Naval Research Laboratory
Washington, DC 20375-5000

NRL Report 9059

Reciprocal Perfect and Asymptotically Perfect Periodic Radar Waveforms and Their Aperiodic Properties

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May 24, 1989

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b DECLASSIFICATION / DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Report 9059	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION Naval Research Laboratory 6b OFFICE SYMBOL (*If applicable*) 5340.1G	7a NAME OF MONITORING ORGANIZATION
6c ADDRESS (*City, State, and ZIP Code*) Washington, DC 20375-5000	7b ADDRESS (*City, State, and ZIP Code*)
8a NAME OF FUNDING / SPONSORING ORGANIZATION Office of the Chief of Naval Research 8b OFFICE SYMBOL (*If applicable*)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c ADDRESS (*City, State, and ZIP Code*) Arlington, VA 22217-5000	10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO: 61153N; PROJECT NO: 021-05-43; TASK NO:; WORK UNIT ACCESSION NO: DN480-006

11 TITLE (*Include Security Classification*)
Reciprocal Perfect and Asymptotically Perfect Periodic Radar Waveforms and Their Aperiodic Properties

12 PERSONAL AUTHOR(S)
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13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (*Year, Month, Day*)	15 PAGE COUNT
	FROM _____ TO _____	1989 May 24	24

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (*Continue on reverse if necessary and identify by block number*)
FIELD	GROUP	SUB-GROUP	Radar Matched filter Waveforms

19 ABSTRACT (*Continue on reverse if necessary and identify by block number*)

Digitally coded radar waveforms can be employed to obtain large time-bandwidth products (pulse compression ratios). A perfect periodic code is defined to be a periodic code whose autocorrelation function has zero sidelobes and whose amplitude is uniform (maximum power efficiency = 1). An asymptotically perfect periodic code has the property that as the number of elements in the code goes to infinity the code's autocorrelation function has zero sidelobes and its power efficiency is one. In this report, we introduce a new class of radar waveforms that are either perfect or asymptotically perfect codes. We call these reciprocal codes because they can be derived through a linear transformation of the known codes. We also examine the reciprocal code's aperiodic performance. This is motivated by the tendency of good periodic codes to yield good aperiodic codes, and hence high pulse compression ratios with low sidelobe responses are attainable.

20 DISTRIBUTION / AVAILABILITY OF ABSTRACT	21 ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL Dr. Karl Gerlach	22b TELEPHONE (*Include Area Code*) (202) 767-3599 22c OFFICE SYMBOL 5340.1G

DD Form 1473, JUN 86 *Previous editions are obsolete.* SECURITY CLASSIFICATION OF THIS PAGE

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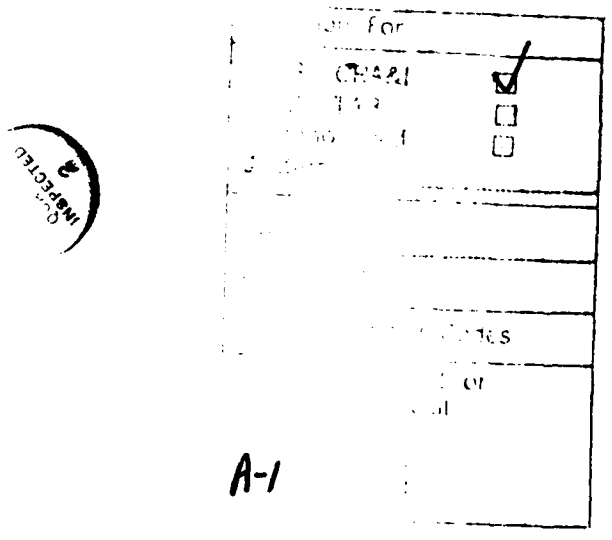


RECIPROCAL PERFECT AND ASYMPTOTICALLY PERFECT PERIODIC RADAR WAVEFORMS AND THEIR APERIODIC PROPERTIES

I. INTRODUCTION

Modern radars generally incorporate pulse compression waveforms to avoid transmitting a pulse having a large peak power. Pulse compression waveforms enable one to transmit a long pulse to obtain sufficient energy on a target for detection and to simultaneously obtain the desired range resolution. This is achieved by modifying the time-bandwidth product *(TB)* of the transmitted waveform. A larger transmit time duration T allows sufficient energy on the target for detection, while $1/B$ determines the resolution of the compressed pulse if no mismatch occurs. The desired signal bandwidth is generally obtained by modulating the signal's phase or frequency while maintaining a constant maximum pulse amplitude. This is illustrated by the linear chirp signal, pseudorandom phase codes, and polyphase pulse compression waveforms. A desirable property of the compressed pulse is that it have low sidelobes to prevent a weak target from being masked in the time sidelobes of a nearby stronger target. It is generally also desired that the compressed pulse does not significantly degrade when the return signal has been Doppler shifted because of target motion.

Digitally coded radar waveforms can be employed to obtain much larger values of *TB* (pulse compression ratios) than are feasible with analog dispersive delay lines. In addition, as it is demonstrated in the following sections, periodic radar waveforms can be defined that have zero sidelobes or almost zero sidelobes. Figure 1 illustrates a digitally coded periodic radar waveform. The complex values of $a_0, a_1, \ldots, a_{N-1}$ make up the code word in a given pulse repetition interval (PRI), where N is the length of the code word. This code word is repeated in succeeding PRIs.

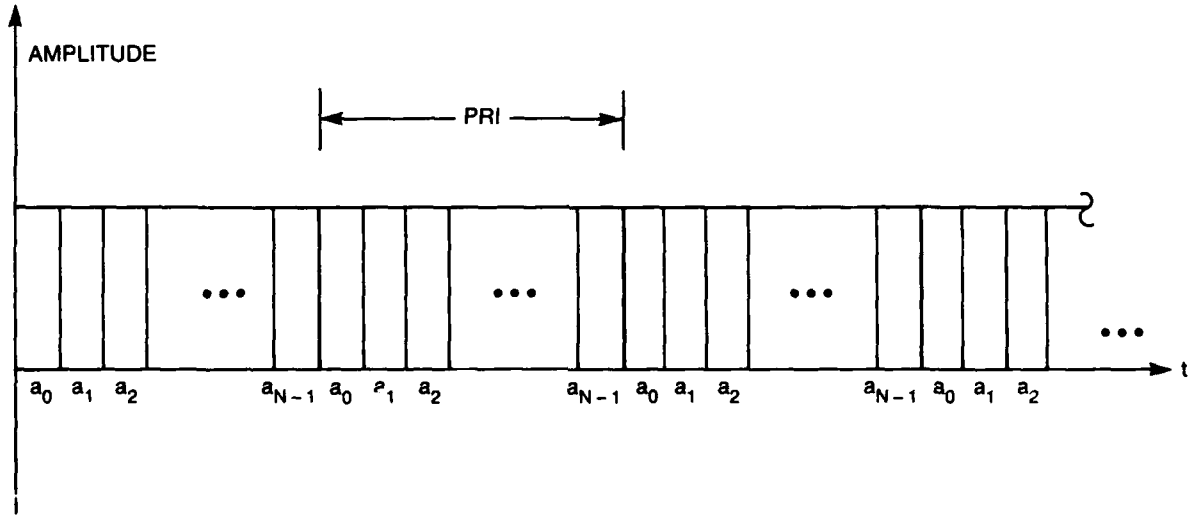


Fig. 1 — Digitally encoded periodic waveform

Manuscript approved May 23, 1988.

