

NPS55-89-04

NAVAL POSTGRADUATE SCHOOL

Monterey, California



CUMULATIVE SEARCH-EVASION GAMES (CSEGs)

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MARCH 1989

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Prepared for:
Naval Postgraduate School
Monterey, CA 93943-5000

FedDocs
D 208.14/2
NPS- 55-89-04

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REPORT DOCUMENTATION PAGE

1 REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b RESTRICTIVE MARKINGS	
2 SECURITY CLASSIFICATION AUTHORITY		3 DISTRIBUTION AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.	
5 DECLASSIFICATION/DOWNGRADING SCHEDULE			
PERFORMING ORGANIZATION REPORT NUMBER(S) NPS55-89-04		5 MONITORING ORGANIZATION REPORT NUMBER(S)	
NAME OF PERFORMING ORGANIZATION Naval Postgraduate School		6b OFFICE SYMBOL (If applicable) Code 55	7a NAME OF MONITORING ORGANIZATION Naval Postgraduate School
ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5000		7b ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5000	
NAME OF FUNDING/SPONSORING ORGANIZATION Naval Postgraduate School		8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN, Direct Funding
ADDRESS (City, State, and ZIP Code) Monterey, CA 93943-5000		10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO PROJECT NO TASK NO WORK UNIT ACCESSION NO	
TITLE (Include Security Classification) CUMULATIVE SEARCH-EVASION GAMES (CSEGS)			
PERSONAL AUTHOR(S) Eagle, James N., and Washburn, Alan R.			
11 TYPE OF REPORT Technical		13b TIME COVERED FROM TO	14 DATE OF REPORT (Year, Month, Day) March 1989
		15 PAGE COUNT 17	
5. SUPPLEMENTARY NOTATION			
COSATI CODES FIELD GROUP SUB-GROUP		18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
9 ABSTRACT (Continue on reverse if necessary and identify by block number) Cumulative search-evasion games (CSEGS) are two-person zero-sum search-evasion games where play proceeds throughout some specified period without interim feedback to either of the two players. Each player moves according to a preselected plan. If (X_t, Y_t) are the positions of the two players at time t , then the game's payoff is the sum over t from 1 to T of $A(X_t, Y_t, t)$. Additionally, all paths must be "connected." That is, the finite set of positions available for a player in any time period depends on the position selected by that player in the previous time period. One player attempts to select a mixed strategy over the feasible T -time period paths to maximize the expected payoff. The other minimizes. Two solution procedures are given. One uses the Brown-Robinson method of fictitious play and the other linear programming. An example problem is solved using both procedures.			
10 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS		21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
2a NAME OF RESPONSIBLE INDIVIDUAL James N. Eagle		22b TELEPHONE (Include Area Code) (408) 646-2654	22c OFFICE SYMBOL 55Er

Cumulative Search-Evasion Games (CSEGs)

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CSEGs are search-evasion games where play proceeds throughout some specified period without any interim feedback to either of the two players, each of whom is assumed to move according to some preselected plan. If (X_t, Y_t) are the positions of the two players at time t , then the payoff is $N = \sum_{t=1}^T A(X_t, Y_t, t)$. That is, the payoff is a cumulative score over the time intervals $1, \dots, T$. One possibility is that $A(X_t, Y_t, t)$ is an indicator of the event $X_t = Y_t$, in which case N is the total number of coincidences. This is the definition that motivated the class, but it is not the only possibility.

Both players are assumed to move among some finite set of cells C . Initial positions in C are determined by probability distributions which are known to both players. One possibility is that the distributions correspond to specific starting cells. This will be the case in a subsequent example, but, again, it is not required.

Given his initial position X_1 and the assumed distribution for player 2's initial cell, player 1 must select a feasible track $X_1, X_2, \dots, X_T$. A track is feasible if X_{t+1} lies in the given set $S(X_t, t)$ for $1 \leq t \leq T-1$. These tracks are player 1's pure strategies. Likewise Y_{t+1} must lie in the given set $E(Y_t, t)$ for $1 \leq t \leq T-1$. Generally $S(i, t)$ and $E(i, t)$ include cell i and some of its neighbors, the idea being that feasible tracks should connect neighboring cells. The payoff is determined once the tracks are selected by both players. Player 1 attempts to maximize the expected payoff, $E[N]$, and player 2 to minimize. Given the interpretation of the problem, it is natural to expect optimal strategies for both sides to be mixed.

1. Discussion and Motivation

One might prefer to consider a similar class of games where the pure strategy payoff is $1 - \epsilon^{-N}$, since that quantity can be interpreted as a detection probability if $A(x, y, t)$ is "detection rate at time t " (Koopman (1980)). Alternatively one might let the payoff be "time to the first detection," as in Ruckle's (1983) Pursuit on a Graph game. Such detection games are of considerable operational interest. Single player versions where player 2's motion is according to a *specified* Markov process have been considered by Stewart (1979, 1980), Eagle (1984), and Trummel and Weisinger (1986), and there is a more extensive literature (Stone (1989)) if the searcher's path is not constrained. It would indeed be satisfying to find an efficient method for solving the corresponding detection games where the evader's path is not probabilistically specified, and where he can thus more completely live up to his title. Unfortunately, the methods to be introduced later are tailored to the payoff N .

rather than $1 - e^{-N}$. Of course $1 - e^{-N}$ is approximately equal to N when N is small, so CSEGs can be regarded as first order approximations to detection games. The scale of $A(\cdot, \cdot, \cdot)$ is immaterial in solving a CSEG, but the validity of the approximation to a detection game will be best when $A(\cdot, \cdot, \cdot)$ is small.

