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Final Scientific Report
NUMERICAL METHODS FOR SINGULARLY PERTURBED
DIFFERENTIAL EQUATIONS WITH APPLICATIONS

For the period 1 June, 1975 - 31 May, 1980

Grant No. AFOSR-75-2818

Rensselaer Polytechnic Institute
Troy, New York 12181

Prepared by Joseph E. Flaherty

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During the course of this grant algorithms were developed and analyzed for the numerical solution of singularly-perturbed (or stiff) initial value (cf. [4], [8], and [9]) and boundary value (cf. [1], [10]) problems for ordinary differential equations and initial-boundary value problems for partial differential equations, [11]. These general purpose methods have been applied to a wide variety of problems arising in several disciplines, including optimization and optimal control (cf. [2], [3]), nonlinear oscillations (cf. [5], [7]), chemical reactions [9], and hydrodynamic stability. [6]. *Keywords: computational, singularly-perturbed, flow, problems*

Five graduate students were partially supported by this grant and assisted Professor Flaherty in his investigations. Four of the students earned MS degrees in either Applied Mathematics or Computer Science and one earned a Ph.D. degree in Mathematics. A paper based on W. Mathon's MS thesis [10] has been accepted for publication in the SIAM J. Sci. and Stat. Comp. and it will appear in print this Fall. Two other MS students (G. M. Heitker and J. M. Coyle) extended Flaherty and Mathon's work to systems of boundary value problems and boundary value problems having rapidly oscillating solutions. Papers based on their theses are being prepared for publication. Both Heitker and Coyle are continuing their studies towards Ph.D. degrees. S. F. Davis recently completed his Ph.D dissertation [11] on adaptive grid finite element methods for time dependent partial differential equations. His method is capable of adapting the finite element mesh in order to follow such sharp

* See the list of References and Abstracts following this report.

transitions as boundary layers, shock layers, wave fronts and melting, boiling, or reaction interfaces. A first paper based on part of Dr. Davis' dissertation is in its final stages of preparation and will be submitted for publication shortly. Additional papers will follow.

Dr. P. T. Boggs of the U. S. Army Research Office was also supported for one month during 1975 and published a paper on optimization techniques [3] due in part to this support.

This grant has also provided support for Professor R. E. O'Malley, Jr. of the University of Arizona to visit R.P.I. for about one month each year and collaborate with Professor Flaherty. This collaboration has been most fruitful and rewarding. Five joint papers have been published (cf. [1], [2], [4], [8], [9]) and two more are being prepared for publication.

During the course of this investigation Professor Flaherty was invited to lecture on material pertaining to this grant at the following institutions: The Weizmann Institute of Science, Rehovot, Israel (1974), Clarkson College (1975), Drexel University (1976), the University of Utah (1976), the University of Melbourne, Melbourne, Australia (1977), the University of Arizona (1977), The Catholic University, Nijmegen, The Netherlands (1978), Stanford University (1980), the University of British Columbia (1980) and the University of California at Davis (1980).

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11. S. F. Davis, "An Adaptive Grid Finite Element Method for Initial-Boundary Value Problems", Ph.D. Dissertation, Rensselaer Polytechnic Institute, March 1980.

THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS FOR STIFF
DIFFERENTIAL EQUATIONS

by

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and

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Abstract

The numerical solution of boundary value problems for certain stiff ordinary differential equations is studied. The methods developed use singular perturbation theory to construct approximate numerical solutions which are valid asymptotically; hence, they have the desirable feature of becoming more accurate as the equations become stiffer. Several numerical examples are presented which demonstrate the effectiveness of these methods.

Published in Mathematics of Computation, Vol. 31, No. 137,
January 1977, pp. 66-93.

ON COMPUTING SINGULAR OPTIMAL CONTROLS

BY

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Abstract

We consider singular optimal control problems consisting of a state equation

$$\dot{x} = Ax + bu$$

for vectors x and scalars u and a cost functional

$$J = \frac{1}{2} \int_0^T (x'Qx + \epsilon^2 u^2) dt$$

to be minimized for $|u| \leq m$ and $\epsilon = 0$. By considering the problem as $\epsilon \rightarrow 0$, singular perturbation concepts can be used to compute solutions consisting of bang-bang controls followed by singular arcs. The procedure further develops a numerical technique proposed by Jacobson, Gershwin, and Lele, as well as additional analytic methods of other authors.

Published in the IEEE Transactions on Automatic Control,
Vol. AC-22, No. 4, August 1977, pp 640-648.