A perfect periodic code (PPC) is defined to be a periodic code whose autocorrelation function (ACF) has zero sidelobes and whose amplitude is uniform (maximum power efficiency = 1); i.e., $|a_1| = |a_2| = \ldots = |a_{N-1}|$, where $|\cdot|$ denotes the magnitude. An asymptotically perfect periodic code (APPC) has the property that as $N \rightarrow \infty$ the code's ACF has zero sidelobes and its power efficiency is one. Known PPCs and APPCs and their properties are discussed in more detail in Refs. 1 and 2.

In this report, we introduce a new class of PPC and APPC that we call reciprocal codes because they can be derived through a linear transformation of the known codes. We also examine the reciprocal code's aperiodic performance. This a motivated by the tendency that good periodic codes yield good aperiodic codes as exemplified by the Frank code [3].

II. PERIODIC CODES

Define a code word **a** such that **a** is a row vector of length N and

$$\mathbf{a} = (a_0, a_1, a_2, \ldots, a_{N-1}), \tag{1}$$

where a_n, $n = 0, 1, 2 \ldots, N-1$ are the elements of the code word. A periodic code is one that repeats the code word **a** indefinitely. Hence if $\mathbf{a}_{pc}$ is the periodic code associated with **a**, then

$$\mathbf{a}_{pc} = \mathbf{a} \circ \mathbf{a} \circ \mathbf{a} \ldots, \tag{2}$$

where the symbol o denotes concatenation.

On reception, a periodic code is match filtered with its code word. The output of the correlation process is also periodic with a period N. Hence, the matched response repeats every N unit time delays as does the sidelobe response.

We form an $N \times N$ circulant matrix A, based on each of the possible unit time delays of the received code:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \ldots & a_{N-2} \\ a_{N-2} & a_{N-1} & a_0 & \ldots & a_{N-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}. \tag{3}$$

Note that the inner product between the first row and the $m+1$th row, $m = 0, 1, \ldots, N-1$ row is equal to the output of the correlation process at the mth unit time delay of any period. Also the inner product of the m_1th and the m_2th row is equal to the inner product of the m_3th and the m_4th row if $m_1 - m_2 = m_3 - m_4$. In fact if r_m is denoted to be the inner product of the first and $m+1$th rows, then

$$AA^t = \begin{bmatrix} r_0 & r_1 & r_2 \cdots & r_{N-1} \\ r_1^* & r_0 & r_1 \cdots & r_{N-2} \\ r_2^* & r_1^* & r_0 \cdots & r_{N-3} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{N-1}^* & r_{N-2}^* & r_{N-3}^* & \cdots r_0 \end{bmatrix}, \tag{4}$$

where t denotes the matrix conjugate transpose and * denotes the complex conjugate. Here the diagonal elements are identical and equal to the matched response, and the off-diagonal elements are associated with the $N - 1$ sidelobe responses. The code word **a** is constrained such that

$$\|\mathbf{a}\|^2 = 1, \tag{5}$$

where $\|\cdot\|$ denotes the vector magnitude. We call a periodic code "perfect" if all the code elements have equal magnitude and all of the sidelobe responses are zero. The first condition implies the code is 100% power efficient or

$$|a_n|^2 = \frac{1}{N}; \quad n = 0, 1, 2, \ldots, N - 1. \tag{6}$$

Zero sidelobe response implies that

$$AA^t = I, \tag{7}$$

where I is the $N \times N$ identity matrix. The code word associated with a PPC is called the perfect periodic code word.

III. GENERATORS FOR PERFECT CODES

A circulant matrix, as given by the form seen in Eq. (3), has the property that it can be written as [4]

$$A = B \Lambda B^*, \tag{8}$$

where B is an $N \times N$ matrix given by

$$B = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 \ldots & 1 \\ 1 & W_N^M & W_N^{2M} \ldots & W_N^{(N-1)M} \\ 1 & W_N^{2M} & W_N^{4M} \ldots & W_N^{2(N-1)M} \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot \\ \cdot & \cdot & \cdot \cdot \cdot & \cdot \\ 1 & W_N^{(N-1)M} & W_N^{(N-1)2M} \cdot \cdot & W_N^{(N-1)(N-1)M} \end{bmatrix}, \tag{9}$$

with

$$W_N = e^{-j2\pi/N},$$

M is an integer that is relatively prime to N, $j = \sqrt{-1}$, and Λ is a $N \times N$ diagonal matrix. We refer to B as the generalized Butler matrix. This matrix has the properties that

$$B^T = B,\ BB^* = I,\ B^3 = B^*,\ B^4 = I, \tag{10}$$

where T denotes the matrix transpose.

From Eq. (8), a circulant matrix has the property that its eigenvectors are equal to the columns of the Butler matrix and its eigenvalues are the diagonal elements of Λ. Furthermore, the diagonal elements of the matrix Λ are defined as the row vector

$$\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \tag{11}$$

and are related to the code word $\mathbf{a}$,

$$\mathbf{a} = \frac{1}{\sqrt{N}}\, \boldsymbol{\lambda}\, B^*. \tag{12}$$

Hence, the code word is obtained directly from the diagonal elements of Λ through the matrix transform B. In a sense, the $\boldsymbol{\lambda}$ vector generates the $\mathbf{a}$ code word. We call $\boldsymbol{\lambda}$ the generating vector and Λ the generating matrix of the periodic code.