Direct motivation of CSEGs is also possible. There are a variety of reasons why the results of search might not be known until it is over. Photographic film might have to be developed or nets hauled in. Another possible application is search planning for autonomous vehicles; for example, an over-the-horizon unmanned aircraft whose track must be specified before launch. Also, there is no real reason in CSEGs for restricting the two sides to consist of a single agent each. The two sides might be teams or even armies, one seeking contact and the other desirous of avoiding it. The “no feedback” restriction might then be viewed as a natural consequence of the difficulties of communication in the field.

Although the payoff in a CSEG has the same form as in a Multi-Stage Game (Thomas(1984)), CSEGs are not MSGs. To make an MSG out of a CSEG one would have to reveal the position of each player to the other after each move, so that the joint position could serve as a “state.” Although such games are interesting, they are not what we have in mind here.

2. Initial Observations

CSEGs are finite, two-person zero-sum games, so solutions certainly exist. The straightforward way to proceed would be to list all feasible tracks and then use linear programming to find the optimal probabilities for each track. The difficulty with this is that the number of feasible tracks explodes rapidly with the size of the problem. If the sets $S(i, t)$ all have three elements, and if the initial distribution for player 1's position is also concentrated on three points, there are 3^T pure strategies for player 1. This kind of exponential growth makes the “brute force” approach impractical for even moderately sized problems. The object must be to take advantage of the special structure of CSEGs to develop more efficient methods.

A mixed strategy for either player is a discrete probability distribution over the possible feasible tracks. Given mixed strategies for players 1 and 2, let $p(i, t)$ be the marginal probability that player 1 visits cell i at time t . Likewise let $q(i, t)$ be the corresponding probability for player 2. Then since the expectation of a sum is the sum of expectations, and since the two players choose their strategies independently,

$$E[N] = \sum_{t=1}^T \sum_{i,j \in C} A(i, j, t) p(i, t) q(j, t).$$

This payoff depends only on the marginal distributions $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$, so there is the possibility of an analysis based directly on them, rather than on the mixed strategies themselves. Furthermore, when $p(\cdot, \cdot)$ is given, player 2's problem in selecting an optimal track is a T -period shortest path

problem, a relatively simple type that can be solved quickly even for large problems. To see this, let $D(j, t) = \sum_{i \in C} A(i, j, t)p(i, t)$ be the penalty associated with visiting cell j at time t . Then player 2 wants to find a feasible track that visits the cells in such a manner as to minimize the sum of all T such penalties, a shortest path problem that can easily be solved using dynamic programming. Given a mixed strategy for player 1, this shortest path solution gives a lower bound on the value of the game. Similar comments hold concerning player 1's selection of a track when $q(\cdot, \cdot)$ is given. The fact that a lower bound on the value of the game is determined by specifying $p(\cdot, \cdot)$ and solving a shortest path problem, and that an upper bound is found by specifying $q(\cdot, \cdot)$ and solving a longest path problem will prove invaluable in the techniques to be discussed in the following sections.

CSEGs often have a "turnpike" property (Whittle (1983)) in the sense that optimal marginal distributions are attracted to a certain equilibrium pair $(p^*(\cdot, \cdot), q^*(\cdot, \cdot))$. More precisely, let $v(t)$ be the value of the one-period matrix game $A(\cdot, \cdot, t)$, and let $p^*(\cdot, t)$ and $q^*(\cdot, t)$ be optimal mixed strategies for the two players, unrestricted except that each must be a discrete probability distribution over the cells in C . If $p^*(\cdot, t)$ and $q^*(\cdot, t)$ are feasible marginal distributions for each time period of a T -period CSEG, then they must also be optimal. Furthermore, the value of the CSEG is $\sum_{t=1}^T v(t)$. In general the feasibility requirement will fail because $p(\cdot, t)$ and $q(\cdot, t)$ are required by the path constraints to resemble the initial distributions for small values of t . However, we can say

Theorem 1. *Suppose $p(\cdot, \cdot)$ and $q(\cdot, \cdot)$ are optimal for the T_1 -period CSEG, suppose $T_2 > T_1$, and let*

$$(\hat{p}(\cdot, t), \hat{q}(\cdot, t)) = \begin{cases} (p(\cdot, t), q(\cdot, t)) & \text{for } t \leq T_1 \\ (p^*(\cdot, t), q^*(\cdot, t)) & \text{for } T_1 < t \leq T_2. \end{cases}$$

If $\hat{p}(\cdot, \cdot)$ and $\hat{q}(\cdot, \cdot)$ are feasible for the T_2 -period CSEG, then they are also optimal.

Proof: Let $E[N(T)]$ be the expected payoff and $V(T)$ be the value of the T -period CSEG. Since $p(\cdot, \cdot)$ is optimal for the T_1 -period game, $E[N(T_1)] \geq V(T_1)$ when player 1 uses $p(\cdot, \cdot)$ and player 2 uses any feasible mixed strategy. Since $\hat{p}(\cdot, \cdot)$ agrees with $p(\cdot, \cdot)$ for $t \leq T_1$, the same can be said of $\hat{p}(\cdot, \cdot)$. Therefore if player 1 uses $\hat{p}(\cdot, \cdot)$,

