AFGL-TR-87-0318
SPECIAL REPORT, NO. 257

Fifth Force Studies for a Layered Earth

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13 November 1987

EARTH SCIENCES DIVISION PROJECT 2309
AIR FORCE GEOPHYSICS LABORATORY
HANSCOM AFB, MA 01731

This document has been reviewed by the ESD Public Affairs Office (PA) and is releasable to the National Technical Information Service (NTIS).

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION
Unclassified

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY

2b. DECLASSIFICATION / DOWNGRADING SCHEDULE

3. DISTRIBUTION / AVAILABILITY OF REPORT
Approved for public release;
Distribution unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
AFGL-TR-87-0318
Special Report, No. 257

5. MONITORING ORGANIZATION REPORT NUMBER(S)

6a. NAME OF PERFORMING ORGANIZATION
Air Force Geophysics Laboratory

6b. OFFICE SYMBOL *(If applicable)*
AFGL/LWG

7a. NAME OF MONITORING ORGANIZATION

6c. ADDRESS *(City, State, and ZIP Code)*
Hanscom AFB
Massachusetts 01731-5000

7b. ADDRESS *(City, State, and ZIP Code)*

8a. NAME OF FUNDING / SPONSORING ORGANIZATION

8b. OFFICE SYMBOL *(If applicable)*

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER

8c. ADDRESS *(City, State, and ZIP Code)*

10. SOURCE OF FUNDING NUMBERS

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
61102F	2309	2309G1	2309G111

11. TITLE *(Include Security Classification)*
Fifth Force Studies for a Layered Earth

12. PERSONAL AUTHOR(S)
Richard C. Walker*

13a. TYPE OF REPORT
Final

13b. TIME COVERED
FROM June 87 TO July 87

14. DATE OF REPORT *(Year, Month, Day)*
1987 November 13

15. PAGE COUNT
34

16. SUPPLEMENTARY NOTATION
*Summer Faculty, Ft. Lewis College, Durango, Colorado

17. COSATI CODES

FIELD	GROUP	SUB-GROUP

18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
Fifth force
Yukawa
Gravitation
Inverse-square law

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*

Yukawa and Newtonian components of gravity over a spherically symmetric layered medium are calculated as superpositions of respective components for uniform spheres of appropriate radii and densities. Outside the surface, both components exhibit attenuation-with-distance characteristics completely describable in terms of relative Yukawa amplitude (α) and the ratio of surface radius to Yukawa "wavelength" (R/λ). Introduction of a parameter Ψ, sensitive to existence of the Yukawa component and calculable from the total field, enables study of layering effects and determination of required measurement precision. Introduction of lateral inhomogeneities exposes limitations of the Ψ-parameter, layered-sphere approach and suggests alternate possibilities.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT
☒ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION
Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL
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22b. TELEPHONE *(Include Area Code)*
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22c. OFFICE SYMBOL
AFGL/LWG

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

Preface

Thanks are due the Air Force Systems Command and the Air Force Office of Scientific Research for sponsorship of this research. The concern, direction, and program administration by Universal Energy Systems is also appreciated.

Special thanks are due Drs. Andrew Lazarewicz, Christopher Jekeli, Thomas Rooney, Donald Eckhardt, and other personnel of the Gravity and Geodesy Branch of the Air Force Geophysics Laboratory (AFGL) for their technical assistance and for general encouragement and support.

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1. INTRODUCTION

The search for a Grand Unified Theory in quantum physics has led to the postulation of a so-called "fifth force" and the so-far-undetected particle associated wth propagation of this field.[1-4] Concurrently, recent geophysical evidence suggests slight deviation of measured "gravitational" attraction from Newton's inverse square formulation.[5] The possibility of detecting this anomalous "force" has spawned the "tower experiment" in progress by the Air Force Geophysics Laboratory (AFGL). In this experiment, deviation of measurements made at various heights on a 600-m tower from values predicted by Newtonian upward continuation of ground-based measurements is the basis for interpretation.

2. OBJECTIVES OF THE RESEARCH EFFORT

Though departure from Newton's Law could be ascribed to various phenomena (for example, spatial variation of the gravitational constant), the field is modeled assuming an anomalous Yukawa contribution consistent with a potential due to an isolated mass of the form

$$\phi(r) = \frac{G_0 m}{r} (1 + \alpha e^{-\mu r})$$

(Received for publication 20 October 1987)

1. Stacey, F.D., Tuck, G.J., Moore, G.I., Holding, S.C., Goodwin, B.D., Zhou, R. (1987) Geophysics and the law of gravity, *Reviews of Modern Physics*, Am. Phys. Soc. 59(No. 1):157-174.

2. Eckhardt, D.H. (1986) Comment on "Reanalysis of the Eotvos Experiment," *Phys. Rev. Letters* 57(No. 22):2868.

3. Fischback, E., Sudarsky, D., Szafer, A., Carrick, T. (1986) Reanalysis of the Eotvos experiment, *Phys. Rev. Letters* 56(No. 1):3-6.

4. Gibbons, G.W., and Whiting, B.F. (1981) Newtonian gravity measurements impose constraints on unification theories, *Nature* 291:636-638.

5. Holding, S.C., Stacey, F.D., Tuck, G.J. (1986) Gravity in mines—An investigation of Newton's Law, *Phys. Rev. D* 33(No. 12):3487.

(Chan and Paik[6]). Objectives are to analyze field behavior to determine necessary precision of measurement and to study effects of depth-related and lateral inhomogeneities using models simple enough to yield such information unambiguously. This effort is designed to complement simultaneous efforts in more detailed design, analysis, and interpretation of the "tower experiment."

3. THE FIELD DUE TO A UNIFORM SPHERE

Assume a sphere of uniform density ρ, volume V, and radius R centered at the origin (Figure 1). Let $\rho \geqslant R$ represent the distance from the origin to a point P. Let us find the Newtonian and Yukawa components of the total field, denoted by $\vec{F}_N(P)$, $\vec{F}_Y(P)$, and $\vec{F}(P)$, respectively.

3.1 General Approach

For each element dV of the sphere, let $\vec{1}_r$ be a unit vector in the direction from P to dV, let r be the distance from P to dV, and let $\vec{1}_r$ be a unit vector in the direction from P to the origin. Then the field can be written as

$$\begin{aligned}\vec{F}(P) &= \vec{F}_N(P) + \vec{F}_Y(P)\\ &= [F_N(P) + F_Y(P)]\ \vec{1}_R\end{aligned}$$

where

$$F_N(P) = \rho\, G_o \int_V \frac{f(r)}{r^2} (\vec{1}_r \cdot \vec{1}_R)\, dV, \tag{1}$$

$$F_Y(P) = -\alpha\, \rho\, G_o \int_V \frac{g(r)}{r^2} (\vec{1}_r \cdot \vec{1}_R)\, dV, \tag{2}$$

and $G_o \frac{f(r)}{r^2} \vec{1}_r$ and $-\alpha\, G_o \frac{g(r)}{r^2} \vec{1}_r$ are the gradients of the Newtonian and Yukawa potentials, respectively, due to each element dV.

Specifically, because these potentials are given by

$$\phi_N(P) = \frac{G_o\, \rho\, dV}{r} \tag{3}$$

$$\phi_Y(P) = -\frac{\alpha\, G_o\, \rho\, dV}{r}\, e^{-\mu r} \tag{4}$$

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6. Chan, H.A. and Paik, H.J. (1984) Experimental Test of a Spatial Variation of the Newtonian Gravitational Constant at Large Distances in *Precision Measurement and Fundamental Constants II*, Taylor, B.N. and Phillips, W.D. (eds.), Natl. Bur. Stand. (U.S.), Spec. Publ. 617, pp. 601–606.

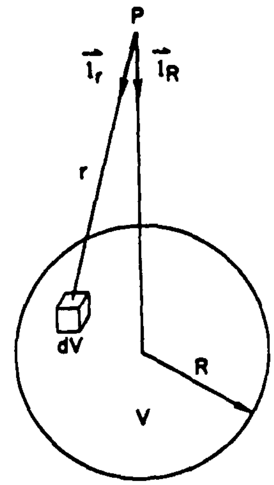

Figure 1. Notation for a Uniform Sphere

their gradients (noting that we have chosen $\bar{1}_r$ to point toward dV) are:

$$\text{grad } \phi_N(P) = \frac{G_o \rho \, dV}{r^2} \bar{1}_r \tag{5}$$

$$\text{grad } \phi_Y(P) = -\frac{\alpha G_o \rho \, dV}{r^2} e^{-\mu r} (1 + \mu r) \bar{1}_r. \tag{6}$$

Hence, the functions f and g are given by

$$f(r) = 1, \tag{7}$$

$$g(r) = e^{-\mu r} (1 + \mu r). \tag{8}$$

In what follows, let h(r) represent f(r) or g(r), as appropriate.