For a PPC, $AA^t = I$, which implies that

$$(B\ \Lambda\ B^*)\ (B\ \Lambda\ B^*)^t = I \tag{13}$$

or

$$B\ \Lambda\ \Lambda^t\ B^* = I \tag{14}$$

or

$$\Lambda\ \Lambda^t = I. \tag{15}$$

Hence, a necessary condition for a PPC is that the magnitudes of the diagonal elements of the generating matrix must equal one. Also, a necessary and sufficient condition for a periodic code to have zero sidelobes is given by Eq. (15).

Now consider two distinct periodic code words $\mathbf{a}_1$ and $\mathbf{a}_2$ and their respective circulant matrices, A_1 and A_2. Then from the preceding discussion,

$$A_1 = B\ \Lambda_1\ B^*; \quad A_2 = B\ \Lambda_2\ B^* \tag{16}$$

where Λ_1 and Λ_2 are the generating matrices of $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively. Now

$$A_1 A_2 = B\ \Lambda_1\ B^*\ B\ \Lambda_2\ B^*, \tag{17}$$

$$= B\ \Lambda_1\ \Lambda_2\ B^*.$$

Set $A = A_1 A_2$ and observe that A is also a circulant matrix whose first row $\mathbf{a}$ is found by circularly convolving $\mathbf{a}_1$ with $\mathbf{a}_2$ or $\mathbf{a} = \mathbf{a} *_c \mathbf{a}_2$, where $*_c$ denotes the circular convolution operation. Now if the magnitudes of the diagonal elements of Λ_1 and Λ_2 are equal to one, then the magnitudes of the diagonal elements of $\Lambda = \Lambda_1 \Lambda_2$ are also one. Hence the new periodic code word $\mathbf{a}$ must have zero sidelobes because of the necessary and sufficient condition given by Eq. (15). As a result, if two known periodic codes have zero sidelobes, we have a mechanism for generating another periodic code with zero sidelobes. In fact, the new codes themselves can be used to generate other zero sidelobe periodic codes by using the same methodology.

If the two periodic codes $\mathbf{a}_1$ and $\mathbf{a}_2$ are perfect the circularly convolved code $\mathbf{a}_1 *_c \mathbf{a}_2$ must have zero sidelobes but not necessarily equal magnitude for all code elements. However, in Section IV, a new code is introduced (which is found by circularly convolving two known PPCs) that not only has zero sidelobes but has equal magnitude for all code elements. Hence the new derived code is also perfect.

IV. RECIPROCAL PERFECT CODES

If $\mathbf{a}$ is a PPC word, then $(1/\sqrt{N})\ \boldsymbol{\lambda}$ is also a PPC word, where $\boldsymbol{\lambda}$ is the generating vector given by Eq. (12). The proof is fairly straightforward in that we demonstrate that the periodic code has constant amplitude and zero sidelobes.

Define the periodic code word

$$\mathbf{a}' = \frac{1}{\sqrt{N}}\ \boldsymbol{\lambda}^*, \tag{18}$$

where $\boldsymbol{\lambda}$ is the generating vector of a PPC. Hence, the elements of $\mathbf{a}'$ all have the same magnitude. We now prove that the sidelobe level is zero.

Associated with $\mathbf{a}'$ is a circulant matrix A' that has a form given by Eq. (3). Hence this matrix can be decomposed as $A' = B\ \Lambda' B^*$, where Λ' is a diagonal matrix. To show that the sidelobes of $\mathbf{a}'$ are zero, we need only show that $\Lambda'\ \Lambda'^{\prime} = I$ or that the associated generating vector $\boldsymbol{\lambda}'$ has elements that are all on the unit circle. This is accomplished by solving for $\boldsymbol{\lambda}^*$ in Eq. (12) and substituting this into Eq. (18) or

$$\mathbf{a}' = \frac{1}{\sqrt{N}}\ (\sqrt{N}\ \mathbf{a}^*)\ B^*. \tag{19}$$

Hence it follows that $\boldsymbol{\lambda}' = \sqrt{N}\ \mathbf{a}^*$, which implies that all the elements of $\boldsymbol{\lambda}'$ are on the unit circle.

Finally, since the complex conjugate of a PPC word is also a PPC word, it follows that $(1/\sqrt{N})\ \boldsymbol{\lambda}$ is a PPC word. In essence for a PPC, the relationship given by Eq. (12) implies that the generating points on the unit circle (given by the λ_n) are mapped into code elements on the $1/\sqrt{N}$ cir-

cle. Furthermore, a reciprocal code is yielded by normalizing the generating points. From Eq. (18) and above it follows that if a reciprocal PPC word is defined as

$$\mathbf{a}' = \mathbf{a}B, \tag{20}$$

then this is also a PPC word.

Note that since the matrix B depends on M, many reciprocal codes can be generated. Since M is relatively prime to N, the number of reciprocal codes is equal to the number of relatively prime integers less than N, which can be found by using Euler's ϕ-function [5]. However, all of these codes for $M > 1$ are merely a reordering of the elements of the $M = 1$ code. In fact, it is elementary to show that if $a'_0, a'_1, \ldots, a'_{N-1}$ are the elements of the reciprocal code for $M = 1$, then $a'_{Mn \bmod N}$, $n = 0, 1, \ldots, N-1$ are the elements of the reciprocal codes for any M relatively prime to N. For example, if $\mathbf{a}' = (a'_0, a'_1, a'_2, a'_3, a'_4)$ and $M = 2$, then $(a'_0, a'_2, a'_4, a'_1, a'_3)$ is also a reciprocal PPC. For $M = 3$, $(a'_1, a'_3, a'_1, a'_4, a'_2)$ is a reciprocal PPC. Hence, all reciprocal codes are related through elementary permutation transformations.

V. RECIPROCAL ASYMPTOTIC PERIODIC PERFECT CODES

An APPC is a periodic code that becomes perfect as the number of code elements N in the periodic code word approaches infinity. For finite N either the sidelobe level is nonzero and/or the power efficiency is less than 100% for these codes. However, as $N \to \infty$, either the sidelobe level is zero or approaches zero and/or the power efficiency is 100% or approaches 100%.

Examples of codes that are APPCs are the shift register codes, the primitive root code, and the quadratic residue code. All of these codes are polyphase and thus 100% efficient and have a relative sidelobe level (voltage) equal to $-1/N$ for all time delays. (Note that the relative sidelobe level is measured with respect to the peak or match point, which is normalized to one.) Hence, as $N \to \infty$ the relative sidelobe level goes to zero and the aforementioned codes become perfect.

It turns out that the reciprocal of these codes (see Eq. (20)) are also APPCs. In this case, the reciprocal codes have zero sidelobes for any N and a percent efficiency that approaches 100% as $N \to \infty$.

