theta)} \qquad (3.1')$$

where $d_{\beta,\gamma}$ is a constant, $\gamma, \beta \in k^0$, $\gamma \neq \beta$, $f_p, H_j \in k$.

Now, we order the β's and the γ's in the following way. We write this set as $\alpha_1,\ldots,\alpha_r$ so that $(\alpha_i \neq \alpha_j,\ i \neq j)$:

$$\sum_{\beta,\gamma} d_{\beta,\gamma}\, D\left(\frac{\theta-\gamma}{\theta-\beta}\right)$$

$$= d_{12}\, D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right) + d_{12}\, D\left(\frac{\theta-\alpha_1}{\theta-\alpha_3}\right) + \ldots + D\left(\frac{\theta-\alpha_1}{\theta-\alpha_n}\right)$$

$$+ d_{23}\, D\left(\frac{\theta-\alpha_2}{\theta-\alpha_3}\right) + d_{24}\, D\left(\frac{\theta-\alpha_2}{\theta-\alpha_4}\right) + \ldots + D\left(\frac{\theta-\alpha_2}{\theta-\alpha_n}\right)$$

$$\vdots$$

$$+ \sum_{j>i} d_{ij}\, D\left(\frac{\theta-\alpha_i}{\theta-\alpha_j}\right)$$

$$\vdots$$

$$+ d_{n-1,n}\, D\left(\frac{\theta-\alpha_{n-1}}{\theta-\alpha_n}\right)$$

$$+ \text{constant} \qquad (3.2)$$

(This is possible because: $D\left((\theta-\alpha_i)/(\theta-\alpha_j)\right)$ $= -D\left((\theta-\alpha_j)/(\theta-\alpha_i)\right) \mod M_{K\circ(\theta)}$.)

We call the above expression reduced, that is, (3.2). For example:

$$d_1\, D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right) + d_2\, D\left(\frac{\theta-\alpha_1}{\theta-\alpha_3}\right) + d_3\, D\left(\frac{\theta-\alpha_2}{\theta-\alpha_3}\right)$$

is reduced, while the expression:

$$d_1\, D\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right) + d_2\, D\left(\frac{\theta-\alpha_1}{\theta-\alpha_3}\right) + d_3\, D\left(\frac{\theta-\alpha_2}{\theta-\alpha_1}\right)$$

is not reduced.

So, (3.1') becomes:

$$\left\{ f = g(\theta) + \sum_{i=1}^{n} S_i(\theta)\log(\theta-\alpha_i) + \sum_{p=1}^{r} s_p(\theta)\log f_p \right.$$

$$+ \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) + \sum_{i=1}^{n-1} \sum_{j>i} d_{ij}\, D\left(\frac{\theta - \alpha_i}{\theta - \alpha_j} \right) \qquad (3.3)$$

(with $\alpha_i \neq \alpha_j$ for all $i \neq j$).

Now, we take the derivative of (3.3), to get:

$$f = g'(\theta) + \sum_{i=1}^{n} S_i(\theta) \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{i=1}^{n} S_i'(\theta) \log(\theta - \alpha_i)$$

$$+ \sum_{p=1}^{r} s_p(\theta) \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'(\theta) \log f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} \left[-\frac{1}{2} \frac{H_{j_0}'}{H_j} \log(1 - H_{j_0}) + \frac{1}{2} \frac{(1 - H_{j_0})'}{(1 - H_{j_0})} \log H_{j_0} \right]$$

$$+ \sum_{i=1}^{n-1} \sum_{j>i} d_{ij} \left[\frac{1}{2} \left(\frac{\theta' - \alpha_i'}{\theta - \alpha_i} - \frac{\alpha_i' - \alpha_j'}{\alpha_i - \alpha_j} \right) \log(\theta - \alpha_j) \right.$$

$$+ \frac{1}{2} \left(\frac{\alpha_i' - \alpha_j'}{\alpha_i - \alpha_j} - \frac{\theta' - \alpha_j'}{\theta - \alpha_j} \right) \log(\theta - \alpha_i)$$

$$\left. + \frac{1}{2} \left(\frac{\theta' - \alpha_j'}{\theta - \alpha_j} - \frac{\theta' - \alpha_i'}{\theta - \alpha_i} \right) \log(\alpha_i - \alpha_j) \right] \qquad (3.4)$$

Identifying the term which multiplies $\log(\theta - \alpha_1)$, we get:

$$S_1'(\theta) + \sum_{j>1} \frac{1}{2} \left(\frac{\alpha_1' - \alpha_j'}{\alpha_1 - \alpha_j} - \frac{\theta' - \alpha_j'}{\theta - \alpha_j} \right) d_{1j} = 0 \qquad (3.5)$$

This is because the $\log(\theta - \alpha_i)$ $(1 \le i \le n)$ are algebraically independent over $k^0(\theta)$ $[\log H_{j_0}$ $(1 \le j_0 \le m)$, $\log(1 - H_{j_0})$ $(1 \le j_0 \le m)$, $\log f_p$ $(1 \le p \le r)$, $\log(\alpha_i - \alpha_j)$ $(i < j)$ $]$, by corollary (3.2).