Our problem now is one of evaluating

$$I(P) = \int_V \frac{h(r)}{r^2} (\bar{1}_r \cdot \bar{1}_R) \, dV. \tag{9}$$

To take advantage of symmetry, for each $r \in [p - R, p + R]$ let S represent the intersection of V with a spherical shell, of radius r and thickness dr, centered on P. Let $\phi = \cos^{-1} (\bar{1}_r \cdot \bar{1}_R)$, and for each ϕ, form a ring in S of radius $r \sin \phi$ and thickness $d\phi$. The volume of the ring is then $2\pi (r \sin \phi)(r d\phi) dr$, the projection $\bar{1}_r \cdot \bar{1}_R$ for each dV in the ring is $\cos \phi$, and r is constant on S; hence

$$I(P) = 2\pi \int_{P-R}^{P+R} h(r) \int_{0}^{\phi_o} \sin\phi \cos\phi \, d\phi \, dr$$

where $\phi_o = \sup \phi$. Evaluating the inner integral yields

$$I(P) = \pi \int_{P-R}^{P+R} h(r) \sin^2 \phi_o \, dr. \tag{10}$$

Using the law of cosines, one obtains

$$\sin^2 \phi_o = \begin{cases} 1 - \left(\dfrac{r^2 + p^2 - R^2}{2\,p\,R}\right)^2 & ;\ p > R, \\ 1 - (r/(2p))^2 & ;\ p = R. \end{cases} \tag{11}$$

3.2 The Newtonian Field

As a check on the above method, the author evaluated the above integral, using $h(r) = f(r) = 1$. The result was

$$I_N(P) = \frac{4\,\pi\,R^3}{3\,p^2}, \text{ or} \tag{12}$$

$$F_N(P) = G_o\,\rho\,\frac{4\,\pi\,R^3}{3\,p^2}, \tag{13}$$

a well-known result more easily obtained via the Divergence Theorem. Note that in the case of a uniform sphere, the Newtonian field outside the sphere is as if the entire mass were concentrated at the center.

3.3 The Yukawa Field

For the Yukawa field, the I(P) integral is given by

$$I_Y(P) = \pi \int_{P-R}^{P+R} e^{-\mu r} (1 + \mu r) \sin^2 \phi_o \, dr. \tag{14}$$

Using Eq. (11) with $A^2 = p^2 - R^2$ gives

$$I_Y(P) = \pi \int_{P-R}^{P+R} e^{-\mu r} (1 + \mu r) \left[1 - A^4/(4p^2r^2) - A^2/(2p^2) - r^2/(4p^2)\right] dr.$$

The Yukawa field does not have a closed-form representation, as multiplying out the integrand gives two terms $e^{-\mu r}/r$ and $e^{-\mu r}/r^2$. Slow convergence of the series representations rendered this approach computationally infeasible for p near R.

However, the integrand of Eq. (14) is well-behaved, consisting of the product of a function dependent only on the nature of the field and the purely geometric factor, $\sin^2 \phi_o$. Rewriting Eq. (14) in terms of the dimensionless parameter r/δ, where $\delta = \mu^{-1}$ is the distance from an isolated point mass at which the Yukawa potential would be 1/e times its value at the point, yields

$$I_Y(P) = \pi\delta \int_{\frac{P-R}{\delta}}^{\frac{P+R}{\delta}} e^{-(r/\delta)} (1 + r/\delta) \sin^2 \phi_o \, d(r/\delta), \tag{15}$$

where $\sin^2 \phi_o$ is given in terms of r/δ by

$$\sin^2 \phi_o = \begin{cases} 1 - \left(\dfrac{(r/\delta)^2 + ((p-R)/\delta)((p+R)/\delta)}{[(p-R)/\delta + (p+R)/\delta]\,(r/\delta)}\right)^2 & ; p > R \\ 1 - \left(\dfrac{r/\delta}{(p-R)/\delta + (p+R)/\delta}\right)^2 & ; p = R. \end{cases} \tag{16}$$

Figures 2 and 3 show plots of the integrand functions; Figure 4 displays $I_Y(r/\delta)/\pi\delta$ as a function of $(p - R)/\delta$. An interpretation of Figure 4 is that, for a given δ, the Yukawa field persists to larger

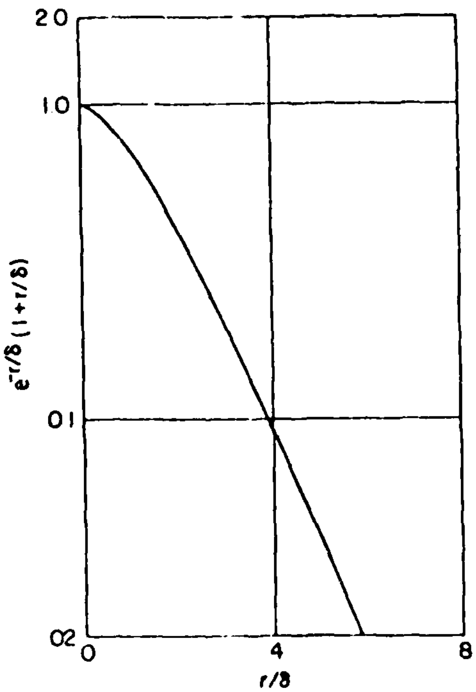

Figure 2. Plot of $e^{-v}(1 + v)$ Versus v

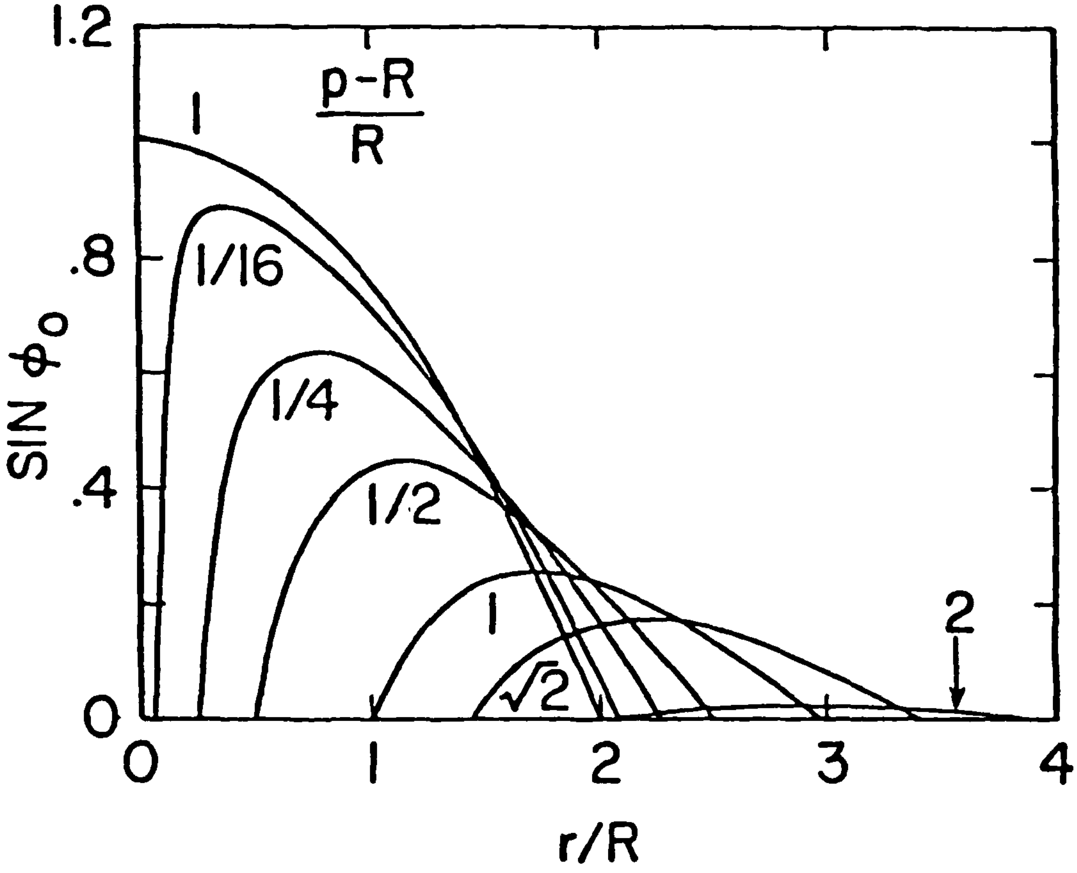

Figure 3. Sin ϕ_0 Versus r/R for Various δ/R (Uniform Sphere)

distances from the surface as the radius of the sphere increases, approaching the limiting condition of a plane surface approximated by the graph corresponding to $\delta/R = 0.001$.