Let ϵ equal the constant relative sidelobes level of a given APPC. Then by definition

$$AA^t = \begin{bmatrix} 1 & \epsilon & \ldots & \epsilon \\ \epsilon & 1 & \ldots & \epsilon \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \epsilon & \epsilon & \ldots & 1 \end{bmatrix} \tag{21}$$

or

$$AA^t = (1-\epsilon)I + \epsilon\, \mathbf{1}\mathbf{1}^T, \tag{22}$$

where

$$\mathbf{1} = (1,1,1,\ldots 1)^T. \tag{23}$$

Since A is a circulant matrix, it has the form given by Eq. (8), which when substituted into Eq. (22) yields

$$B\ \Lambda\ B^*\ B\ \Lambda^t\ B^* = (1-\epsilon)\ I\ +\ \epsilon\ 11^T \tag{24}$$

or

$$\Lambda\ \Lambda^t = (1-\epsilon)\ I\ +\ \epsilon(B^*1)\ (B1)^T. \tag{25}$$

It can be shown that

$$B1 = (\sqrt{N}, 0, 0 \ldots, 0)^T, \tag{26}$$

so that

$$(B1)\ (B1)^T = \begin{bmatrix} N & 0 \ldots & 0 \\ 0 & 0 \ldots & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & \ldots & 0 \end{bmatrix}. \tag{27}$$

If $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ are the diagonal elements of Λ, it follows from Eqs. (25) and (27) that

$$|\lambda_0|^2 = 1\ +\ \epsilon\ (N\ -\ 1) \tag{28}$$

and

$$|\lambda_k|^2 = 1\ -\ \epsilon \qquad k\ =\ 1, 2, \ldots, N\ -\ 1. \tag{29}$$

If $\epsilon = -\ 1/N$, which is the relative sidelobe level of the binary psuedorandom, primitive root, and quadrative residue codes, then it is found that

$$|\lambda_0|^2 = \frac{1}{N} \tag{30}$$

and

$$|\lambda_k|^2 = 1\ +\ \frac{1}{N}, \qquad k\ =\ 1, 2, \ldots, N\ -\ 1. \tag{31}$$

Now if **a** is polyphase, the reciprocal code must have zero sidelobes because of the necessary and sufficient condition given by Eq. (15). The efficiency of the new code is calculated by averaging over the squared magnitudes (or powers) of the code elements and dividing by the maximum squared magnitude (or power) of the code elements. From Eqs. (30) and (31), it is seen that the efficiency of the reciprocal APPC can be written as

$$\text{eff} = \frac{\frac{1}{N^2} + (N-1)\left(1 + \frac{1}{N}\right)^2}{N\left(1 + \frac{1}{N}\right)^2} \tag{32}$$

or

$$\text{eff} \approx 1 - \frac{1}{N}. \tag{32}$$

Hence from Eq. (33), as $N \to \infty$ the percent efficiency goes to 100%. For example for $N = 10$, the reciprocal code is 91% efficient; for $N = 100$, the reciprocal code is 99% efficient.

VI. APERIODIC PROPERTIES

A. Reciprocal PPCs

Two known PPCs are the Frank code [3] and the Lewis and Kretschmer P4 [6,7] code. The Frank code word is formed by concatenating the rows of the conjugated Butler matrix:

$$\mathbf{a} = \frac{1}{N}\left(1,1,\ldots,1,W_N^{-L},\ldots W_N^{-L(N-1)},1,W_N^{-2L},\ldots,W_N^{-2L(N-1)},\ldots,1,W_N^{-L(N-1)},\ldots,W_N^{-L(N-1)^2}\right). \tag{34}$$

Note that the code word length of a Frank code is always a squared integer: N^2. The elements of the Frank code can be rewritten simply as

$$a_n = \frac{1}{N} W_N^{-Ln[n/N]}, \qquad n = 0, 1, \ldots, N^2 - 1, \tag{35}$$

where $[\cdot]$ is the least integer function (the greatest integer less than or equal to the argument).

From Eq. (20), the kth element of a reciprocal PPC is given by

$$a'_k = \frac{1}{N^2} \sum_{n=0}^{N^2-1} W_{N^2}^{-Ln[n/N^2] + Mnk}, \qquad k = 0, 1, \ldots, N^2 - 1. \tag{36}$$

If $L = M$, Appendix A shows that

$$a'_k = \frac{1}{N} W_N^{-Mk[k/N]} W_{N^2}^{Mk^2}. \tag{37}$$

Note that the reciprocal of the Frank code is merely the element-by-element product of the original Frank code and the diagonal of the B matrix. Note also, for $L \neq M$, that the resultant reciprocal code elements are permutations of the code elements given by Eq. (37) as discussed in Section IV.

The Lewis and Kretschmer P4 code has code word elements having the form

$$a_n = \frac{1}{\sqrt{N}} W_N(1/2)\, Ln(n+N \bmod 2)+nl, \qquad n = 0,1\,2, \ldots, N-1 \tag{38}$$

where N is the code word length, l is an arbitrary integer, L is an integer relatively prime to N, and N mod 2 equals 0 if N is even, or 1 if N is odd. Again by using Eq. (20), the kth element of a reciprocal PPC is found to be

$$a_k' = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{(1/2)\, Ln(n+N \bmod 2) + n(l+Mk)}, \qquad k = 0, 1, \ldots, N-1. \tag{39}$$

If $L = M$, Appendix B shows that

$$a_k' = \frac{1}{\sqrt{N}} W_N^{-(1/2)Mk(k+1) - kl} \cdot e^{j\phi}, \tag{40}$$

where ϕ is constant phase (given in Appendix B). The reciprocal P4 code is seen to be merely the complex conjugate of the original P4 code with a constant phase rotation. Hence, the B matrix transformation of the P4 code is isomorphic.