$$E[N(T_2)] \geq V(T_1) + \sum_{t=T_1+1}^{T_2} v(t).$$

Likewise if player 2 uses $\hat{q}(\cdot, \cdot)$,

$$E[N(T_2)] \leq V(T_1) + \sum_{t=T_1+1}^{T_2} v(t).$$

The theorem follows. Furthermore, the value of the T_2 -period CSEG is

$$V(T_2) = V(T_1) + \sum_{t=T_1+1}^{T_2} v(t). \blacksquare$$

If (i) $A(\cdot, \cdot, t)$ does not actually depend on t , then neither will $p^*(\cdot, t)$ nor $q^*(\cdot, t)$. Additionally, if (ii) the path constraints allow both players to remain stationary, then these two “equilibrium” distributions will be feasible at $t + 1$ if they are feasible at t . Finally, if (i) and (ii) hold plus $p^*(\cdot, \cdot)$ and $q^*(\cdot, \cdot)$ are feasible at time t , then $p^*(\cdot, \cdot)$ and $q^*(\cdot, \cdot)$ are feasible and optimal marginal distributions from t onward. Solving the CSEG can then be viewed as programming the two sides to move from the given initial position distributions to equilibrium distributions. Only the transient phase presents any computational difficulty; once the equilibrium distributions are encountered, they are feasible and optimal from that point on. We now turn to methods for solving specific CSEGs.

3. The Brown-Robinson Method

In Robinson (1951), the method of fictitious play was shown to iteratively solve two-person zero-sum matrix games. This procedure had been suggested earlier by G. W. Brown. To describe fictitious play, let player 1 be the row (maximizing) player and player 2 be the column (minimizing) player. Rows and columns correspond to the pure strategies (tracks) described earlier. If player 1 selects row i and player 2 selects column j , then reward a_{ij} is paid from player 2 to player 1. In each fictitious play of the game (except the first), the players select the best pure strategy response to the empirical mix of the opponent’s pure strategies observed so far. So at play $k \geq 2$, player 1 chooses the pure strategy x_k (a vector where every component but one is 0) that is a best response to

$$\bar{y}_k = \frac{1}{k-1} \sum_{t=1}^{k-1} y_t = \bar{y}_{k-1} + \frac{1}{k-1}(y_{k-1} - \bar{y}_{k-1}),$$

where y_t is the pure strategy played by player 2 at time t . Then player 2 chooses the pure strategy y_k , which is the best response to the updated row average

$$\bar{x}_k = \frac{1}{k} \sum_{t=1}^k x_t = \bar{x}_{k-1} + \frac{1}{k}(x_k - \bar{x}_{k-1}).$$

Any limit points of the sequences $\{\bar{x}_k\}$ and $\{\bar{y}_k\}$ are solutions to the game. Also upper and lower bounds on the value of the game, v , are determined at each game play. Specifically, at game play k ,

$$\underline{v}_k = (\bar{x}_k)^t A y_k \leq v \leq (x_k)^t A \bar{y}_k = \bar{v}_k,$$

and both $\underline{v}_k$ and $\bar{v}_k$ converge to v . Fictitious play begins with the players selecting arbitrary strategies (pure or mixed) $x_1 = \bar{x}_1$ and $y_1 = \bar{y}_1$.

We note that to solve a matrix game by fictitious play, each player need only be able to select a best pure strategy response to any mixed strategy and evaluate the expected return. For CSEGs, this means that for fictitious play number $k \geq 2$, player 1 must be able to first update the running

average of the previously observed $k - 1$ pure strategies played by player 2, and then solve the T -period longest path problem giving the best pure strategy response for player 1. Similarly, player 2 must be able to update the running average of the previously observed k pure strategies played by player 1, and then solve the shortest path problem giving the best pure strategy response for player 2. The procedure begins with both players selecting arbitrary T -period strategies.

The Brown-Robinson method is notorious for converging very slowly to the optimal solution. However the simplicity of the updating procedure, which allows solution of moderate sized problems on microcomputers, makes it appealing for CSEGs.

4. The Linear Programming (LP) Method

It has been mentioned that CSEGs could conceivably be solved with LP methods if all pure strategies are enumerated. In this section an LP formulation is presented which does not require this explicit enumeration yet, unlike fictitious play, solves the game exactly.

To set up the LP, first let $g(j, t)$ be the smallest possible payoff accumulated over periods $t, t + 1, \dots, T$, given that player 2 starts in cell j at time t and that player 1's mixed strategy has marginals $p(\cdot, \cdot)$. Then

$$g(j, t) = \sum_{i \in C} A(i, j, t) p(i, t) + \min_{k \in E(j, t)} g(k, t + 1). \quad (1)$$

Since player 2's location at time 1 is specified by the distribution $q(\cdot)$, player 1's object is to maximize $E[N] = \sum_{i \in C} q(i) g(i, 1)$.

The feasibility (i.e., path) constraints are incorporated by introducing $u(i, j, t)$ as the probability that player 1 visits cell i at time t and cell j at time $t + 1$. Then the marginal variables $p(\cdot, \cdot)$ can be dispensed with because

$$p(i, t) = \sum_{j \in S(i, t)} u(i, j, t); \quad i \in C, t = 1, \dots, T - 1; \quad (2)$$

or alternatively,

$$p(i, t) = \sum_{j \in S^*(i, t)} u(j, i, t - 1); \quad i \in C, t = 2, \dots, T. \quad (3)$$