3.4 Yukawa and Newtonian Fields Compared

According to the definition of the dimensionless parameter $r/\delta = \mu r$ for the Yukawa field, Eqs. (1) and (2) take the forms:

$$F_N(P) = \rho\, G_o\, \pi\, \delta \int_{\frac{P-R}{\delta}}^{\frac{P+R}{\delta}} (1)\, \sin^2 \phi_0\, d(r/\delta), \tag{17}$$

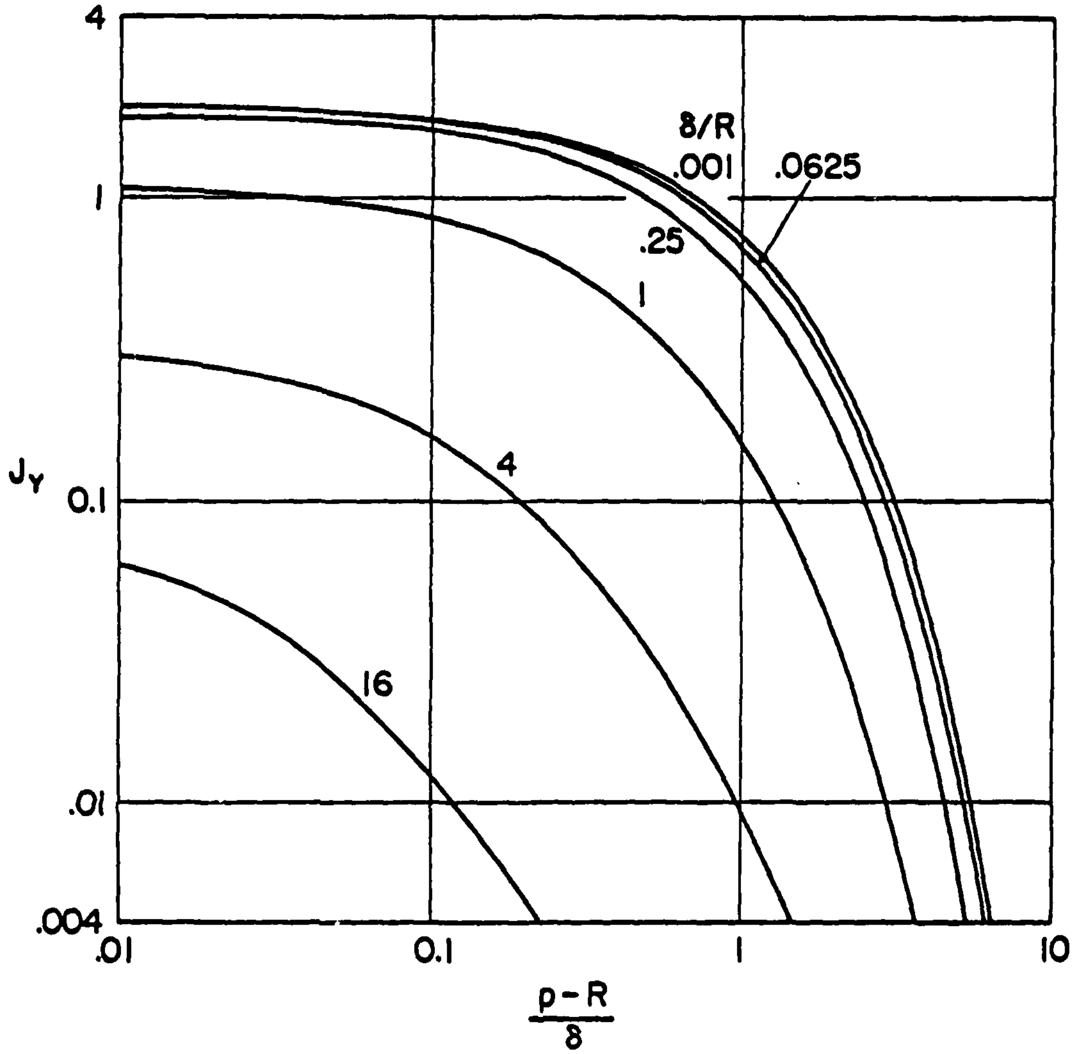

Figure 4. J_Y Versus $(p - R)/\delta$ for Various δ/R (Uniform Sphere)

$$F_Y(P) = \alpha\rho\, G_o\, \pi\, \delta \int_{\frac{P-R}{\delta}}^{\frac{P+R}{\delta}} e^{-r/\delta}\,(1 + r/\delta)\,\sin^2 \phi_o\, d(r/\delta). \tag{18}$$

Define

$$J_N(P) = F_N(P)/(\rho\, G_o\, \pi\, \delta), \tag{19}$$

$$J_Y(P) = F_Y(P)/(\alpha\rho\, G_o\, \pi\, \delta), \tag{20}$$

corresponding to the values of the respective integrals. Eqs. (17) and (13) together give

$$J_N(P) = \frac{4}{3(p/R)^2\,(\delta/R)} \tag{21}$$

corresponding to the Newtonian field.

Perhaps the most obvious comparison of the Yukawa and Newtonian field components is via their ratio, specifically, by

$$\frac{1}{\alpha}\frac{F_Y(P)}{F_N(P)} = \frac{J_Y(P)}{J_N(P)}, \tag{22}$$

as depicted in Figure 5. The range of the Yukawa field compared to the Newtonian field is limited; the ratio J_Y/J_N for each curve drops by three decades from its surface value within 10 δ units of distance from the surface. Rollover for each curve begins at or before $(p - R)/\delta = 1$, but in contrast to Figure 4, the smaller δ/R values correspond to slightly earlier and more rounded rollovers. This behavior is explained by J_N being more nearly constant near the surface for larger values of R/δ, as the surface corresponds to points farther out on the inverse square dependence. The main feature of Figure 5 is the overall decrease in the height of the curves, in particular the surface values of J_Y/J_N, as δ/R decreases. Clearly, the J_N becomes much greater than J_Y as R increases, because J_N is proportional to the volume whereas J_Y is effectively more locally defined for $R >> \delta$.

Because F_Y and F_N cannot be measured individually at present, interpretation from actual measurements must be based on some other parameter, definable in terms of the total field. Such a parameter will be presented shortly.

4. THE FIELD DUE TO A LAYERED SPHERE

Consider a set of concentric spheres $S_1, S_2, \ldots, S_N$ of radii $R_1 > R_2 \ldots > R_N > 0$, respectively. Define

$$\rho_i = \text{density of} \begin{cases} \text{shell } S_i \cap S_{i+1};\ i = 1, 2, \ldots, N-1, \\ \text{sphere } S_N \qquad ;\ i = N. \end{cases}$$

Because

$$\rho_i = \begin{cases} \rho_1 & ;\ i = 1 \\ (\rho_i - \rho_{i-1}) + \rho_{i-1} & ;\ i = 2, \ldots, N, \end{cases} \tag{23}$$

