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Analysis of a Limiting-Amplitude Problem in Acousto-Elastic Interactions, I

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Time-harmonic scattering and radiation by an elastic body into an inviscid fluid is studied as a problem of limiting amplitude. By supposing that one knows all about the exterior Neumann-radiation problem and a certain interior traction eigenvalue problem, the interface problem is reduced to an interior one with a nonlocal boundary condition, which is shown to be well-posed in a weak sense, *modulo* the satisfaction of a solvability condition. The convergence of a solution-approximating Galerkin procedure is established. It is shown that one can compute the acoustic field without examining the mentioned eigenvalue problem.

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ANALYSIS OF A LIMITING-AMPLITUDE PROBLEM IN ACOUSTO-ELASTIC INTERACTIONS, I

1. INTRODUCTION

We are concerned here with the correct formulation and the solution of a boundary and interface problem in elliptic partial differential equations that models the physical setting of certain steady-state, or time-harmonic, fluid-elastic interactions. Specifically, we consider an elastic shell-like body immersed in a homogeneous, inviscid fluid of unbounded extent in all directions. The entire continuum is assumed to be driven by the action of certain known, externally applied sources of energy generating an acoustic disturbance propagating in the fluid and forces applied to the surface(s) and the bulk of the elastic body. These sources sustaining the motion are supposed to begin at some instant and thereafter approach harmonic time-dependence, with a common angular frequency, and we assume that the induced fields of response in the fluid and elastic media exhibit the same qualitative sort of temporal behavior as the process evolves. We set the problem of determining the time-harmonic parts of these resultant fields, paying particular attention to the solution for the field in the fluid and its far-field pattern. To this end, we must first ensure that we formulate a mathematical model problem having as its unique solution precisely the desired limiting-amplitude fields and then show how to set up a convergent scheme for their numerical approximation.

Thus, we are examining a simplified version of the problem that must be addressed whenever one wishes to predict the sound field that is both radiated and scattered by, say, a vibrating metal structure submerged in an ocean in which there is also propagating an acoustic wave originating from other sources. That is, the setting described is essentially a first approximation to that prevailing in a broad range of technical questions of fundamental significance to the Navy, including direct problems of design and inverse problems of detection. It is evident that a thorough investigation of the solution of the simplified problem posed will be invaluable as a guide in attacking the study of more realistic models incorporating the effects of spatially varying fluid properties and the presence of air-fluid and earth-fluid interfaces.

While the developments of the subsequent sections are conducted in a systematic manner, to avoid obscuring the basic issues in this introductory section on motivation and orientation we shall be intentionally rather cavalier *vis-à-vis* matters of precision and complete specification of hypotheses. Consider a homogeneous and isotropic elastic medium, with Lamé parameters $\lambda \geq 0$ and $\mu > 0$ and density ϱ_o, occupying the closure $\overline{\Omega}_o$ of a bounded and (for simplicity) connected open set Ω_o in $\mathbf{R}^3$. The restrictions specified for the Lamé parameters are equivalent to the reasonable requirements that Young's modulus be positive and Poisson's ratio be nonnegative and less than one half. We suppose for now that the boundary $\Gamma_o := \partial\Omega_o$ of Ω_o is "smooth," and denote by $\mathbf{n}_o$ the normal field of unit magnitude on Γ_o that is "directed into the exterior of Ω_o." The complement $\mathbf{R}^3 \setminus \overline{\Omega}_o$, which need not be connected, is decomposed into its open connected components and written as $\Omega_+ \cup \Omega_-$, with Ω_+ representing the single unbounded component and Ω_- denoting the union of the (finite number of) bounded components. In case Ω_- is the empty set, or, as we prefer to say here (for a reason that will soon be apparent), "absent," one should make the obvious adjustments in the reasoning. We write $\Gamma_- := \partial\Omega_-$ and $\Gamma_+ := \partial\Omega_+$, so that $\Gamma_o = \Gamma_- \cup \Gamma_+$, and indicate by $\mathbf{n}_-$ and $\mathbf{n}_+$ the restrictions of $\mathbf{n}_o$ to Γ_- and Γ_+, respectively. A homogeneous fluid is supposed to fill Ω_+; while this fluid is modeled as inviscid, it is assumed that it may possess some mechanism for internal damping, so that the