Here $S^*(i, t) = \{j \in C | i \in S(j, t - 1)\}$ for i in C and $t = 2, \dots, T$ is "the set of cells player 1 might have come from." This is distinguished from $S(i, t)$, which is "the set of cells to which player 1 might go." As long as the right hand sides of (2) and (3) are equal, the common value is a feasible marginal distribution for player 1. Using only the $u(\cdot, \cdot, \cdot)$, $g(\cdot, \cdot)$, and $p(\cdot, T)$ variables, player 1's problem is the following LP (the indicated dual variables will later be associated with player 2's LP):

$$\text{maximize } \sum_{i \in C} q(i)g(i, 1)$$

subject to:

dual variables

$$\sum_{k \in S(i, 1)} u(i, k, 1) = p(i); \quad i \in C \quad h(i, 1) \quad (4)$$

$$-\sum_{j \in S^*(i, t)} u(j, i, t-1) + \sum_{k \in S(i, t)} u(i, k, t) = 0; \quad i \in C, t = 2, \dots, T-1 \quad h(i, t) \quad (5)$$

$$-\sum_{j \in S^*(i, T)} u(j, i, T-1) + p(i, T) = 0; \quad i \in C \quad h(i, T) \quad (6)$$

$$-\sum_{k \in C} A(i, k, T)p(k, T) + g(i, T) \leq 0; \quad i \in C \quad q(i, T) \quad (7)$$

$$-\sum_{i \in C} A(i, j, t) \sum_{l \in S(i, t)} u(i, l, t) - g(k, t+1) + g(j, t) \leq 0; \quad j \in C, k \in E(k, t), \quad v(j, k, t) \quad (8)$$

$$t = 1, \dots, T-1$$

$$u(i, j, t) \geq 0; \quad i, j \in C, \quad t = 1, \dots, T-1$$

$$p(i, T) \geq 0; \quad i \in C$$

Constraints (4) enforce the starting condition $p(i, 1) = p(i)$; constraints (5) enforce the equality of (2) and (3); constraints (6) and (7) are the appropriate terminal conditions for $p(i, T)$ and $g(i, T)$; and constraints (8) are implied by (1). A proof that (8) and (1) are actually equivalent, and that the solution of the LP is therefore the solution of the game, could be based on an inductive argument that the objective function cannot be maximal unless at least one (8)-type constraint is tight for each (j, t) . However, it is simpler to merely observe that the solution of this LP is in any case a lower bound on the value of the game, and to conclude equality from the fact that the dual of this LP is the corresponding minimization problem for player 2.

This duality relationship will also allow us to identify the optimal solution for one player from the optimal dual variables in his opponent's LP. To see this, let $v(i, j, t)$ be player 2's counterpart to $u(i, j, t)$, and let $h(i, t)$ be the maximum obtainable expected total reward when player 1 starts in cell i at time t and player 2 uses $v(\cdot, \cdot, \cdot)$. Then the problem player 2 must solve, which is the dual of player 1's LP, is

$$\text{minimize } \sum_{i \in C} p(i)h(i, 1)$$

subject to:

dual variables

$$\sum_{k \in E(i, 1)} v(i, k, 1) = q(i); \quad i \in C \quad g(i, 1) \quad (9)$$

$$-\sum_{j \in E^*(i, t)} v(j, i, t-1) + \sum_{k \in E(i, t)} v(i, k, t) = 0; \quad i \in C, t = 2, \dots, T-1 \quad g(i, t) \quad (10)$$

$$-\sum_{j \in E^*(i, T)} v(j, i, T-1) + q(i, T) = 0; \quad i \in C \quad g(i, T) \quad (11)$$

$$-\sum_{k \in C} A(i, k, T)q(k, T) + h(i, T) \geq 0; \quad i \in C \quad p(i, T) \quad (12)$$

$$-\sum_{j \in C} A(i, j, t) \sum_{l \in E(j, t)} v(j, l, t) - h(k, t+1) + h(i, t) \geq 0; \quad i \in C, k \in S(i, t), \quad u(i, k, t) \quad (13)$$

$$t = 1, \dots, T-1$$

$$v(i, j, t) \geq 0; \quad i, j \in C, \quad t = 1, \dots, T-1$$

$$q(i, T) \geq 0; \quad i \in C$$

Player 1's LP can be made smaller by using (6) to solve for $p(i, T)$ and then substituting into (7). This eliminates constraints (6) and variables $p(i, T)$. Likewise constraints (11) and variables $g(i, T)$ can be eliminated from player 2's LP. After these simplifications, the number of variables in player 1's LP is the number of nodes plus the number of arcs in the T -period network specified by constraints (4) and (5). Similarly, the number of variables in player 2's smaller problem is the number of nodes plus arcs defined by constraints (9) and (10). Furthermore, the number of constraints in one player's LP is equal to the number of variables in his opponent's problem. So for both players, the number of variables and constraints expands linearly with T rather than exponentially. Thus for other than very small problems, solving these LPs is less burdensome than the "brute force" LP procedure mentioned earlier.

When compared to fictitious play, the LP procedure's primary advantage is that exact answers are produced. One would expect to resort to fictitious play only when the LP problem size exceeds the capability of available LP solvers.

5. The One-Dimensional Game

Consider a CSEG where $2n$ cells ($n \geq 1$) are arranged linearly with the searcher (player 1) initially in cell 1 and the evader (player 2) initially in cell $2n$. Transitions to neighboring cells are possible, or either party may remain stationary. Thus, except for end cells 1 and $2n$, $E(i, t) = S(i, t) = E^*(i, t) = S^*(i, t) = \{i-1, i, i+1\}$ for all t . The payoff at time t is 1 if searcher and evader are in the same cell, otherwise 0. The equilibrium distributions $p^*(\cdot, t)$ and $q^*(\cdot, t)$ are easily demonstrated

to be uniform, so for large T we expect the value of the T -period game to be $v_n(T) = T/2n - K_n$ for some K_n . Questions of interest are:

- Is K_n predictable, and what does “large T ” mean?
- What is the nature of the optimal strategies?