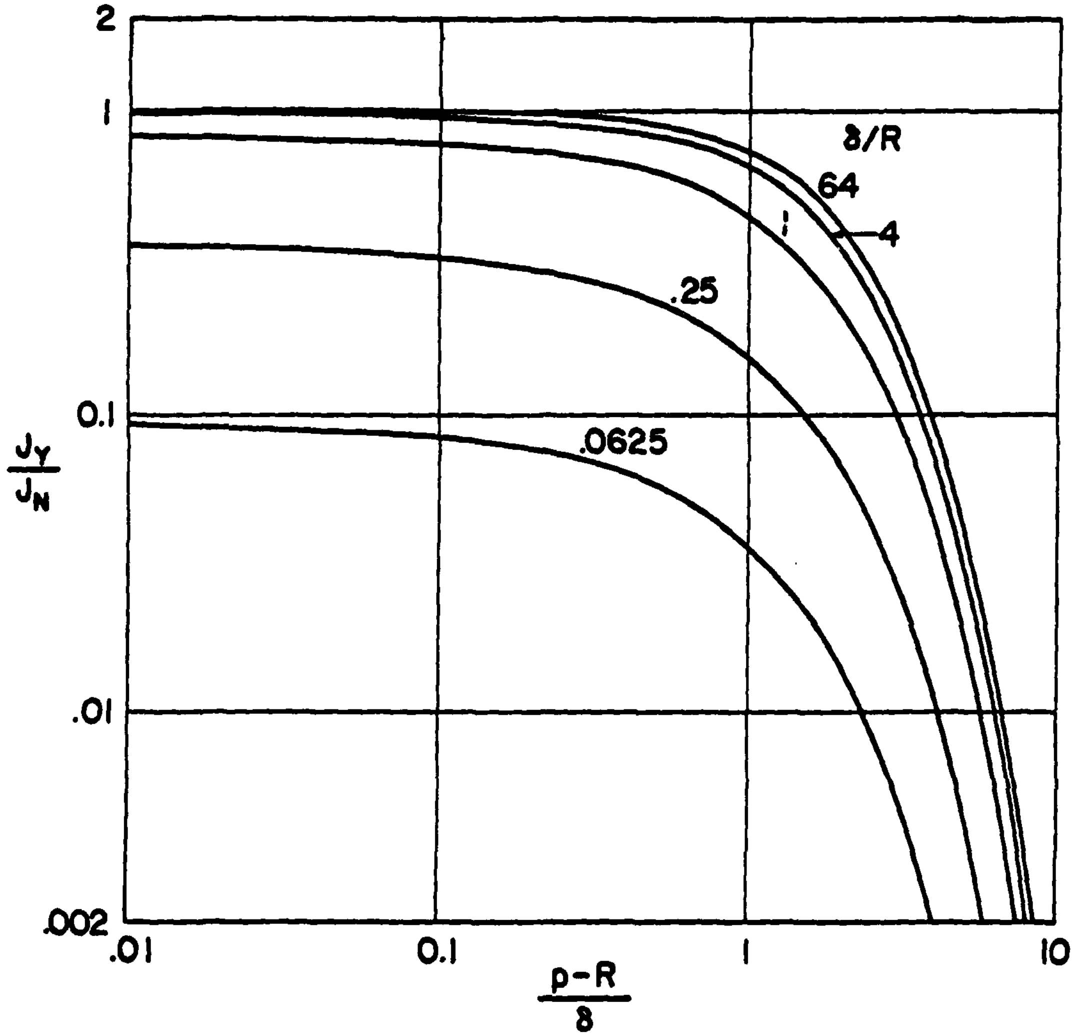

Figure 5. J_Y/J_N Versus $(p-R)/\delta$ for Various δ/R (Uniform Sphere)

the density distribution can as well be duplicated by superposing the spheres themselves, having corresponding densities

$$\Delta_i = \begin{cases} \rho_1 & ; i = 1 \\ \rho_i - \rho_{i-1} & ; i = 2, \ldots, N, \end{cases} \tag{24}$$

The field due to a layered sphere is then the superposition of fields due to individual spheres whose radii correspond to the layer boundaries and whose densities are given in terms of the layer densities by

Eq. (24). Specifically, if J_N and J_Y are defined as in Eqs. (19) and (20), where now F_N and F_Y are for the layered case and ρ_1 replaces ρ in the definition, one obtains

$$J*(P) = \sum_{i=1}^{N} \gamma_i J*_i(P,S_i) \tag{25}$$

where

$$\gamma_i = \Delta_i/\rho_1,$$
$$* = \text{N or Y},$$

and $J*_i(P, S_i)$ represents the value of $J*$ at point P due to a single sphere of density Δ_i. According to the previous section, the $J*_i(P, S_i)$ quantities are given by

$$J_{Ni}(P, S_i) = \frac{4}{3(P/R_i)^2 (\delta/R_i)}, \tag{26}$$

$$J_{Yi}(P,S_i) = \int_{\frac{P-R_i}{\delta}}^{\frac{P+R_i}{\delta}} e^{-v} (1 + v) \sin^2 \phi_{oi}\, dv \tag{27}$$

where

$$\sin^2 \phi_{oi} = \begin{cases} 1 - \dfrac{v^2 + [(p-R)/\delta]\,[(p+R)/\delta]^2}{v[(p-R)/\delta + (p+R)/\delta]}; & p > R, \\[2ex] 1 - \left(\dfrac{v}{(p-R)/\delta + (p+R)/\delta}\right)^2 & ; p = R. \end{cases} \tag{28}$$

5. SENSING THE YUKAWA COMPONENT

This section defines a parameter sensitive to the Yukawa field. The effects of changing model and field parameters are then treated. Finally, the topic of required measurement accuracy is addressed. Layered or uniform sphere models are assumed throughout this section.

5.1 Interpretation Parameters

A primary reason for the layered sphere model is to enable study of the effects of inhomogeneities on the interpretation parameters available from field measurements. Hence, let us define such parameters according to the criteria that the parameter must be (1) calculable from field data, and (2) sensitive to the existence of the Yukawa field.

If G_0, ρ_1, and α were known percisely, a parameter such as $F^o_{Na}(P)$ called "apparent Newtonian surface field," defined as

$F^o_{Na}(P)$ = the value at the surface of a Newtonian field having the observed value F(P) at point P, given the model, (29)

would contain all the information available from the measured total field. If this parameter is "normalized" by dividing by the total surface field actually observed, this new parameter has the advantage of being immune to inaccuracies in the gravitational constant and the first layer density. Mathematically, this parameter is described by

$$\frac{F^o_{Na}}{F^o} = \frac{F(P)\,(p/R)^2}{F^o} \tag{30}$$

in terms of observed field values, and by

$$\frac{F^o_{Na}}{F^o} = \frac{[J_N(P) - \alpha\, J_Y(P)]\,(p/R)^2}{[J^o_N - \alpha\, J^o_Y]} \tag{31}$$

in terms of the J_N and J_Y integrals. If the field were totally Newtonian, this parameter would be 1 for all $p \geq R$, as the numerator of Eq. (30) then is exactly F^o_N, according to Eq. (21). Hence, deviation of this parameter from unity is indicative of non-Newtonian behavior, assuming perfect measurements over the Earth of the model.

Because departure from "1" is of interest, a new parameter, equivalent to this departure, is what will be called the ψ parameter:

$$\psi(P) = \frac{F^o_{Na}}{F^o} - 1. \tag{32}$$

In terms of the J_N and J_Y integrals, one obtains

$$\psi(P) = \frac{-\alpha\,[J_Y(P)(p/R)^2 - J^o_Y]}{J^o_N - \alpha\, J^o_Y} \tag{33}$$

by substitution from Eq. (30), the key step being to note that $J_N(P)(P/R)^2 - J^o_N$, which arises in the numerator, is zero.

A useful feature of the ψ parameter is that for $\delta/R << 1$ (as for the Earth) and for $\alpha < 0.1$, αJ^o_Y is much less than J^o_N in the denominator. Thus, in practical cases, ψ can be studied via ψ/α, which is virtually α-independent (see Table 1).

Table 1. ψ/α Versus $(p - R)/\delta$ for $\alpha = 0.004$, 0.1 ($\delta/R = 1/10000$)

$(p - R)/\delta$	δ/α ($s = 0.004$)	δ/δ ($\alpha = 0.1$)
0.0095	1.42359×10^{-6}	1.43261×10^{-6}
0.0560	8.17235×10^{-6}	8.17246×10^{-6}
0.3147	4.04967×10^{-5}	4.04973×10^{-5}
1.7798	1.24694×10^{-4}	1.24696×10^{-3}
10.0624	1.49994×10^{-3}	1.49996×10^{-3}

5.2 Influence of Inhomogeneity With Depth

Consider a two-layer sphere of radius R_1. Changes in the first-layer thickness $H_1 = R_1 - R_2$ and in the resistivity contrast ρ_2/ρ_1 clearly change the scale, but not the shape of the Newtonian dependence for $p > R_1$, as the Newtonian field for $p > R_1$ is the sum of two terms having $1/p^2$ as a common factor. Considering the fact that the density contrast does not appear inside the J_Y integral of Eq. (27), it follows that, for fixed δ/R_1 and $(R_1 - R_1)/\delta$, variations in ρ_2/ρ_1 affect only the vertical positioning and not the shape of the ψ/α curve (Figure 6). Comparing these curves (for $\delta/R_1 = 1/10000$) with those of Figure 7 (for $\delta/R_1 = 1$) shows that this dependence on ρ_2/ρ_1 tends to disappear as R_1 becomes less than a few multiples of δ. Also note from Figure 6 that the ρ_2/ρ_1 dependence tends to disappear as R_1 becomes less than a few multiples of δ. Also note from Figure 6 that the ρ_2/ρ_1 dependence of curve position decreases as ρ_2/ρ_1 becomes greater than 1.

Less mathematically obvious, but computationally verified to four significant figures for the curves of Figure 8, is that for fixed δ/R_1 and $\rho/_2/\rho_1$, the shape of the ψ/α curve is invariant with respect to first-layer thickness. Again, the R_2/R_1 dependence is subdued for R less than a few multiples of δ. Note from Figure 8 that, for fixed $(P - R_1)/\delta$ and $R_1 >> \delta$, $\log(\psi/\delta)$ varies nearly linearly with $(R_1 - R_2)/\delta$ for $(R2 - R_1/\delta < 1$ whereas for $(R_2 - R_1)/\delta > 4$ $\log(\psi/\alpha'1)$ is virtually independent of variations in this parameter.

In summary, then, layer thickness and density variations with depth affect only the relative magnitudes of the Newtonian and Yukawa components on or outside the sphere. Each component has its own falloff-with-distance characteristic, whose shape is invariant with respect to layer parameters, given a specific outer radius R_1 and the field parameters α and δ. For models of practical interest ($\delta/R << 1$), the above is essentially true independent of α, which effectively enters only as a scaling factor.