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propagation of small disturbances in the fluid is governed by the "damped" wave equation. The density of the quiescent fluid is designated by ϱ_+, its characteristic phase speed of small disturbances by c_+, and the damping constant by γ_+, with $\gamma_+ \geq 0$. Actually, we shall carry along the hypothesis that γ_+ can take on certain complex values by supposing that

$$\gamma_+ \in \{ z \in \mathbf{C} \mid \text{either } z = 0 \text{ or } \operatorname{Re} z > 0 \},$$

although an appropriate physical interpretation is not clear when $\operatorname{Im} \gamma_+ \neq 0$. The set Ω_-, accounting for any "cavities" within the elastic medium, is taken as evacuated (*i.e.*, containing no material); if some other continuum were postulated to lie in Ω_-, there would be introduced certain additional complexities that we wish to avoid in the present exposition.

Prior to some time t_0, say, $t_0 = 0$, we suppose that the system rests in a state of equilibrium in which the pressure in the fluid has the constant value p_0 throughout Ω_+ and no body-force field is applied to any portion of either medium. Then, reckoned relative to the *in vacuo* position of the elastic body, the static, or quiescent, elastic displacement field $\mathbf{U}_0 : \Omega_o \to \mathbf{R}^3$ satisfies

$$\mathbf{\Delta}^{\star}_{\lambda,\mu}\mathbf{U}_0 = \mathbf{0}, \qquad \text{in } \Omega_o, \tag{1.1}$$

$$\mathbf{T}^{\mathbf{n}_+}[\mathbf{U}_0] = -p_0\mathbf{n}_+, \qquad \text{on } \Gamma_+, \tag{1.2}$$

and

$$\mathbf{T}^{\mathbf{n}_-}[\mathbf{U}_0] = \mathbf{0}, \qquad \text{on } \Gamma_- \tag{1.3}$$

(conditions serving to characterize $\mathbf{U}_0$ only to within an added rigid-displacement field). Here, the operator $\mathbf{\Delta}^{\star}_{\lambda,\mu}$ is given by

$$\mathbf{\Delta}^{\star}_{\lambda,\mu} := -\mu \operatorname{curl}\operatorname{curl} + (\lambda + 2\mu)\operatorname{grad}\operatorname{div};$$

in terms of the Cartesian components (U_j) of a vector-valued field $\mathbf{U}$, the Cartesian components of $\mathbf{\Delta}^{\star}_{\lambda,\mu}\mathbf{U}$ then appear as

$$(\mathbf{\Delta}^{\star}_{\lambda,\mu}\mathbf{U})_j = \mu\Delta U_j + (\lambda + \mu)U_{k,kj},$$

with $\Delta := \operatorname{div}\operatorname{grad}$ denoting the Laplacian (acting on scalar fields) and a comma signifying partial differentiation with respect to the Cartesian coördinates indicated by the following subscripts. Here and in the sequel, the conventions of the index notation are in force, with the range of all free and summed indices to be understood as $\{1, 2, 3\}$. By introducing the (Cartesian components of the) strain tensor corresponding to a field $\mathbf{U}$ in Ω_o according to

$$\varepsilon_{jk}[\mathbf{U}] := \frac{1}{2}(U_{j,k} + U_{k,j}) \tag{1.4}$$

and, in turn, the (Cartesian components of the) stress tensor generated by $\mathbf{U}$ as

$$\sigma_{jk}[\mathbf{U}] := \lambda\varepsilon_{ll}[\mathbf{U}]\delta_{jk} + 2\mu\varepsilon_{jk}[\mathbf{U}], \tag{1.5}$$

δ_{jk} being the Kronecker symbol, *the traction field* $\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}]$ *induced by* $\mathbf{U}$ *on* Γ_o is defined as having the Cartesian components given by

$$(\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}])_j := \sigma_{jk}[\mathbf{U}]|_{\Gamma_o} n_{ok}, \tag{1.6}$$

(n_{o1}, n_{o2}, n_{o3}) denoting Cartesian components of $\mathbf{n}_o$, when the traces $\sigma_{jk}[\mathbf{U}]|_{\Gamma_o}$ exist in an appropriate sense. Recall that $\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}]$ represents physically the tractions exerted on the elastic material in $\overline{\Omega}_o$ by agents lying outside that set. In (1.2) and (1.3) we have indicated by $\mathbf{T}^{\mathbf{n}_+}[\mathbf{U}_0]$ and $\mathbf{T}^{\mathbf{n}_-}[\mathbf{U}_0]$ the restrictions of $\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}_0]$ to Γ_+ and to Γ_-, respectively.