We now introduce a PPC that is found by circularly convolving a Frank code with a P4 code of length N^2 where N must be odd. Let $\mathbf{a}_F$ and $\mathbf{a}_{P4}$ be the code words associated with the Frank and P4 codes. The new PPC is then given by

$$\mathbf{a} = (\mathbf{a}_F *_c \mathbf{a}_{P4}) \cdot \frac{1}{N}, \tag{41}$$

where the $1/N$ normalizes the code's power to one. The proof that the convolved Frank-P4 is perfect is given in Appendix C. It can be shown that the elements of the convolved Frank-P4 are given by

$$a_n = \frac{1}{N^2} \sum_{m=0}^{N^2-1} W_{N^2}^{-m[m/N] + (1/2)L(n-m)(n-M+1) + (n-m)l}, \tag{42}$$

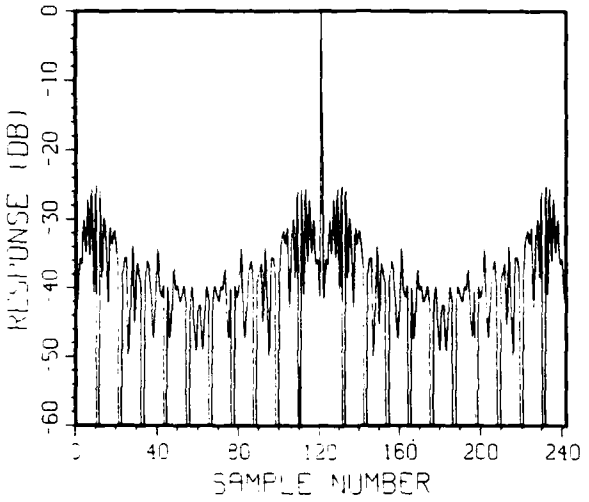
$$n = 0, 1, 2, \ldots N^2 - 1.$$

Figure 2 is a plot of the aperiodic ACF and the ambiguity function (see the next subsection for the definition) of the convolved Frank-P4 code. It is observed that the sidelobe level of the convolved Frank-P4 code is comparable to the P4 code, and like the P4 or Frank codes, it is Doppler tolerant.

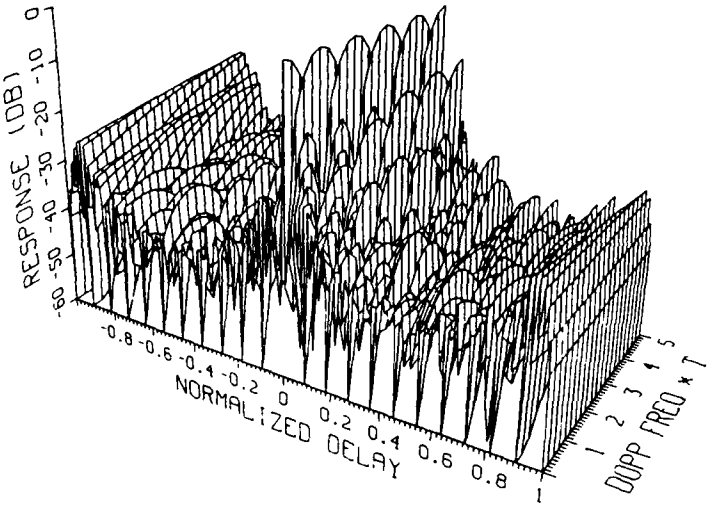
The reciprocal PPC elements are given by

$$a_k' = \frac{1}{N^2} \sum_{n=0}^{N^2-1} \sum_{m=0}^{N^2-1} W_{N^2}^{-m[m/N] + (1/2)L(n-m)(n-m+1) + (n-m)l + Mnk}$$

$$k = 0, 1, 2, \ldots N^2 - 1. \tag{43}$$



(a)



(b)

Fig. 2 — (a) Aperiodic ACF of the convolved Frank-P4 code; (b) ambiguity function, $N = 121$

B. Reciprocal Asymptotically Perfect Periodic Codes

In this subsection, the reciprocal codes associated with two codes that have a periodic autocorrelation function with all of the sidelobes equal to -1 are discussed (see Ref. 2 for more details). These codes are based on Number Theory considerations and are called the primitive root and quadratic residue codes. Some examples of the ACFs and ambiguity functions of these codes are presented.

We first define the discrete ambiguity function (DAF). Let $f_d T$ be the product of the target Doppler frequency and the code time duration, and let **a** be a code word defined by Eq. (1). Note that T is equal to the time length of **a**. The DAF $A(mf_d T)$ is defined as

$$A(m,f_dT) = \begin{cases} \left| \sum_{n=0}^{N-m-1} a_n^* \, a_{n+m} \, e^{j(2\pi/N)n \, f_dT} \right|^2, & m = 0, 1, \ldots, N-1 \\ \left| \sum_{n=0}^{N+m-1} a_n \, a_{n-m}^* \, e^{j(2\pi/N)n \, f_dT} \right|^2, & m = -1, -2, \ldots, -(N-1) \\ 0, & |m| \geq N \end{cases} \quad (44)$$

where the index m is associated with the range cells (or time delays) in the sidelobes. The ACF is simply equal to $A(m,0)$.

1. Primitive Root Code

The N code words of the primitive root code [2] are defined as

$$a_n = \frac{1}{\sqrt{N}} W_{N+1}^{\alpha^n}, \qquad n = 0, 1, 2, \ldots, N-1 \quad (45)$$

where $N+1$ must be a prime number and α is a primitive root modulo $N+1$ [5]. The reciprocal code is found by using Eq. (20) or

$$a'_k = \frac{1}{N} \sum_{n=0}^{N-1} W_{N+1}^{\alpha^n} \, W_N^{Mkn}, \qquad k = 0, 1, \ldots, N-1. \quad (46)$$

Figures 3 and 4, respectively, are example plots of the aperiodic ACFs of the primitive root code and its reciprocal for $N = 100$, $M = 1$, and $\alpha = 2$ ($N + 1 = 101$ is a prime number). Note that the sidelobe structure and level are very similar for the two codes. Figures 5 and 6, respectively, show the DAF of the primitive root code and its reciprocal. For the DAF plots, we have normalized the time delay axis to T. Hence for $m = N-1$, normalized delay equals 1. From these figures, again note the similar sidelobe structure. Also, it is observed that these codes have ambiguity functions that are similar to those of the binary shift register codes.

2. Quadratic Residue Code

For this code, which is binary, the Legrendre symbol [5] (q/p) is introduced. This symbol is defined for all q that are not divisible by p; it is equal to 1 if q is a quadratic residue of p; otherwise, it is equal to -1. Note that q is a quadratic residue of p if the congruence

$$z^2 = q \bmod p \quad (47)$$

has a solution.