As a specific example, Figure 9 shows ψ/α curves for two-layer models, where R_1 is the radius of the Earth (6371 km) and $\delta = 200$ m. The curves all have the same shape. Thus, if the Earth were a layered sphere, measuring its field and calculating the ψ parameter at various heights would ideally give either a curve similar to Figure 9 (scaled by a factor of α), indicating presence of a nonzero Yukawa component, or else essentially zero values for the parameter, indicating that any Yukawa component present was too small to be reliably detected.

The fact that the curves roll over around H = 600 m means that, if δ is about 200 m, measurements to a height of about 600 m should be made to ensure seeing the characteristic features of the curve (ramp followed by rollover). That the curves for various values of h_1 are spread only over a range of a factor of 2 suggests that ability to resolve the curve is affected little by first-layer thickness. A similar argument applies to density.

Note that the effects of lateral inhomogeneities have not yet been considered. Also, curves for multi-layer spheres have the same shape as those for two- and one-layer models.

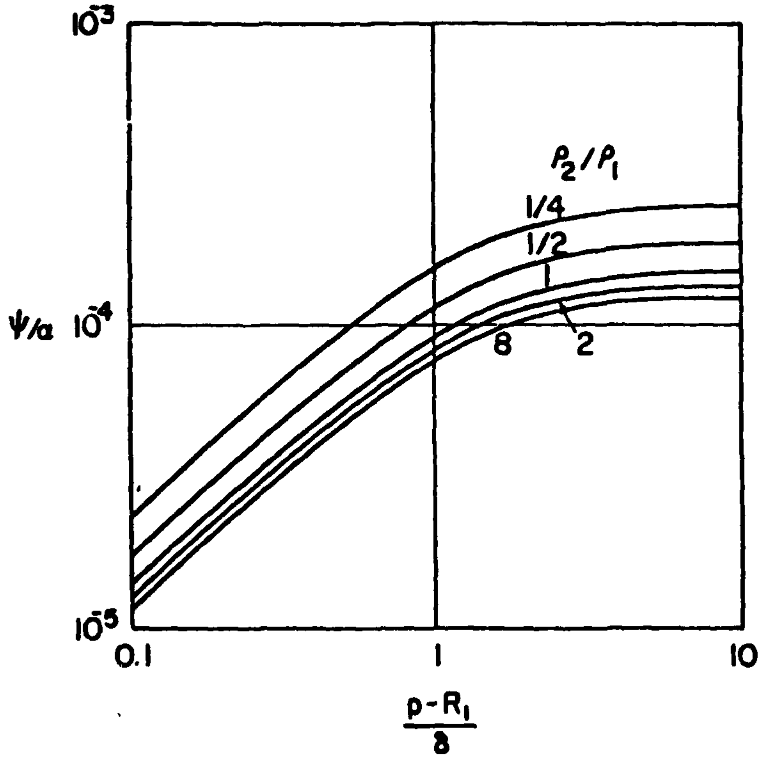

Figure 6. ψ/α Versus $(p - R_1)/\delta$ for Various ρ_2/ρ_1 ($H_1/\delta = 1/4$, $R_1/\delta = 1/10000$)

5.3 Influence of Field Parameters α and δ

From the previous discussion, it follows that the shape of the log-log plot of ψ versus h is determined completely by the value of δ/R. For $\delta/R << 1$:

1. If R is fixed, lateral positioning of the curve of ψ versus h is determined solely by δ. Specifically, the rollover portion of the curve occurs around $0.6\delta < h < 5\delta$.
2. That ψ/α is virtually independent of α allows the effects of α and δ on vertical positioning to be studied independently of each other. Clearly, if ψ/α is independent of α, then the value ψ_a appproached by ψ as h increases is proportional to α. If α is held fixed, ψ_a turns out to be proportional to δ.

For example, for a uniform Earth of radius 6371 km, and for δ set to 1, ψ_a/α has the value 2.36×10^{-4}. Hence, for this Earth and any α and δ, one can calculate

$$\psi_a = \alpha\,\delta\,(2.36 \times 10^{-4}). \qquad (34)$$

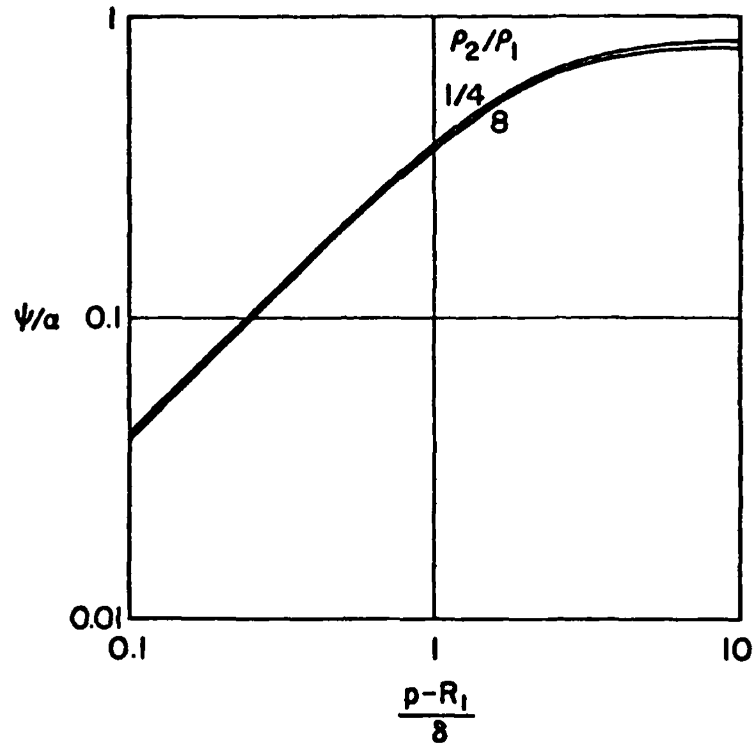

Figure 7. ψ/α Versus $(p - R_1)/\delta$ for Various $\rho_2/\rho_1 (H_1/\delta = 1/4, R_1/\delta = 1)$

Because the influence of layer parameters on vertical positioning of the ψ curve is relatively minor, the above equation will be a useful guideline in determining required accuracy of measurement in the next section.

A clear implication of (1) and (2) above is that δ could theoretically be determined (by curve-matching or geometric construction, for example) from horizontal positioning of the rollover. Knowing δ, one could then obtain α according to

$$\psi_a = \alpha \, \delta \, \psi_a^*,$$

where ψ_a^* is a reference value for the particular model type. However, it seems likely that the estimate of δ from a measurement-derived curve would be much more prone to error than the $\alpha\delta$ product due to uncertainty in curve location associated with measurement errors. Again, geologic noise due to deviation of the real Earth from that of the model has not yet been considered.

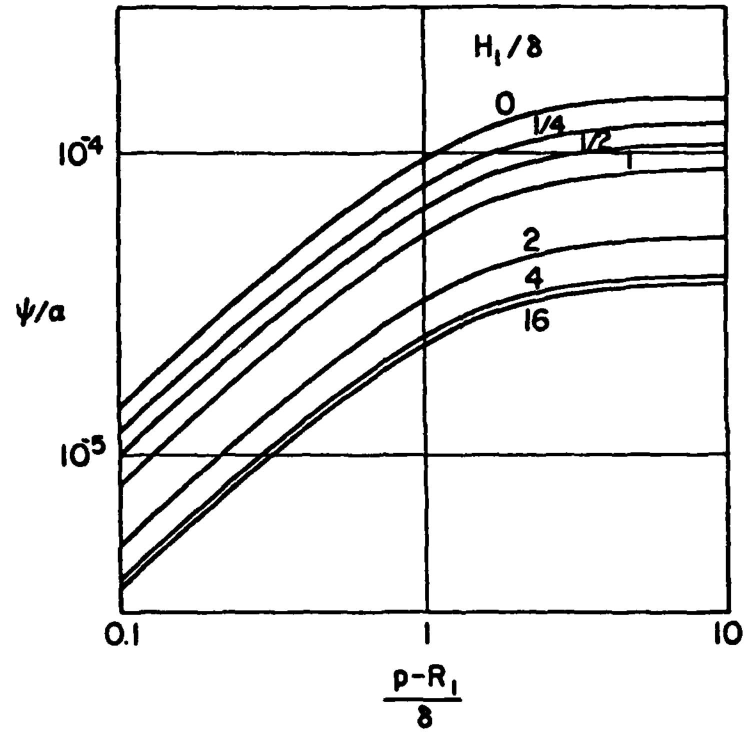

Figure 8. ψ/α Versus $(p - R)/\delta$ for Various H_1/δ ($\rho_2/\rho_1 = 4$, $R_1/\delta = 1/10000$)

5.4 Required Accuracy of Measurement

Assume measurements of the total field are to be taken at various altitudes above the surface of a layered sphere with $\delta/R << 1$. How is inaccuracy in the measurements related to resolvability of the ψ-parameter curve obtained?

