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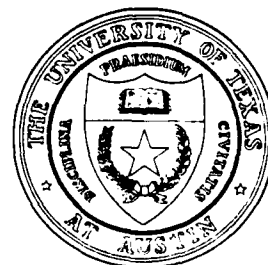# Cone Quasi-Concave Multi-Objective Programming Theory and Dominance Cone Constructions

by
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# CENTER FOR CYBERNETIC STUDIES

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## Abstract

Some basic theory of "cone quasi-concave multi-objective programming" is developed. This new class of vector extremal problems with quasi-concave multiple objectives employs ideas of non-dominated solutions associated with dominance cones. Necessary as well as sufficient conditions for optimal solutions to such problems are provided. A simple example illustrates the concepts involved. In addition, for general applications in economics, it is shown how to establish dominance cones to realize producer priorities, consumer preferences, and other concerns exogenously determined.

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## 1. Introduction

In the multi-objective programming literature, many authors presume the concavity of objective functions in seeking Pareto-optimal solutions. Following the basic ideas of goal programming introduced by Charnes and Cooper [3] , [4], and [5] in the early fifties and developed through the sixties, Yu [22], and Beregstresser, Charnes and Yu [2] generalized Pareto-optimal solutions to nondominated solutions and developed them in the objective space of multi-objective problems. Charnes, Cooper, Wei , and Huang [8] studied the properties of nondominated solutions in decision spaces which were normed vector spaces. They further developed new approaches and applied them to extensions of game theory [9]. Charnes, Huang, Rousseau, and Wei [10] also initiated and developed "T-non-dominated efficiency" for multi-payoff n-person games with interacting "cross-constrained" strategy sets as vector extremal principles for solutions of such games.

A great deal of research effort has been expended on the theory of quasi-concave functions for economic studies. Many utility functions and production functions are quasi-concave rather than concave. Especially for their economic applications, Diewert, Avriel, and Zang [13] and others have achieved important research results on the properties of general quasi-concave functions. Arrow and Enthoven [1], Mangasarian [15], Ferland [14] and others focused on single objective quasi-concave programming with quasi-concave objective and constraint functions. Their methods, however, do not extend in any immediate way to multi-objective programming, since a non-negative linear combination of quasi-concave functions is not necessarily quasi-concave. Craven [12] considered a special case of multi-objective programming assuming that the weighted sum of the objective functions is pseudo-concave for each suitably chosen set of weights. This approach does not apply either, since a pseudo-concave function must be a quasi-concave function [15]. So far as we know, with the exception of the discussion in [10], little has been touched upon in quasi-concave multi-objective programming, especially as related to new ideas of solutions of games with interacting or "cross-constrained" strategy sets [10].

In the present paper we develop some basic theory of "cone quasi-concave multi-objective programming", a new class of multi-criteria decision problems, which incorporates nondominated solutions associated with dominance cones. Necessary as well as sufficient conditions for

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optimal solutions to such problems are also obtained. With dominance cones appropriately constructed, the resultant non-dominated solutions, which are actually a subset of the Pareto-optimal solutions, should be more desirable in the sense that they are better suited to the needs, obligations, and preferences of decision makers. A simple example illustrates the concepts involved. In addition, we show how to establish dominance cones to realize producer priorities, consumer preferences, and other concerns exogenously determined.

## 2. Generalized Cone Concavity and Nondominated Solutions

In this section we review some relevant results regarding cones, their polar cones, generalized cone concavity and nondominated solutions for later use in our development. We also derive some properties of generalized cone concavity.

A set S in $E^n$ is convex iff $x_1 x_2 \in S$ implies that $\lambda x_1 + (1-\lambda) x_2 \in S$ for all $0 \leq \lambda \leq 1$. A set S is a cone iff $x \in S$ and $\lambda \geq 0$ imply that $\lambda x \in S$. S is a convex cone iff S is a cone and is convex. Thus, S is a convex cone iff $x_1 x_2 \in S$ and $\lambda_1, \lambda_2 \geq 0$ imply that $\lambda_1 x_1 + \lambda_1 x_2 \in S$.

For an arbitrary set S in $E^n$, let $\bar{x} \in \bar{S}$, where $\bar{S}$ denotes the closure of the set S. Denote the "tangency cone" of S at $\bar{x}$ by $T(S, \bar{x})$, where $T(S, \bar{x}) = \{h \in E^n$: there exists a sequence $\{x^k\}$ and a sequence $\{\lambda^k\}$ such that $h = \lim_{k \to \infty} \lambda^k (x^k - \bar{x})$, with $x^k \in S, \lambda^k > 0$, and $\lim_{k \to \infty} x^k = \bar{x}\}$.

Further, denote the (**negative**) polar cone of S by $S^*$ where $S^* = \{y \in E^n : x^t y \leq 0$ for all $x \in S\}$ (the superscript "t" denotes transpose). A cone $\Lambda$ in $E^n$ is said to be acute if there exists an open half-space $H = \{x \in E^n : a^t x > 0, a \neq 0\}$ such that $\bar{\Lambda} \subset H \cup \{0\}$.

Cones, polyhedral cones and polar cones in $E^n$ are discussed further in Rockefellar [17], Stoer and Witzgall [21] and Yu [22]. More general results in normed linear spaces are derived in Charnes, Cooper, Wei and Huang [8].

Proof of the following lemma may be found in [8], [17], [21] and [22].

**Lemma 2.1:** Let $\Lambda$ and $\Lambda_1$ be cones in $E^n$.

(i)    If $\Lambda \subset \Lambda_1$ then $\Lambda^* \supset \Lambda_1$

(ii)   Int $\Lambda^* \neq \emptyset$ if and only if $\Lambda$ is acute

(iii)  If $\Lambda$ is a convex cone then $(\Lambda^*)^* = \bar{\Lambda}$

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(iv)  When $\Lambda$ is acute,

$$\text{Int } \Lambda^* = \{y \in E^n : x^t y < 0 \quad \text{for all } x \in \overline{\Lambda}, x \neq 0\} \text{ and}$$

$$\Lambda \cap (-\Lambda) = \{0\}$$

**Definition 2.1:**  Let S be a convex set in $E^n$ and $\Lambda$ be a convex cone in $E^m$. A real-valued vector function $g : S \rightarrow E^m$ is called "$\Lambda$ – concave on S" if

$$g(\lambda x^1 + (1-\lambda) x^2) - (\lambda g(x^1) + (1-\lambda) g(x^2)) \in \Lambda$$

for all $x^1, x^2 \in S$ and $\lambda \in (0,1)$.

**Definition 2.2:**  Let S be a convex set in $E^n$ and $\Lambda$ be a convex cone in $E^m$. A real-valued vector function $g : S \rightarrow E^m$ is called "$\Lambda$–quasiconcave on S" if

$$g(\lambda x^1 + (1-\lambda) x^2) - \text{Min}\{g(x^1), g(x^2)\} \in \Lambda \quad \text{for all } x^1, x^2 \in S \text{ and } \lambda \in (0,1).$$

and is called "strictly $\Lambda$–quasiconcave on S" if

$$g(\lambda x^1 + (1-\lambda) x^2) - \text{Min}\{g(x^1), g(x^2)\} \in \text{Int } \Lambda \quad \text{for all } x^1 \neq x^2 \in S \text{ and } \lambda \in (0,1),$$

where

$$\text{Min}\{g(x^1), g(x^2)\} = \begin{pmatrix} \min(g_1(x^1), g_1(x^2)) \\ \cdot \\ \cdot \\ \cdot \\ \min(g_m(x^1), g_m(x^2)) \end{pmatrix}$$

From the definition above, a real-valued function g defined on a convex set S is quasiconcave [15] iff

$$g(\lambda x^1 + (1-\lambda) x^2) - \text{Min}\{g(x^1), g(x^2)\} \geq 0 \text{ for all } x^1, x^2 \in S \text{ and } \lambda \in (0,1),$$

and is strictly quasiconcave [14] iff

$$g(\lambda x^1 + (1-\lambda) x^2) - \text{Min}\{g(x^1), g(x^2)\} > 0 \quad \text{for all } x^1, x^2 \in S, x^1 \neq x^2, \text{ and}$$

$\lambda \in (0,1)$.

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*(The name "pit-free" for quasi-concave and "strictly pit-free" for strictly quasi-concave has been suggested in the past as a mnemonically better rendition of these properties.)*

**Lemma 2.2:[15]** Let S be a convex set in $E^n$ and g be a differentiable real-valued function defined on S. g is quasi-concave on S iff for any $x^1, x^2 \in S$, $g(x^2) \geq g(x^1)$ implies that $\nabla g(x^1)(x^2 - x^1) \geq 0$, or equivalently, for any $x^1, x^2 \in S$, $\nabla g(x^1)(x^2 - x^1) < 0$ implies that $g(x^2) < g(x^1)$.

**Definition 2.3:** Let S be a convex set in $E^n$ and $\Lambda$ be a convex cone in $E^m$. A real-valued vector function $g : S \to E^m$ is called "$\Lambda$ (i) –strictly quasiconcave on S" if g is $\Lambda$ –quasiconcave and $p_i\, g_i$ is strictly quasiconcave on S for any nonzero $p \in -\Lambda^*$ where $p = (p_1, \ldots, p_i, \ldots, p_m)^t$.

**Lemma 2.4 [10]** : Let S be a convex set in $E^n$, $\Lambda$ be a convex cone in $E^m$, and $g : S \to E^m$ be a real-valued vector function. If, for all $\alpha \in E^m$, the set $S_\alpha = \{x \in S : g(x) \in \alpha + \Lambda\}$ is convex, then g is $\Lambda$–quasiconcave on S.

We now give our definition of the class of "vector extremal" or multi-objective programming problems we shall consider.

