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RELAXATION OSCILLATIONS IN AIRCRAFT CRUISE-DASH OPTIMIZATION

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Abstract

Periodic solutions in energy approximation are sought for aircraft optimal cruise-dash problems. The cost functional is the weighted sum of the time taken and the fuel used averaged over one cycle. It is known from previous work that in energy-state approximation, relaxed-steady-state controls give lower costs than steady-state solution. However, this control is not implementable. Higher approximations to this are sought via averaging oscillations. The "fast" dynamics (path-angle / altitude / throttle / lift-coefficient) is modeled in terms of periodic solutions in a boundary-layer-like motion which does not die out, but moves along with the progression of the "slow" state, energy. This is shown not to help the situation. A better approximation in terms of relaxation oscillations is proposed. Unlike earlier models, the energy is allowed to vary. However, the net change in energy per cycle is zero. Fast, constant-energy climbs and descents and slow energy transitions are "spliced" together in zeroth order approximation to obtain the periodic solutions. The energies in question are determined as part of the problem. This technique is shown to produce a more practical solution, but still needs improvement for practical application.

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Lis t of Symbols

C_D . . . Drag coefficient

C_L . . . Lift coefficient

E . . . . Specific energy (m)

H . . . Hamiltonian

h . . . . Altitude (m)

J . . . . Cost Functional (N/m)

E . . . . Specific energy (m)

g . . . . Acceleration due to gravity (m/s^2)

H . . . Pseudo-Hamiltonion

h . . . . Altitude (m)

J . . . . Cost functional (N/m)

J . . . . Total cost of segment of relaxation oscillation (N)

L . . . . Lift (N)

Q . . . Maximum fuel flow rate (N/s)

R . . . Range (m)

R_F . . . Final range, wavelength (m)

t . . . . Time (s)

T . . . . Maximum thrust (N)

V . . . Airspeed (m/s)

W . . . Aircraft weight (N)

W_F . . . Fuel used (N)

Greek Symbols.

ε . . . . Perturbation parameter

η . . . . Throttle setting

γ . . . . Flight path-angle (rad.)

λ . . . . Co-state vector

λ_x . . . Co-state variable

μ . . . . Lagrange parameter

θ . . . . Weight in cost functional (N/s)

ρ . . . . Air density (kg/cu.m)

σ . . . . TSFC (N/N/s)

Introduction.

Before solving the cruise-dash solution in point-mass modeling, it is instructive to study the nature of the solutions, by restricting the problem through assumptions. The energy approximation has a long history in flight-path optimization [3]. This is a good starting point for the discussion of the cruise-dash problem. The steady-state analysis of the cruise-dash problem in energy approximation has been studied by several researchers [1,7,10]. For the cruise problem, Ref. [10] showed that the hodograph is non-convex and that "chattering" solutions can attain lower costs. Figure 1 shows a generic cruise-dash non-convex energy hodograph. The cost is plotted against the energy-rate for all available controls. The relaxed-steady-state solution (D) achieves lower costs. However, this solution has several shortcomings. The solution cannot be practically implemented. The aircraft cannot fly steady and level at points A and B. The altitude transitions are unpenalized. The gain over steady-state solutions justifies further analysis based on the relaxed-steady-state solution.

Averaging Oscillations

In the averaging oscillations, the idea is to model the "fast" (path-angle / altitude / throttle / lift-coefficient) motion in terms of periodic control in a boundary-layer-like motion which does not die out, but moves with the progression of the "slow" state (energy). This averaging type of approximation is familiar in initial-value problems for uncontrolled dynamic systems (the Bogoliubov -type averaging theory [9]). Reference [5] has a low thrust-orbit transfer example. This represents a contrast to the usual singular perturbation procedure, where the transients die out. The equations of motion are the energy-state equations

$$J = \min_u \int_0^{R_f} \frac{\eta Q + \theta}{V \cos \gamma}\, dR \tag{1}$$

In this form of the cruise-dash problem, q represents the cruise problem (minimum-fuel). As $\theta \rightarrow \infty$, the emphasis on time goes high and the problem can be identified as the minimum-time or the dash problem. Between these two extremes are the cruise-dash problems. The system of equations are:

$$\begin{aligned} \varepsilon h' &= \tan \gamma \\ \varepsilon \gamma' &= \frac{g}{V^2}\left[\frac{L}{W \cos \gamma} - 1\right] \\ E' &= \frac{\eta T - D}{W \cos g} \end{aligned} \tag{2}$$

In this analysis, the slow variable is frozen in the study of the fast motions. But unlike the normal perturbation approach, the net change in energy is achieved only on the average. Also, the fast motion has to be solved for first, for a range of the slow variables. Only then can the slow motion be solved for.

Fast Motions

The equations of motion for the fast motions are

$$\begin{aligned} \varepsilon h' &= \tan \gamma \\ \varepsilon \gamma &= \frac{g}{V^2}\left[\frac{L}{W \cos \gamma} - 1\right] \end{aligned} \tag{3}$$

$$\Delta E = \frac{1}{R_f} \int_0^{R_f} \frac{\eta T - D}{W \cos \gamma} dR$$

Here, V = √2g(E-h), and the energy, E is frozen. The first two are the usual differential equations. The last equation is the statement that the energy-change be obtained only on the average. For a steady cruise-dash problem involving no net energy-changes, $\Delta E = 0$.

Periodic solutions satisfying :

$$h(R_f) = h(0) \quad (\text{open})$$

$$\gamma(R_f) = \gamma(0) \quad (=0) \qquad (4)$$

are sought. The wavelength R_f of the oscillations is unknown and has to be determined as part of the problem. A trivial solution consisting of constant values of h and g rules out sustained oscillations and reduces to the standard singular perturbation problem. Nontrivial solutions satisfying these conditions are sought here.

Necessary and boundary conditions for periodic solutions are found in the literature. However, the current situation is a little different because of the isoperimetric constraint. In Ref.[13], the equations and boundary conditions for a optimal periodic control problem with isoperimetric constraints are derived. The equations of motion were given in (2).

The isoperimetric constraint to augmented to the cost function. Then the Hamiltonian is:

$$H = \frac{\eta Q + \theta}{V \cos \gamma} + \lambda_h \tan \gamma + \frac{\lambda_\gamma g}{V^2} \left[\frac{L}{W \cos \gamma} - 1 \right] + \frac{\mu (\eta T - D)}{W \cos \gamma} \qquad (7)$$

The co-states satisfy:

$$\lambda_h' = -\frac{\partial H}{\partial h} = \frac{\eta Q + \theta}{V^2 \cos \gamma} V_h - \frac{\eta (Q_h + Q_V V_h)}{V \cos \gamma} + \frac{2 \lambda_\gamma g V_h}{V^3} - \frac{\lambda_\gamma \gamma \rho_h S C_L}{2 W \cos \gamma} - \frac{\mu (\eta T_h - D_h + D_V V_h)}{W \cos \gamma}$$

where, $V_h = \frac{\partial V}{\partial h} = -\frac{g}{V}$

$$\lambda_\gamma' = -\frac{\sin\gamma}{V\cos^2\gamma}\left[\eta Q + \theta + \frac{\lambda_\gamma g L}{V^2 W} + \frac{\mu(\eta T - D)}{W}\right] + \frac{\lambda_h}{\cos^2\gamma}$$

The lift-coefficient is obtained from:

$$\frac{\partial H}{\partial C_L} = 0 \Rightarrow \lambda_\gamma g - V^2\mu\frac{\partial C_D}{\partial C_L} = 0$$

The throttle is given by:

$$\eta = 1 \quad \text{if} \quad S < 0$$
$$= 0 \quad \text{if} \quad S > 0$$
$$= ? \quad \text{if} \quad S \equiv 0$$

where, $S = \frac{\partial H}{\partial \eta} = \frac{Q}{V\cos\gamma} + \frac{\mu T}{W\cos\gamma}$

The periodic boundary conditions are (Ref. thesis)

$$h(X_f) = h(0)$$
$$\gamma(X_f) = \gamma(0)$$
$$\lambda_h(X_f) = \lambda_h(0)$$
$$\lambda_\gamma(X_f) = \lambda_\gamma(0)$$

The condition for the Lagramge parameter is:

$$\int_0^{R_f} \frac{\eta T - D}{V\cos\gamma}\, dR = 0$$

The condition fo r the free wavelength is:

$$H = J = \frac{1}{R_f}\int_0^{R_f} \frac{\eta Q + \theta}{V \cos\gamma} dR$$

The Slow Motions

The fast motions have to be solved for a range of values of the show parameter, here E. Using the value of the Lagrange parameter :f.mu:ef., that emerges from the analysis of the fast motions, as a control, the slow motions have to be analyzed as:

$$\dot{E} = \frac{\Delta E}{X_f}(E, \mu)$$

Energy is the state variable and the parameter, m is the control.

For the steady cruise-dash problem, the average energy-rate is zero. Thus, $\Delta E = 0$ for one cycle and the cost per cycle is (from the fast-motion analysis):

$$J = \frac{1}{R_f}\int_0^{R_f} \frac{\eta Q + \theta}{V \cos\gamma} dR$$

This means that the slow motion has, for steady cruise-dash, a trivial solution where, the "frozen" energy in the fast solution remains constant. This motion consistes of continuous fast oscillations superimposed on a steady cruise-dash. The average cost the complete maneuver is the the average cost the fast oscillations.

This average copst is a function of the (frozen) energy. The cost is thus a one parameter family, with energy as the parameter. To get the lowest cost, a one dimensional search has to be conducted and the best cruise-dash energy thus determined by selection.

Numerical Results

This approach was applied to the F4 aircraft model . To solve the resulting NL2PBVP numerically, the multiple shooting technique of refBulirsch] was used. Initially, the simplest case of cruise ($\theta = 0$) was studied.