Subsequent to the time $t_0 = 0$, we assume that various externally impressed disturbing agents begin smoothly to act on, and in a neighborhood of, the elastic body, *viz.*, tractions applied to both the vacuum-elastic boundary Γ_- and the fluid-elastic interface Γ_+, a body-force field within Ω_o, and an "incident field" propagating in the fluid contained in Ω_+. Henceforth, we shall consider all scalar fields as $\mathbf{C}$-valued ($\mathbf{C}$ denoting the complex numbers), all vector fields as $\mathbf{C}^3$-valued; the corresponding fields of physical significance are then to be obtained by taking the real parts (when γ_+ is real). Points of $\mathbf{R}^4$ we write in the form $(\mathbf{x}, x_4)$, with $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ and $x_4 \in \mathbf{R}$; the operators curl, grad, and div, when applied to appropriate fields defined on a cylinder such as $\Omega_o \times \mathbf{R}$ in $\mathbf{R}^4$, shall be interpreted as acting in the "spatial variables" only. Thus, we are given vector-valued fields $\mathbf{T}_o : \Gamma_o \times \mathbf{R} \to \mathbf{C}^3$ and $\mathbf{F}_o : \Omega_o \times \mathbf{R} \to \mathbf{C}^3$, smooth in their fourth arguments, and satisfying $\mathbf{T}_o(\cdot, t) = 0$ on Γ_o and $\mathbf{F}_o(\cdot, t) = 0$ in Ω_o for $t < 0$. Consistently, as we have done for $\mathbf{n}_o$ and $\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}_0]$, when f_o is some (scalar- or vector-valued) field on Γ_o, we shall denote by f_- and f_+ its restrictions to Γ_- and Γ_+, respectively, *i.e.*, $f_\pm := f_o|\Gamma_\pm$, the notation $g|F$ being reserved throughout to indicate the restriction of the function g to the subset F of its domain. Thus, we write, *e.g.*, $\mathbf{T}_-(\cdot, t) := \mathbf{T}_o(\cdot, t)|\Gamma_-$ and $\mathbf{T}_+(\cdot, t) := \mathbf{T}_o(\cdot, t)|\Gamma_+$. To describe the fluid incident field, we suppose that there is specified an open subset Ω^ι of $\mathbf{R}^3$ containing the elastic body along with its cavities, *i.e.*, containing $\Omega_- \cup \overline{\Omega}_o$, and a smooth scalar field $\Phi^\iota : \Omega^\iota \times \mathbf{R} \to \mathbf{C}$, the velocity potential of a disturbance that would propagate in Ω^ι if no elastic medium were present and the fluid were to fill all of Ω^ι. Then Φ^ι is to satisfy $\Phi^\iota(\cdot, t) = 0$ in Ω^ι whenever $t < 0$ and

$$-\Delta\Phi^\iota + \frac{\gamma_+}{c_+^2}\Phi^\iota_{,4} + \frac{1}{c_+^2}\Phi^\iota_{,44} = 0, \qquad \text{in } \Omega^\iota \times \mathbf{R}. \tag{1.7}$$

We are implying here that any "sources" sustaining the disturbance represented by Φ^ι are situated outside Ω^ι (and so, in particular, outside the elastic medium). In terms of this velocity potential, the incident particle-velocity field $\mathbf{V}^\iota$ and the incident acoustic, or excess, pressure field P^ι are defined in $\Omega^\iota \times \mathbf{R}$ by

$$\mathbf{V}^\iota := -\frac{1}{\varrho_+}\operatorname{grad}\Phi^\iota \tag{1.8}$$

(again, with the gradient operator acting in the spatial variables alone), and

$$P^\iota := \Phi^\iota_{,4} + \gamma_+\Phi^\iota; \tag{1.9}$$

we shall use the corresponding rules of calculation when defining the velocity and pressure fields induced in the fluid by any other velocity potential, as well.