With these preliminaries, the code is defined as

$$a_n = \frac{1}{\sqrt{N}} \left(\frac{n}{N} \right), \qquad n = 0, 1, \ldots, N-1 \quad (48)$$

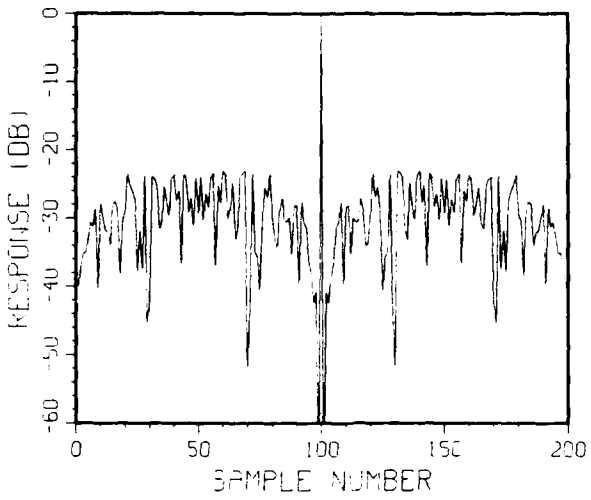


Fig. 3 — Aperiodic ACF of the primitive root code, $p = 101$, $\alpha = 2$

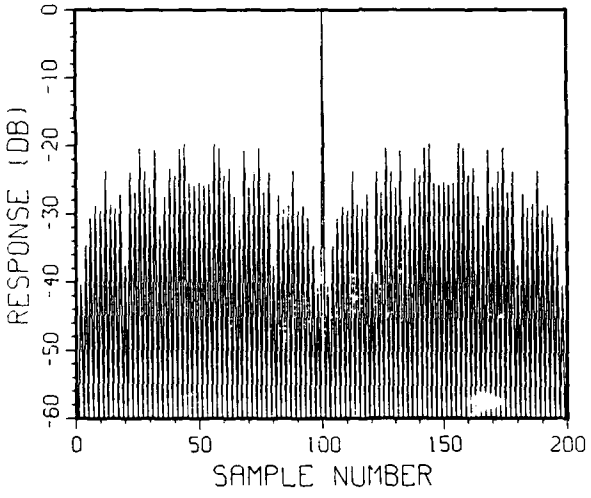


Fig. 4 — Aperiodic ACF of the reciprocal primitive root code, $p = 101$, $\alpha = 2$

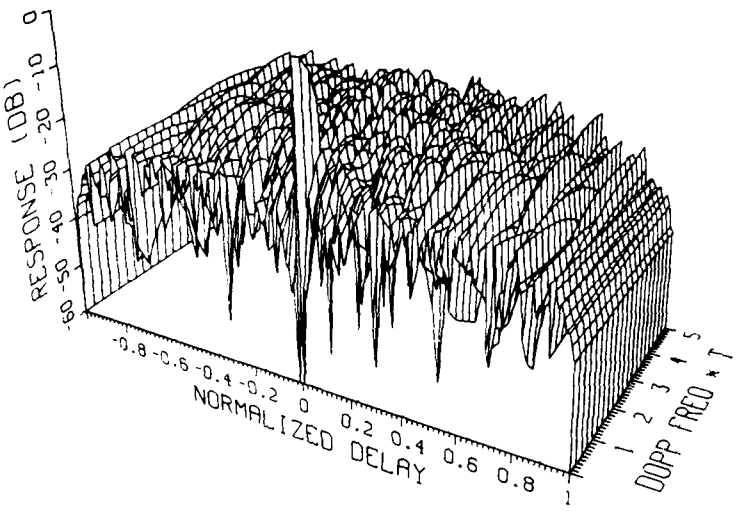


Fig. 5 — Ambiguity function of the primitive root code,
$p = 101$, $\alpha = 2$

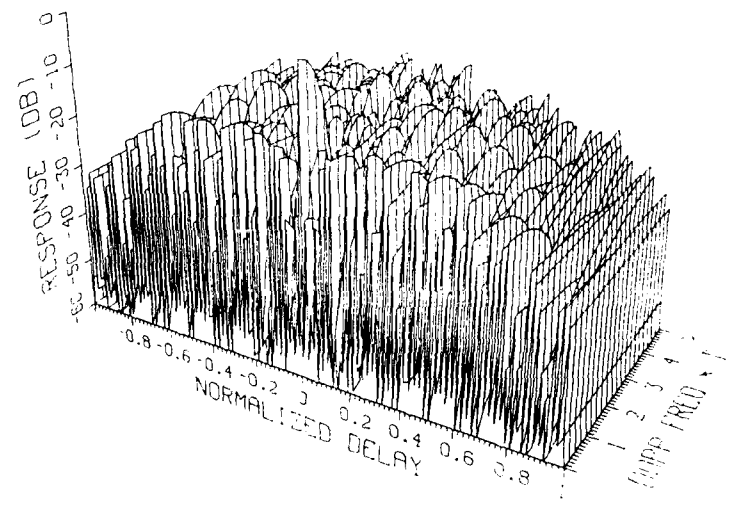


Fig. 6 — Ambiguity function of the reciprocal primitive root code,
$p = 101$, $\alpha = 2$

where N is prime number of the form: $4m-1$. We define $(0/N) = 1$. For example, for $N = 11$, the code word **a** is given by

$$\mathbf{a} = \frac{1}{\sqrt{N}} (1,1,-1,1,1,1,-1,-1,-1,1,-1). \tag{49}$$

The reciprocal code is found by using Eq. (18) or

$$a_k' = \frac{1}{N} \sum_{n=0}^{N-1} \left[\frac{n}{N}\right] W_n^{Mnk}, \quad k = 0, 1, \ldots, N-1. \tag{50}$$

Figures 7 and 8, respectively, are example plots of the aperiodic ACFs of the quadratic residue code and its reciprocal for $N = 103$ and $M = 1$. Again, the sidelobe structure and levels of the two codes are similar. This is also observed of the DAFs of the two codes shown in Figs. 9 and 10. Again, note that the ambiguity functions are similar to those of the binary shift register codes.