First, the value of energy was arbitrarily chosen (E = 14 km). The problem was solved for various wavelengths. Figure 2 shows the altitude histories of the oscillations against the range normalized by the wavelength, for a range of wavelengths. Figure 3 shows the path-angle histories against the same variable. It can be seen from these figures, that as the wavelength increases, there is a marked dwell at an upper and a lower altitude. It was found that these altitudes were the "chattering" altitudes for that energy. Thus, as the final range increased, the averaging oscillations approach the time-sharing solution.

Figure 4 shows the costs averaged over one period and the (constant) Hamiltonian as a function of the wavelengths. It shows that the cost approaches a minimum only asymptotically as the wavelength becomes very large. The condition H = J is only met :t a point at infinity. This means that the best maneuver is the time-shared maneuver. The best energy is then the one picked by the chattering solution. This is disappointing in that the averaging solution does no better than the time-sharing solution. All the shortcomings of the time-sharing solution are still present.

The same trends were seen at a different energy.

A higher approximation or a better technique is needed. A possible answer is a "relaxation oscillation" described in the next section.

Relaxation Oscillations for Cruise-Dash

One of the shortcomings of the averaging method is that the energy is constant. As was seen, the best averaging oscillation turned out to be the time-sharing solution of Ref.Houlihan]. At constant energy, sustained level flight is not possible at zero or full throttle. A possible remedy to this is offered in the form of relaxation oscillations.

The major step in the development of the theory is to remove the restriction of constant energy. As a chattering control achieves lower costs than the steady-state solution, this aspect of the problem cannot be ignored. The solution has to start from there.

This section analyzes the proposed relaxation oscillations in detail. The theory is applied to the numerical model. Figure 5 ilustrates the proposed modelling in the altitude - airspeed chart for an aircraft. A family of constant energy is shown. At each energy, there are two points between which the chattering takes place. One is a lower-altitude higher-speed thrusting point and the other a higher-altitude lower-speed coasting point. The loci of such points are lines for zero thrust and for full thrust .

A relaxation oscillation is shown in the chart by the sequences A-B-C-D. The individual segments of the flight can be described as:

A Constant energy climb; full throttle with presumed transition to zero

B Glide to a lower energy; throttle zero

C Constant energy descent ; the throttle is assumed to switch from 0 to 1.

D Climb back to the original states; throttle is at a maximum.

The motion takes place at different time-scales. Segments A and C are similar to the inner layers in the singular perturbation analysis of cruise-dash Shankar]. These are the fast motions (hence they will be referred to as transients). Segments B and D are similar to the slow outer solutions to the same problem. The two kinds of motions are assumed to alternate: constant energy transients and energy transitions. The total cost for one cycle is computed as the sum of the costs for each segment. The average cost per cycle is the ratio of the total cost to the total range obtained in one cycle. The two energies are parameters to be determined as part of the problem.

The next two sections analyze the two different types of motion in detail.

Energy Transition

The general equations of motion are (in singular perturbation notation)

$$\varepsilon h' = \tan\gamma$$

$$\varepsilon\gamma' = \frac{g}{V^2}\left[\frac{L}{W\cos\gamma} - 1\right]$$

$$E' = \frac{\eta T - D}{W\cos\gamma}$$

The average cost is given by:

$$J = \frac{1}{R_f}\int_0^{R_f}\frac{\eta Q + \theta}{V\cos\gamma}\,dR$$

The energy transitions are the slow motions. In the spirit of the singular perturbation theory, taking the limit as $\varepsilon\to 0$,

$$h' = 0 \Rightarrow \gamma = 0$$

$$\gamma' = 0 \Rightarrow L = W \quad \text{(which determines } C_L \text{)}$$

Then, the sole state equation is

$$E' = \frac{\eta T - D}{W}$$

where, the drag is evaluated at lift equals weight. The controls are the throttle (η), and the altitude (h). The Hamiltonian is given by the cost integrand plus E' with the costate λ_E:

$$H = \frac{\eta Q + \theta}{V} + \lambda_E\frac{\eta T - D}{W}$$

As in the singular perturbation analysis of the cruise-dash problem Cliff,Chichka], two sets of controls (h_c, η_c) emerge, one for increasing energy (for segment D) and one for decreasing the energy (segment B). As the Hamiltonian is linear in h, the two throttle positions are, obviously, zero and full for decreasing and increasing the energy respectively. Then :

$$\dot{E} = -D/W < 0 \quad \text{for } \eta = 0$$

$$\dot{E} = (T-D)/W > 0 \quad \text{for } \eta = 1$$

One way to obtain the altitudes corresponding to these is from the energy hodograph. The two tangency points of the convex hull (points A and B in Figure 1) are points in question. The graphical method is, of course, not accurate. These points are also singular points in the boundary-layer state-space. So analytical methods can be used to determine the altitudes. This is discussed in the next section. The important point to be made here is that there are two commanded altitudes, one corresponding to $\eta = 0$ and the other to $\eta=1$. For now, these are presumed to be known functions of the energy.

The total cost for this segment is given by:

$$\tilde{J}_i = \int_0^{R_f} \frac{\eta Q + \theta}{V} dR$$

The boundary conditions are the two specified energies :

$E(0) = E_0$; $E(R_f) = E_f$

The unknown final range is determined from the second boundary condition. This is a fairly simple two-point boundary-value problem and is easily solved using any method.

The multiple shooting method [4] was used to solve the problem numerically. Figure 6 shows the downrange obtained for energy transition between 6 km. and 17 km. for the cruise case ($\theta = 0$). Both case of increasing and decreasing energies are shown on the same plot. Figure 7 shows the cost of increasing the energy between the same limits. Note the decreasing energy costs nothing in the cruise case. For a cruise-dash problem, there is a non-zero cost for decreasing energy, because of the non-zero time taken.

This is all the information that is needed to analyze this motion for any transitional energies in the range [6 km., 17km.]. This is because

$$\tilde{J}(E_0, E_2) = \tilde{J}(E_0, E_1) + \tilde{J}(E_1, E_2)$$

Constant Energy Transient

In this section the constant energy segments of the relaxation oscillations (A and C) are discussed. This transient is the boundary-layer motion from one chattering point to the other, (h_{c0}, h=0) to (h_{c1}, h=1) and vice-versa. Before analyzing the problem, the command altitudes are determined analytically.

Command Altitudes

The command altitudes occur at the singular points in the state space of the boundary-layer at that energy for h=0 and h=1. Proceeding formally, the equations in SPT notation are those of equation 17. The energy, E, and its co-state, λ_E are frozen in the boundary layer. The hamiltonian is given by equation 7. λ_E: has to be determined. For this, consider the steady-state singular arc:

$$\dot{h} = 0 \Rightarrow \gamma = 0$$

$$\dot{\gamma} = 0 \Rightarrow L = W \quad (\text{determines } C_L)$$

$$\dot{E} = 0 \Rightarrow \eta = D/T\ .$$

Moreover for a singular arc,

$$\frac{\partial H}{\partial \eta} = 0 \Rightarrow \lambda_E = -\frac{\sigma W}{V}$$

where, σ and V are evaluated at the steady-state conditions:

$$\frac{\partial H}{\partial h} = \frac{\partial H}{\partial \gamma} = \frac{\partial H}{\partial C_L} = 0$$

The states are the altitude and path-angle governed by:

$$h' = \tan\gamma$$
$$\gamma' = \frac{g}{V^2}\left[\frac{L}{W\cos\gamma} - 1\right]$$

The singular points points in this state-space with $\eta = 0$ or 1 are determined by the following equations:

$$h' = 0$$

$$\gamma' = 0$$

$$\lambda_h' = \frac{\partial H}{\partial h} = 0$$

$$\lambda_\gamma' = -\frac{\partial H}{\partial \gamma} = 0$$

where the Hamiltonian is given by equation 6. Also, by the Pontryagin principle,

$$H_{C_L} = 0$$

Note that these derivatives are taken at constant energy. These three equations can be solved for the unknown variables, in particular for the altitudes. The command altitudes for the two thrust levels are now known.

Equations for the Transient Motion

The optimal control problem to be solved for the transients is:

$$\text{Min } \tilde{J}_i = \int_0^{R_f} \frac{\eta Q + \theta}{V \cos \gamma} \, dR$$

for i =A or C. The states satisfy the differential equations ( ). The net change in energy is zero on the average and so the following parametric constraint must be satisfied:

$$\int_0^{R_f} \frac{\eta T - D}{W \cos\gamma} \, dR = 0$$

The cost is augmented by the constraint with a suitable Lagrange parameter m in the usual way [2]. Then the Hamiltonian is given by equation ( ). The Euler-lagrange equations for the co-states are given by equations 8 and 9. The controls C_L and η are given by equations 10 and 11. The throttle is assumed to be bang-bang. At one end, η=0 and at the other, η=1. Therefore, there must be an odd number of throttle switchings.

The boundary conditions for the problem are:

$$h(0) = h_{c0} \quad ; \quad h(R_f) = h_{cf}$$

$$\gamma(0) = 0 \quad ; \quad \gamma(R_f) = 0$$

$$\text{and} \quad \int_0^{R_f} \frac{\eta T - D}{W \cos \gamma} \, dR = 0$$

h_{c0} and h_{cf} depend on the energy and whether the aircraft is climbing (segment A) or descending (segment C).

In fact, only one of the segments needs to be investigated as the trajectory for the other segment can be determined from the following transformations:

$$h(R)\,|_1 = h(R_f - R)\,|_2$$

$$\gamma(R)\,|_1 = -\gamma(R_f - R)\,|_2$$

$$\lambda_h(R)\,|_1 = -\lambda_h(R_f - R)\,|_2$$

$$\lambda_\gamma(R)\,|_1 = \lambda_\gamma(R_f - R)\,|_2$$

The cost for either trajectory is the same. The final range R_f is free and has to be determined. The application of the optimality conditions has reduced the problem to a nonlinear two-point boundary-value problem.