In the full space-time formulation, for the determination of the resultant motion of the entire continuum in response to these specified disturbances, we would then wish to find a vector (elastic-displacement) field $\mathbf{U} : \Omega_o \times \mathbf{R} \to \mathbf{C}^3$ and a scalar (fluid-velocity-potential) field $\Phi : \Omega_+ \times \mathbf{R} \to \mathbf{C}$ satisfying, in an appropriate sense, the hyperbolic partial differential equations

$$-\Delta\Phi + \frac{\gamma_+}{c_+^2}\Phi_{,4} + \frac{1}{c_+^2}\Phi_{,44} = 0, \qquad \text{in } \Omega_+ \times \mathbf{R}, \tag{1.10}$$

and

$$-\boldsymbol{\Delta}^*_{\lambda,\mu}\mathbf{U} + \varrho_o\mathbf{U}_{,44} = \mathbf{F}_o, \qquad \text{in } \Omega_o \times \mathbf{R}, \tag{1.11}$$

the initial conditions

$$\left.\begin{aligned} \Phi(\cdot, t) &= 0 \qquad \text{in } \Omega_+ \\ \mathbf{U}(\cdot, t) &= \mathbf{0} \qquad \text{in } \Omega_o \end{aligned}\right\} \quad \text{for } t < 0, \tag{1.12}$$

the interface conditions

$$\left.\begin{aligned} \Phi_{,\mathbf{n}_+} + \varrho_+ \mathbf{U}_{,4}|_{\Gamma_+} \cdot \mathbf{n}_+ &= -\Phi^{\iota}_{,\mathbf{n}_+}, \\ \mathbf{T}^{\mathbf{n}_+}[\mathbf{U}] + (\Phi_{,4}|_{\Gamma_+} + \gamma_+ \Phi|_{\Gamma_+})\mathbf{n}_+ &= \mathbf{T}_+ - (\Phi^{\iota}_{,4}|_{\Gamma_+} + \gamma_+ \Phi^{\iota}|_{\Gamma_+})\mathbf{n}_+, \end{aligned}\right\} \quad \text{on } \Gamma_+ \times \mathbf{R}, \tag{1.13}$$

and the boundary condition

$$\mathbf{T}^{\mathbf{n}_-}[\mathbf{U}] = \mathbf{T}_-, \qquad \text{on } \Gamma_- \times \mathbf{R}. \tag{1.14}$$

Here, $(\cdot)|_{\Gamma_+}$ denotes a trace, and $(\cdot)_{,\mathbf{n}_+}$ denotes a normal derivative on Γ_+ (taken with respect to the spatial variables at a fixed time), while $\mathbf{a} \cdot \mathbf{b} := a_k b_k$ for elements $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ of $\mathbf{C}^3$. The equalities in (1.10) through (1.14) result upon first setting down the conditions to be fulfilled by the state of the medium when the elastic-displacement field is calculated relative to the *in vacuo* position of the elastic body and then employing (1.1) through (1.3) to remove the appearance of the state of prestress induced by the constant pressure p_0. Thus, $\mathbf{U}$ represents the perturbation-displacement field of the elastic medium, taken relative to its quiescent position of equilibrium at times preceding t_0, the displacements relative to its *in vacuo* position being given by $\mathbf{U} + \mathbf{U}_0$, with $\mathbf{U}_0$ as described above. The velocity potential Φ has no intrinsic physical interpretation when Φ^{ι} is nonzero. Rather, it is the sum $\Phi + \Phi^{\iota}$ in $(\Omega_+ \cap \Omega^{\iota}) \times \mathbf{R}$ that describes the disturbance propagating in the fluid (the state of the fluid being obtained by superposition of this perturbation onto the quiescent state with constant pressure p_0); when there are neither tractions $\mathbf{T}_o$ impressed upon Γ_o nor a body force $\mathbf{F}_o$ acting in Ω_o, it is customary to refer to Φ as the velocity potential of the "scattered field" corresponding to the incident-field velocity potential Φ^{ι}. The fulfilment of the first of conditions (1.13) ensures the continuity of the normal component of the continuum velocity on Γ_+ (which is the most that can be demanded, in view of the inviscid nature of the fluid), while the satisfaction of the second condition in (1.13) provides for the continuity of the continuum tractions on Γ_+. Of course, with $\mathbf{a} \times \mathbf{b}$ denoting the vector product of $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{C}^3$, from (1.13) we have