Assume that possible inaccuracy of measurement is independent of height, so that the measured value $\hat{F}_T$ of the field at point P at or above the surface is described by

$$\hat{F}_T = F_T \pm \epsilon \tag{35}$$

Then, according to

$$\psi = \frac{F_T\,(p/R)^2}{F_T^o} - 1 \tag{36}$$

the calculated value of ψ at P satisfies

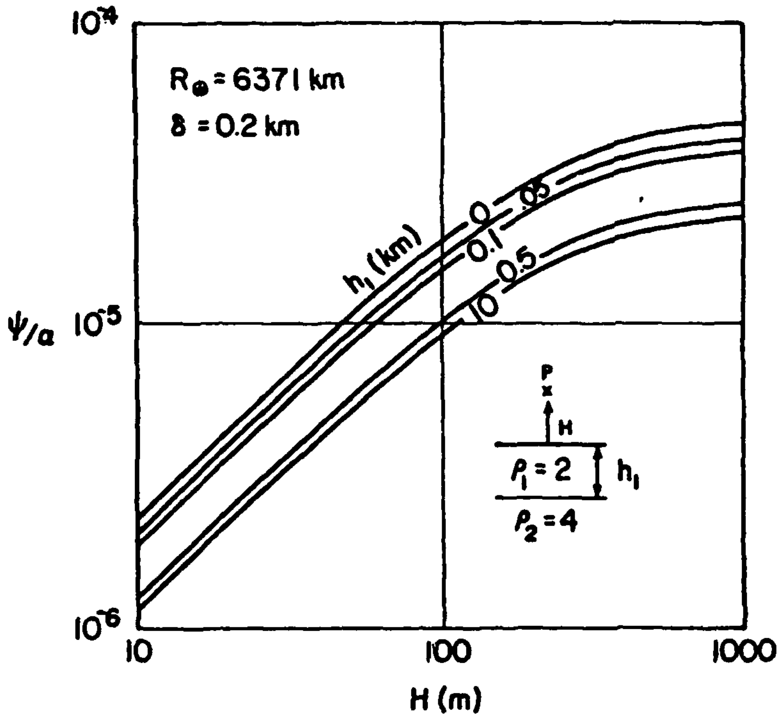

Figure 9. ψ/α Versus Height for Various Thicknesses

$$\frac{(p/R)^2\,(F_T - \epsilon)}{F_T^o + \epsilon} - 1 \le \hat{\psi} \le \frac{(p/R)^2\,(F_T + \epsilon)}{F_T^o - \epsilon} - 1. \tag{37}$$

Assuming $\epsilon << F_T^o$ and using the Maclaurin series for $1/(F_T^o \pm \epsilon)$ gives

$$\frac{(p/R)^2\,(F_T - \epsilon)}{F_T^o}\,[1 - (\epsilon/F_T^o) + \ldots] - 1 \le \hat{\psi} \le$$
$$\frac{(p/R)^2\,(F_T + \epsilon)}{F_T^o}\,[1 + (\epsilon/F_T^o) + \ldots] - 1. \tag{38}$$

Substituting via Eq. (14) and retaining only terms containing ϵ/F_T to the 0 and 1 powers gives

$$\psi - (p/R)^2 \left[\frac{F_T}{F_T^o}\left(\frac{-\epsilon}{F_T^o}\right) - \frac{\epsilon}{F_T^o}\right] \le \hat{\psi} \le \psi + (p/R)^2 \left[\frac{F_T}{F_T^o}\left(\frac{+\epsilon}{F_T^o}\right) + \frac{\epsilon}{F_T^o}\right]. \tag{39}$$

For measurements within a few kilometers of the surface of the Earth, $p/R \approx 1$ and $F_T/F_T^o \approx 1$; thus, one obtains

$$\psi - \frac{2\epsilon}{F_T^o} < \hat{\psi} < \psi + \frac{2\epsilon}{F_T^o}. \tag{40}$$

Next, consider the log-log plot of ψ as a function of height h (the solid curve of Figure 10). According to inequality (40), pairs $(h_i, \hat{\psi}_i)$ calculated from measurements will lie in the envelope between curves 1 and 2, which correspond to the right and left sides of the inequality. Note that, because $\psi \to 0$ as $h \to 0$, there exists some minimum value of h, shown as h_c, at which the value of ψ becomes smaller than the possible error $2\epsilon/F_T^o$ in its estimated value. However, points to the right of h_c should contain more information about the location of the "actual" curve. How well the curve is likely to be approximated depends on the tightness of the envelope between curves 1 and 2 and the length of its leftmost extent, determined by $2\epsilon/F_T^o$.

Because all curves are the same shape, all have rollovers that occur at the same multiple of δ, and the asymptotic value $\psi_a = \lim_{h\to\infty} \psi$ depends primarily on the parameters α and δ, constraints on accuracy can be formulated largely in terms of plot geometry.

To begin, ramp slopes are very near 45 degrees, and all curves are shaped such that a vertical line $h = 5\delta$ lies to the right of the rollover and intersects the extrapolated ramp a factor of 5 higher. A reasonable criterion for acceptable curve resolution would be to specify factors u and v such that

$$\psi/[\psi - (2\epsilon/F_T^o)] \le v \text{ for } 5\delta/u \approx h \le 5\delta; \tag{41}$$

specifically,

$$\psi/[\psi - (2\epsilon/F_T^o)] = v \text{ when } h = 5\delta/u, \tag{42}$$

in view of the fact that the envelope between curves 1 and 2 widens toward the left.

From Figure 10, one obtains

$$\psi = (2\epsilon/F_T^o)\left(\frac{5\delta/u}{h_c}\right) \text{ at } h = 5\delta/u; \tag{43}$$

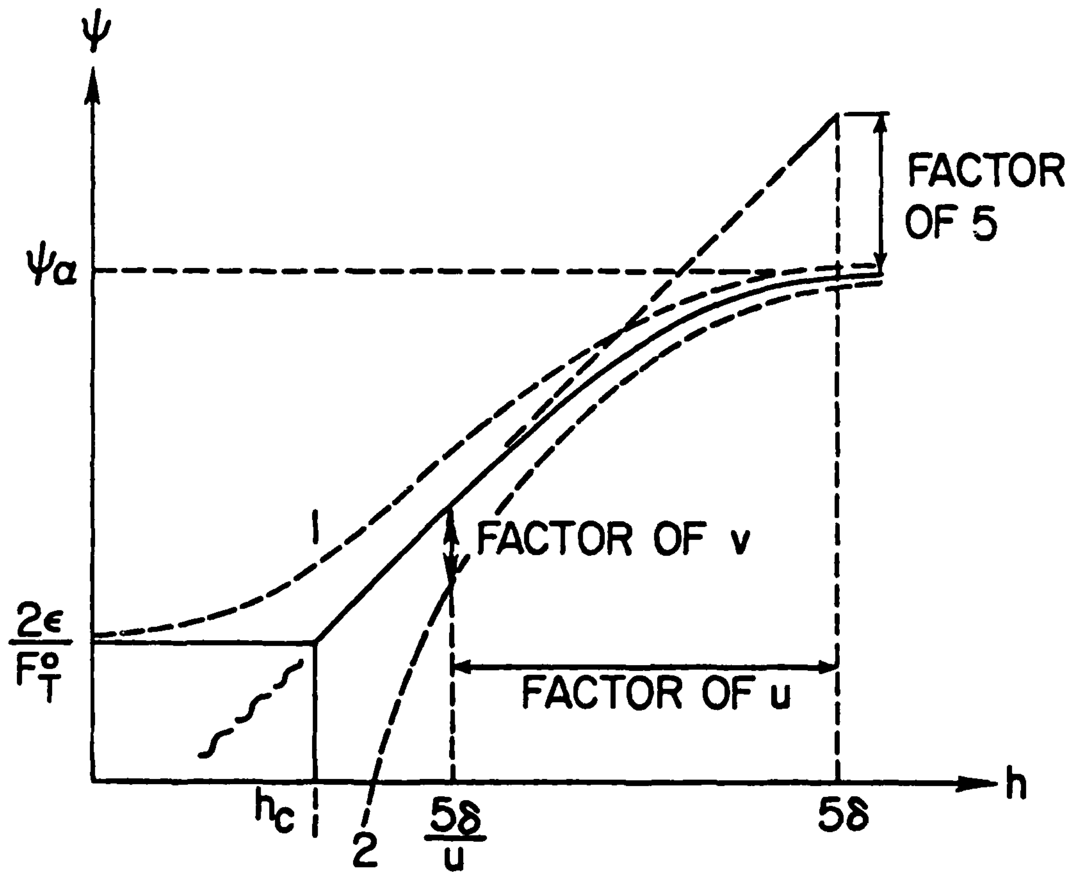

Figure 10. ψ-Curve Resolution Determination (Curve 1: Ordinate = $\psi + 2\epsilon/F_T^o$; Curve 2: Ordinate = $\psi - 2\epsilon/F_T^o$)

hence the constraint (42) becomes

$$v = [5\delta/(uh_c)]/[5\delta/uh_c) - 1] \tag{44}$$

and furthermore,

$$5\psi_a/(2\epsilon/F_T^o) = 5\delta/h_c. \quad (45)$$

Eliminating h_c from the above equations and solving for ϵ gives

$$\epsilon = \frac{5\,\psi_a\,(v - 1)}{2\,u\,v}\,F_T. \quad (46)$$

For example, suppose R = 6371 km, and δ and α are postulated as 0.2 km and 0.008, respectively. Assuming a uniform Earth, one obtains from Eq. (34)

$$\psi_a \approx (0.008)(0.2)(2.36 \times 10^{-4}) = 3.76 \times 10^{-7}.$$

Suppose it is desired that, over the two decades of h to the left of h = 5δ, estimated values of $\hat{\psi}$ are not to depart from ψ by more than a factor of 1.2. Then u = 100, v = 1.2, and

$$\epsilon/F_T^o = \frac{5(3.76 \times 10^{-7})(1.2 - 1)}{2(100)(1.2)}$$
$$\approx 1.57 \times 10^{-9}.$$