The NL2PBVP for the aircraft was solved numerically using the multiple-shooting method of Bulirsch]. Figure 8 shows the altitude against the downrange normalized by R_f. for several values of the final range, for the cruise case (h = 0). Figure 9 shows the path-angle history for the same case. The (constant) value of the energy was 10 kilometers.

It is important to remember that these transients take place at a different time-scale than the outer-layer energy transitions. In this layer, the downrange is frozen. In the zeroth order analysis currently being pursued, the constant energy transients receive no credit for downrange achieved. However, there is a finite non-zero cost of transition. To find this cost of transition at zero range, one has to find the costs for several final ranges and extrapolate the results to zero final range. This transition cost must be determined for all values of transition energy. The cost of the ascent trajectory is exactly equal to the cost of the descent trajectory. Then the transition costs are:

$$\tilde{J}_i \equiv \tilde{J}_i(E_i) \quad \text{for } i = A \text{ or } C$$

Figure 10 shows the total transition costs for the cruise case at an energy of 10 kilometers.as a function of the final range. Th· ..ansition cost seems to be fairly linear. The solid line in the figure is the least-squares linear fit. The cost at zero final range is easily determined numerically. Figure 11 shows the costs for constant-energy transients for several values of energy, for the cruise case. Using cubic-spline interpolation , the transient costs can be easliy found for any energy in that range.

The Best Relaxation Maneuver

So far, the individual segments of the relaxation oscillations have been discussed at depth. When, the segments are "spliced" together, these oscillations form a two-parameter family, the parameters being the two energies. The average cost per oscillation (in N/m) is then given by:

$$J(E_1, E_2) = \frac{\tilde{J}_A + \tilde{J}_B + \tilde{J}_C + \tilde{J}_D}{R_{fB} + R_{fD}}$$

where, the individual components represent the total cost of segment i, determined by the appropriate equations (5.9) or (5.33). R_B and R_D are the ranges achieved by segments B and D respectively. Let $E_1 < E_2$ (without loss of generality) and :f. $\Delta E = E_2 - E_1$. Using these two parameters rather than E_1 and E_2, the cost function is:

$$J \equiv J(E_1, \Delta E)$$

Now, $\Delta E = 0$ represents oscillations at one constant energy, which is similar to the averaging oscillations of the previous chapter. This is to be avoided and so it is necessary that $\Delta E > 0$.

Figure 12 shows the average cost of relaxation oscillations for the cruise case as a function of the energy increase for different lower energies. Figure 13 shows the same quantities for a cruise-dash case (h = 1.6 N/s). It is now a simple matter to find the values of E_1 and ΔE that minimize the cost function (equation ...). To find the solution numerically, a sequential linear least-squares method was used to solve the nonlinear programming problem. The cost for arbitrary values of E_1 and ΔE was determined by a cubic-spline interpolation between computed data points. With this, it is a simple matter to obtain the gradients of the cost function with respect to the independent variables. The following table summarizes the results: Figure 14 shows the resulting trajectory for one case (θ=0). It is seen that there are four corners in the trajectory. This is due to the fact that the lift-coefficient jumps at these points. Such inconsistencies are to be expected in a zeroth-order analysis. Higher order approximations should reveal the true nature of the trajectories. At q=0, the resulting costs are a little higher than the costs achieved by the chattering solution. This is due to the fact that the relaxation oscillations make some allowances for the transient costs.

Figure 15 is a summary of the whole exercise. Here, average costs are shown against energy for the steady-state, chattering and the relaxation oscillation solutions. Also shown is the point-mass periodic solution of Ref.Grimm]. Figure 16 shows the same quantities for a cruise-dash case (θ = 1.6 N/s). It is seen from Figure 16 that for the case, θ = 1.6 N/s :ef., relaxation costs are higher than steady-state (even though the chattering solutions are not). This fact gives support to the argument given in Refs[1,13] that if there is enough emphasis on time, the steady-state solution is globally optimal.

Table 1. Averaging Costs

Theta (N/s)	E1 (km)	ΔE (km)	Cost (N/km)
0.0	11.8425	3.155	2.3545
1.6	15.5376	0.8274	8.3775

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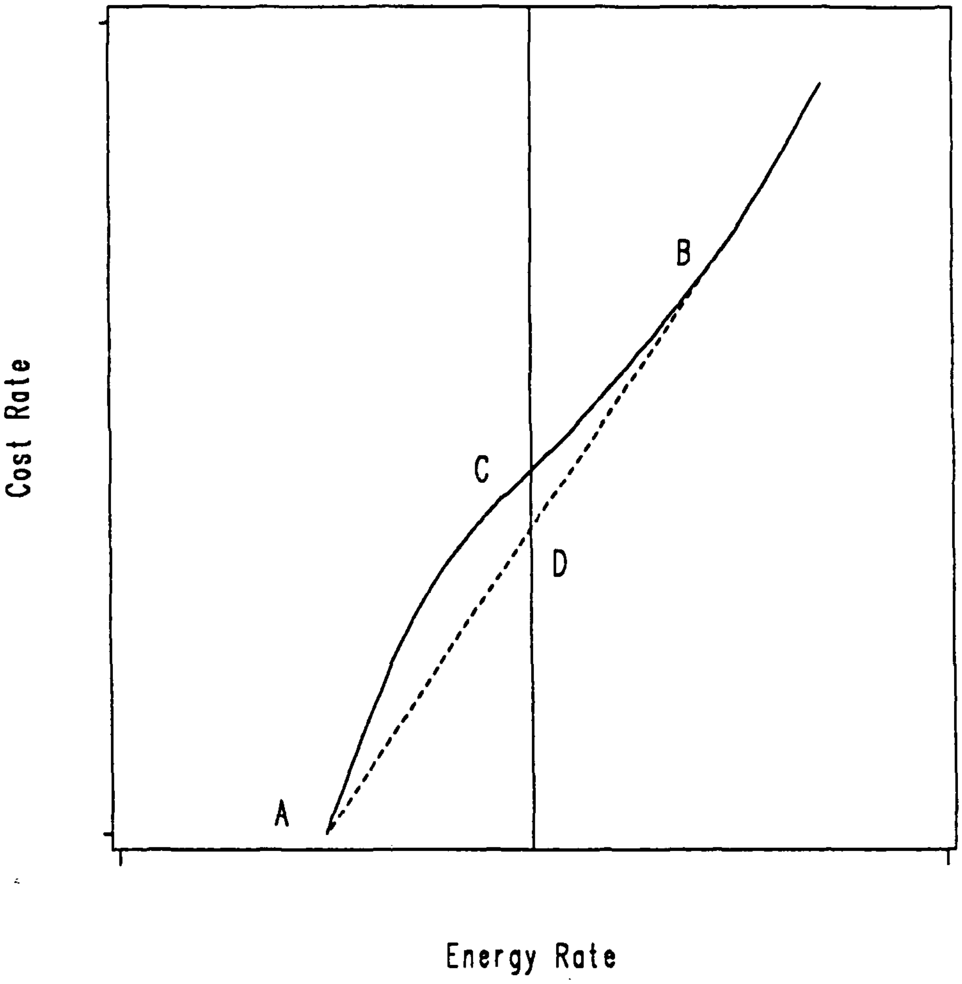

Figure 1. Generic Non-convex Energy Hodograph.

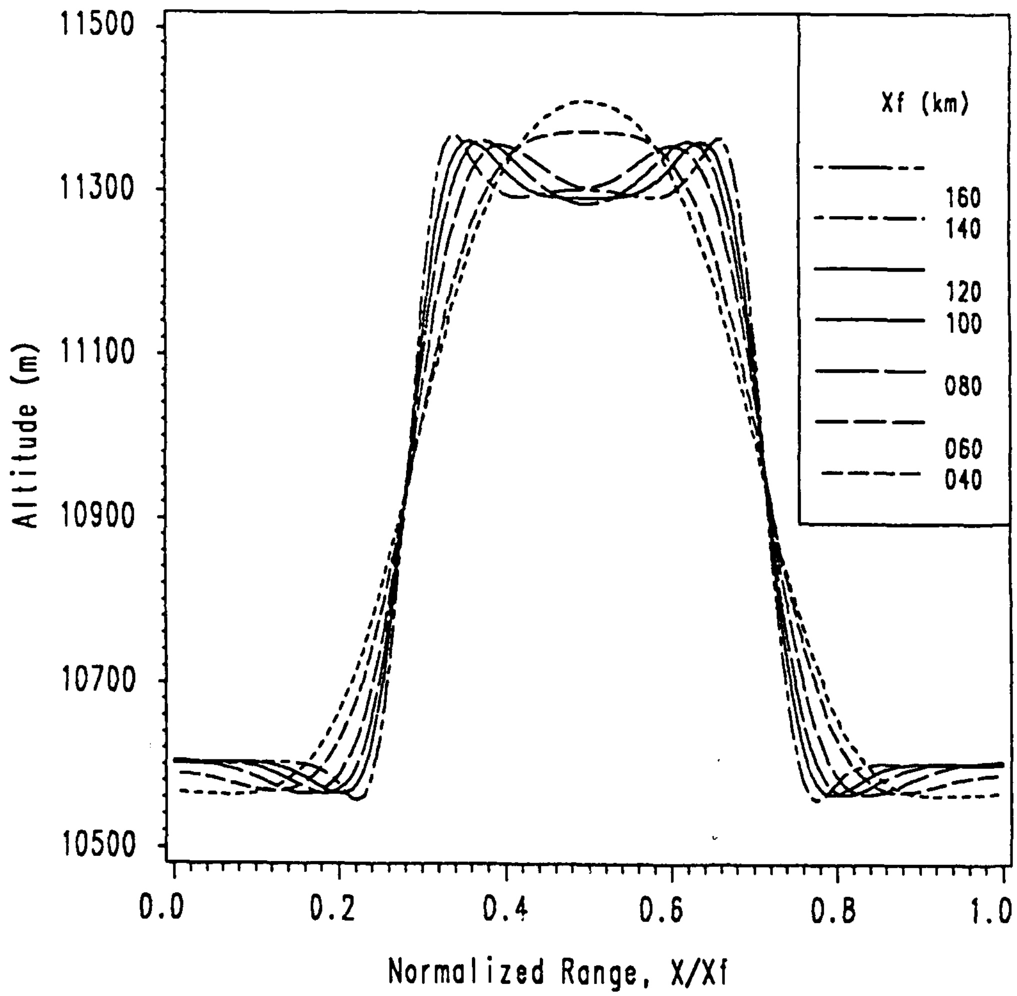

Figure 2. Averaging Oscillations: Altitude vs. Normalized Range.