$$\mathbf{n}_+ \times \mathbf{T}^{\mathbf{n}_+}[\mathbf{U}] = \mathbf{n}_+ \times \mathbf{T}_+, \qquad \text{on } \Gamma_+ \times \mathbf{R},$$

so that the tangential component of $\mathbf{T}^{\mathbf{n}_+}[\mathbf{U}]$ on Γ_+ vanishes if $\mathbf{T}_+$ is a normal field on Γ_+; this result is also due to the inviscidity of the fluid.

Now, to pose the problem of present interest, let us consider the special circumstance in which the impressed traction and body-force fields asymptotically approach harmonic time-dependence, of angular frequency $\omega > 0$, on the respective spatial parts of their domains of definition as $t \to \infty$, while the fluid-incident-field velocity potential displays such behavior in a neighborhood of $\Omega_- \cup \overline{\Omega}_o$. In such a case, we can write

$$\mathbf{T}_o(\cdot, t) = \mathbf{T}_o^{\tau}(\cdot, t) + \mathbf{t}_o(\cdot)e^{-i\omega t} \qquad \text{on } \Gamma_o \text{ for } t \in \mathbf{R}, \tag{1.15}$$

$$\mathbf{F}_o(\cdot, t) = \mathbf{F}_o^{\tau}(\cdot, t) + \mathbf{f}_o(\cdot)e^{-i\omega t} \qquad \text{in } \Omega_o \text{ for } t \in \mathbf{R}, \tag{1.16}$$

and

$$\Phi^{\iota}(\cdot, t) = \Phi^{\iota\tau}(\cdot, t) + \varphi^{\iota}(\cdot)e^{-i\omega t} \qquad \text{in } \tilde{\Omega}^{\iota} \text{ for } t \in \mathbf{R}, \tag{1.17}$$

wherein $\Omega_- \cup \overline{\Omega}_o \subset \tilde{\Omega}^{\iota} \subset \Omega^{\iota}$, and the functions $\mathbf{T}_o^{\tau}(\cdot, t)$, $\mathbf{F}_o^{\tau}(\cdot, t)$, and $\Phi^{\iota\tau}(\cdot, t)$, along with sufficiently many of their derivatives, vanish on their respective domains in the limit as $t \to \infty$, while $\mathbf{t}_o$, $\mathbf{f}_o$, and φ^{ι} are certain known (spatially dependent) complex amplitudes. Again, we write $\mathbf{T}_{\pm}^{\tau}(\cdot, t) := \mathbf{T}_o^{\tau}(\cdot, t)|\Gamma_{\pm}$ and $\mathbf{t}_{\pm} := \mathbf{t}_o|\Gamma_{\pm}$.

Then, from (1.7), under appropriate conditions of transience of this sort for $\Phi^{\iota T}$ and of smoothness for φ^{ι}, the latter function must satisfy

$$\Delta\varphi^{\iota} + \left(\frac{\omega(\omega + i\gamma_+)}{c_+^2}\right)\varphi^{\iota} = 0 \qquad \text{in } \tilde{\Omega}^{\iota},$$

or

$$\Delta\varphi^{\iota} + \kappa_+^2\varphi^{\iota} = 0 \qquad \text{in } \tilde{\Omega}^{\iota}, \tag{1.18}$$

in which

$$\kappa_+ \equiv \kappa_+(\omega; c_+, \gamma_+) := \sqrt{\frac{\omega(\omega + i\gamma_+)}{c_+^2}}. \tag{1.19}$$