One reasonable strategy for the evader is what we will call “spreading.” The idea is to achieve the equilibrium distribution as fast as possible, and while doing so to assure that every cell feasible for the searcher contains at most the equilibrium probability. Spreading is not feasible in every CSEG, but the evader has no trouble employing it in the game under consideration. Figure 1 shows a spreading strategy when there are four cells.

Cell				
1	2	3	4	
			1.00	1
		.25	.75	2
	.25	.25	.50	3
.25	.25	.25	.25	4
.25	.25	.25	.25	5

Figure 1. Evader “spreads” unit probability over 4 cells. Cells not feasible for the searcher are shaded.

Since the searcher can obtain nothing on the first 2 opportunities and at most .25 per look on the third and subsequent opportunities, $v_2(T) \leq (T - 2)/4$ for $T \geq 2$. Therefore $K_2 \geq .5$. In fact $v_n(T) \leq (T - n)/2n$ for $T \geq n$ because evader spreading is feasible for any n , so $K_n \geq .5$ for $n \geq 1$.

Searcher spreading is also feasible here. Against searcher spreading the evader’s best strategy is to simply remain stationary, in which case there is no payoff for the first $2n - 1$ time periods. Therefore $v_n(T) \geq (T - (2n - 1))/2n$ for $T \geq 2n - 1$, and hence $K_n \leq (2n - 1)/2n$. Thus

$$.5 \leq K_n \leq \frac{2n - 1}{2n} \leq 1. \quad (14)$$

For all T , spreading is optimal for both sides when $n = 1$. It also turns out to be optimal for the evader when $T = 2n$, a game that is of some interest because $2n$ is the smallest value of T such that the solution is not trivial. To see this, note first that we have already established that $v_n(2n) \leq .5$. The searcher can also guarantee a payoff of .5, but by “rushing” rather than spreading. In rushing, the searcher essentially charges from one end to the other at top speed, except that for all t such that $2 \leq t \leq 2n$ he must be equally likely to occupy cells t and $t - 1$; the split is required to

prevent the possibility that the evader might pass by without coincidence. By rushing, the searcher guarantees that the probability of a coincidence somewhere in the first $2n$ periods is at least .5, so $v_n(2n) \geq .5$. Therefore $v_n(2n) = .5$, since the opposite inequality has already been established.

Obviously the searcher could continually rush from one end to the other, obtaining a payoff of .5 for every $2n$ time periods. This is not attractive when T is large, however, since a uniform distribution will in the long run obtain a payoff of $1/2n$ per time period. The searcher's dilemma is that rushing and spreading each have their attractions. Unfortunately the two strategies are incompatible in that rushing retains a concentrated distribution, whereas spreading aims for uniformity. This dilemma does not exist for the evader, since spreading is optimal for $T = 2n$ and also attractive in the long run. One might therefore expect that K_n would be closer to .5 than to 1 in (12). This turns out to be the case. Table 1 shows K_n for $1 \leq n \leq 6$ as established with linear programming formulations generated by the General Algebraic Modeling System (GAMS) and solved with MINOS (Modular In-core Nonlinear Optimization System) on the NPS IBM 3033AP mainframe computer.

n	K_n	T_n
1	.5000	2
2	.5357	6
3	.5431	10
4	.5440	13
5	.5459	14
6	.5459	19

Table 1. K_n and T_n for $n = 1, \dots, 6$.

Additionally T_n is listed, which is the first time both probability distributions become uniform. T_n is remarkably close to $3n$, but is not $3n$ exactly. When $n = 3$, player 1 can only force a payoff of $9/6 = .5433$ at time 9 if uniformity at time 9 is forced.

Figure 2 shows how the searcher's probability distribution $p(\cdot, \cdot)$ evolves with time when there are $2n = 12$ cells and T is 19 or larger. The first six time periods are not shown because $p(t, t) = 1$ for time $t \leq 6$; the searcher moves forward at top speed as long as contact with the evader is physically impossible. Equilibrium first appears at time 19. The searcher's motion might reasonably be characterized as a compromise between rushing and spreading.

Figure 3 shows the evader's probability distribution $q(\cdot, \cdot)$. The highest probability is in cell 12 at time 11, the last time at which the searcher is guaranteed not to be there. That probability (.306) is evenly divided between cells 11 and 12 at time 12, and then spreads out from there. The equilibrium distribution first appears at time 18. Note that the probability in low numbered cells goes through a maximum. This also happens with the searcher ($p(1, 17) = .0899 > p(1, 19) = .0833$), but much more weakly.