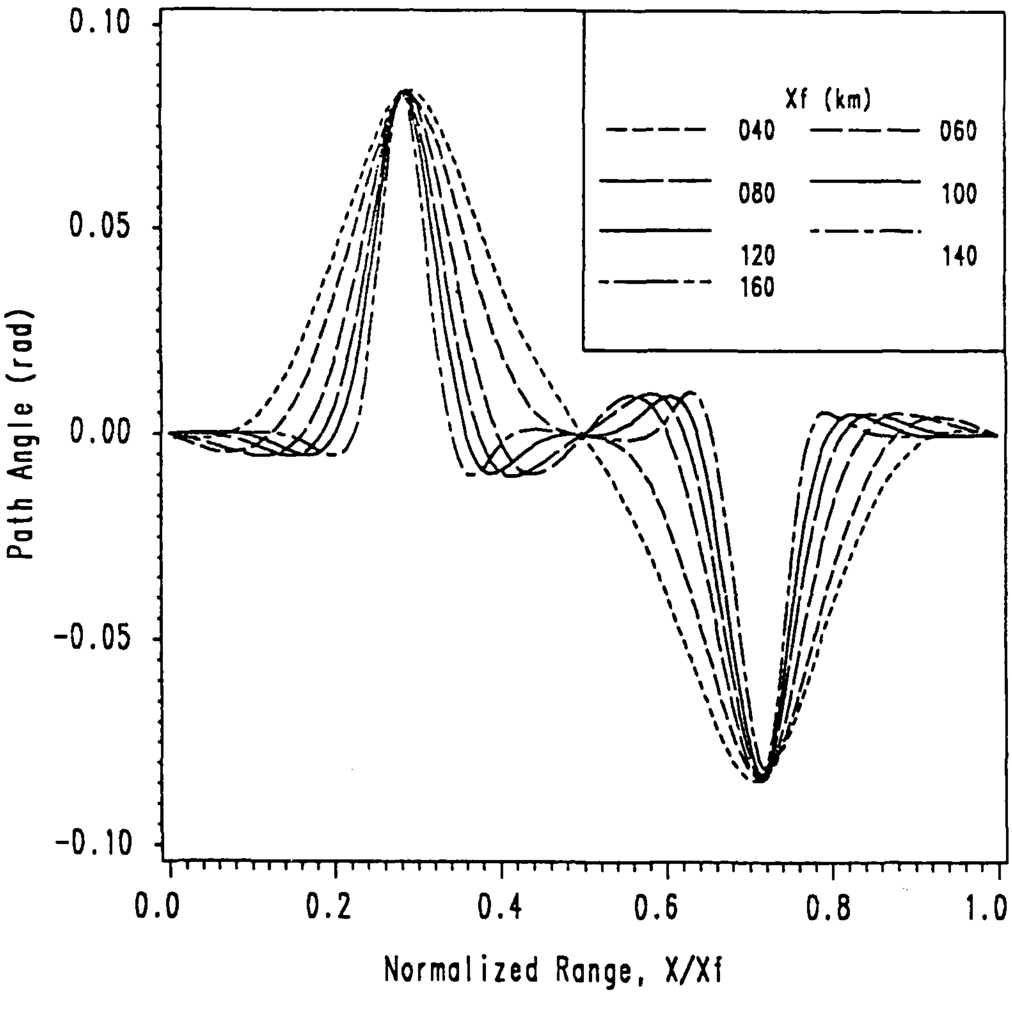

Figure 3. Averaging Oscillations: Path-Angle vs. Normalized Range.

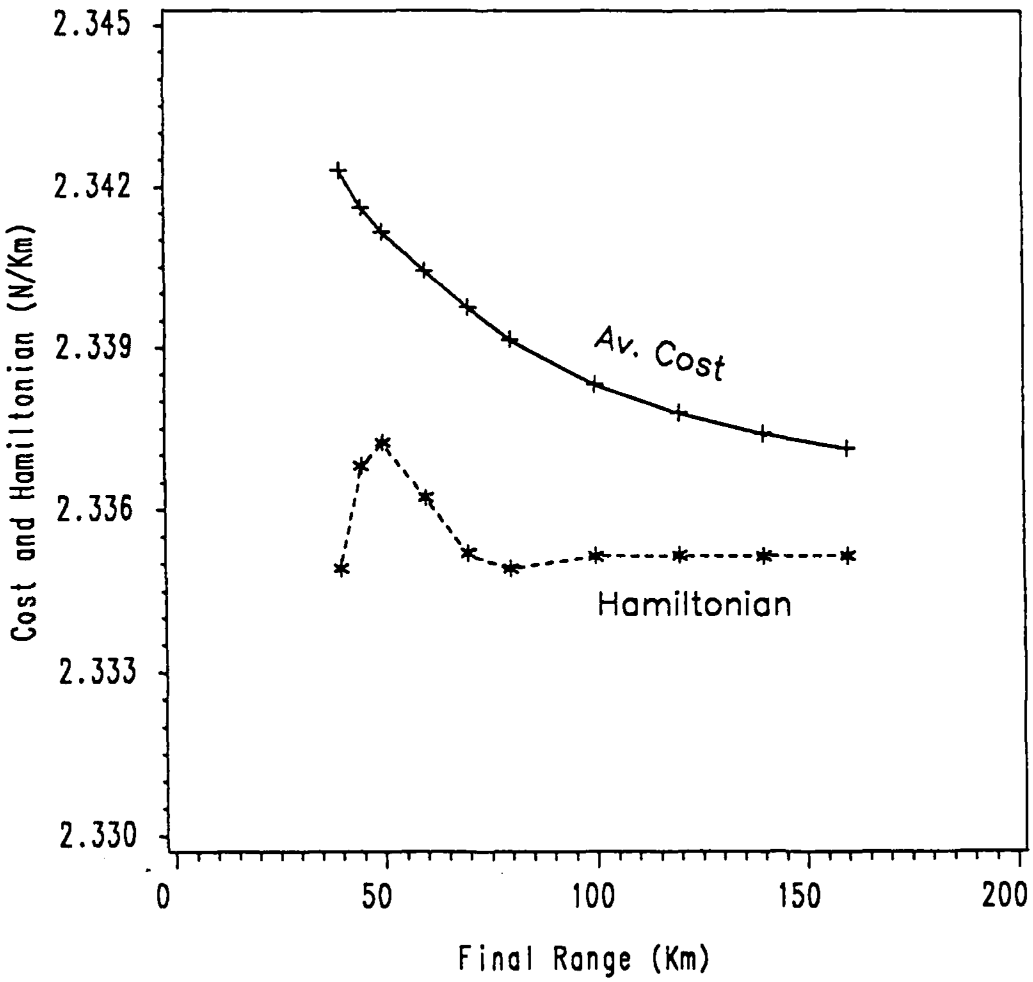

Figure 4. Averaging Oscillations: Average Cost vs. Wavelength.

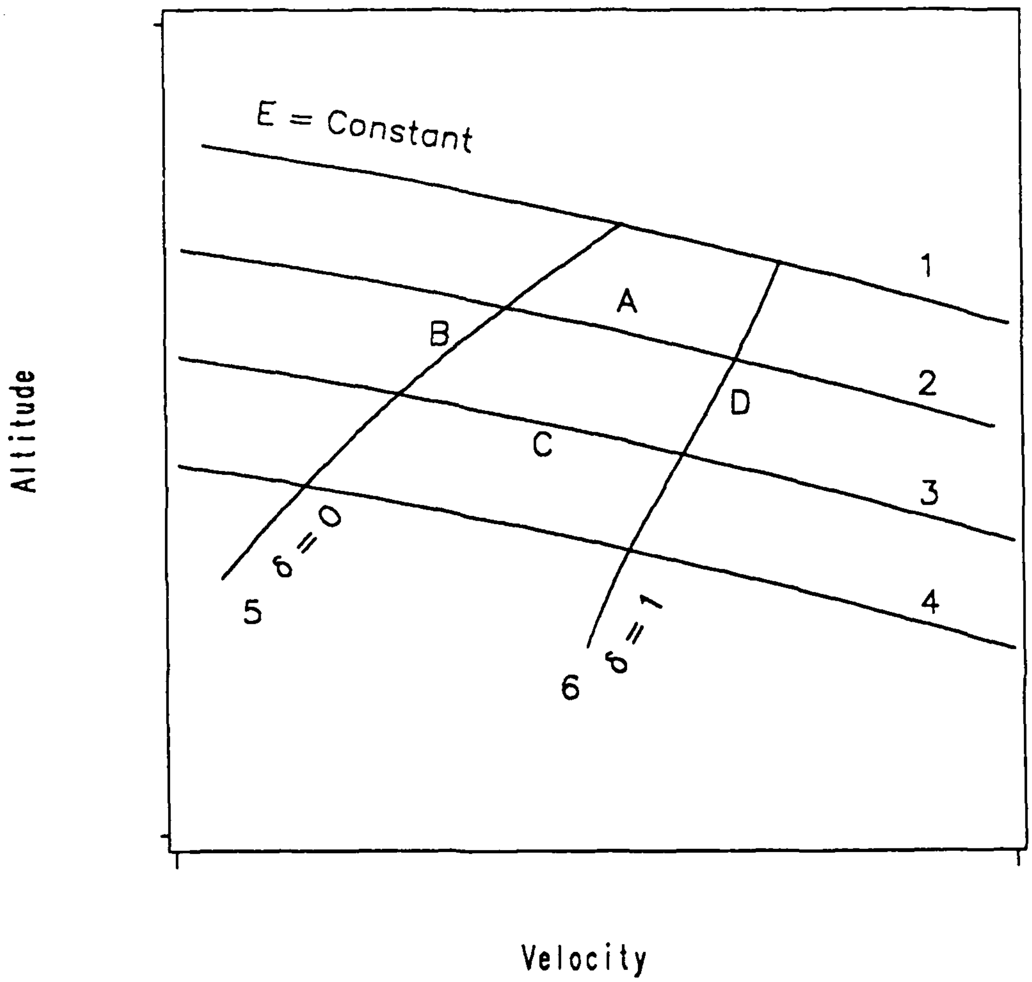

Figure 5. Generic Relaxation Oscillations.