Now, (3.5) implies:

$$S_1(\theta) + \sum_{j>1} \frac{1}{2} \left(\log(\alpha_1 - \alpha_j) - \log(\theta - \alpha_j) \right) d_{1j} + \text{constant} = 0$$

which, by proposition 3, gives $d_{1j} = 0$ for $j > 1$, and $S_1(\theta) = s_1$ is a constant. By induction we prove easily that $d_{ij} = 0$ for all i,j and that $S_i(\theta) = s_i$ is a constant.

$$\int f = g(\theta) + \sum_{i=1}^{n} s_i \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p + \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) \tag{3.6}$$

where d_{j_0}, s_i are constants, $\alpha_i \in k^o$, and f_p, $H_{j_0} \in k$.

At this point, we distinguish two cases:

Case 1-a: θ is algebraic over k^o ($\log f_p$ $(1 \leq p \leq r)$, $\log H_{j_0}$ $(1 \leq j_0 \leq m)$, $\log(1 - H_{j_0})$ $(1 \leq j_0 \leq m)$). So, by lemma 4, we get:

$$\theta = \sum c_p \log f_p + \sum c_{j_0} \log H_{j_0} + \sum a_{j_0} \log(1 - H_{j_0}) + g$$

where c_p, c_{j_0}, a_{j_0} are constants, and $g \in k^o$.

So, $L_i = \theta - \alpha_i$ is a linear logarithmic expression over k^o ($\log f_p$, ... , $\log H_{j_0}$, ... , $\log(1 - H_{j_0})$), and (3.6) can be written as:

$$\int f - \sum_{j_0=1}^{m} d_{j_0} \ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0 \cdot \ell_2(1 - H_{j_0}) - \sum_{p=1}^{r} 0 \cdot \ell_2(1 - f_p) - \sum s_K \log L_i \in F$$

which implies, by proposition 1, that $\int f$ is a simple elementary-dilogarithmic expression over k and our theorem is proved in this case.

Case 1-b: θ is transcendental over $F = k^o$ ($\log f_p$ $(1 \leq p \leq r)$, $\log H_{j_0}$ $(1 \leq j_0 \leq m)$, $\log(1 - H_{j_0})$ $(1 \leq j_0 \leq m)$). (3.6) can be written as:

$$\int f - \left[\sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) \right]' = g(\theta) + \sum_{i=1}^{n} s_i \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p \tag{3.8}$$

From this, and as in the proof of Liouville's theorem, we deduce that $s_i = 0$ for all $1 \leq i \leq n$.

Also, by proposition 5, we deduce that there exists c, a constant, and $v \in F$ such that:

$$g(\theta) + \sum_{p=1}^{r} s_p(\theta) \log f_p = c\theta + v \quad (\theta = \log a)$$

so:

$$\int f - \sum_{j_0=1}^{n} d_{j_0} \ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0 \cdot \ell_2(1 - H_{j_0}) - \sum_{p=1}^{r} 0 \cdot \ell_2(1 - f_p)$$

$$- c \log a \in F$$

==> by proposition 1 that $\int f$ is a simple elementary-dilogarithmic expression over k, and the theorem is proved in case 1.

Case 2: $K_1 = k(\theta, \theta')$ and $\theta = \ell_2(a)$, where $a \in k$. Let $k_1 = k(\log(1 - a))$. So, $\theta' \in k_1$. If θ is algebraic over k_1, then, by lemma 4, $\theta \in k_1$. So, writing (3.1) again, we have:

$$\int f = g + \sum_{i=1}^{m} s_i \log v_i + \sum_{j=1}^{n} d_j D(h_j) \tag{3.9}$$

where $g, s_i, v_i, h_j \in k_1$.

Then, using case 1 (the logarithmic case), we deduce that $\int f$ is a simple elementary-dilogarithmic expression over k.