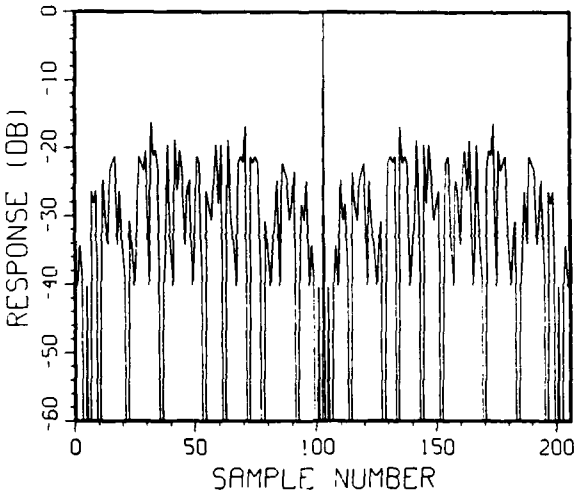


Fig. 7 — Aperiodic ACF of the quadratic residue code, $p = 103$

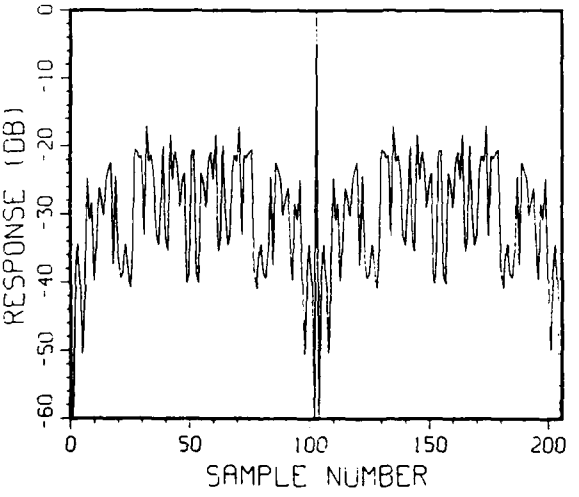


Fig. 8 — Aperiodic ACF of the reciprocal quadratic residue code, $p = 103$

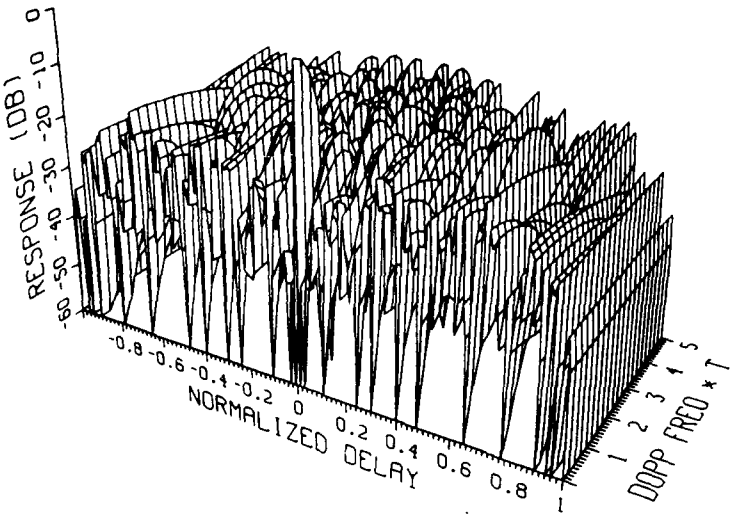


Fig. 9 — Ambiguity function of the quadratic residue code, $p = 103$

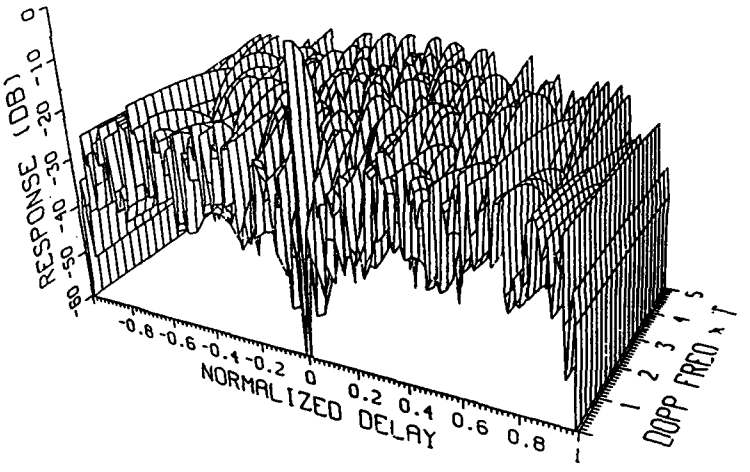


Fig. 10 — Ambiguity function of the reciprocal quadratic residue code, $p = 103$

VII. SUMMARY

In this report, new waveforms are described that have been recently investigated for use in radar systems. Of particular interest are digital coded polyphase waveforms having very low sidelobes after processing. Low sidelobes are desired to prevent the masking of weak targets in the sidelobes of strong targets or clutter returns. Processing a polyphase waveform consists of digital matched filtering whereby the output of this filtering is the ACF of the waveform.

New PPCs are investigated and presented whose ACFs have zero sidelobes and whose amplitude is uniform (maximum power efficiency = 1). In addition, new APPCs, which have the property that as the number of elements in the code goes to infinity the code's ACF has zero sidelobes and its power efficiency is one, are discussed. These codes were called reciprocal codes because they can be derived through the Butler matrix transformation of the known codes. It is shown that the reciprocal code of a PPC is also a PPC and that the reciprocal code of an APPC is also an APPC. Also, we have presented a new PPC that results from circularly convolving the Frank code with an odd square ordered Lewis and Kretschmer P4 code.

We have also examined the reciprocal code's aperiodic performance. This is motivated by the tendency of good periodic codes to yield good aperiodic codes. It is found that many of the reciprocal codes have aperiodic sidelobe levels similar to the low sidelobe levels exhibited by well-known codes (such as the binary shift register codes).

VIII. REFERENCES

1. R. Turyn, "Sequences with Small Correlation," in *Error Correcting Codes*, H.B. Mann, ed. (Wiley, 1969), pp. 195-228.

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Appendix A

RECIPROCAL FRANK CODE

The code elements of the reciprocal Frank code ($M = L$) are given by

$$a_k' = \frac{1}{N^2} \sum_{n=0}^{N^2-1} W_n^{-Mn[n/N]} W_{N^2}^{Mnk}, \qquad k = 0, 1, \ldots, N^2 - 1. \tag{A1}$$

Set $n = n_1 N + n_2$, $k = k_1 N + k_2$ where $n_1, n_2, k_1, k_2 = 0, 1, \ldots, N - 1$. Equation (A1) then can be rewritten as

$$a_k' = \frac{1}{N^2} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} W_N^{-(n_1 n_2 + k_1 n_1 + k_2 n_2)M} W_{N^2}^{Mn_2 k_2}, \tag{A2}$$

$$= \frac{1}{N^2} W_N^{Mk_1 k_2} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} W_N^{-m(n_1 - k_1)(n_2 - k_2)} W_{N^2}^{Mn_2 k_2}.$$

Set $n_1' = n_1 - k_1$, $n_2' = n_2 - k_2$. Thus

$$a_k' = \frac{1}{N_2} W_N^{Mk_1 k_2} W_{N_2}^{Mk_2^2} \sum_{n_1'=-k_1}^{N-1-k_1} W_{N_2}^{Mn_1' k_2} \left[\sum_{n_2'=k_2}^{N-1-k_2} W_N^{-Mn_1' n_2'} \right]. \tag{A3}$$