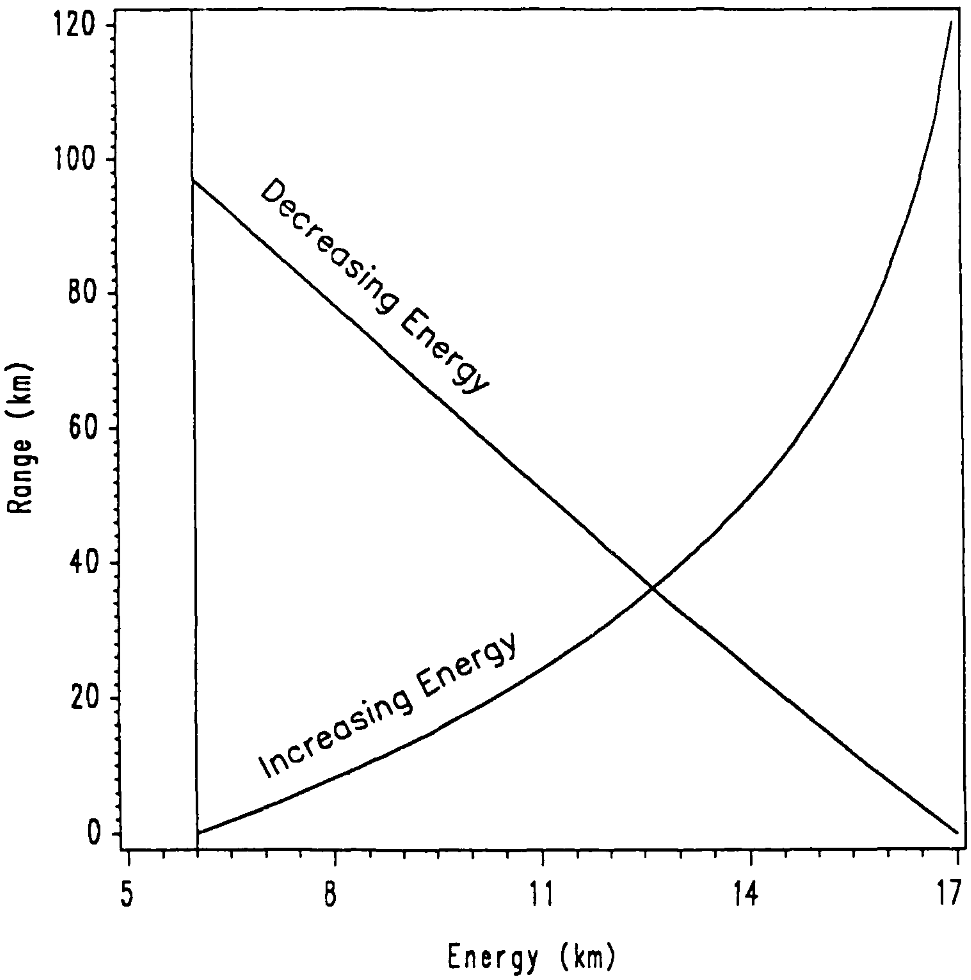

Figure 6. Energy Transitions: Range vs. Energy for Decreasing and Increasing Energy ($\theta = 0$)

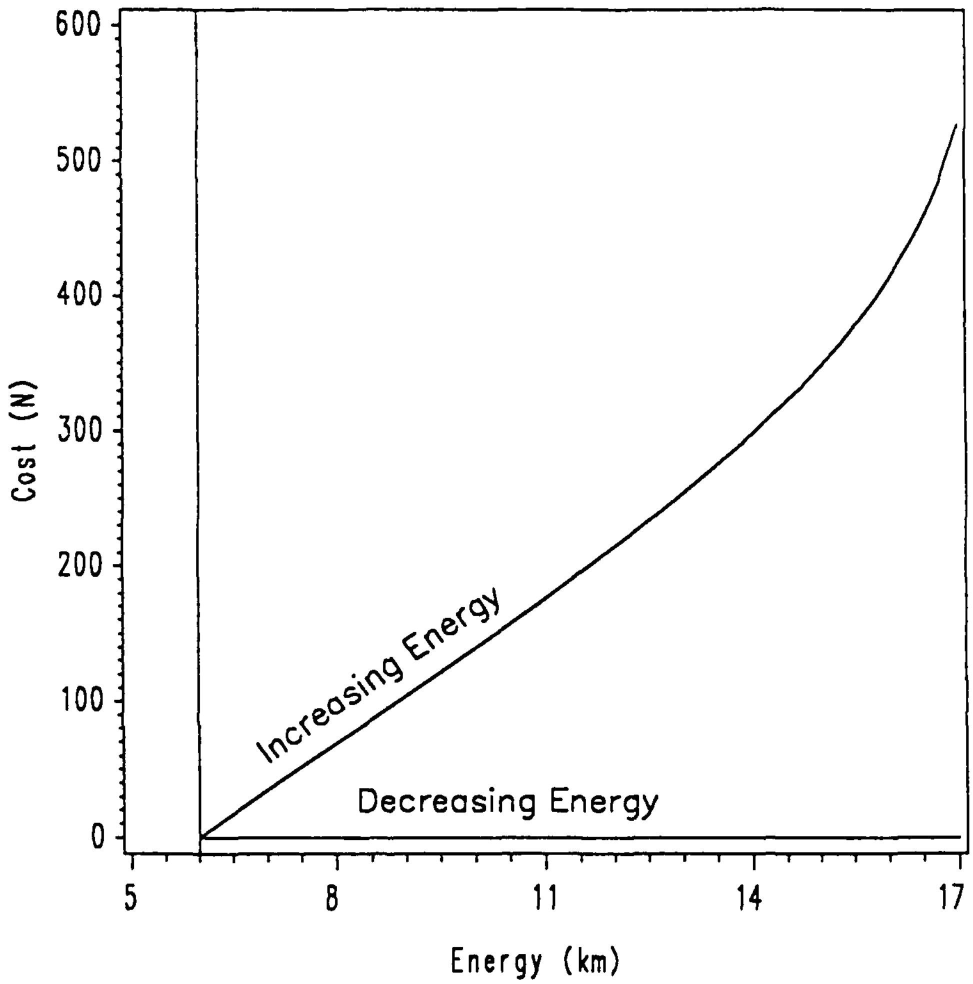

Figure 7. Energy Transitions: Cost vs. Energy
for Decreasing and Increasing Energy ($\theta = 0$)