Thus, if $F_T^o \approx 1000$ cm/s^2, the tolerable error in measurement of F_T is

$$\epsilon \approx 1.6 \times 10^{-6}\ \text{cm/s}^2. \quad (47)$$

5.5 Atmospheric Corrections

Suppose our spherical layered Earth has an atmosphere. If a measurement is made at height h above the surface, what corrections should be applied to account for the field due to the air?

To correct the Newtonian component to the value that would have been measured if no atmosphere were present, one simply subtracts the field due to the portion of the atmosphere that lies below the alititude of the measurement. That this procedure works is due to model symmetry and to the fact that, at each point in space, the divergence of $\bar{F}_N$ is proportional to the mass density there. Hence, the volume integral in the Divergence Theorem is essentially the mass of the volume.

However, div $\bar{F}_Y$ can be nonzero even in free space. Thus, if points at which div $\bar{F}_Y = 0$ are called "sources" of the Yukawa field, one could say that these sources are "smeared" spatially, relative to the density distributions associated with them. An implication is that the atmosphere both above and below the point in question must be considered in calculating the Yukawa field due to the atmosphere.

Suppose the required precision in measurement (say $\hat{F} = F \pm \epsilon$) is obtainable and the Newtonian atmosphere correction is made. Whether a Yukawa atmosphere correction is necessary depends on whether

$$F_{Y\,air} << F_{Y\,Earth}, \text{ and} \tag{48}$$

$$F_{Y\,air} << \epsilon. \tag{49}$$

To answer the first criterion, let us construct a crude model of the atmosphere and calculate $F_{Y\,air}/F_{Y\,Earth}$ at a height of 0.6 km (if our measurements were on a 0.6-km tower, the specified quantity would be largest at the top). Using $\rho = kP/T$ and $\rho_o = 0.00129$ g/cm^3 gives

$$\rho = 0.391\ P/T \tag{50}$$

where ρ = air density (ρ_o at sea level), P = pressure (atm.), and T = temperature (deg K). Using $dT/dh = -6.5$ deg K/km and $P = \exp[-h\,(cm)/(2930\,T)]$ (Allen[7]), with T = 303 deg K gives

$$\rho(h) = 0.0013 - 1.18 \times 10^{-4}\ h\ g/cm^3, \tag{51}$$

with h in kilometers, as a linear approximation for air density as a function of height. Using $\delta = 0.2$ km and layer boundaries at $0.6 \pm 2^n(\delta/8)$ (stopping above at $0.6 + 8\delta$ and below at the surface) and assigning to each layer its mean air density gives

$$\overline{J_Y} = 1.92978,\ \underline{J_Y} = 1.95857$$

corresponding to the air above and below 0.6 km, respectively. A uniform Earth has $J_{Y\,Earth} = 0.098$ at $h = 0.6$ km. Then, using

$$\frac{F_{Y\,air}}{F_{Y\,earth}} = \frac{\overline{\rho_1}\,\overline{J_Y} + \underline{\rho_1}\,\underline{J_Y}}{\rho_1\,J_{Y\,earth}}$$

where $\bar{\rho} = 0.001231$ g/cm^3 and $\underline{\rho_1} = 0.001228$ g/cm^3 are the densities of the first layers above and below 0.6 km, respectively, and $\rho_1 = 2.57$, one obtains

$$F_{Y\,air}/F_{Y\,earth} \approx 1.17 \times 10^{-4} \tag{52}$$

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7. Allen, C.W. (1981) *Astrophysical Quantities*, 3rd ed., Univ. London, p. 120.

at h = 0.6 km. Thus, it appears that no correction for the Yukawa field due to the atmosphere is necessitated by the first criterion.

For the second criterion, $F_{Y\,air} << \epsilon$, let us calculate $F_{Y\,air}$ at the surface, at which its value is largest. Using a similar layered air model gives $J^o_{Y\,air} = 1.9618$. The total field is about 1000 cm/s^2 and $J^o_{N\,Earth} = 4.274 \times 10^4$; hence,

$$F^o_Y \approx 1000\,\alpha \frac{\rho_{1\,air} J^o_{Y\,air}}{\rho_{earth} J^o_{N earth}} = 1.8 \times 10^{-7} \text{ cm/s}^2, \tag{53}$$

where F^o is approximated by F^o_N. This value is about a factor of 10 less than $\epsilon = 1.6 \times 10^{-6}$ cm/s^2 of the previous section, so for these examples, one need not correct for the Yukawa component due to the atmosphere.

6. EFFECTS OF LATERAL INHOMOGENEITIES

The extent to which lateral variations in density affect reliable sensing of the Yukawa field will now be considered. To enable basic understanding of field behavior in terms of a few model parameters, the models are deliberately simple.

6.1 The Field Due to a Uniform Rod

Assume a point P whose distances from the center of a rod are q units measured perpendicular to the rod and p units measured parallel to the rod (Figure 11). Let the rod be of length L and be described by its linear density $\sigma = \rho A$, where A is its cross-sectional area. The Yukawa *component* parallel to the rod is

$$F_{Y\parallel}(P) = -\alpha\, G_o\, \sigma \int_{-L/2}^{L/2} \frac{e^{-\mu r}(1 + \mu r)\cos\phi}{r^2}\, dz \tag{54}$$

where

$$\cos\phi = \sqrt{1 - (q/r)^2}, \tag{55}$$

and ϕ is as shown in Figure 11. Letting $v = \mu r = [r]m/\delta$ then yields

$$K_{Y\parallel}(P) = \int_{\sqrt{(p-L/2)^2+q^2}/\delta}^{\sqrt{(p+L/2)^2}/\delta} \frac{e^{-v}(1 + v)}{v^2}\, dv, \tag{56}$$

where

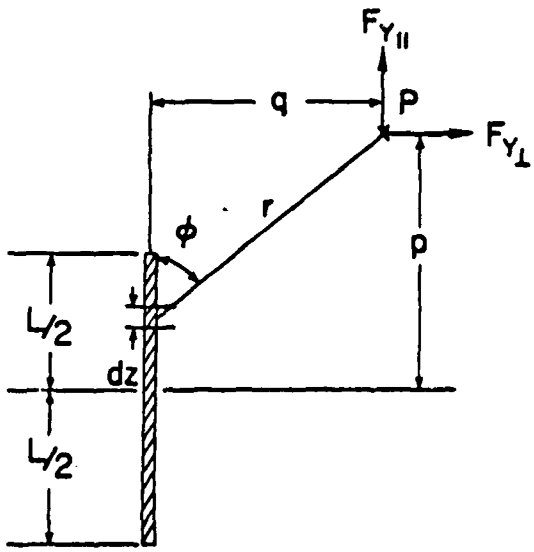

Figure 11. Yukawa Components of Field due to a Rod

$$F_{Y_{\parallel}}(P) = \frac{-\alpha G_0 \sigma}{\delta} K_{Y_{\parallel}}(P). \tag{57}$$

The component perpendicular to the rod is given by

$$F_{Y_{\perp}}(P) = -\alpha G_0 \sigma \int_{-L/2}^{L/2} \frac{e^{-\mu r}(1 + \mu r)\sin\phi}{r^2} dz, \tag{58}$$

where $\sin\phi = q/r$. The substitution $v = r/\delta$ yields

$$K_{Y_{\perp}}(P) = (q/\delta) \int_{\sqrt{(p-L/2)^2+q^2}/\delta}^{\sqrt{(p+L/2)^2+q^2}/\delta} \frac{e^{-v}(1 + v)}{v^2\sqrt{v^2 - (q/\delta)^2}} dv \tag{59}$$

where

$$F_{Y_{\perp}}(P) = \frac{-\alpha G_0 \sigma}{\delta} K_{Y_{\perp}}(P). \tag{60}$$

The corresponding Newtonian components are described by

$$K_{N_{\parallel}} = (B - A)/AB, \tag{61}$$

$$K_{N_\perp} = \frac{A\sqrt{B^2 - Q^2} - B\sqrt{A^2 - Q^2}}{Q\,A\,B}, \tag{62}$$

where $A = \sqrt{(p - L/2)^2 + q^2}/\delta$, $B = \sqrt{(p + L/2)^2 + q^2}/\delta$, and $Q = q/\delta$.