So, we consider the case θ transcendental over k_1. As in the previous case, (3.9) can be written:

$$\int f = g(\theta) + \sum_{i=1}^{n} S_i(\theta) \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) + \sum_{i=1}^{n-1} \sum_{j>i} d_{ij} D\left(\frac{\theta - \alpha_i}{\theta - \alpha_j}\right)$$

where $f_p, H_{j_0} \in k_1$, $\alpha_i \neq \alpha_j$, $i \neq j$, and $\alpha_i \in k_1^0$, a normal finite extension of k containing the roots of v_i, h_j, and $(1 - h_j)$ for all i,j.

Now, we use the same argument as in case 1 ($\theta = \log a$) and proposition 3 to deduce:

$$\int f = g(\theta) + \sum_{i=1}^{n} s_i \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p + \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0})$$

where f_p, $H_{j_0} \in k_1$, $\alpha_i \in k_1^0$, and s_i, d_{j_0} are constants.

We also distinguish two cases:

<u>Case 2-a</u>: θ is algebraic over $F_1 = k_1^0$ ($\log f_p$ $(1 \le p \le r)$, $\log(H_{j_0})$ $(1 \le j_0 \le m)$, $\log(1 - H_{j_0})$ $(1 \le j_0 \le m)$).

We apply again the same argument as in case 1-a (using lemma 4), and obtain: $\int f$ is a simple elementary-dilogarithmic expression over k_1 ==> by case 1 and since $f \in k$ that $\int f$ is a simple elementary-dilogarithmic expression over k.

<u>Case 2-b</u>: θ is transcendental over $F_1 = k_1^0$ ($\log f_p$ $(1 \le p \le r)$, $\log H_{j_0}$ $(1 \le j_0 \le m)$, $\log(1 - H_{j_0})$ $(1 \le j_0 \le m)$).

Then, from (3.10) and as in case 1-b ($\theta = \log a$), we deduce that:

$s_i = 0$, for all $1 \le i \le N$

and that there exists c, a constant, and $v \in F_1$, such that:

$$g(\theta) + \sum_{p=1}^{r} s_p(\theta) \log f_p = c\theta + v \quad (\theta = \ell_2(a))$$

(3.10) ==>

$$\int f - \sum_{j_0=1}^{n} d_{j_0}\ell_2(H_{j_0}) - \sum_{j_0=1}^{m} 0 \cdot \ell_2(1 - H_{j_0}) - \sum_{p=1}^{r} 0 \cdot \ell_2(1 - f_p) - c\ell_2(a) \in F_1 = F_1 \left(\log(1 - a) \right)$$

(since $\log(1 - a) \in k_1$) ==> by proposition 1 that $\int f$ is simple elementary-dilogarithmic over k_1 ==> by case 1 that $\int f$ is simple elementary-dilogarithmic over k.

Case 3: $K_1 = k(\theta)$, $\theta = \exp a$, $a \in k$, and θ transcendental over k. As seen before, we can write (3.1) as:

$$\int f = g(\theta) + \underbrace{\sum_{i=1}^{h-1} S_i(\theta)\log(\theta - \alpha_i)}_{\text{only } (n-1) \text{ terms}} + \sum_{p=1}^{r} s_p(\theta)\log f_p$$

$$+ \sum_{j_o=1}^{m} d_{j_o} D(H_{j_o}) + \sum_{i=1}^{n-1}\sum_{j>i} d_{ij}\, D\left(\frac{\theta - \alpha_i}{\theta - \alpha_j}\right) \qquad (3.11)$$

$(\alpha_i \in k^{\circ})$.

In this case, we assume that $\alpha_N = 0$, and $\log\theta \in k$ ($\alpha_i \neq 0$ for $i \neq N$).

The derivative of (3.11) is exactly (3.4), from which we extract the coefficient of $\log(\theta - \alpha_1)$ and use corollary (3.2) to obtain:

$$S_1'(\theta) + \sum_{\substack{j>1 \\ j\neq n}} \frac{1}{2}\left(\frac{\alpha_1' - \alpha_j'}{\alpha_1 - \alpha_j} - \frac{\theta' - \alpha_j'}{\theta - \alpha_j}\right) d_{1j} + \frac{1}{2} d_{1n}\left(\frac{\alpha_i'}{\alpha_i} - \frac{\theta'}{\theta}\right) = 0$$

$$\Longrightarrow \quad S_1(\theta) + \sum_{\substack{j>1 \\ j\neq N}} \frac{1}{2}\left(\log(\alpha_1 - \alpha_j) - \log(\theta - \alpha_j)\right) d_{1j}$$

$$+ \frac{1}{2} d_{1N}\log\alpha_1 - \frac{1}{2} d_{1N} a = \text{constant}$$

(since $\log\theta = a \in k$) ==> (by proposition (4):