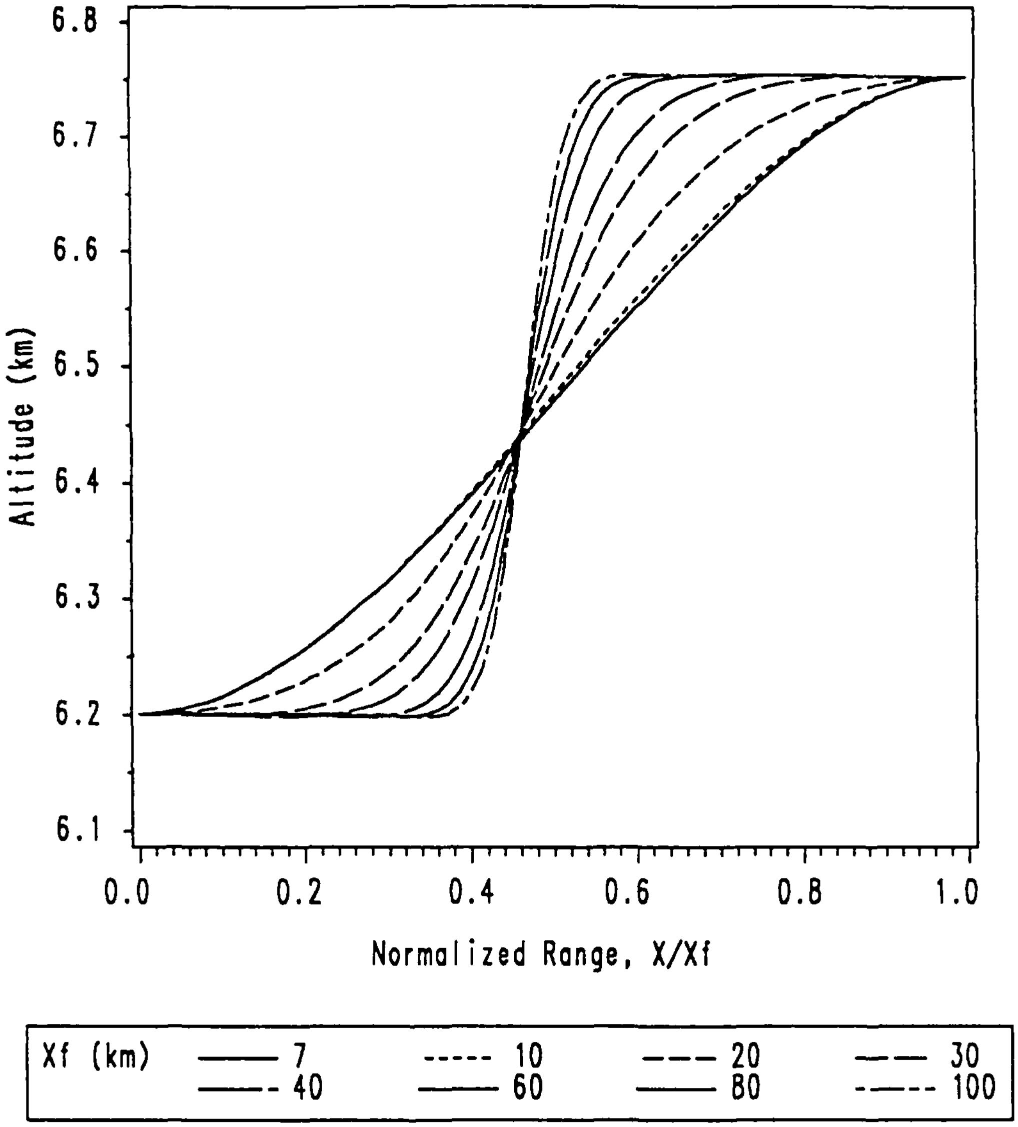

Figure 8. Constant Energy Transients: E = 10 km.
Altitude vs.Normalized Range ($\theta = 0$)

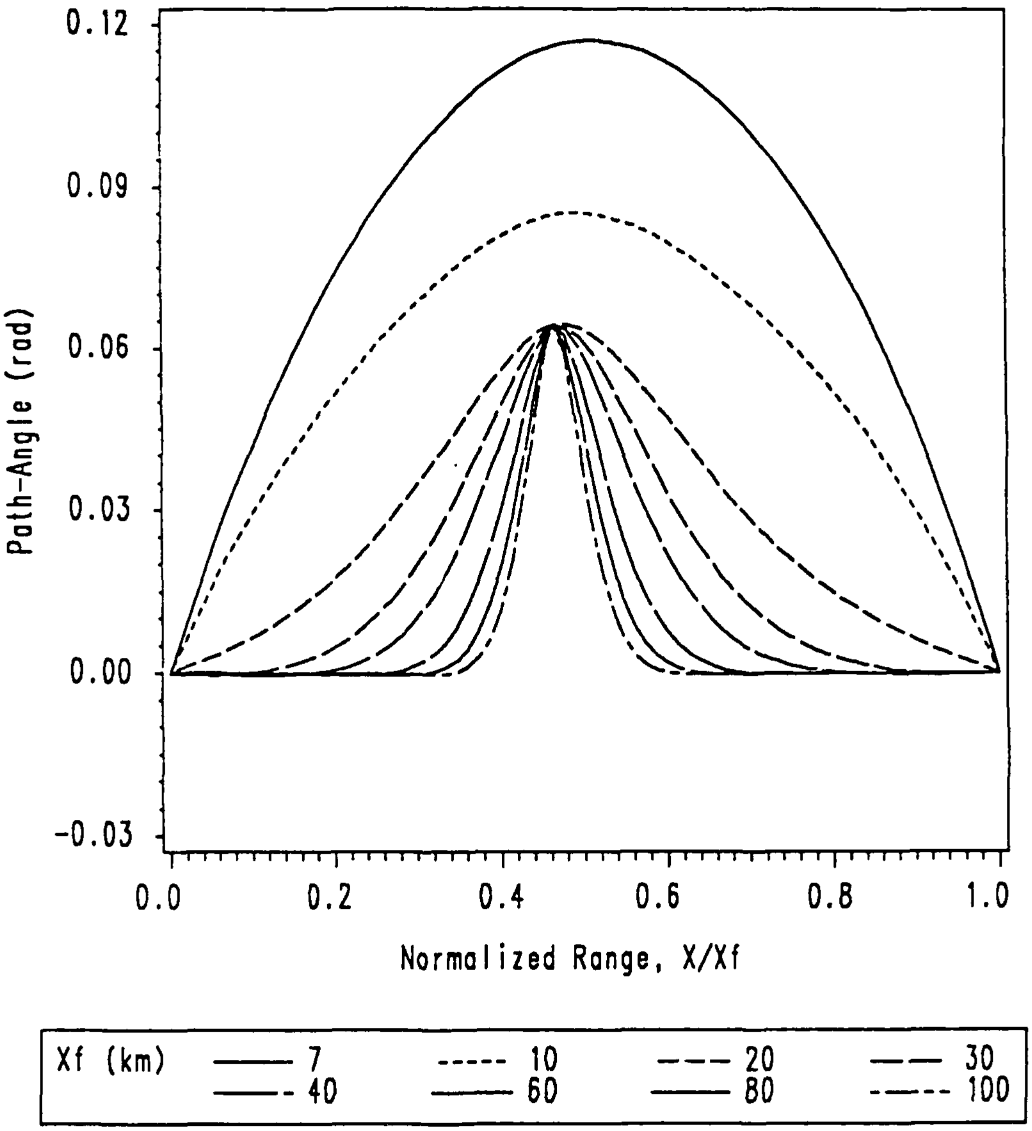

Figure 9. Constant Energy Transients: E = 10 km.
Path-angle vs.Normalized Range ($\theta = 0$)

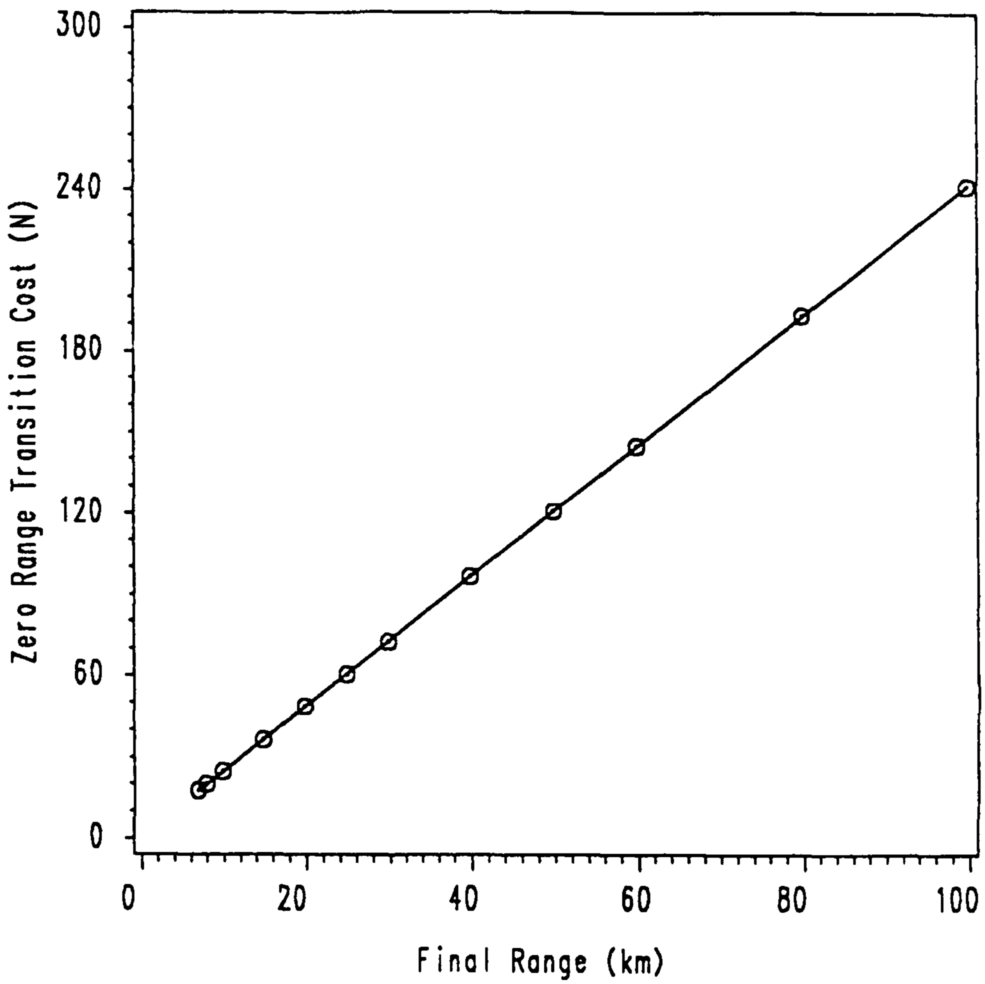

Figure 10. Constant Energy Transients: E = 10 km.
Total Cost vs. Final Range ($\theta = 0$)