**Definition 2.4:** The multi-objective programming problem is defined in terms of a set of objectives $L = \{1, 2, \ldots, \ell\}$ ; $\ell$ real-valued functions $f_j$, $j \in L$, the objective functions; a real-valued vector function $g = (g_1, \ldots, g_m)$, the constraint function; a convex set S, the domain of all $f_j$ and g; a convex cone $K$ in $E^m$, the constraint cone; a convex cone $W$ in $E^\ell$, the dominance cone; and

$X(K) = \{x = (x_1, \ldots, x_n) : g(x) \in K, x \in S\}$, the constraint set. The multi-objective programming problem is therefore formulated as

$$(W\text{-}K\text{-}P) \quad \begin{cases} \text{Max} \ (f_1(x), \ldots, f_\ell(x)) \\ \quad x \in X(K) \end{cases}$$

**Definition 2.5:** A point $\bar{x} \in X(K)$ is called a nondominated solution of (W-K-P) associated with W if there does not exist any point $y \in X(K)$ such that

$$(\ldots, f_j(y), \ldots)^t \in (\ldots, f_j(\bar{x}), \ldots)^t + W$$

$$f_{j_0}(y) \neq f_{j_0}(\bar{x}) \text{ for some } j_0 \in L.$$

The corresponding point $(f_1(\bar{x}), \ldots, f_\ell(\bar{x}))^t$ in the objective space is called a nondominated point associated with W.

For many purposes in economics and elsewhere, what we seek are Pareto-optimal solutions to such multi-objective programming problems.  (See, Charnes, Huang, Rousseau and Semple [9] and Charnes, Huang, Rousseau and Wei [10].)  Note that if we set the dominance cone $W = E^\ell_+$, the nonnegative orthant, then the nondominated solutions of (W-K-P) associated with $E^\ell_+$ are precisely the Pareto-optimal solutions.  The particular subset of Pareto-optimal solutions from which a final choice is to be effected (for example, by a regulatory agency) will depend on the preferences of the decision-making body over the outcomes in the objective function space (i.e., the nondominated or Pareto-optimal points).  By applying different dominance cones $W \supset E^\ell_+$, the set of solutions can be further restricted in accordance with such preferences.

To illustrate what is involved, consider  the simple two-variable case where two objective functions $f_1 (x_1, x_2)$ and $f_2 (x_1, x_2)$ are to be maximized subject to a certain constraint set.  Figure 2.1 depicts the decision space and its mapping into the objective function space.  The (nondominated) Pareto-optimal solutions given by the sets $I_1$, $I_2$ and $I_3$ are mapped into  (nondominated) Pareto-optimal points denoted by the sets $J_1$, $J_2$ and $J_3$, respectively, so that
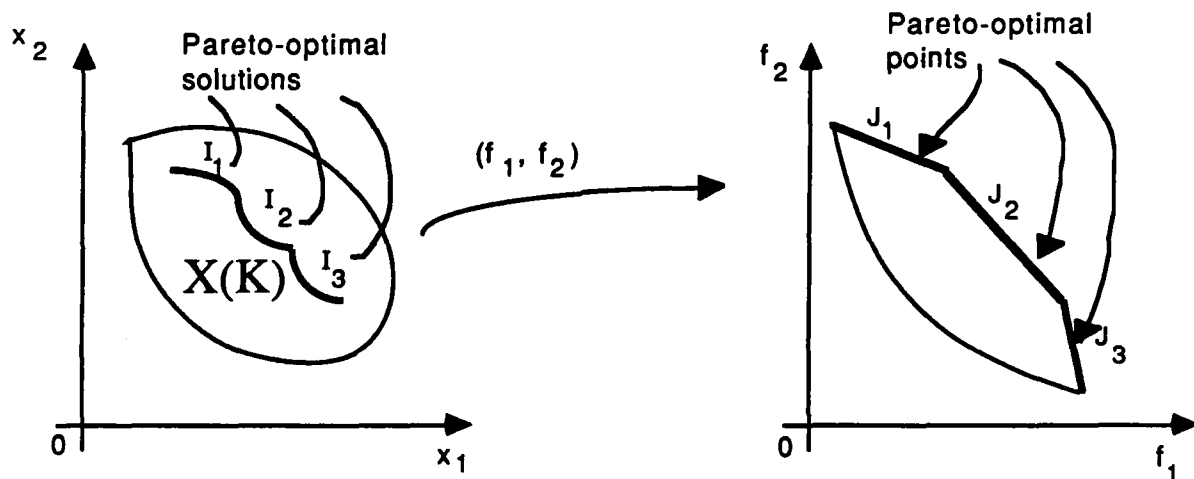
$$J_i = (f_1 (I_i), f_2 (I_i)), \ i = 1, 2, 3.$$



Figure 2.1

A preference for objective $f_1$ over objective $f_2$ will lead the decision maker to focus on the points of $J_2$ and $J_3$. Accordingly, a dominance cone $W_1$ can be established as shown in Figure 2.2. Associated with $W_1$, the points of $J_1$ are no longer nondominated. Consider, for example, the point $(f_1 (\bar{x}) , f_2 (\bar{x}))$ which lies in $J_1$ and is Pareto-optimal. Associated with the dominance cone $W_1$, $(f_1 (\bar{x}) , f_2 (\bar{x}))$ is dominated by the point $(f_1 (y) , f_2 (y))$ (which is also Pareto-optimal) since $(f_1 (y) , f_2 (y))$ can be expressed as $(f_1 (\bar{x}) , f_2 (\bar{x})) + w$, where $w$ is a vector in $W_1$. Thus, $(f_1 (y) , f_2 (y)) \in f_1 (\bar{x}) , f_2 (\bar{x}))$ + $W_1$. Associated with $W_1$, only the points of $J_2$ and $J_3$ remain nondominated, and the possible solutions are therefore restricted to those of $I_2$ and $I_3$. Similarly, a preference for $f_2$ over $f_1$ will lead to elimination of the points of $J_3$ by using the dominance cone $W_3$. Associated with $W_3$, only the points of $J_1$ and $J_2$ are nondominated, and the solutions are this time confined to those of $I_1$ and $I_2$. A balance between the two objectives will focus attention on the points of $J_2$. This will require a dominance cone $W_2 = W_1 \cup W_3$, in which case only the points of $J_2$ and solutions of $I_2$ are nondominated.
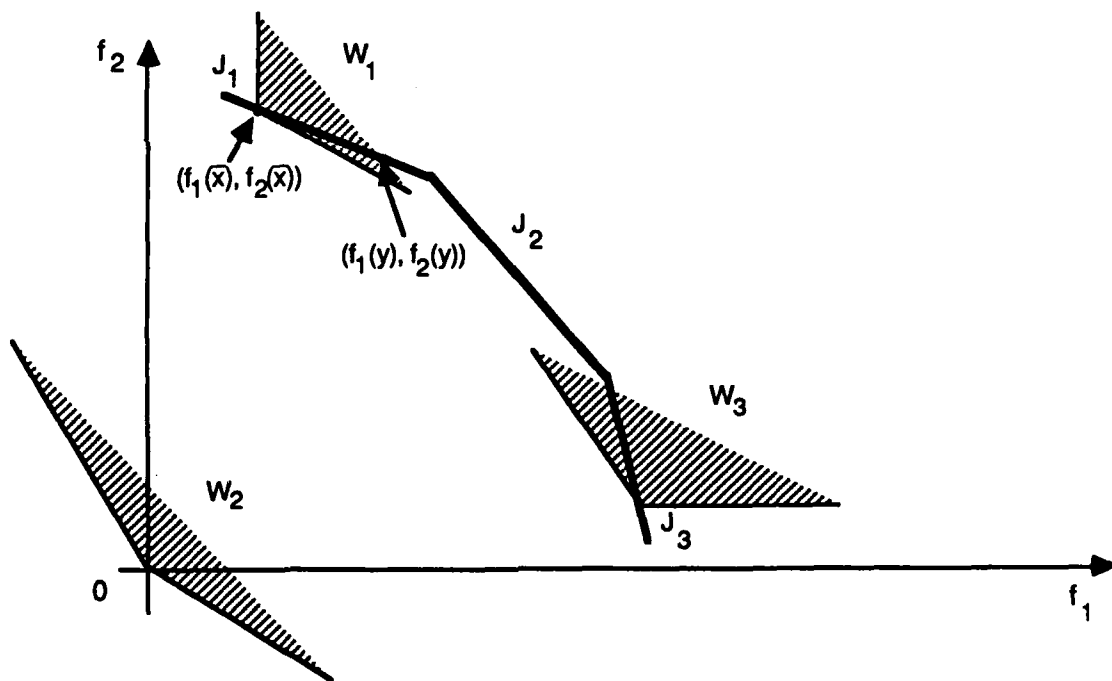


Figure 2.2

Note that in each case the dominance cone contains $E_+^\ell$. Otherwise, some of the nondominated solutions would not be Pareto-optimal. This is illustrated in Figure 2.3, where $J_1 \cup J_2$ is the set of Pareto-optimal points, but $J_0 \cup J_1 \cup J_2$ is the set of nondominated points associated with the dominance cone W which is smaller than the nonnegative orthant. In the present paper we shall focus only on dominance cones which contain the nonnegative orthant $E_+^\ell$, and in Section 4 we show how dominance cones can be constructed to achieve what is required.
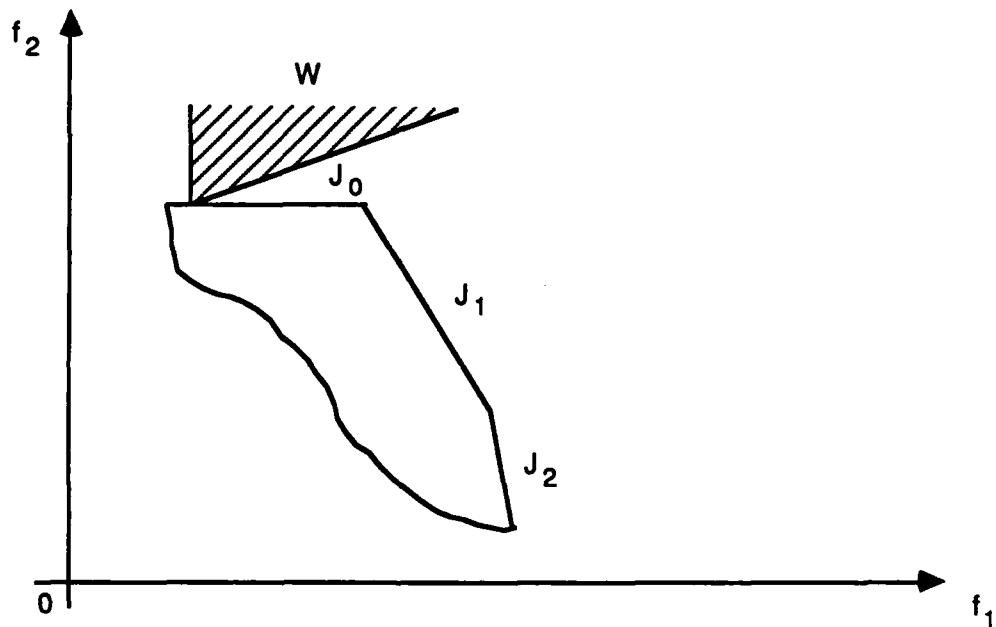