An Algorithm Based on Singular Perturbation Theory,
For Ill-Conditioned Minimization Problems

by

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Abstract

The problem of finding an unconstrained local minimum of a function f is considered. Ill-conditioned minimization problems are characterized by long, very narrow valleys which tend to trap many numerical procedures, i.e., methods applied to such problems tend to "jam". To overcome this difficulty, the problem is reformulated as that of integrating the differential equation $x' = -\nabla f(x)$. The ill-conditioning of the minimization problem carries over to the differential equation by making it stiff. Thus methods for integrating stiff systems of equations are investigated and adapted to this special problem. In particular, a recent method of Miranker based on singular perturbation theory is modified and shown to be applicable to this problem. Some numerical results are presented which indicate that the resulting algorithms are quite competitive.

In SIAM J. Numer. Anal., Vol. 14, No. 5, Oct. 1977, pp. 830-843.

SINGULAR SINGULAR-PERTURBATION PROBLEMS

by

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Abstract

We consider the nonlinear system

$$(1) \quad \epsilon \dot{z} = f(z, t, \epsilon)$$

of M nonlinear equations on some finite subinterval of $t \geq 0$ subject to the initial condition

$$(2) \quad z(0) = z^0(\epsilon)$$

in the limit as the small positive parameter $\epsilon \rightarrow 0$. Any acquaintance with singular perturbations would lead one to expect that the unique solution of this initial value problem might converge as $\epsilon \rightarrow 0$ to a solution z_0 of the limiting equation

$$(3) \quad f(z_0, t, 0) = 0$$

away from any "boundary layer" regions of nonuniform convergence. In the singular situation when $f_z(z_0, t, 0)$ is a singular matrix, this limiting equation has an infinite number of solutions, so the limiting equation (3) is not adequate for the purpose of determining the reduced problem satisfied by the limiting solution (assuming that such a limit exists).

We consider the initial value problem (1) - (2) under the primary assumptions that the matrix

$$(4) \quad f_z(z, t, 0) \quad f_{0z}(z, t)$$

has constant rank k , $0 < k < M$, that its null space is spanned by $M - k$ linearly independent eigenvectors, and that f_z has k stable eigenvalues (counting multiplicities) for all z and t . Assuming sufficient differentiability of f and asymptotic expansions for both f and z^0 , we construct an asymptotic solution of the form

$$(5) \quad z(t, \varepsilon) = Z(t, \varepsilon) + \Pi(\tau, \varepsilon)$$

where Z and Π have asymptotic power series expansions in ε the terms of Π all tend to zero as the stretched variable $\tau = t/\varepsilon$ tends to infinity.

Published in Lecture Notes in Mathematics, No.594,
Singular Perturbations and Boundary Layer Theory, Lyon 1976,
A.-Dold and B. Eckman, Eds., Springer-Verlag, Berlin, 1977,
pp 422-436.

FREQUENCY ENTRAINMENT OF A FORCED VAN DER POL OSCILLATOR

by

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and

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ABSTRACT

A van der Pol relaxation oscillator that is subjected to external sinusoidal forcing can exhibit stable and unstable periodic and almost periodic responses. For some forcing amplitudes it even happens that two stable subharmonics having different periods may coexist. We investigate here the stable responses of such forced oscillators. By numerically computing the rotation number of stable oscillations for various values of the forcing amplitude and oscillator tuning, we obtain descriptions of regions of phase locking, successive bifurcation of stable subharmonic and almost periodic oscillations, and overlap regions where two distinct stable oscillations can coexist.

Published in Studies in Applied Mathematics, Vol. 58, 1978
pp 5-15.

NUMERICAL INVESTIGATION OF THE
EFFECT OF A CORIOLIS FORCE ON THE STABILITY OF PLANE
POISEUILLE FLOW

by

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ABSTRACT

The stability of the viscous flow between two parallel horizontal plates due to a constant reduced pressure gradient in a system rotating about a vertical axis is studied. The critical value of the Reynolds number R , based on the reduced pressure gradient, is a function of a dimensionless rotation parameter T , the Taylor number. A numerical solution of the eigenvalue problem shows that (i) the minimum point for the viscous instability mode associated with plane Poiseuille flow at $T = 0$ disappears at a value of T of about $T \approx 0.4$, and (ii) for $T \neq 0$ a new instability mode appears as a result of the Coriolis effect on the basic flow and in the perturbation equations. This new instability gives the critical value of R for values of T as small as 0.06.