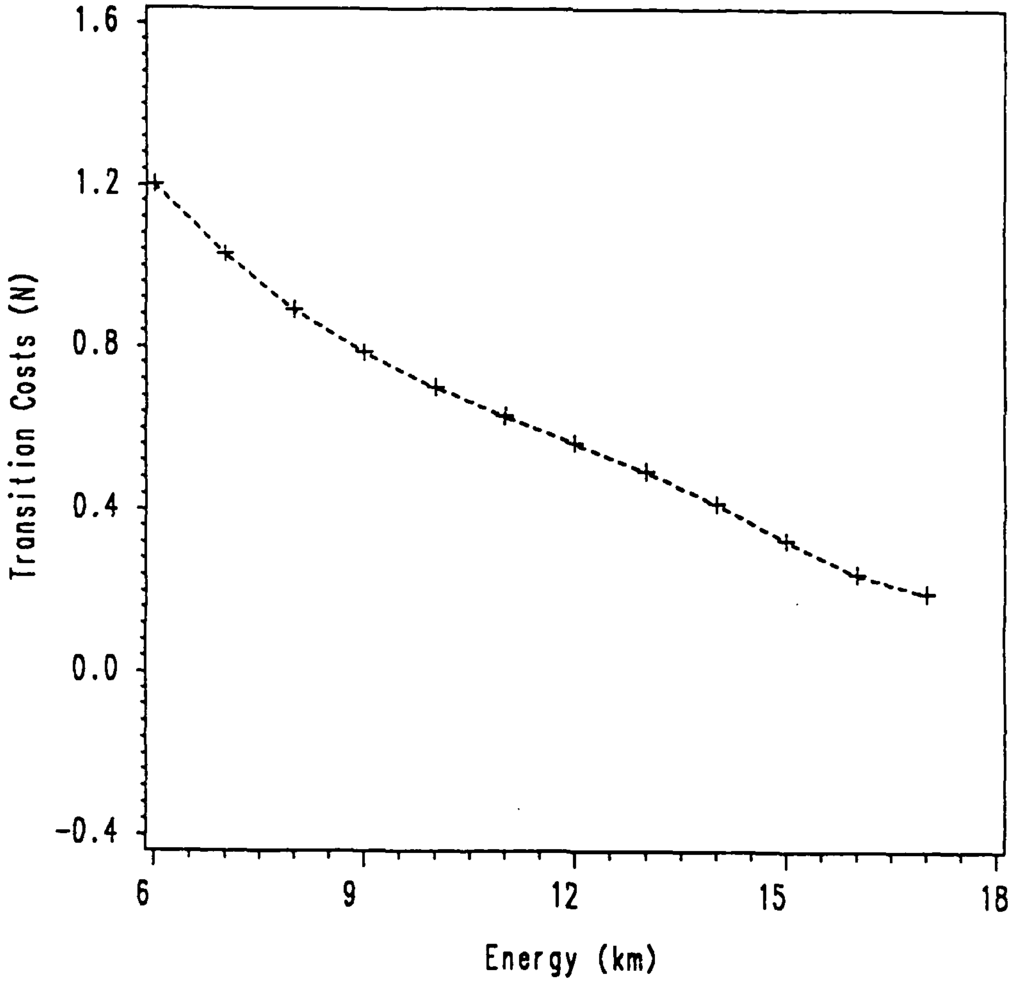

Figure 11. Constant Energy Transients: E = 10 km.
Transient Costs vs. Energy ($\theta = 0$ and $1.6N/s$)

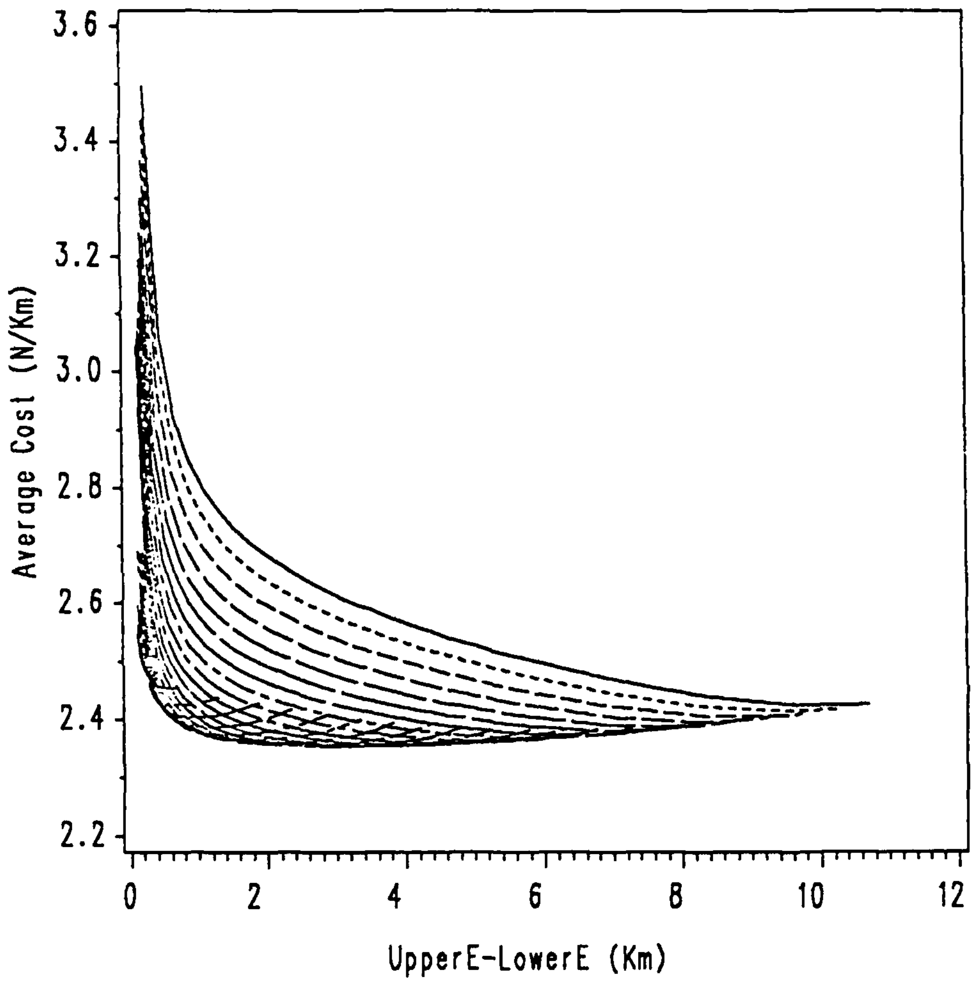

Figure 12. Relaxation Oscillation Costs
($J(E_o, \Delta E)$ for $\theta = 0$)

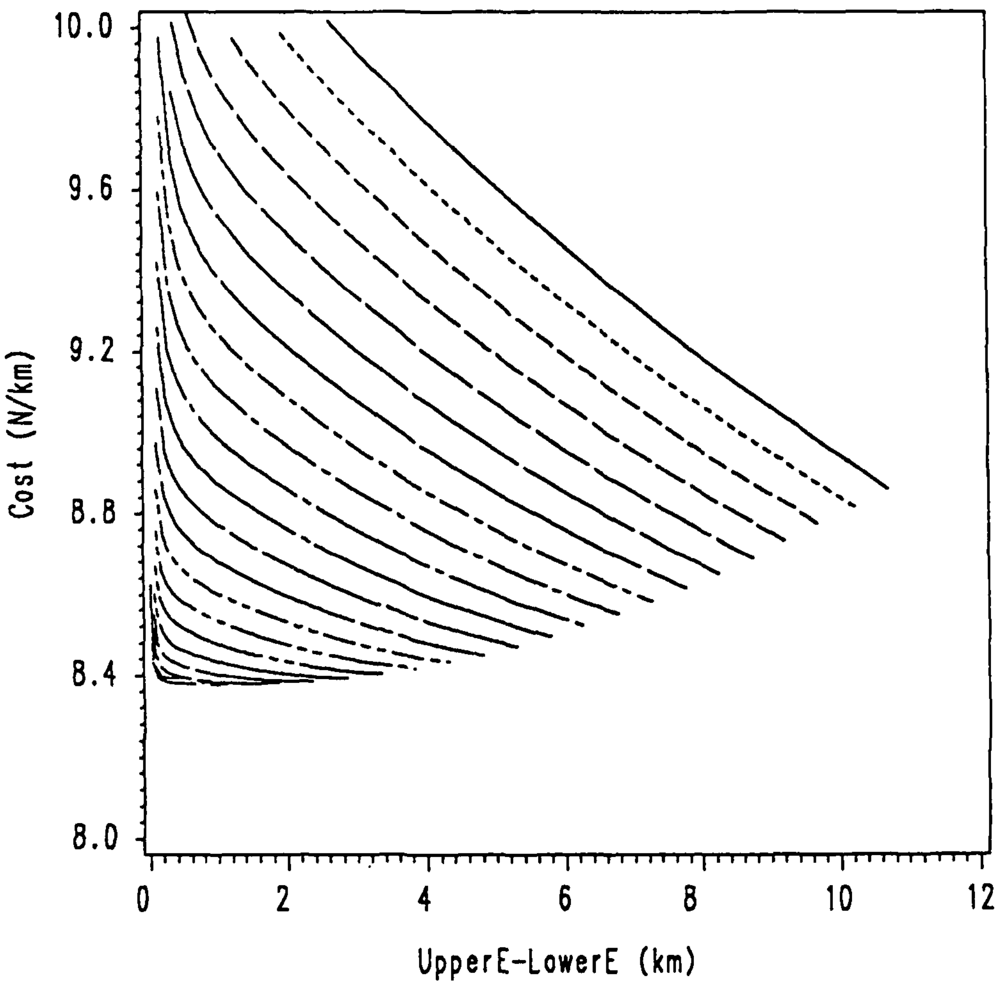

Figure 13. Relaxation Oscillation Costs
($J(E_o, \Delta E)$ for $\theta = 1.6 N/s$)