Figure 2.3

## 3. Cone Quasiconcave Multi-objective Programming

Arrow and Enthoven [1] considered the single objective programming problem where the objective function and constraint functions are quasiconcave. However, their method does not extend to multi-objective quasiconcave programming since a nonnegative linear combination of quasiconcave functions is not necessarily quasiconcave. For example, $f_1(x_1, x_2) = x_1^3$ and $f_2(x_1, x_2) = x_2^3$ are both quasiconcave, but $f_1(x_1, x_2) + f_2(x_1, x_2) = x_1^3 + x_2^3$ is not quasi-concave.

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Craven [12] considered a special case of multi-objective programming in which he assumed that $\sum_{j \in L} p_j f_j$ is pseudoconcave for each $p = (p_1, \ldots, p_\ell)^t \in -W^*$. This approach is not suitable either, since pseudoconcave functions must be quasiconcave [15] and sums of quasiconcave functions need not be quasiconcave.

We first provide (in Theorem 3.1) sufficient conditions for a solution to the problem

$$\text{Max} \sum_{j \in L} f_j(x)$$

$x \in X$, where $X \subset E^n$ is a convex set. This is accomplished by partitioning $L = \{1, 2, \ldots, \ell\}$ into $k$ sub-groups $T_i$, $i = 1 \ldots, k$ (i.e., $T_i \cap T_j = \varnothing$, $i \neq j$ and $\bigcup_{i=1}^{k} T_i = L$), such that $\sum_{j \in T_i} f_j$ is quasi-concave for each $i = 1, \ldots, k$.

We then develop several theorems for solutions to cone quasi-concave multi-objective programming problems, a new class of multicriteria decision problems, which incorporates non-dominated solutions associated with dominance cones. Necessary conditions for solutions to such problems are given in Theorems 3.2 and 3.4, while sufficient conditions are given in Theorems 3.5, 3.6 (based on Theorem 3.1), 3.7, 3.8, and 3.9.

**Theorem 3.1:** Let $D$ and $X$ be convex sets with $D \supset X$ and $\{f_j(x)\}_{j \in L}$ be differentiable on an open set containing $D$. Let $\{T_i : i = 1, \ldots, k\}$ be a partition of $L$ such that for each $i$,

$\sum_{j \in T_i} f_j(x)$ is quasi-concave on $D$. Let $\bar{x} \in X$.

(i) If for each $i$ $(1 \leq i \leq k)$, $\sum_{j \in T_i} \nabla f_j (\bar{x})(x-\bar{x}) \leq 0$ for all $x \in X$, and there exists at least one

$x^i \in D$ such that $\sum_{j \in T_i} \nabla f_j (\bar{x})(x^i - \bar{x}) < 0$

or

(ii) If $\{f_j(x)\}_{j \in L}$ are twice differentiable on $D$ and for each $i$ $(1 \leq i \leq k)$ $\sum_{j \in T_i} \nabla f_j (\bar{x})(x-\bar{x}) = 0$ for all

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$x \in X$ and there exists at least one $x^i \in D$ such that $\sum\limits_{j \in T_i} \nabla f_j (\overline{x}) (x^i - \overline{x}) > 0$, then $\overline{x}$ is a solution of

$$\text{Max} \sum_{j \in L} f_j (x) \quad \text{s.t. } x \in X.$$

**Proof:**

(i) For any $x \in X$ and $0 \leq \theta \leq 1, 0 \leq \theta_m \leq 1, m = 1, \ldots, k$ with $\theta + \sum\limits_{m=1}^{k} \theta_m = 1$.

Let $x(\theta) = \theta x + \sum\limits_{m=1}^{k} \theta_m x^m$. Then

$$\sum_{j \in T_i} \nabla f_j (\overline{x}) (x(\theta) - \overline{x}) = \sum_{j \in T_i} \nabla f_j (\overline{x}) (\theta (x - \overline{x}) + \sum_{m=1}^{k} \theta_m (x^m - \overline{x}))$$

$$= \theta \sum_{j \in T_i} \nabla f_j (\overline{x}) (x - \overline{x}) + \sum_{m=1}^{k} \theta_m \sum_{j \in T_i} \nabla f_j (\overline{x}) (x^m - \overline{x})$$

$$= \theta \sum_{j \in T_i} \nabla f_j (\overline{x}) (x - \overline{x}) + \theta_i \sum_{j \in T_i} \nabla f_j (\overline{x}) (x^i - \overline{x})$$

$$+ \sum_{\substack{m=1 \\ m \neq i}}^{k} \theta_m \sum_{j \in T_i} \nabla f_j (\overline{x}) (x^m - \overline{x})$$

$$< 0 \quad \text{for } \theta_i > 0$$

Since $\sum\limits_{j \in T_i} f_j (x)$ is quasiconcave, by Lemma 2.2, we have

$$\sum_{j \in T_i} f_j (x(\theta)) < \sum_{j \in T_i} f_j (\overline{x}) \quad \text{for } \theta_i > 0$$

Letting $\theta \to 1^+$, we have $x(\theta) \to x$, by continuity of $\{f_j\}_{j \in L}$, we have $\sum\limits_{j \in T_i} f_j (x(\theta)) \to$

$\sum\limits_{j \in T_i} f_j (x)$, hence

$$\sum_{j \in T_i} f_j (x) \leq \sum_{j \in T_i} f_j (\overline{x}) \quad \text{for every } x \in X$$

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Then $$\sum_{j \in L} f_j(x) = \sum_{i=1}^{k} \sum_{j \in T_i} f_j(x) \leq \sum_{i=1}^{k} \sum_{j \in T_i} f_j(\bar{x}) = \sum_{j \in L} f_j(\bar{x}) \quad \text{for all } x \in X.$$

(ii) Assume to the contrary that $\bar{x}$ is not a solution i.e., there exists some $x \in X$ such that

$$\sum_{j \in T_i} f_j(x) > \sum_{j \in T_i} f_j(\bar{x})$$

Then there exists at least one $i \in \{1, \ldots, k\}$ such that $\displaystyle\sum_{j \in T_i} f_j(x)) > \sum_{j \in T_i} f_j(\bar{x})$

By quasiconcavity of $\displaystyle\sum_{j \in T_i} f_j$, we have

$$\sum_{j \in T_i} f_j(\bar{x} + \mu(x - \bar{x})) \geq \sum_{j \in T_i} f_j(\bar{x}) \quad \text{for all } 0 \leq \mu \leq 1$$

By the continuity and quasiconcavity of $\displaystyle\sum_{j \in T_i} f_j$, there exists $\mu^* \in [0, 1)$ such that

$$\sum_{j \in T_i} f_j(\bar{x} + \mu(x - \bar{x})) = \sum_{j \in T_i} f_j(\bar{x}) \quad \text{for } 0 \leq \mu \leq \mu^* \qquad (3.1)$$

$$\sum_{j \in T_i} f_j(\bar{x} + \mu(x - \bar{x})) > \sum_{j \in T_i} f_j(\bar{x}) \quad \text{for } \mu^* < \mu \leq 1 \qquad (3.2)$$

Since $\displaystyle\sum_{j \in T_i} \nabla f_j(\bar{x})(x^i - \bar{x}) > 0$ with (3.1) and (3.2), there exist two sequence $\{\mu_n\}$ with $\mu_n > \mu^*$ and

$\mu_n \to \mu^*$, and $\{v_n\}$ with $v_n > 0$ and $v_n \to 0$ such that

$$\sum_{j \in T_i}(\bar{x} + \mu_n(x - \bar{x})) = \sum_{j \in T_i} f_j(\bar{x} + v_n(x^L - \bar{x})) \qquad (3.3)$$

First, suppose $\mu^* > 0$, let $\theta_n = 1 - \dfrac{\mu^*}{\mu_n}$, it is clear that $\theta_n \to 0^+$ as $n \to \infty$.

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Since $\bar{x} + \mu^* (x - \bar{x}) + \theta_n \, v_n \, (x^i - \bar{x}) = (1 - \theta_n)(\bar{x} + \mu_n(x - \bar{x})) + \theta_n \, (\bar{x} + v_n \, (x^i - \bar{x}))$.