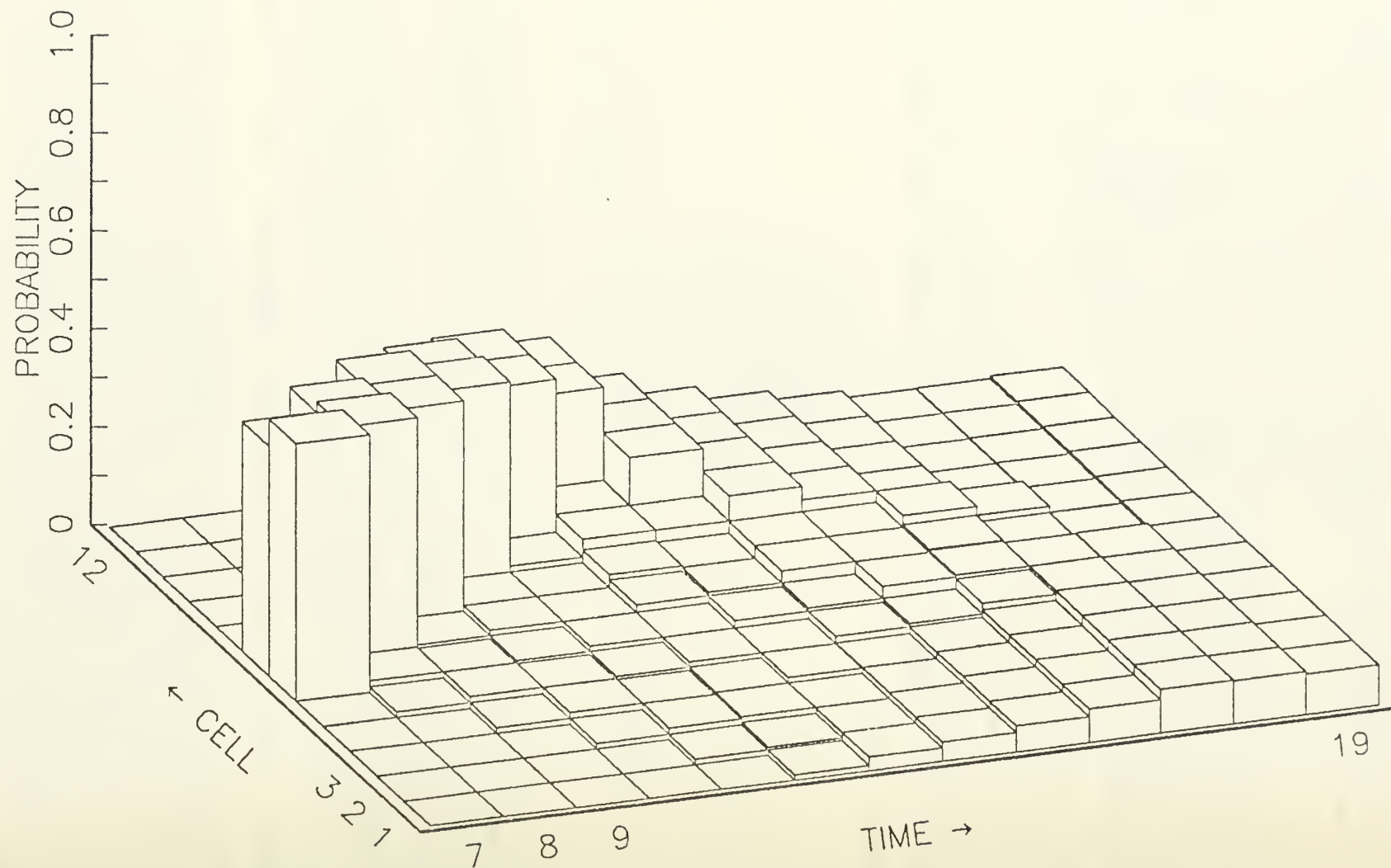


Figure 2. Searcher's Probability Distribution Evolving with Time.