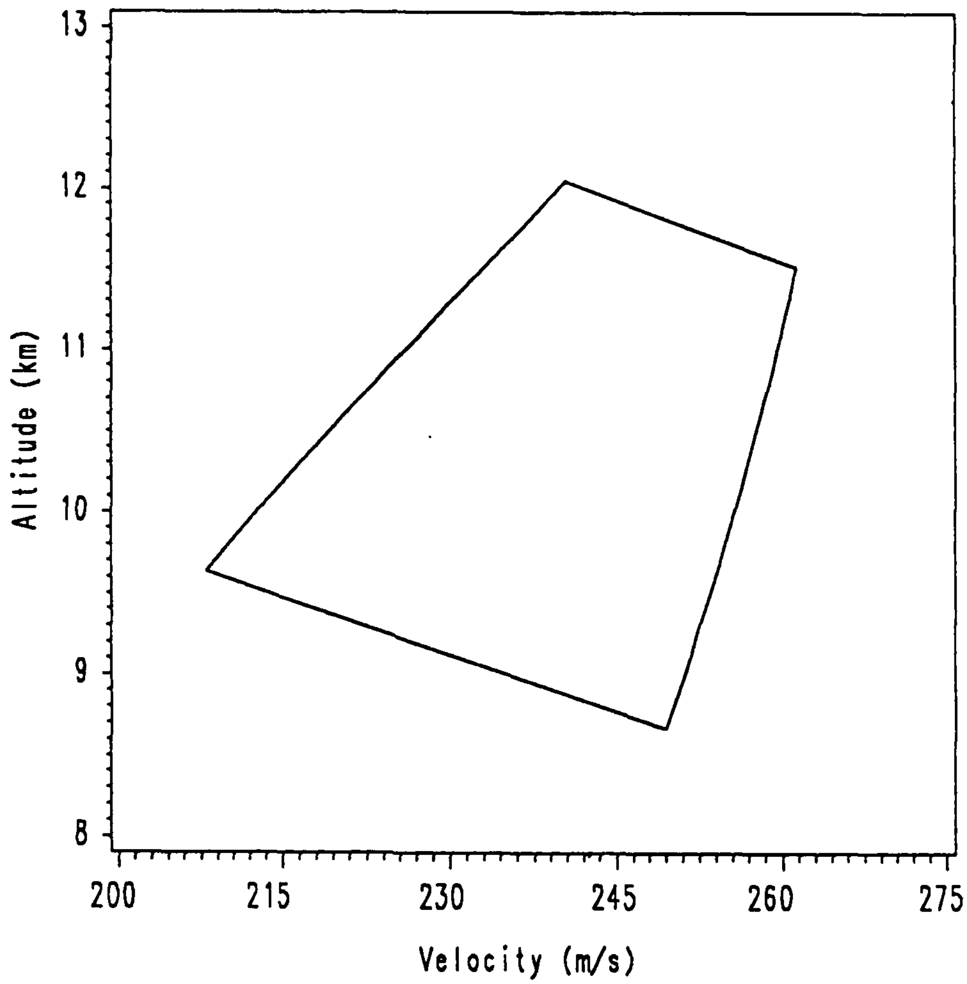

Figure 14. Best Relaxation Oscillation for $\theta = 0$

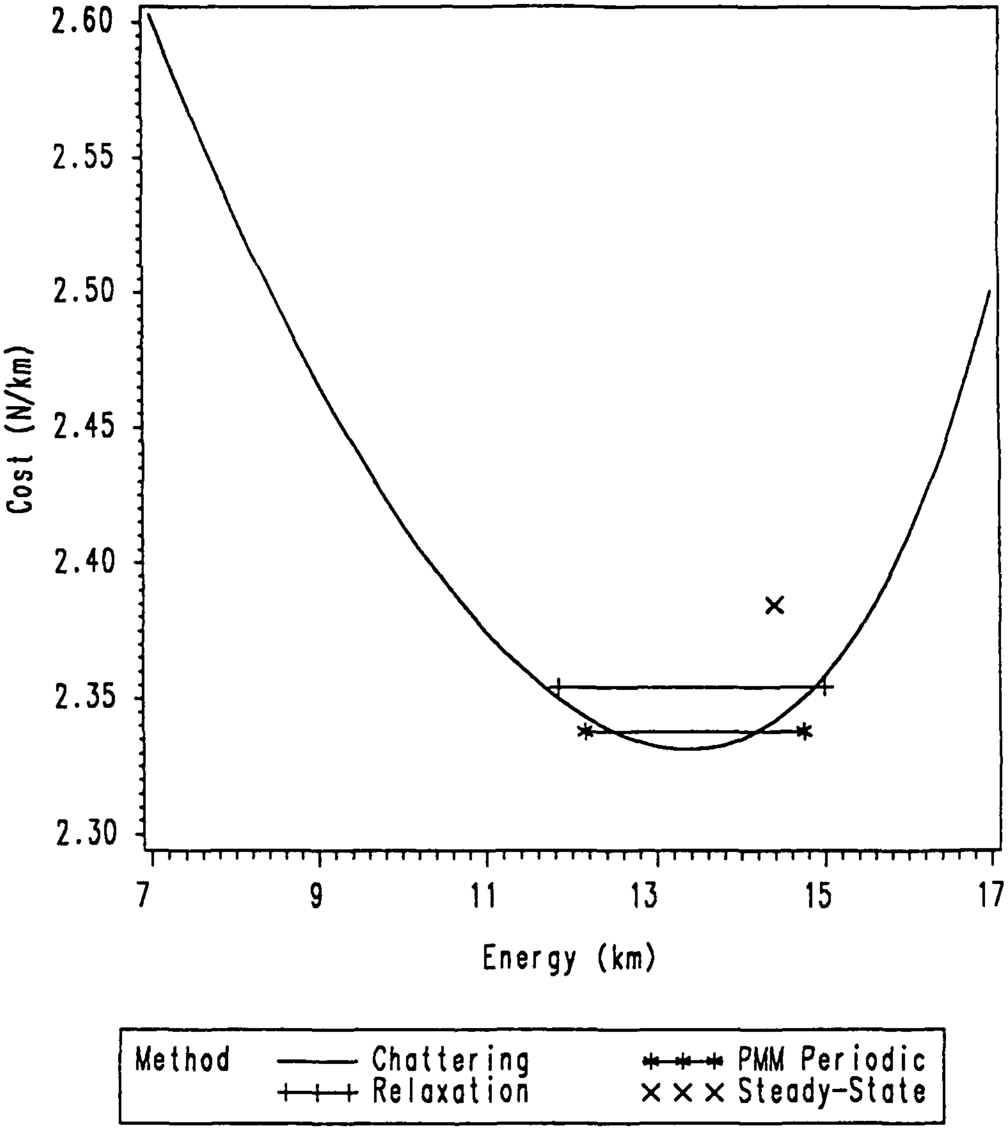

Figure 15. Average Costs with Various Methods.($\theta = 0$)

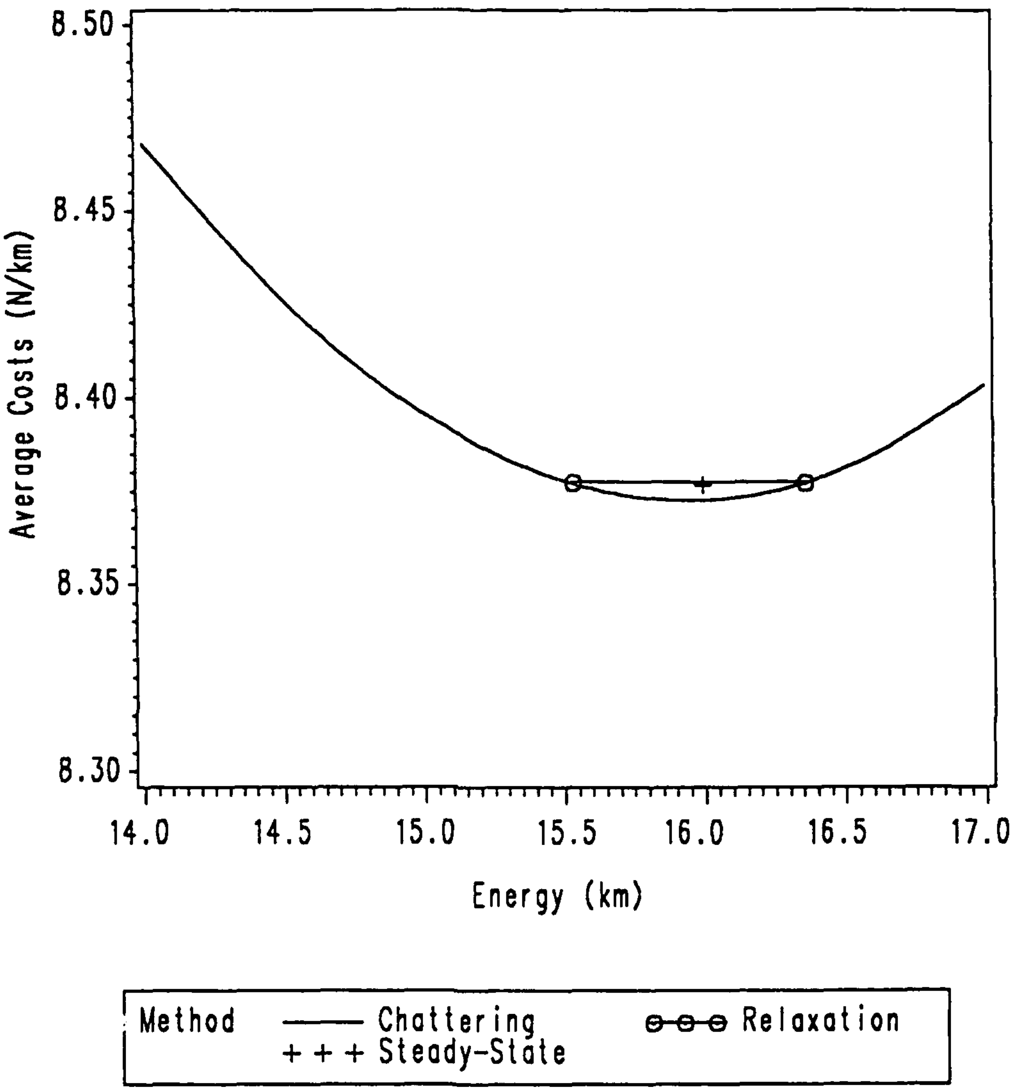

Figure 16. Average Costs with Various Methods.($\theta = 1.6\ N/s$)