6.2 A Rectangular Block in a Layered Medium

Consider a rectangular parallelepiped of uniform density ρ_D imbedded in a much larger layered sphere such that the "top" of the parallelepiped is essentially parallel to the locally nearly planar surface (Figure 12).

Subdivide the block into n identical, vertically oriented parallelepipeds. Further subdivide each smaller block into m sections defined by the intersection of the block with various layers. If each section is assigned an appropriate density difference and then replaced by a vertical rod of similar height and mass passing through the section's center, the Yukawa field at point P (vertical component), given by

$$F_Y(P) = -\alpha\, G_o\, \rho_1\, \pi\, \delta\, [J_{Y_L}(P) + J_{Y_D}(P)], \tag{63}$$

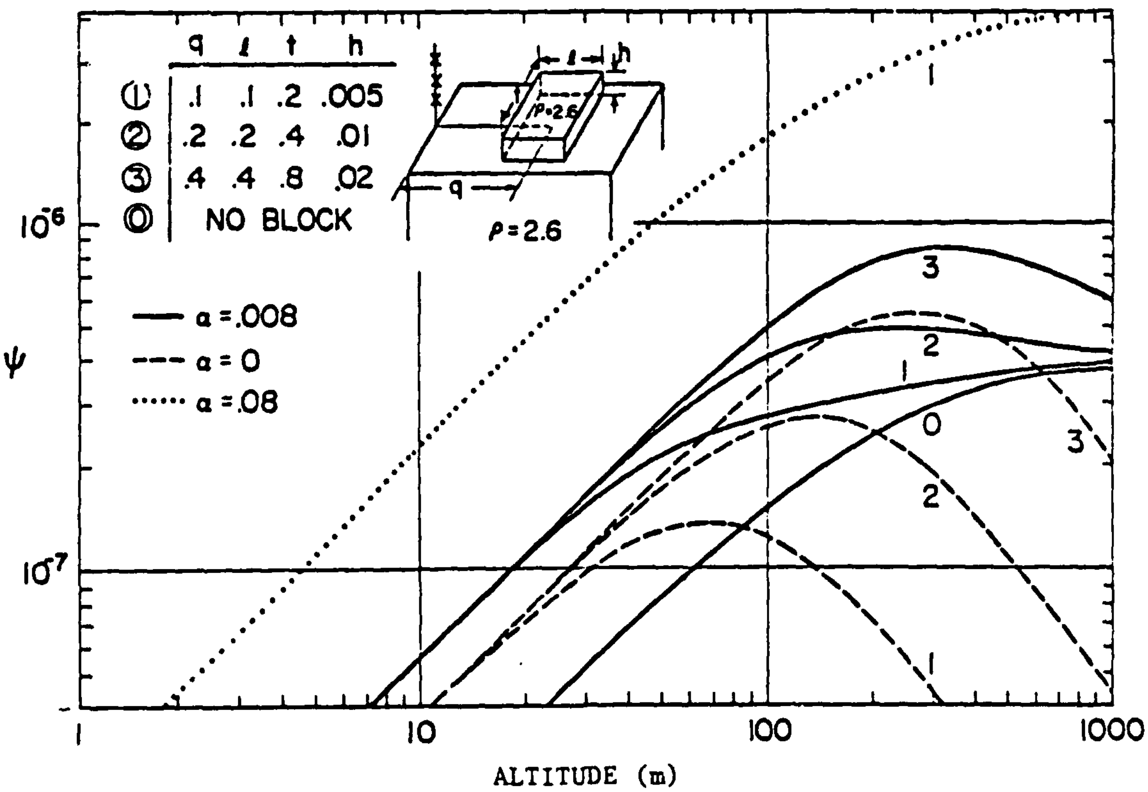

Figure 12. ψ Curves in Presence of Offset Block

requires

$$J_Y(P) = \frac{A}{\pi\,\delta^2} \sum_{j=1}^{m} \frac{\rho_D - \rho_j}{\rho_1} \sum_{i=1}^{n} K_{Y_{ij}}(P), \tag{64}$$

where J_{Y_L} and J_{Y_D} are for the field of the layered sphere and for that of the block in excess of that of the sphere, respectively, A is the cross-sectional area of each section, ρ_j is the density of the layer intersected by the J^{th} section of a block, and $K_{Y_{ij}}$ is the expression of Eq. (56) calculated for the j^{th} section of the i^{th} rod. The expression for $J_{N_{ij}}(P)$ is similar except that $K_{N_{ij}}(P)$, corresponding to Eq. (57), appears in place of $K_{Y_{ij}}(P)$.

6.3 The Parameter ψ

The parameter ψ, defined previously consistent with

$$\psi = \frac{F_T\,(p/R)^2}{F_T^o} - 1, \tag{65}$$

is calculated for the block-and-sphere model as

$$\psi = \frac{J_{N_o}(p/R)^2 - J_{N_o}^o - \alpha\,(J_Y\,(p/R)^2 - J_Y^o)}{J_N^o - \alpha\,J_Y^o}, \tag{66}$$

where

$J_* = J_{*L} + J_{*D}$,
L designates due to the layered sphere,
D designates due to density difference between block and sphere,
* is "N" for Newtonian or "Y" for Yukawa component, and
superscript "o" indicates surface value.

Lateral inhomogeneity distorts the location and shape of the ψ curve from that of a layered Earth. The models of Figure 12 are crudely similar to the case of a broad hill near the measurement site. For α = 0.008, the typical rollover is masked by the occurrence of higher ψ values near the surface, relative to those for the homogeneous model. These curves lie only a factor of 1.5 above those for $\alpha = 0$ (no Yukawa field) to the left of the rollover; hence, for small α, the shape of the curve can be primarily a geometric effect caused by distortion of the Newtonian field near the inhomogeneity. The curve for $\alpha = 0.08$ shows behavior more typical of a layered model and lies everywhere at least a factor of 6 above that for $\alpha = 0$ due to greater strength of the Yukawa component.

These results indicate that a reliable interpretation of an experiment using the ψ parameter requires that the portion of the field due to lateral inhomogeneities be subtracted before the ψ curve is calculated. An approach that avoids using a parameter defined in terms of one specific model type would consist essentially of comparing measurements at various heights with the upward-continued surface field, assumed to be Newtonian.

7. CONCLUSIONS

For a sphere or layered sphere model, the shape of the ψ- parameter curve is determined by the values of δ/R and α; changes in density or layer thickness affect only the vertical positioning of the curve. For $\delta/R << 1$, as for the Earth, the shape of the ψ curve is virtually α-independent as well, as α enters into ψ essentially as a scaling factor. Hence, such models enable study of the relationship between ψ curve discernibility (Yukawa field detection) and measurement precision, exclusive of effects of lateral inhomogeneities.

The major limitation of these models is that the Earth is not free from lateral density and thickness variations. Thus, although the Yukawa field is a rather local effect, distortion of the much larger Newtonian component due to lateral density variations can render the ψ curve uninterpretable in raw form.

The effect of atmospheric mass on the Yukawa component appears negligible, and the effect on the Newtonian component is removed easily.

8. RECOMMENDATIONS

One might explore the degree to which the ψ-parameter approach in the presence of lateral inhomogeneities could be salvaged. In the experimental environment, such an approach might involve modeling of the subsurface followed by determination of the residual field in excess of that of a "best-fit" spherically layered model. Subtracting the residual field at measurement points on the tower should then yield a vertical field profile as if no lateral variations were present. A major concern with this method is that the modeling phase (which must assume α and δ) could result in a mass distribution that would yield erroneous corrections for tower measurements while still reproducing the ground measurements. This phenomenon would be due to the Newtonian and Yukawa fields having different ranges of influence.

Alternatively, one might discard the ψ-parameter and employ a modeling scheme similar to that above, but using tower measurements along with the ground measurements in the modeling phase. This procedure might be most powerful if α and δ are the entities to be modeled, given a fairly restricted selection of possible Earth models. A question in this case is whether the Earth models can be reasonably constrained sufficiently to enable reliable estimates of α, δ, and/or $\alpha\delta$.

Another possibility, partially explored by Christopher Jekeli and the author, is to model the field above the surface directly from the boundary values on the surface and/or on the tower without reference to an Earth model. Whereas avoiding the Earth model circumvents questions concerning mass-distribution equivalence, a new difficulty arises in that a gauge potential enabling workable differential equations relies on the as-yet-unknown Yukawa field in its boundary conditions. Thus, the price of more efficient and elegant data usage appears to be increased difficulty in engineerng such a scheme.

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