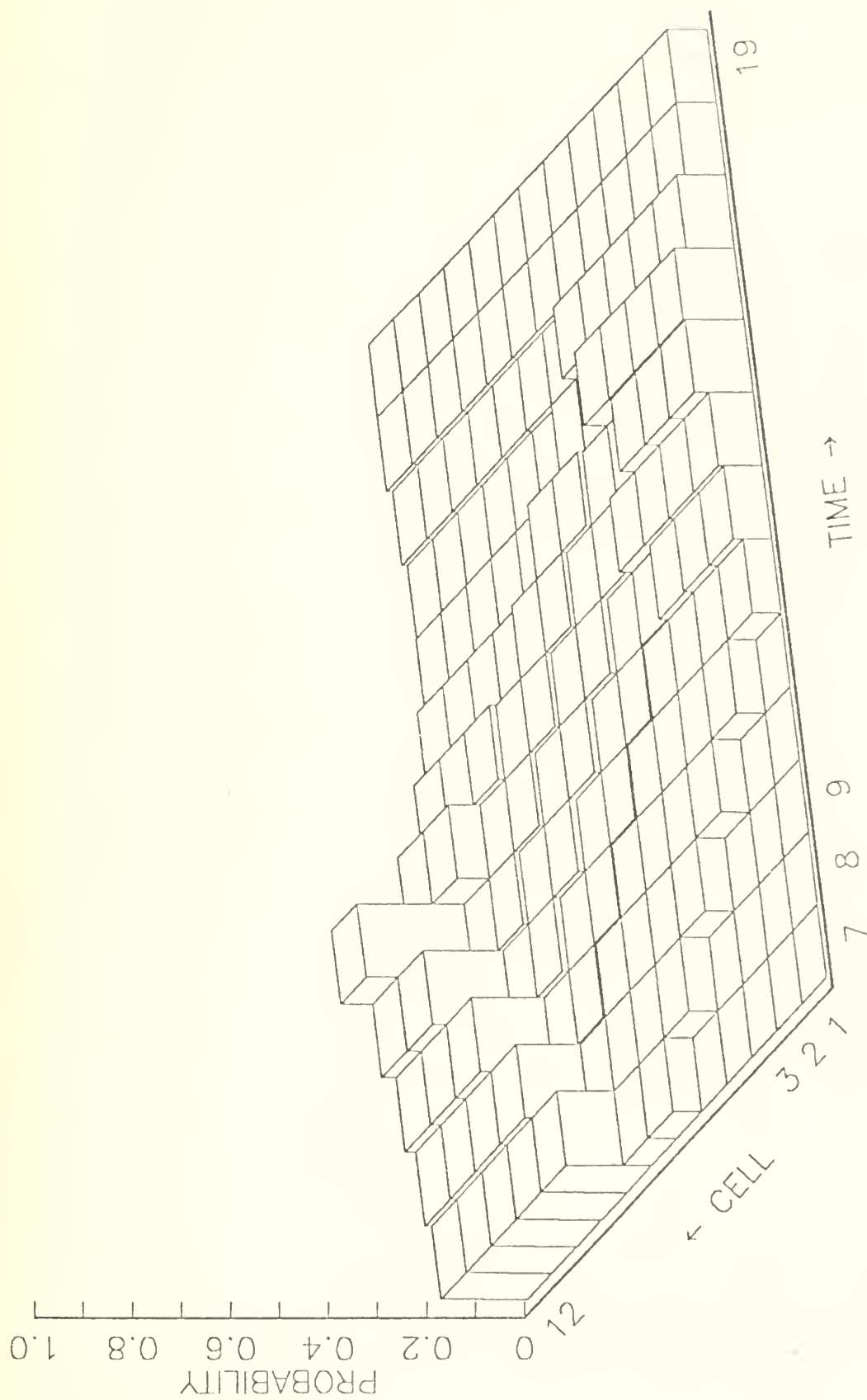


Figure 3. Evader's Probability Distribution Evolving with Time.

6. A Two-Dimensional Example

Now consider an 8-time period problem where a searcher and a evader move over a 5×5 grid of cells. The searcher begins in the upper left cell and the evader begins in the lower right cell. The searcher detects the evader with certainty if they share the same cell. Both players can move between cells in a single time period if the cells share a side or a corner. This problem has approximately 381,000 pure strategies (i.e.; feasible paths) for each player. It can be solved with linear programming but is large enough to make the Brown-Robinson method attractive—especially if a microcomputer solution is desired.

The Brown-Robinson procedure for this problem was programmed in Fortran 77 on a Macintosh IIx computer. After 40,000 fictitious plays, mixed strategies for both sides were generated which bounded the value of the game between .1845 and .1938. On this microcomputer, approximately 5 fictitious plays per second were accomplished. Figure 4 indicates the rate of convergence of the bounds.

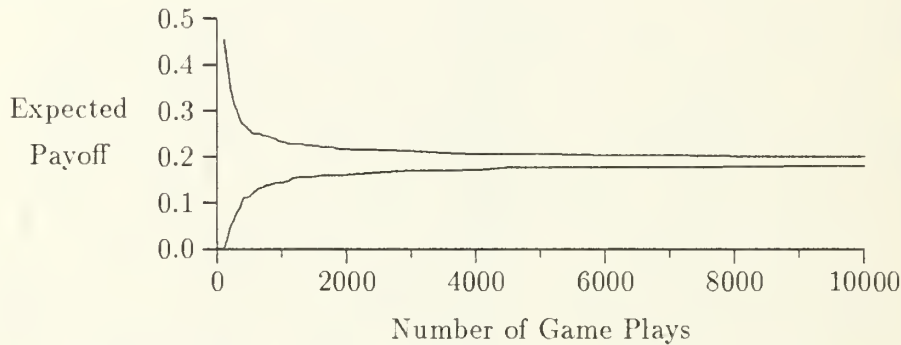


Figure 4. Bounds on the Value of the Game Generated by Fictitious Play.

The same problem was solved exactly using linear programming. Required were 1383 variables and the same number of constraints. An optimal solution was obtained after 2385 pivots and used approximately 410 CPU seconds on the NPS mainframe. The value of the game is .1891. Optimal marginal distributions for the searcher and evader ($\times 1000$) are shown in Figures 5 and 6.

Since any $u(\cdot, \cdot, \cdot)$ and $v(\cdot, \cdot, \cdot)$ will be optimal if they satisfy the path constraints and have optimal marginal distributions, it is reasonable to suspect that this problem might have many optimal solutions. This, in fact, is the case. Even the marginals are not unique. For example, any marginal distribution for the evader at time 2 is optimal if it “connects” optimal marginals at times 1 and 3. Figures 5 and 6 show optimal solutions with diagonal symmetry, but this symmetry was forced for esthetic reasons by adding additional constraints.

In this problem, the equilibrium distribution of .04 in each cell is reached at time 8 for both players. For the evader, this distribution is a feasible extension of his optimal marginal distribution at time 6. Were that true at time 6 for the searcher as well, then equilibrium would have been reached one time period earlier at time 7. Instead the evader concentrates his effort at time 7