However, because of the periodicity of W_N,

$$\sum_{n_2'=-k_2}^{N-1-k_2} W_N^{-Mn_1' n_2'} = \sum_{n_2'=0}^{N-1} W_N^{-Mn_1' n_2'} = \begin{cases} N, \text{ if } n_1' = 0 \\ 0, \text{ otherwise} \end{cases}. \tag{A4}$$

Hence, it follows that Eq. (A3) reduces to

$$a_k' = \frac{1}{N} W_N^{Mk_1 k_2} W_{N^2}^{Mk_2^2}. \tag{A5}$$

Now

$$W_{N^2}^{Mk^2} = W_{N^2}^{M(k_1 N + k_2)^2} = W_N^{2Mk_1 k_2} W_{N^2}^{Mk_2^2} \tag{A6}$$

or

$$W_{N^2}^{Mk_2^2} = W_N^{-2Mk_1k_2} W_{N^2}^{Mk^2}. \tag{A7}$$

Thus

$$a_k' = \frac{1}{N} W_N^{-Mk_1k_2} W_{N^2}^{Mk^2}, \tag{A8}$$

$$= \frac{1}{N} W_N^{-Mk[k/N]} W_{N^2}^{Mk^2}.$$

Appendix B

RECIPROCAL P4 CODE

The code elements of the reciprocal P4 code are given by the expression

$$a'_k = \frac{1}{N} \sum_{n=0}^{N-1} W_N^{(1/2)\,Mn(n\,+\,N\ \mathrm{mod2})\,+\,(l\,+\,Mk)n}. \tag{B1}$$

Now

$$\frac{1}{2}Mn(n + N \bmod 2) + (l + Mk)n = \frac{1}{2}M\,[(n + k)^2 - k^2 + nN \bmod 2] + ln. \tag{B2}$$

Set $m = n + k$ so that

$$a'_k = \frac{1}{N}\, W_n^{-(1/2)\,Mk^2} \sum_{m=k}^{N-1+k} W_N^{(1/2)M[m^2\,+\,mN\ \mathrm{mod2}\,-\,kN\ \mathrm{mod2}]\,+\,l(m-k)} \tag{B3}$$

or

$$a'_k = \frac{1}{N}\, W_N^{-(1/2)\,Mk(k\,+\,N\ \mathrm{mod2})\,-\,lk} \cdot \sum_{m=0}^{N-1} W_N^{(1/2)\,Mn(m\,+N\ \mathrm{mod2})\,+\,ln}. \tag{B4}$$

Note that the summation term in Eq. (B4) is independent of k. In fact, it can be shown [B1] that the magnitude of this summation term equals $\sqrt{N}$. Thus if we set

$$\phi = \arg \left\{ \sum_{m=0}^{N-1} W_N^{(1/2)\,Mn(m\,+\,N\ \mathrm{mod2})\,+\,ln} \right\}, \tag{B5}$$

then

$$a'_k = \frac{1}{\sqrt{N}} W_N^{-(1/2)\,Mk(k\,+\,N\ \mathrm{mod2})\,-\,lk}\, e^{j\phi}. \tag{B6}$$

REFERENCE

B1. I.M. Vinogradov, *Elements of Number Theory* (Dover Publications, 1954).

Appendix C

THE CONVOLVED FRANK-P4 CODE

In this appendix, it is shown that the circularly convolved Frank-P4 code is perfect. The new PPC is given by

$$\mathbf{a} = (\mathbf{a}_F *_c \mathbf{a}_{P4}) \cdot \frac{1}{N}, \tag{C1}$$

where $\mathbf{a}_F$ and $\mathbf{a}_{P4}$ are Frank and P4 code words, respectively. In addition N must be odd. We prove the $\mathbf{a}$ is perfect by showing that the reciprocal of $\mathbf{a}$, denoted by $Rec\{\mathbf{a}\}$, is perfect. Hence, if $Rec\{\mathbf{a}\}$ is perfect, then $\mathbf{a}$ is perfect (see Section IV).

Now $Rec\{\mathbf{a}\}$ is simply

$$Rec\ \{\mathbf{a}\} = Rec\{\mathbf{a}_F\}\ \mathbf{x}\ Rec\{\mathbf{a}_{P4}\} \cdot N, \tag{C2}$$

where $\mathbf{x}$ denotes the product of the corresponding elements of $Rec\{\mathbf{a}\}$ and $Rec\{\mathbf{a}_{P4}\}$. For example,

$$(b_0,b_1,b_2,b_3)\ \mathbf{x}\ (c_0,c_1,c_2,c_3) = (b_0c_0,b_1c_1,b_2c_2,b_3c_3). \tag{C3}$$

It is shown in Appendixes A and B that

$$Rec\{\mathbf{a}_F\} = \left\{\frac{1}{N}\ W_N^{-Mk[k/N]} W_{N^2}^{Mk^2}\right\}, \qquad k = 0,1,\ldots,N^2-1 \tag{C4}$$

and for a P4 code of length N^2,

$$Rec\{\mathbf{a}_{P4}\} = \left\{\frac{1}{N}\ W_{N^2}^{-(1/2)\,Mk(k\,+\,1)\,-\,lk}\ e^{j\phi}\right\}, \qquad k = 0,\ 1,\ ,\ldots\ ,\ N^2\ -\ 1. \tag{C5}$$

Hence, from Eq. (C2)

$$Rec\{\mathbf{a}\} = \left\{\frac{1}{N}\ W_N^{-Mk[k/N]}\ W_{N^2}^{(1/2)Mk(k\,+\,1)\,-\,(m\,+\,l)k}\ e^{j\phi}\right\}, \qquad k = 0,\ 1,\ \ldots\ ,\ N^2\ -\ 1. \tag{C6}$$

It is shown in Ref. C1 that $\mathbf{a}_F\ \mathbf{x}\ \mathbf{a}_{P4}$ is a perfect code. Note that the form of $Rec\ \{\mathbf{a}\}$ is the same as $\mathbf{a}_F\ \mathbf{x}\ \mathbf{a}_{P4}$. Hence $Rec\ \{\mathbf{a}\}$ is a PPC, from which it follows that $\mathbf{a}$ also is a PPC.

REFERENCE

C1. F.F. Kretschmer, Jr. and Karl Gerlach, "Radar Waveforms Derived from Orthogonal Matrices," NRL Report 9080, Feb. 1989.