By quasiconcavity of $\displaystyle\sum_{j \in T_i} f_j$, (3.3) implies

$$\sum_{j \in T_i} f_j \, (\bar{x} + \mu^* (x - \bar{x}) + \theta_n \, v_n \, (x^i - \bar{x})) \geq \sum_{j \in T_i} f_j \, (\bar{x} + v_n \, (x^i - \bar{x})) \qquad (3.4)$$

By (3.1) and (3.4), we have

$$\left( \sum_{j \in T_i} f_j \, (\bar{x} + \mu^*(x - \bar{x}) + \theta_n \, v_n \, (x^i - \bar{x})) - \sum_{j \in T_i} f_j \, (\bar{x} + \mu^*(x - \bar{x})) \right) / \theta_n \, v_n$$

$$= \left( \sum_{j \in T_i} f_j \, (\bar{x} + \mu^*(x - \bar{x}) + \theta_n \, v_n \, (x^i - \bar{x})) - \sum_{j \in T_i} f_j \, (\bar{x}) \right) / \theta_n \, v_n$$

$$\geq \left( \sum_{j \in T_i} f_j \, (\bar{x} + v_n(x^i - \bar{x})) - \sum_{j \in T_i} f_j \, (\bar{x}) \right) / \theta_n \, v_n$$

$$= (1 / \theta_n) \cdot \left( \sum_{j \in T_i} f_j \, (\bar{x} + v_n(x^i - \bar{x})) - \sum_{j \in T_i} f_j \, (\bar{x}) \right) / v_n$$

Since $\displaystyle\sum_{j \in T_i} \nabla f_j (\bar{x}) \, (x^i - \bar{x}) > 0$ is finite, we have

$$\lim_{n \to \infty} (1/\theta_n) \left( \sum_{j \in T_i} f_j \, (\bar{x} + v_n(x^i - \bar{x})) - \sum_{j \in T_i} f_j \, (\bar{x}) \right) / v_n$$

$$= \lim_{n \to \infty} (1/\theta_n) \sum_{j \in T_i} \nabla f_j \, (\bar{x}) \, (x^i - \bar{x})$$

$$= + \infty$$

and

$$\lim_{n \to \infty} \left( \sum_{j \in T_i} f_j \left( \bar{x} + \mu^* (x - \bar{x}) + \theta_n v_n (x^i - \bar{x}) \right) - \sum_{j \in T_i} f_j \left( \bar{x} + \mu^* (x - \bar{x}) \right) \right) / \theta_n v_n$$

$$= \sum_{j \in T_i} \nabla f_j \left( \bar{x} + \mu^* (x - \bar{x}) \right) (x^i - \bar{x})$$

$$< \infty$$

This contradicts (3.5).

Now suppose $\mu^* = 0$. By (3.3) and quasiconcavity of $\sum_{j \in T_i} f_j$, we have

$$\sum_{j \in T_i} \nabla f_j \left( \bar{x} + v_n (x^i - \bar{x}) \right) \left( \mu_n (x - \bar{x}) - v_n (x^i - \bar{x}) \right) \geq 0$$

i.e.,

$$\mu_n \sum_{j \in T_i} \nabla f_j \left( \bar{x} + v_n (x^i - \bar{x}) \right) (x - \bar{x}) / v_n \geq \sum_{j \in T_i} \nabla f_j \left( \bar{x} + v_n (x^i - \bar{x}) \right) (x^i - \bar{x}) \qquad (3.6)$$

Since $\sum_{j \in T_i} \nabla f_j (\bar{x}) (x - \bar{x}) = 0$ and

$$\lim_{n \to \infty} \left( \sum_{j \in T_i} \nabla f_j (\bar{x} + v_n (x^i - \bar{x})) (x - \bar{x}) \right) / v_n$$

$$= \lim_{n \to \infty} \left( \sum_{j \in T_i} \nabla f_j (\bar{x} + v_n (x^i - \bar{x})) (x - \bar{x}) - \sum_{j \in T_i} \nabla f_j (\bar{x}) (x - \bar{x}) \right) / v_n$$

$$= \sum_{j \in T_i} (x^i - \bar{x})^t \nabla^2 f_j (\bar{x}) (x - \bar{x})$$

$$< \infty$$

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we have

$$\lim_{n \to \infty} [\mu_n (\sum_{j \in T_i} \nabla f_j (\bar{x} + \nu_n (x^i - \bar{x}))) / \nu_n] (x - \bar{x}) = 0$$

But

$$\lim_{n \to \infty} \sum_{j \in T_i} \nabla f_j (\bar{x} + \nu_n (x^i - \bar{x})) (x^i - \bar{x}) = \sum_{j \in T_i} \nabla f_j (\bar{x}) (x^i - \bar{x}) > 0$$

This contradicts (3.6)                                                  Q.E.D.

**Corollary 1:** Let S be a convex set in $E^n$. Let $\{f_j (x)\}_{j \in L}$ be differentiable functions, $g(x)$ be an m-dimensional differentiable $E_+^m -$ quasiconcave function, all defined on an open set which contains S. Let $\sum_{j \in L} f_j(x)$ be quasi-concave on S and let $(\bar{x}, \bar{\lambda})$ satisfys

$$\sum_{j \in L} \nabla f_j (\bar{x}) + \bar{\lambda}^t \nabla g (\bar{x}) = 0$$

$$\hspace{12cm} (3.7)$$

$$g (\bar{x}) \geq 0, \quad \bar{x} \in S \hspace{6cm} (3.8)$$

$$\bar{\lambda}^t g (\bar{x}) = 0, \bar{\lambda} \geq 0 \hspace{6cm} (3.9)$$

(i) if there exists some $x^1 \in S$ such that

$$\sum_{j \in L} \nabla f_j (\bar{x}) (x^1 - \bar{x}) < 0$$

or

(ii)    if there exists some $x^1 \in S$ such that

$$\sum_{j \in L} \nabla f_j (\bar{x}) (x^1 - \bar{x}) \neq 0$$

and $\{f_j\}_{j \in L}$ are twice differentiable in S, then $\bar{x}$ is a solution of Max $\sum_{j \in L} f_j(x)$,  s.t. $g(x) \geq 0$, $x \in S$

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**Proof:** Since $g_j$ is quasiconcave in $S$ for each $j$ $(1 \leq j \leq m)$, it is easy to check that $X = \{x \in S : g(x) \geq 0\}$ is convex set. Let $T_1 = L$ and $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^t$. By (3.8) and (3.9), for $\bar{\lambda}_i > 0$ we have $g_i(\bar{x}) = 0$, hence $\nabla g_i(\bar{x})(x-\bar{x}) \geq 0$ for all $x \in X$ by quasiconcavity of $g_i$. Then we have

$$\sum_{j \in L} \nabla f_j(\bar{x})(x-\bar{x}) = -\sum_{i=1}^{m} \bar{\lambda}_i^t \nabla g_i(\bar{x})(x-\bar{x})$$

$$\leq 0 \qquad \text{for all } x \in X \tag{3.10}$$

(i): By (i) of the Theorem 3.1, we know that $\bar{x}$ is a solution of $\text{Max} \sum_{j \in L} f_j(x)$,

s.t. $g(x) \geq 0, x \in S$.

(ii): if we exclude case (i), we have

$$\sum_{j \in L} \nabla f_j(\bar{x})(x-\bar{x}) \geq 0 \qquad \text{for all } x \in S \tag{3.11}$$

In view of the condition (ii), we can write

$$\sum_{j \in L} \nabla f_j(\bar{x})(x^1-\bar{x}) > 0 \qquad \text{for some } x^1 \in S \tag{3.12}$$

Combining (3.11) and (3.10), we have

$$\sum_{j \in L} \nabla f_j(\bar{x})(x-\bar{x}) = 0 \qquad \text{for all } x \in X$$

By (ii) of Theorem 3.1, we know that $\bar{x}$ is a solution of $\text{Max} \sum_{j \in L} f_j(x)$ s.t. $g(x) \geq 0, x \in S$.

Q. E. D.

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**Assumption (A):** For any $x \in X(K)$, $y \in X(K)$, let $f_j^c = \min(f_j(x), f_j(y))$ for each $j \in L$. Then

$$\lambda(\{f_j^c\}_{j \in L} - \{f_j(x)\}_{j \in L}) + (1-\lambda)(\{f_j^c\}_{j \in L} - \{f_j(y)\}_{j \in L}) \in W$$

**Theorem 3.2[10]:** Consider (W-K-P). Let $W \supseteq E^{\ell}_+$ be acute, $\{f_j(x)\}_{j \in L}$ be W-quasiconcave, and $g(x)$ be such that $X(K)$ is a convex set. If $\bar{x} \in X(K)$ is a nondominated solution of (W-K-P) associated with W, and Assumption (A) holds, then there exists nonzero $p \in -W^*$ such that

$$\sum_{j \in L} p_j f_j(\bar{x}) \geq \sum_{j \in L} p_j f_j(x) \quad \text{for all } x \in X(K).$$

**Theorem 3.3 [8]:** Let $\bar{x} \in X(K)$. If there exists $p \in -W^*$ such that $\displaystyle\sum_{j \in L} p_j f_j(\bar{x}) \geq \sum_{j \in L} p_j f_j(x)$ for all $x \in X(K)$

then

$$\left(\sum_{j \in L} p_j \nabla f_j(\bar{x})\right)^t \in T^*(X(K), \bar{x})$$

**Lemma 3.1 [8]:** Let $\bar{x} \in X(K)$ and $g(x)$ be Fréchet differentiable at $\bar{x}$, then $T(X(K), \bar{x}) \subset -C^*(\bar{x})$, where

$$C(\bar{x}) = \{(\nabla g(\bar{x}))^t \gamma : \gamma \in -K^* \text{ with } \gamma^t g(\bar{x})) = 0\}.$$

By Lemma 2.1, $T^*(X(K), \bar{x}) \supset -\overline{C(\bar{x})}$. Thus we can make the following definition.

**Definition 3.1:** A point $\bar{x} \in X(K)$ is said to be a "regular point" of the constraint set $X(K)$ if $T^*(X(K), \bar{x}) \subset -C(\bar{x})$.