NONUNIQUE STABLE RESPONSES OF EXTERNALLY
FORCED OSCILLATORY SYSTEMS

by

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ABSTRACT

A population model, based on a system of difference equations, is presented which possesses a stable relaxation oscillation. When this system is subjected to external periodic forcing, it exhibits phase locking of subharmonics and for some parameter values the co-existence of distinct stable periodic solutions. Successive bifurcations of subharmonics can be described in terms of their rotation numbers.

In Annals of the New York Academy of Sciences, Vol. 316, Bifurcation Theory and Applications in Scientific Disciplines, O. Gurel and O. E. Rössler, Eds., 1979.

THE NUMERICAL SOLUTION OF SINGULAR
SINGULARLY-PERTURBED INITIAL VALUE PROBLEMS

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ABSTRACT

We consider the vector initial value problem $\dot{y} = f(y, t, \epsilon)$, $y(0) = y^0(\epsilon)$ in the situation when the $m \times m$ matrix $f_y(y, t, 0)$ is singular with constant rank $k < m$ and has k stable eigenvalues. We show how to determine the unique limiting solution Y_0 of the reduced problem $f(Y_0, t, 0) = 0$ and how to obtain a uniform asymptotic expansion of the solution which is valid for small values of ϵ on finite t intervals. A numerical technique is developed to calculate the limiting solution and the results of some examples are compared with an existing code for stiff differential equations.

In Numerical Analysis of Singular Perturbation Problems, P. W. Hemker and J. J. H. Miller, Eds., Academic Press, London, 1979, pp. 143-180.

ANALYTICAL AND NUMERICAL METHODS FOR NONLINEAR
SINGULAR SINGULARLY-PERTURBED INITIAL VALUE PROBLEMS

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ABSTRACT

This paper considers the vector problem $\varepsilon \dot{y} = f(y, t, \varepsilon)$, $y(0)$ prescribed, when the $m \times m$ matrix $f_y(y, t, 0)$ is singular and of rank k , $0 \leq k < m$, with k stable eigenvalues. The reduced system $f(Y_0, t, 0) = 0$ is, then, insufficient to determine the limiting solution Y_0 for $t > 0$. Instead, depending on the row-reduced form of f_y , a differential equation is obtained for Y_0 and an initial vector $Y_0(0)$ is obtained as the steady-state solution of an inner problem. Alternative analytical and numerical possibilities are explored, and analytical and numerical solutions are obtained for representative problems.

In SIAM J. Appl. Math., Vol. 38, No. 2, April 1980, pp. 225-248.

COLLOCATION WITH POLYNOMIAL AND TENSION
SPLINES FOR SINGULARLY PERTURBED
BOUNDARY VALUE PROBLEMS*

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and

William Mathon
IBM
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ABSTRACT

Collocation methods using both cubic polynomials and splines in tension are developed for second order linear singularly-perturbed two-point boundary value problems. Rules are developed for selecting tension parameters and collocation points. The methods converge outside of boundary layer regions without the necessity of using a fine discretization. Numerical examples comparing the methods are present.

To appear in SIAM J. Sci. & Stat. Comp., 1980

AN ADAPTIVE GRID FINITE ELEMENT METHOD FOR
INITIAL-BOUNDARY VALUE PROBLEMS

by

Stephen F. Davis

A Thesis Submitted to the Graduate
Faculty of Rensselaer Polytechnic Institute
in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
Major Subject: Mathematics

ABSTRACT

A finite element method is developed to solve initial-boundary value problems for vector systems of parabolic partial differential equations in one space dimension and time. The methods adapt the computational mesh as the solution progresses in time and are thus able to follow and resolve such sharp transitions as boundary layers, shock layers, or wave fronts. This permits an accurate solution to be calculated with fewer mesh points than would be necessary with a uniform mesh.

The overall method contains two parts, a solution algorithm and a mesh selection algorithm. The solution algorithm is a finite element-Galerkin method on trapezoidal space-time elements, using either piecewise linear or cubic polynomial approximations.

(continued)

An efficient and stable algorithm is developed to solve the resulting block tridiagonal nonlinear discrete system.

The mesh selection algorithm builds upon the work of de Boor, Jupp and others on variable knot spline interpolation.

Both the solution and the mesh selection algorithm are analyzed in detail. Error estimates are obtained for the solution algorithm. The mesh selection algorithm is shown to produce a mesh which minimizes the L_2 error of the solution for a given number of mesh points.

A computer code implementing these algorithms has been written and applied to a number of problems. These computations confirm that the theoretical error estimates are attained and demonstrates the utility of variable mesh methods for partial differential equations.