$d_{1j} = 0$, for all $j > 1$, $j \neq N$

and:

$$S_1'(\theta) = \frac{1}{2} d_{1n}\left(\frac{\theta'}{\theta} - \frac{\alpha_1'}{\alpha_1}\right)$$

By induction on i, we can now deduce that:

$d_{ij} = 0$, for all i and for all $j > 1$, $j \neq N$

and:

$$S_i'(\theta) = \frac{1}{2} d_{1n} \left(\frac{\theta'}{\theta} - \frac{\alpha_i'}{\alpha_i} \right) \quad (1 \le i \le n-1) \tag{3.13}$$

So, (3.11) becomes:

$$\int f = g(\theta) + \sum_{i=1}^{n-1} S_i(\theta) \log(\theta - \alpha_i) + \sum_{p=1}^{r} s_p(\theta) \log f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} D(H_{j_0}) + \sum_{i=1}^{n-1} d_{1n}\, D\left(\frac{\theta - \alpha_i}{\theta} \right)$$

$$\Longrightarrow f = g'(\theta) + \sum_{i=1}^{n} S_i(\theta) \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{i=1}^{n} S_i'(\theta) \log(\theta - \alpha_i)$$

$$+ \sum_{p=1}^{r} s_p(\theta) \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p'(\theta) \log f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} \left[-\frac{1}{2} \frac{H_{j_0}'}{H_{j_0}} \log(1 - H_{j_0}) + \frac{1}{2} \frac{(1 - H_{j_0})'}{(1 - H_{j_0})} \log H_{j_0} \right]$$

$$+ \sum_{i=1}^{n-1} d_{in} \left[\frac{1}{2} \left(\frac{\theta' - \alpha_i'}{\theta - \alpha_i} - \frac{\alpha_i'}{\alpha_i} \right) (a + c) \right.$$

$$\left. + \frac{1}{2} \left(\frac{\alpha_i'}{\alpha_i} - \frac{\theta'}{\theta} \right) \log(\theta - \alpha_i) + \frac{1}{2} \left(\frac{\theta'}{\theta} - \frac{\theta' - \alpha_i'}{\theta - \alpha_i} \right) \log \alpha_i \right] \tag{3.14}$$

(c is a constant such that $\log \theta = a + c$). In the above expression, the coefficient of $\log(\theta - \alpha_i)$ is zero, as we have seen before.

Now, by corollary 3.1, θ is transcendental over:

$F_0 = k^0 \left(\log \alpha_i \ (1 \le i \le n-1),\ \log H_{j_0} \ (1 \le j_0 \le m),\ \log(1 - H_{j_0}) \ (1 \le j_0 \le m),\ \log f_p \ (1 \le p \le r) \right)$.

On the other hand, we choose the $\log f_p$ $(1 \le p \le r)$ in such a way that they are linearly independent and transcendental over k^0. Then, by lemma 3 and corollary (3.1), they are algebraically independent over $k^0(\theta)$.

From (3.14), we deduce that there exist subsets J_p, I_p, T_p such that:

$$s_p'(\theta) + \sum_{j_0 \varepsilon J_p} \left(-\frac{1}{2} \frac{H_{j_0}'}{H_{j_0}} d_{j_0} \right) + \sum_{j_0 \varepsilon I_p} \frac{1}{2} \frac{(1 - H_{j_0})'}{(1 - H_{j_0})} d_j$$

$$+ \sum_{i \varepsilon T_p} \frac{1}{2} d_{in} \left(\frac{\theta'}{\theta} - \frac{\theta' - \alpha_i'}{\theta - \alpha_i} \right) = 0 \qquad (3.15)$$

(this is the coefficient of log f_p; j_p, I_p, T_p exist because log a_i, log H_{j_0}, log$(1 - H_{j_0})$ could depend on log f_p).

By proposition 4, we deduce that $d_{in} = 0$ for all $i \varepsilon T_p$. So, $s_p'(\theta) \varepsilon k \Longrightarrow s_p(\theta) = s_p \varepsilon k$ by proposition 5 (for all p).