**Theorem 3.4:** Consider (W-K-P). Let $\{f_j(x)\}_{j \in L}$ be W-quasiconcave and $g(x)$ be such that $X(K)$ is convex set. If $\bar{x} \in X(K)$ is a nondominated solution of (W-K-P) associated with W and a regular point of $X(K)$, then there exist nonzero $p \in -W^*$ and $\gamma \in -K^*$ such that

$$\sum_{j \in L} p_j \nabla f_j(\bar{x}) + \gamma^t \nabla g(\bar{x}) = 0$$

$$\gamma^t g(\bar{x}) = 0$$

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**Proof:** By Theorem 3.2, there exists a nonzero $p \in -W^*$ such that

$$\sum_{j \in L} p_j f_j(\bar{x}) \geq \sum_{j \in L} p_j f_j(x) \quad \text{for all } x \in X(K)$$

By Theorem 3.3, we have

$$\left( \sum_{j \in L} p_j f_j(\bar{x}) \right)^t \in T^*(X(K), \bar{x})$$

Since $\bar{x}$ is a regular point, we have that

$$\left( \sum_{j \in L} p_j f_j(\bar{x}) \right)^t \in -C(\bar{x})$$

Then there exists $\gamma \in -K^*$ with $\gamma^t g(\bar{x}) = 0$ such that

$$\left( \sum_{j \in L} p_j f_j(\bar{x}) \right)^t = -(\nabla g(\bar{x}))^t \gamma$$

that is

$$\sum_{j \in L} p_j \nabla f_j(\bar{x}) + \gamma^t \nabla g(\bar{x}) = 0$$

$$\gamma^t g(\bar{x}) = 0$$

<div align="center">Q.E.D.</div>

**Theorem 3.5:** Consider $(E^\ell_+ - K - P)$. Let $g$ be such that $X(K)$ is a convex set, $p \in -(E^\ell_+)^*$ and $I = \{i : p_i \neq 0, 1 \leq i \leq \ell\} \neq \varnothing$. Let $\{f_j\}_{j \in L}$ be $E^\ell_+(k)$ – strictly quasiconcave for some $k \in I$, and $\bar{x} \in X(K)$ be a local solution of Max $\sum_{j \in L} p_j f_j(x)$, s.t. $x \in X(K)$, then $\bar{x}$ is a Pareto-optimal solution of $(E^\ell_+ - K - P)$.

**Proof:** Assume to the contrary that $\bar{x}$ is not a Pareto-optimal solution of $(E^\ell_+ - K - P)$, i.e., there exists some $\hat{x} \in X(K)$ such that

$$f_j(\hat{x}) \geq f_j(\bar{x}) \quad \text{for all } j \in L$$

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then for any $0 < \mu < 1$, we have

$$f_j(\bar{x} + \mu (\hat{x} - \bar{x})) \geq f_j(\bar{x}) \qquad \text{for all } j \in L$$

and

$$f_k(\bar{x} + \mu (\hat{x} - \bar{x})) > f_k(\bar{x})$$

hence

$$\sum_{j \in L} p_j f_j(\bar{x} + \mu(\hat{x} - \bar{x})) > \sum_{j \in L} p_j f_j(\bar{x}) \qquad \text{for all } 0 < \mu < 1.$$

We have a contradiction.

Q.E.D.

**Theorem 3.6:** Consider $(E^\ell_+ - K - P)$. Let $g$ be such that $X$ (K) S is a convex set, $p \in -(E^\ell_+)^*$ and $I = \{ i: p_i \neq 0, 1 \leq i \leq \ell \} \neq \emptyset$. Let $\{f_j\}_{j \in L}$ be $E^\ell_+$ (k) – strictly quasiconcave for some $k \in I$, and $\{T_i = i = 1, \ldots, \ell \}$ be a partition of L such that for each i, $\sum_{j \in T_i} p_j f_j(x)$ is quasi-concave. Let $\bar{x} \in X$ (K)

(i)  if for each i $(1 \leq i \leq \ell)$, $\sum_{j \in T_i} p_j \nabla f_j(\bar{x})(x - \bar{x}) \leq 0$ for

all $x \in X$ (K), and there exists at least one $x^i \in$ S such that $\sum_{j \in T_i} p_j \nabla f_j(\bar{x})(x^i - \bar{x}) < 0$

or

(ii) If $\{f_j(x)\}_{j \in L}$ are twice differentiable on S and for each i $(1 \leq i \leq k)$,

$\sum_{j \in T_i} p_j \nabla f_j(\bar{x})(x - \bar{x}) = 0$  for all $x \in X$ (K) and there exists at least are $x^i \in$ S such that

$\sum_{j \in T_i} p_j \nabla f_j(\bar{x})(x^i - \bar{x}) > 0$, then $\bar{x}$ is a Pareto optimal solution of $(E^\ell_+ - K - P)$.

**Proof:** The proof follows directly from Theorem 3.1 and Theorem 3.5.

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**Theorem 3.7:** Consider $(E^\ell_+ - E^m_+ - P)$. Let $p \in -(E^\ell_+)^*$ and $I = \{i : p_i \neq 0, 1 \leq i \leq l\} \neq \varnothing$. Let $\{f_j\}_{j \in L}$ be $E^\ell_+(k)$ – strictly quasiconcave for some $k \in I$, $\sum_{j \in T_i} p_j f_j$ be quasiconcave, and $g$ be $E^m_+$ –quasi-concave. Let $(\overline{x}, \overline{\lambda})$ satisfy

$$\sum_{j \in L} p_j \nabla f_j(\overline{x}) + \overline{\lambda}^t \nabla g(\overline{x}) = 0$$

$$g(\overline{x}) \in E^m_+, \quad \overline{x} \in S$$

$$\overline{\lambda}^t g(\overline{x}) = 0, \quad \overline{\lambda} \geq 0$$

(i) If there exists some $x^1 \in S$ such that $\sum_{j \in L} p_j \nabla f_j(\overline{x})(x^1 - \overline{x}) < 0$

or

(ii) if there exists some $x^1 \in S$ such that

$$\sum_{j \in L} p_j \nabla f_j(\overline{x})(x^1 - \overline{x}) \neq 0$$

and $\{f_j\}_{j \in L}$ is twice differentiable then $\overline{x}$ is a Pareto-optimal solution of $(E^\ell_+ - E^m_+ - P)$.

**Proof:** The proof follows from corollary 1 of Theorem 3.1 and Theorem 3.5.


**Theorem 3.8:** Consider $(E^\ell_+ - E^m_+ - P)$. Let $p \in -(E^\ell_+)^*$ and $I = \{i : p_i \neq 0, 1 \leq i \leq l\}$ contain at least two elements. Let $\{f_j\}_{j \in L}$ be $E^\ell_+(k)$ –strictly quasiconcave for some $k \in I$, $g$ be such that $X(K)$ is a convex set $\overline{x} \in \text{Int } X(K)$. If

$$\sum_{j \in L} p_j \nabla f_j(\overline{x}) = 0 \qquad\qquad (3.14)$$

then $\overline{x}$ is a Pareto-optimal solution of $(E^\ell_+ - K - P)$.

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Proof: Assume to the contrary that $\bar{x}$ is not a Pareto-optimal solution of $(E^\ell_+ - K - P)$, i.e., there exists some $x \in X(K), x \neq \bar{x}$ such that

$$f_j(x) \geq f_j(\bar{x}) \qquad \text{for all } j \in L$$

and

$$f_{j_0}(x) > f_{j_0}(\bar{x}) \qquad \text{for some } j_0$$

Then for any $0 < \hat{\mu} < 1$,

let $\hat{\chi} = \chi + \hat{\mu}(x - \bar{x})$, we have

$$f_j(\hat{\chi}) \geq f_j(\bar{x}) \qquad \text{for all } j \in L \tag{3.15}$$

and

$$f_k(\hat{\chi}) > f_k(\bar{x}), \qquad \text{by strict quasi-concavity}$$

Hence

$$\nabla f_k(\bar{x})(\hat{x} - \bar{x}) > 0$$

Using Theorem (3.14), we have

$$\sum_{\substack{j \in L \\ j \neq k}} p_j \nabla f_j(\bar{x})(\hat{x} - \bar{x}) = -p_k \nabla f_j(\bar{x})(\hat{x} - \bar{x})$$

$$< 0$$

Then there exists $i \in I, i \neq k$, such that

$$\nabla f_i(\bar{x})(\hat{x} - \bar{x}) < 0$$

we have

$$f_i(\hat{x}) < f_i(\bar{x}).$$

This contradicts Theorem (3.15)            Q.E.D.

We close this section with the following theorem which was originally given in [8].

**Theorem 3.9:** Consider (W-K-P). Let W be acute and closed. If for some $p \in -\text{Int } W^*$,

$\bar{x} \in X$ (K) is a solution of $\text{Max}_{j \in} \sum_{i \in L} p_j f_j (x)$, s.t. $x \in X$ (K), then $\bar{x}$ is a nondominated solution of

(W-K-P) associated with W.

## 4. Construction of Dominance Cones

We now provide two theorems which indicate how dominance cones can be constructed for the more general problem (W- $E_+^m$ -P). Let $a^i \in E_+^\ell$, $i = 1, \ldots, k$, all nonzero. The corresponding half-spaces are given by $H_i = \{Z: Z^t a^i \leq b_i\}$ and the bounding hyperplanes by $J_i = \{Z: Z^t a^i = b_i\}$ .

Let $H = \bigcup_{i=1}^{k} H_i$, $D_j = \bigcap_{\substack{i=1 \\ i \neq j}}^{k} H_i$ and $C_j = \bigcap_{\substack{i=1 \\ i \neq j}}^{k} \overline{H_i^c}$, where $\overline{H_i^c}$ denotes the closure of the complement $H_i^c$ of $H_i$.

**Theorem 4.1:** Let $V \subset E_+^\ell$ be a closed convex cone. If for some j, $1 \leq j \leq k$, $a^j \notin V$, then for every $Z^0 \in J_j \cap \text{Int } C_j$ there exists a $Z \in H$ such that

$$z \in z^0 + \text{Int} (-V^*).$$

**Proof:** Assume to the contrary that there is no $Z \in H$ such that $z \in z^0 + \text{Int} (-V^*)$.

Let $S = \{s: s \in Z^0 - z + \text{Int} (-V^*), \text{for some } Z \in \overline{H^c} \cap J_j \}$. It is straightforward to show that $S$ is a convex set with $0 \notin S$. Hence, by the separation theorem, there exists a nonzero $p \in E^\ell$ such that $p^t s \leq 0$ for all $s \in S$.