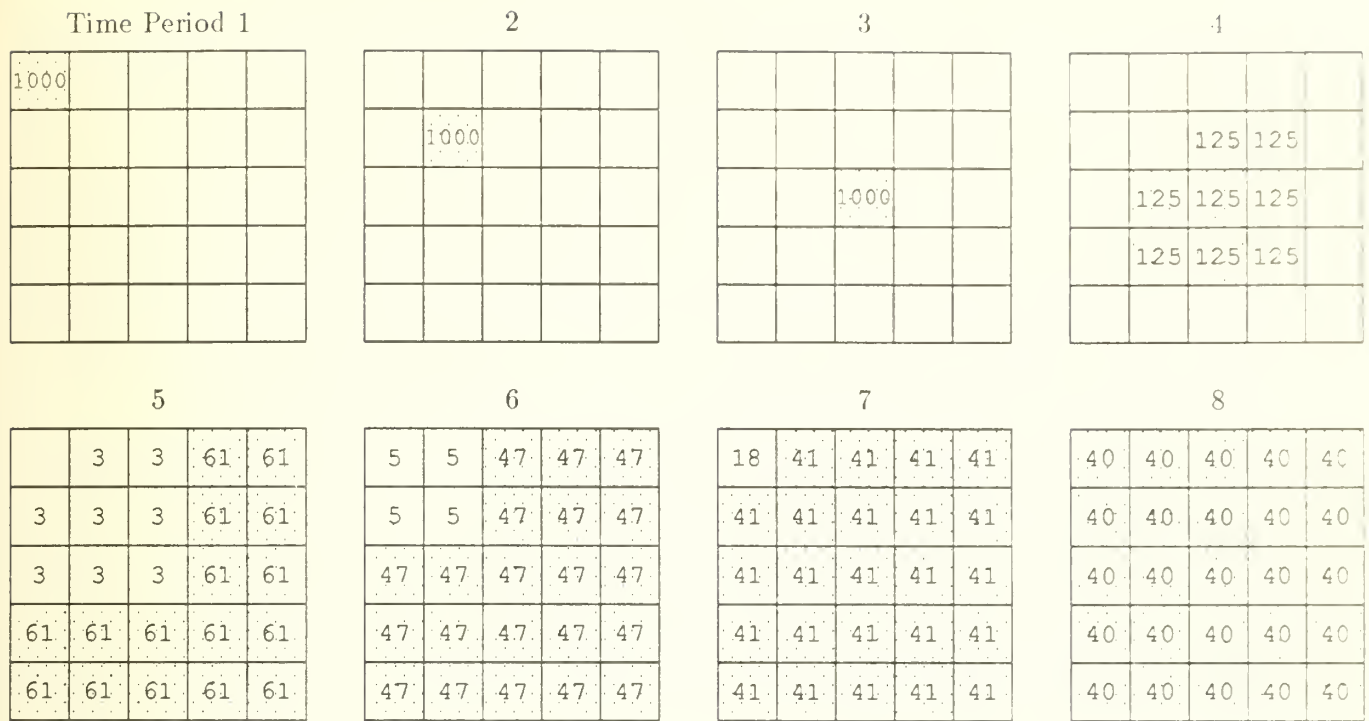


Figure 5. Searcher's Marginal Distribution (×1000).

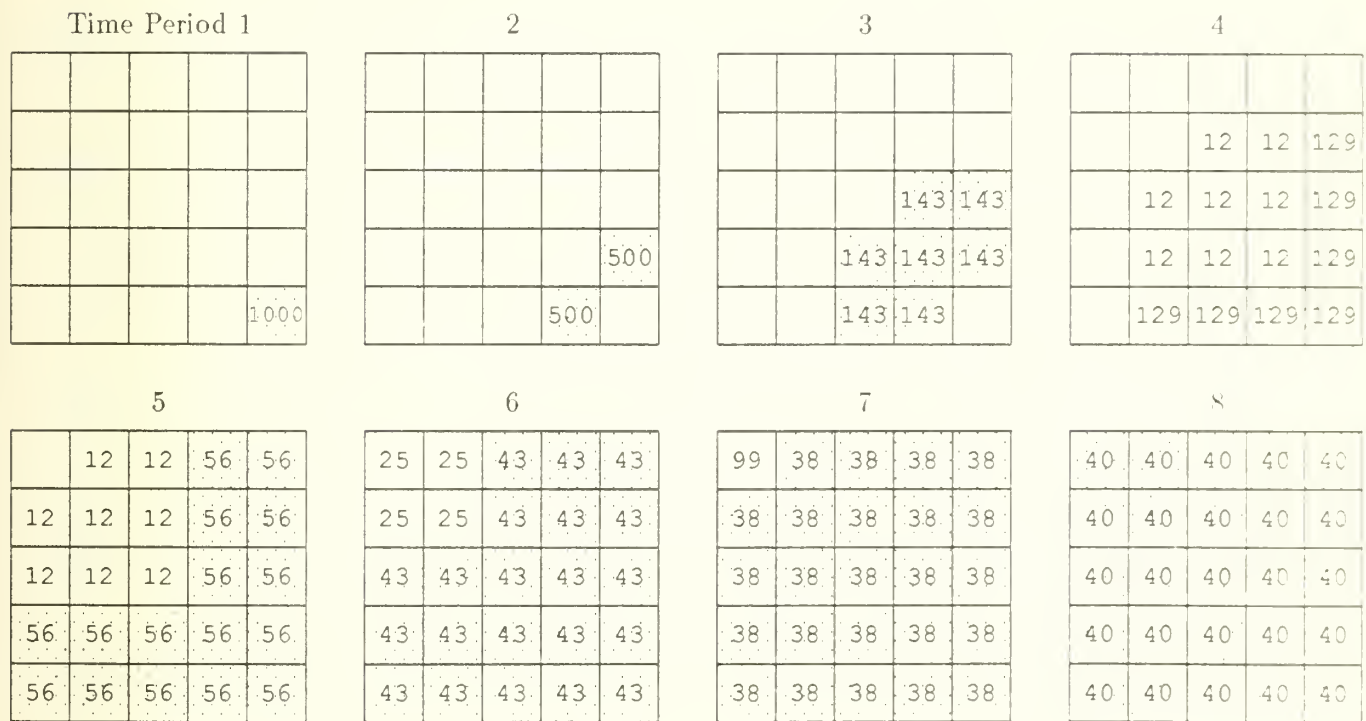


Figure 6. Evader's Marginal Distribution (×1000).

in the upper left-hand corner, taking advantage of a low searcher marginal level there. For all times $t \geq 8$, the equilibrium distribution is optimal for both players, and the value of the game is $.1891 + .04(t - 8)$.

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