So, (3.14) becomes:

$$f = g'(\theta) + \sum S_i(\theta) \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{p=1}^{r} s_p \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p' \log f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} \left[-\frac{1}{2} \frac{H_{j_0}'}{H_{j_0}} \log(1 - H_{j_0}) + \frac{1}{2} \frac{(1 - H_{j_0})'}{(1 - H_{j_0})} \log H_{j_0} \right]$$

$$+ \sum_{i=1}^{n-1} d_{in} \left[\frac{1}{2} \left(\frac{\theta' - \alpha_i'}{\theta - \alpha_i} - \frac{\alpha_i'}{\alpha_i} \right) (a + c) + \frac{1}{2} \left(\frac{\theta'}{\theta} - \frac{\theta' - \alpha_i'}{\theta - \alpha_i} \right) \log \alpha_i \right] \qquad (3.16)$$

But, from (3.13), we had:

$$S_i'(\theta) = \frac{1}{2} d_{in} \left(\frac{\theta'}{\theta} - \frac{\alpha_i}{\alpha_i} \right)$$

$$\Longrightarrow \quad S_i(\theta) = \frac{1}{2} d_{in} (a - \log \alpha_i) + c_i, \quad c_i \text{ is a constant} \qquad (3.17)$$

So, $S_i \varepsilon F_0 = k^0$ (log α_i $(1 \le i \le n-1)$, log H_{j_0} $(1 \le j_0 \le m)$, log$(1 - H_{j_0})$ $(1 \le j_0 \le m)$, log f_p $(1 \le p \le r)$).

Computing the coefficient of $(\theta - \alpha_i)'/(\theta - \alpha_i)$ in (3.16), we get:

$$g'(\theta) + \sum_{i=1}^{n-1} \left(S_i + \frac{1}{2} d_{in} \left[(a + c) - \log \alpha_i \right] \right) \frac{\theta' - \alpha_i'}{\theta - \alpha_i} \varepsilon F_0$$

Considering the partial fraction of $g(\theta)$, we can prove, as in the proof of Liouville's theorem, that (since $\alpha_i \neq 0$):

$$S_i + \frac{1}{2} d_{in} \left[(a + c) - \log \alpha_i \right] = 0, \quad \text{for all } i \leq n - 1 \tag{3.18}$$

Comparing with (3.17), we deduce that:

$$d_{in} \left[a - \log \alpha_i \right] = \text{constant}, \quad \text{for all } i \leq n - 1 \tag{3.19}$$

We claim that $d_{in} = 0$, otherwise we would have:

$$a' - \frac{\alpha_i'}{\alpha_i} = 0 \implies \frac{\theta'}{\theta} - \frac{\alpha_i'}{\alpha_i} = 0 \implies N_0 \frac{\theta'}{\theta} - \frac{\text{Norm}(\alpha)'}{\text{Norm}(a)} = 0 \tag{3.20}$$

where $N_0 = [k^0:k]$, and Norm is the usual norm from k^0 to k.

So, (3.20) implies:

$$(\theta^{N_0} \text{Norm}(\alpha))' = 0 \implies \theta^{N_0} \in k \implies \text{contradiction}$$

and:

$$d_{in} = 0, \quad \text{for all } 1 \leq i \leq n-1$$

which implies that $S_i' = 0$ by (3.18) $\implies S_i = \text{constant} \implies$

$$f = g'(\theta) + \sum_{i=1}^{n-1} S_i \frac{(\theta - \alpha_i)'}{(\theta - \alpha_i)} + \sum_{p=1}^{r} s_p \frac{f_p'}{f_p} + \sum_{p=1}^{r} s_p' \log f_p$$

$$+ \sum_{j_0=1}^{m} d_{j_0} \left(- \frac{1}{2} \frac{H_{j_0}'}{H_{j_0}} \log(1 - H_{j_0}) + \frac{1}{2} \frac{(1 - H_{j_0})'}{(1 - H_{j_0})} \log H_{j_0} \right)$$

Let $F_{00} = k^0 \left(\log f_p \ (1 \leq p \leq r), \log H_{j_0} \ (1 \leq j_0 \leq m), \log(q - H_{j_0}) \ (1 \leq j_0 \leq m) \right)$. θ is transcendental over F_{00}, and, as in the proof of Liouville's theorem, we get $S_i = 0$, for all i. Also, we get $g(\theta) = g \in F_{00}$, by proposition 5.

$$\int \left[f - \left[\sum_{j_o=1}^{m} d_{j_o} D(H_{j_o}) \right]' \right] = g + \sum s_p \log f_p$$

$$\Longrightarrow \int f - \sum_{j_o=1}^{m} d_{j_o} \ell_2(H_{j_o}) - \sum_{j_o=1}^{m} 0 \cdot \ell_2(1 - H_{j_o})$$

$$- \sum_{p=1}^{r} 0 \cdot \ell_2(1 - f_p) = g \in F_{oo}$$

==> $\int f$ is a simple elementary-dilogarithmic expression over k by proposition 1, so the theorem is proved.

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