For any $Z \in \overline{H^c} \cap J_j$ , $w \in \text{Int} (-V^*)$ and $\lambda > 0$, we have

$$z^0 - z + \lambda w \in S$$

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so that

$$P^t Z^o \leq P^t Z - \lambda P^t w.$$

Hence

$$P^t Z^o \leq P^t Z \qquad \text{for every} \quad Z \in \overline{H}^c \cap J_j \qquad (4.1)$$

and

$$P^t w \leq 0 \qquad \text{for every} \quad w \in \text{Int}(-V^*) \qquad (4.2)$$

Thus we have that

$$p \in (\text{Int}(-V^*))^* = -V, \quad \text{i.e.,} \; p \leq 0 \qquad (4.3)$$

Now consider the following system:

$$a^{jt} Z = 0 \qquad (4.4)$$

$$p^t Z > 0 \qquad (4.5)$$

We claim that (4.4) and (4.5) have at least one solution. To substantiate this, suppose to the contrary that the system has no solution. That is, for all $Z$ satisfying $a^{jt} Z = 0$, we must have $p^t Z \leq 0$. By Farkas' lemma, there exists some nonzero number $\mu$ such that $\mu a^j = p$. If $\mu > 0$, we have $p \geq 0$, which contradicts (4.3). If $\mu < 0$, we have $a^j = \mu^{-1} p \in V$, which contradicts $a^j \notin V$. Hence, (4.4) and (4.5) have at least one solution, say $\overline{Z}$.

Consider the point $Z^o + \alpha \overline{Z}$. We have that $(Z^o + \alpha \overline{Z})^t a^j = Z^{ot} a^j + \alpha \overline{Z} a^j = Z^{ot} a^j = b_j$, for every $\alpha$. That is $Z^o + \alpha \overline{Z} \in J_j$ for every $\alpha$.

Since $Z^o \in \text{Int} C_j$, there exists some $\overline{\alpha} < 0$ such that

$$Z^o + \alpha \overline{Z} \in C_j \qquad \text{for all} \; \alpha \in (\overline{\alpha}, 0].$$

Hence:

$$Z^o + \alpha \overline{Z} \in \overline{H}^c \cap J_j \quad \text{for all} \; \alpha \in (\overline{\alpha}, 0]$$

and

$$p^t(Z^o + \alpha \overline{Z}) = p^t Z^o + \alpha p^t \overline{Z} < p^t Z^o, \qquad \text{for all} \; \alpha \in (\overline{\alpha}, 0]$$

which contradicts (4.1).

Q.E.D.

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We illustrate this Theorem in the following example diagrammed in Fig. 4.1 . $-V^*$ shown in Fig. 4.1, is the dominance cone. $a^2$ and $a^3$, the respective normals of the segments $J_2$ and $J_3$ , are elemtns of $V$. On the other hand, $a^1$ and $a^4$, the normals of the segments $J_1$ and $J_4$ respec-tively are not in $V$. The Pareto-optimal points on $J_1$ and $J_4$ are no longer nondominated points associated with $-V^*$. For example, see $z^0$ in the diagram.. $z^0$ is apparently dominated by $z^1$ associated with $-V^*$. Only the points on $J_2$ and $J_3$ remains nondominated.
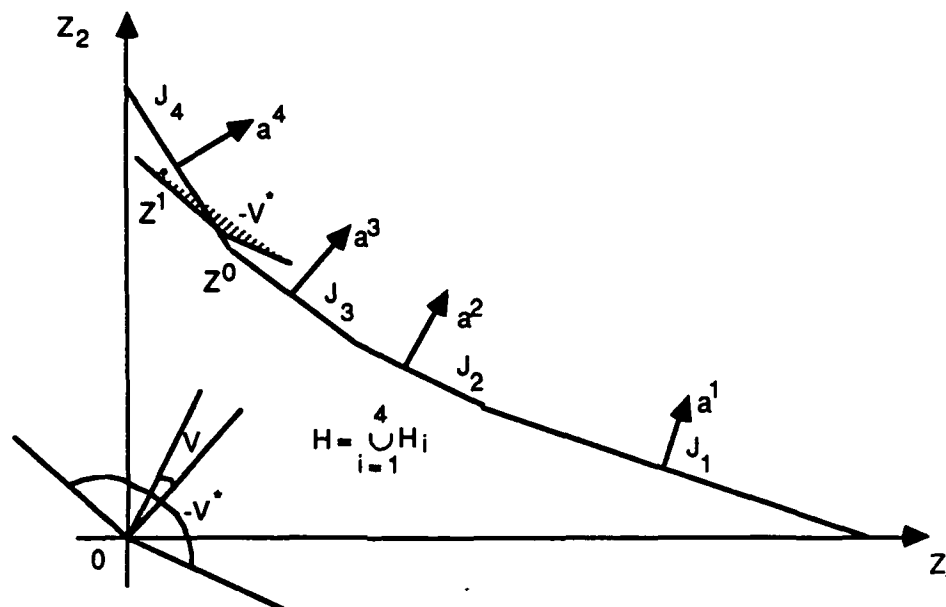


Figure 4.1

Next we present the following theorem which was proved in [11] and [19], and can be used to handle other cases as shown in Figure 4.2

**Theorem 4.2** [11, 19] : Let $V \subset E^{\ell}_+$ be a closed convex cone. If for some $j$ $(1 \leq j \leq k)$, $a^j \notin V$, then for every $z^0 \in J_j \cap \text{Int } D_j$ there exists a $z \in \overset{k}{\underset{i=1}{\cap}} H_i$ such that $z \in z^0 + \text{Int}(-V^*)$.

In Figure 4.2 the dominance cone is given by $-V^*$. $a^1$ and $a^2$, the normals to segments $J_1$ and $J_2$ , respectively, are elements of $V$, whereas $a^3$ and $a^4$ are not, $z^0$ is in $J_3$ . Clearly, $z^0$ is dominated by $z'$ associated with $-V^*$. Furthermore, only the points on $J_1$ and $J_2$ are nondominated points. The Pareto-optimal points on $J_3$ and $J_4$ are no longer nondominated points associated with $-V^*$.
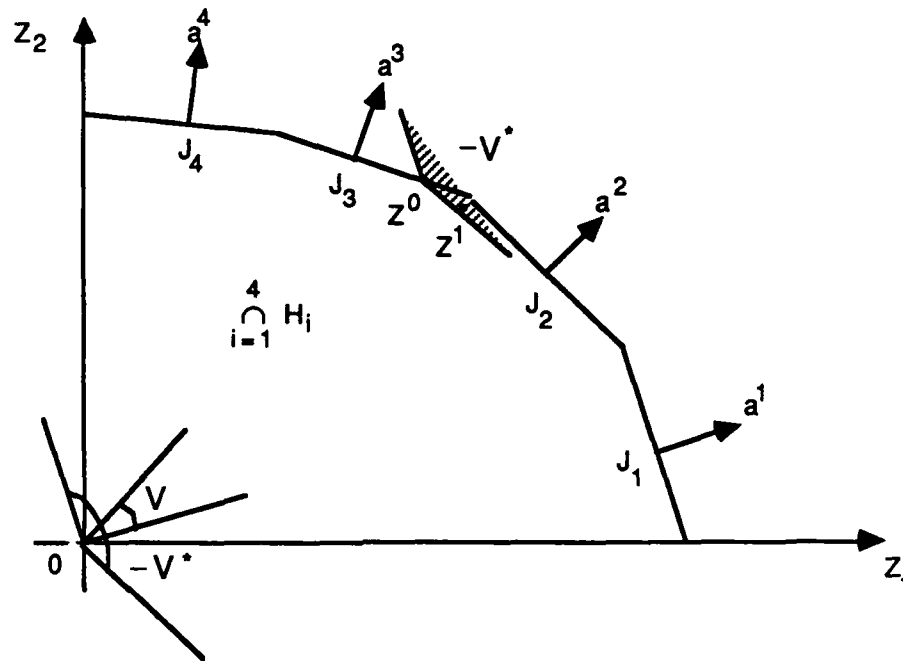
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Figure 4.2

Theorem 4.1 and 4.2 thus provide a means for constructing dominance cones that further restrict the set of Pareto-optimal points in accordance with specified preferences and priorities over outcomes. (See also the "goal-focusing" ideas and usages of [6], [7] ). For computational practical usages one may employ hyperplanes to approximate the Pareto-optimal frontier. Objective functions all are linear. The Pareto-optimal frontier consists of hyperplanes when all the objective functions are linear. The normals of those hyperplanes corresponding to preferred outcomes are then used to span the cone $V$ and thereby to obtain the dominance cone $W = -V^*$. This is an evident alternative way to do goal focusing [6], [7]. The sufficient conditions given in Theorem 3.9 can then determine the nondominated solutions associated with W.

## 5. An Illustrative Example

Consider the following vector extremal or multi-objective optimization problem in two variables in which there are three objective functions

$$f_1(x_1, x_2) = 30x_1 - \tfrac{1}{4}x_1^2 - 2x_1 x_2$$

$$f_2(x_1, x_2) = -\tfrac{1}{6}x_1^2 - \tfrac{1}{7}x_2^2 + \tfrac{1}{2}x_1 + \tfrac{6}{7}x_2 + \tfrac{199}{14}$$

$$f_3(x_1, x_2) = 66x_2 - \tfrac{1}{4}x_2^2 - 6x_1 x_2$$

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and two constraints

$$g_1(x_1, x_2) = 15 - x_1 - x_2$$

$$g_2(x_1, x_2) = 38 - 3x_1 - 2x_2$$

The domain of the object and constraint functions is given by the convex set

$$S = \{ (x_1, x_2) : 0 \leq x_1 \leq 10, 0 \leq x_2 \leq 10 \},$$

and the constraint cone $K = E_+^2$.

The problem may be written as

$$(W\text{–}E_+^2\text{–}P) \quad \begin{cases} \text{Max } (f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2)) \\ g_i(x_1, x_2) \geq 0, i = 1, 2 \\ (x_1, x_2) \in S \end{cases}$$

We first consider the special case where $W = E_+^3$. The respective Hessian matrices for the three objective functions are given by

$$\nabla^2 f_1 = \begin{bmatrix} \frac{-1}{2} & -2 \\ -2 & 0 \end{bmatrix}, \quad \nabla^2 f_2 = \begin{bmatrix} \frac{-1}{4} & 0 \\ 0 & \frac{-2}{7} \end{bmatrix}, \quad \nabla^2 f_3 = \begin{bmatrix} 0 & -6 \\ -6 & \frac{-1}{2} \end{bmatrix}.$$

Since $\nabla^2 f_2$ is a negative definite matrix, $f_2$ is strictly concave and, hence, strictly quasi concave.

$\nabla^2 f_1$ and $\nabla^2 f_3$ are neither positive semi-definite nor negative semi-definite, so $f_1$ and $f_3$ are neither convex nor concave functions. However, $f_1$ and $f_3$ are quasi concave on S. Their level sets on S

$$S_\alpha(f_1) = \{ (x_1, x_2) \in S: f_1(x_1, x_2) \geq \alpha \}$$

$$= \{ (x_1, x_2) \in S: \quad x_2 \leq 15 - \frac{1}{8} x_1 - \frac{\alpha}{2x_1} \}$$

$$S_\alpha(f_3) = \{ (x_1, x_2) \in S: f_3(x_1, x_2) \geq \alpha \}$$

$$= \{ (x_1, x_2) \in S: x_i \leq 11 - \frac{1}{24} x_2 - \frac{\alpha}{6x_2} \}$$

are convex for all $\alpha > 0$.

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Since $f_1$ and $f_3$ are nonnegative on S, the level sets $S\alpha$ $(f_1)$ and $S\alpha$ $(f_3)$ are also convex for all $\alpha \leq 0$.

Furthermore, $f_1$ and $f_3$ are strictly quasiconcave. To see this, consider the indifference sets for $f_1$ and $f_3$.

$$I_\alpha (f_1) = \{ (x_1, x_2) \in S: f_1 (x_1, x_2) = \alpha \}$$

$$= \{ (x_1, x_2) \in S: \ 30x_1 \ -\frac{1}{4} x_1^2 - 2x_1 x_2 = \alpha \}$$

$$I_\alpha (f_3) = \{ (x_1, x_2) \in S: f_3 (x_1, x_2) = \alpha \}$$

$$= \{ (x_1, x_2) \in S: 66 x_2 - \frac{1}{4} x_2^2 - 6x_1 x_2 = \alpha \}$$

Since neither indifference set contains any straight line segments for any $\alpha$, $f_1$ and $f_3$ are strictly quasi concave. [13]

By Theorem 3.8, the set of Pareto-optimal solutions constitutes the intersection of the constraint set with the set $D = \{(x_1, x_2): \lambda_1 \nabla f_1 (x_1, x_2) + \lambda_2 \nabla f_2 (x_1, x_2) + \lambda_2 \nabla f_3 (x_1, x_2) = 0,$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$ with at least two $\lambda_j$ nonzero$\}$. But D is the region bounded by three distinct "contract curves", one curve for each of the three possible pairs of objective functions:

$$\Gamma_1 = \{(x_1, x_2): \lambda_1 \nabla f_1 (x_1, x_2) + \lambda_2 \nabla f_2 (x_1, x_2) = 0, \ \lambda_1, \lambda_2 > 0\}$$

$$\Gamma_2 = \{(x_1, x_2): \eta_1 \nabla f_1 (x_1, x_2) + \eta_2 \nabla f_3 (x_1, x_2) = 0, \ \eta_1, \eta_2 > 0, \}$$

$$\Gamma_3 = \{(x_1, x_2): \mu_1 \nabla f_2 (x_1, x_2) + \mu_2 \nabla f_3 (x_1, x_2) = 0, \ \mu_1, \mu_2 > 0, \}.$$

Eliminating $\lambda_1, \lambda_2, \eta_1, \eta_2, \mu_1, \mu_2$, we obtain
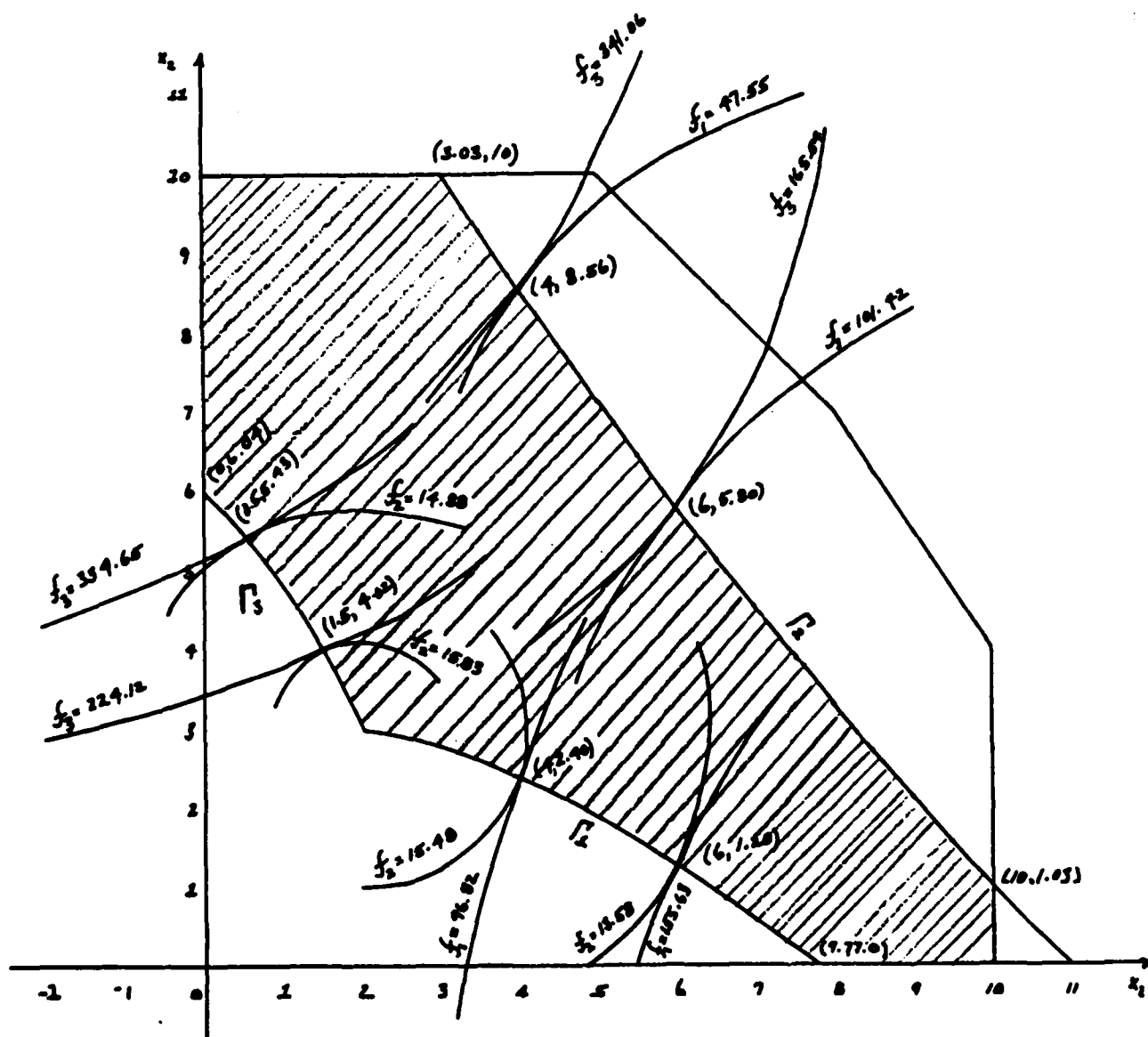
$$\Gamma_1 = \{ (x_1, x_2): 8 x_2^2 - 7x_1^2 + 2 x_1 x_2 + 8 x_1 - 144 x_2 + 360 = 0\}$$

$$\Gamma_2 = \{(x_1, x_2): 4x_2^2 + 12x_1^2 + x_1 x_2 - 588 x_2 - 852 x_1 + 7920 = 0\}$$

$$\Gamma_3 = \{(x_1, x_2): 96 x_2^2 - 84 x_1^2 - 7 x_1 x_2 - 274 x_2 + 1092 x_1 - 1848 = 0\}.$$

Hence, the set of Pareto-optimal solutions is given by the intersection of the constraint set with the set bounded by the three contract curves $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ as shown by the shaded region in Figure 5.1.

Figure 5.1

Next illustrate our approach water procedure for constructing dominance cones, we restrict attention to two objective functions, $f_1$ and $f_3$, in order that the Pareto-optimal frontier may be displayed in two dimensions. We shall employ the following variant of our preceding example.

$$f_1(x_1, x_2) = 30 x_1 - \frac{1}{4} x_1^2 - 2 x_1 x_2$$

$$f_3(x_1, x_2) = 66 x_2 - \frac{1}{4} x_2^2 - 6 x_1 x_2$$

$$g_1(x_1, x_2) = 15 - x_1 - x_2$$

$$g_2(x_1, x_2) = 38 - 3 x_1 - 2 x_2$$

$$S = \{ (x_1, x_2)^t : 0 \leq x_1 \leq 10, 0 \leq x_2 \leq 10 \}$$

$$X\left(E_+^2\right) = \{ (x_1, x_2)^t \in S : g_1(x_1, x_2) \geq 0, g_2(x_1, x_2) \geq 0 \}.$$

and

$$\left(W - E_+^2 - P\right) \begin{cases} \text{Max } (f_1(x_1, x_2), f_3(x_1, x_2)) \\ (x_1, x_2)^t \in X\left(E_+^2\right) \end{cases}$$

Figure 5.2 depicts the set of Pareto-optimal solutions of $(W - E_+^2 - P)$ given by the curve

$$\Gamma = \Gamma_1 \cup \{ (x_1, x_2)^t : x_2 = 10, 0 \leq x_1 \leq 3.03 \} \cup \{ (x_1, x_2)^t : x_1 = 10, 0 \leq x_2 \leq 1.05 \} \cup X\left(E_+^2\right)$$

Figure 5.2

FIGURE 5.2

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Consider a sequence of solutions in $\Gamma$, say {A, B, C, D, E, F, G, H, Q, U, V, W, Z} = {(0, 10), (2,10), (3.25, 9.66), (4.25, 8.2), (4.5, 7.84), (5.5, 6.48), (6, 5.8), (7, 4.52), (7.5, 3.9), (8.25, 3.01), (8.25, 3.01), (8.75, 2.43), (9.25, 1.87), (10, 0)}.
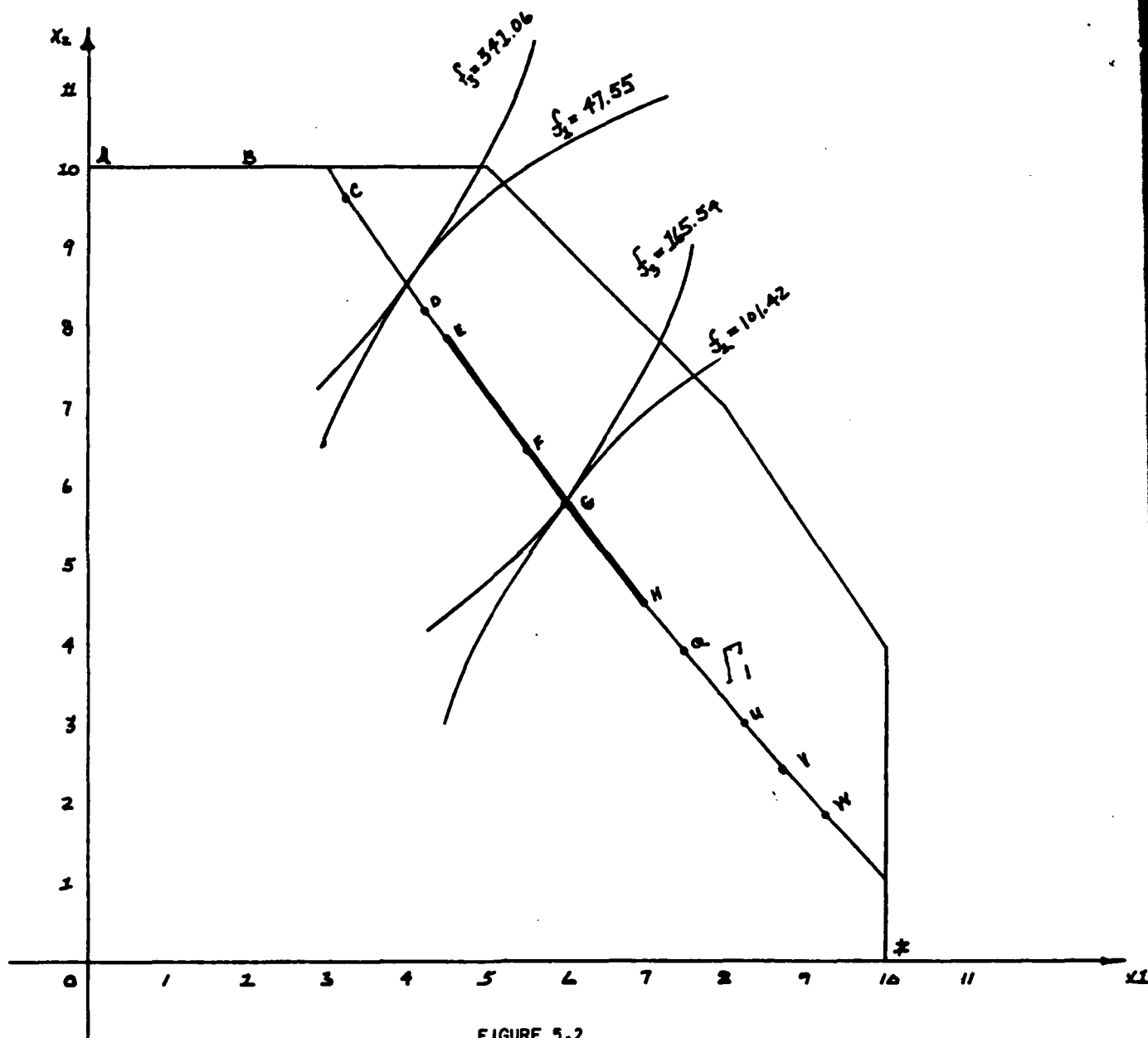
The corresponding sequence of Pareto-optimal points in objective space is given by {A', B', C', D', E', F', G', H', Q', U', V', W', Z'} = {(0, 635), (19, 515), (32.04, 426.05), (53.32, 315.15), (59.37, 290.41), (86.36, 202.8), (101.42, 165.54), (134.48, 103.36), (152.39, 78.16), (180.85, 47.37), (200.84, 31.32), (221.6, 18.72), (275, 0)}, as shown in Figure 5.3 where the shaded region is the range of ($f_1$, $f_3$) under the domain $X \left( E_+^2 \right)$ .

Figure 5.3

The Pareto-optimal frontier in Figure 5.3 is approximated by six line segments $J_1, \ldots, J_6$ given by

$$J_1 = \{ (f_1, f_3)^t: \ 5.8\, f_1 + f_3 = 635, \quad 0 \leq f_1 < 59.37 \}$$

$$J_2 = \{ (f_1, f_3)^t: 2.97\, f_1 + f_3 = 467, \qquad 59.37 \leq f_1 \leq 101.42 \}$$

$$J_3 = \{ (f_1, f_3)^t: 1.9\ f_1 + f_3 = 356, \quad 101.42 < f_1 \leq 134.48 \}$$

$$J_4 = \{ (f_1, f_3)^t: 1.2\ f_1 + f_3 = 265.7, \quad 134.48 < f_1 \leq 180.85 \}$$

$$J_5 = \{ (f_1, f_3)^t: 0.7\ f_1 + f_3 = 175, \quad 180.85 < f_1 \leq 221.5 \}$$

$$J_6 = \{ (f_1, f_3)^t: 0.4\ f_1 + f_3 = 96.4, \qquad 221.6 < f_1 \leq 275 \}$$

Now suppose that the decision maker's preferences or priorties are for outcomes within the vicinity of $G' \in J_2 \cap J_3$. To ensure such outcomes, a cone $V$ can be constructed such that the normals to $J_2$ and $J_3$ lie in $V$, but the normals of $J_1, J_4, J_5$, and $J_6$ are excluded from $V$. For example,

$$V = \{ \lambda\,(1.5, 1) + \mu\,(4, 1): \ \lambda \geq 0, \mu \geq 0 \} \text{ will fulfill this requirement.}$$

Now let $W = -V^*$. By Theorem 4.1, **none** of the points in $J_1 \cup J_4 \cup J_5 \cup J_6$ is a nondominated point associated with W, and $J_2 \cup J_3$ is the set of **all** nondominated points associated with W. The corresponding nondominated solution set is that part of the curve $\Gamma$ of Figure 5.2 extending from E = (4.5, 7.84) to H = (7, 4.52). Notice particularily how much of the Pareto-soptimal curve through A, B, C, . . . , Z is excluded by this dominance (or "goal-focusing") cone.

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Consider a sequence of solutions in $\Gamma$, say $\{A, B, C, D, E, F, G, H, Q, U, V, W, Z\}$ = $\{(0, 10),$ $(2,10),$ $(3.25, 9.66),$ $(4.25, 8.2),$ $(4.5, 7.84),$ $(5.5, 6.48),$ $(6, 5.8),$ $(7, 4.52),$ $(7.5, 3.9),$ $(8.25, 3.01),$ $(8.25, 3.01),$ $(8.75, 2.43),$ $(9.25, 1.87),$ $(10, 0)\}$.

The corresponding sequence of Pareto-optimal points in objective space is given by $\{A', B',$ $C', D', E', F', G', H', Q', U', V', W', Z'\}$ = $\{(0, 635),$ $(19, 515),$ $(32.04, 426.05),$ $(53.32, 315.15),$ $(59.37,$ $290.41),$ $(86.36, 202.8),$ $(101.42, 165.54),$ $(134.48, 103.36),$ $(152.39, 78.16),$ $(180.85, 47.37),$ $(200.84, 31.32),$ $(221.6, 18.72), (275, 0)\}$, as shown in Figure 5.3 where the shaded region is the range of $(f_1, f_3)$ under the domain $X$ $(E_+^2)$ .

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Now suppose that the decision maker's preferences or priorties are for outcomes within the vicinity of $G' \in J_2 \cap J_3$. To ensure such outcomes, a cone $V$ can be constructed such that the normals to $J_2$ and $J_3$ lie in $V$, but the normals of $J_1, J_4, J_5$, and $J_6$ are excluded from $V$. For example,

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## 6. Concluding Remarks

The theory of quasiconcave functions, as a generalization of concave functions, has been the focus of much research effort for applications in economics, where many utility and production functions are quasi-concave but not concave. See [20] for the most recent accumulation of research results. However,

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neither they nor earlier important results (e.g., [1], [12], [13], [14], and [15] extend in any immediate way to multi-objective programming where the needs, obligations, and preferences of **different** decision makers are to be addressed. Beginning in the current paper [10], a start was made with new ideas of "T-nondominated efficiency" and nondominated solutions for multi-payoff n-person games with interacting or "cross-constrained" strategy sets.

The present paper building on the results in [8], [9], and [10] has developed some basic ideas and theory for "cone quasiconcave multiobjective programming", including necessary as well as sufficient conditions for optimal solutions to such problems.

Thus, with the special method provided for construction of needed dominance cones, this new instrument can be applied, for example, to secure more apt and improved analysis of conflicting interests of multiple economic actors (e.g., firms) and synthesis of better policies by their regulating agencies.

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Some basic theory of "cone quasi-concave multi-objective programming" is developed. This new class of vector extremal problems with quasi-concave multiple objective employs ideas of nondominated solutions associated with dominance cones. Necessary as well as sufficient conditions for optimal solutions to such problems are provided. A simple example illustrates the concepts involved. In addition, for general applications in economics, it is shown how to establish dominance cones to realize producer priorities, consumer preferences, and other concerns exogenously determined.

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