Here and throughout, $(\cdot)^{1/2} \equiv \sqrt{(\cdot)}$ denotes the principal branch of the square-root function: when z is real and nonnegative, $z^{1/2}$ is the nonnegative square root of z, and then, in general, $z^{1/2} := |z|^{1/2}\exp(i \arg z/2)$, with $\arg z \in (-\pi, \pi]$ indicating the principal argument of $z \in \mathbf{C}$. Thus, $\kappa_+(\omega; c_+, 0) = \omega/c_+$. If $\operatorname{Re}\gamma_+ > 0$, then the principal argument of $\omega(\omega + i\gamma_+)/c_+^2$ lies in $(0, \pi)$, since

$$\omega(\omega + i\gamma_+) = \omega(\omega - \operatorname{Im}\gamma_+) + i\omega\operatorname{Re}\gamma_+,$$

and so its principal square root $\kappa_+(\omega; c_+, \gamma_+)$ has positive real and imaginary parts. Explicitly, setting

$$R_+ \equiv R_+(\omega, \gamma_+) := \frac{\operatorname{Re}\gamma_+}{(\omega - \operatorname{Im}\gamma_+)} \qquad \text{if } (\omega - \operatorname{Im}\gamma_+) \neq 0,$$

we find that

$$\kappa_+(\omega; c_+, \gamma_+) = \frac{1}{c_+}\sqrt{\frac{\omega(\omega - \operatorname{Im}\gamma_+)}{2}}\left\{\{1 + [1 + R_+^2]^{1/2}\}^{1/2} + \frac{iR_+}{\{1 + [1 + R_+^2]^{1/2}\}^{1/2}}\right\}$$

$$\text{if } (\omega - \operatorname{Im}\gamma_+) > 0, \tag{1.20$_1$}$$

$$\kappa_+(\omega; c_+, \gamma_+) = \frac{1}{c_+}\sqrt{\frac{\omega(\operatorname{Im}\gamma_+ - \omega)}{2}}\left\{\frac{-R_+}{\{1 + [1 + R_+^2]^{1/2}\}^{1/2}} + i\{1 + [1 + R_+^2]^{1/2}\}^{1/2}\right\}$$

$$\text{if } (\omega - \operatorname{Im}\gamma_+) < 0, \tag{1.20$_2$}$$

and

$$\kappa_+(\omega; c_+, \gamma_+) = \frac{(1 + i)}{c_+}\sqrt{\frac{\omega\operatorname{Re}\gamma_+}{2}} \qquad \text{if } (\omega - \operatorname{Im}\gamma_+) = 0. \tag{1.20$_3$}$$

In view of (1.8) and (1.9), $\mathbf{V}^{\iota}$ and P^{ι} then have in $\tilde{\Omega}^{\iota} \times \mathbf{R}$ the same form of a time-transient superimposed upon a time-harmonic contribution, the complex amplitude of the latter deriving from φ^{ι}: for every t, in $\tilde{\Omega}^{\iota}$ we have

$$\mathbf{V}^{\iota}(\cdot, t) = -\frac{1}{\varrho_+}\operatorname{grad}\Phi^{\iota T}(\cdot, t) - \frac{1}{\varrho_+}\operatorname{grad}\varphi^{\iota}(\cdot)e^{-i\omega t}$$

and

$$P^{\iota}(\cdot, t) = \Phi^{\iota T}_{,4}(\cdot, t) + \gamma_+\Phi^{\iota T}(\cdot, t) + (\gamma_+ - i\omega)\varphi^{\iota}(\cdot)e^{-i\omega t}.$$

Pursuant to these hypotheses, and supposing that there exists a corresponding unique pair $(\mathbf{U}, \Phi)$ satisfying (1.10) through (1.14), we further make the fundamental assumption that the latter functions have the forms

$$\mathbf{U}(\cdot, t) = \mathbf{U}^{\tau}(\cdot, t) + \mathbf{u}(\cdot)e^{-i\omega t} \qquad \text{in } \Omega_o \text{ for } t \in \mathbf{R} \tag{1.21}$$

and

$$\Phi(\cdot, t) = \Phi^{\tau}(\cdot, t) + \varphi(\cdot)e^{-i\omega t} \qquad \text{in } \Omega_+ \text{ for } t \in \mathbf{R}, \tag{1.22}$$

wherein $\mathbf{U}^{\tau}$ and Φ^{τ} are "sufficiently weak transients," *i.e.*, vanish, along with certain of their derivatives, in a sufficiently strong sense as $t \to \infty$. In this setting, our objective is the determination of the assumed-to-exist "complex limiting amplitudes" $\mathbf{u}$ in Ω_o and φ in Ω_+; in particular, we wish first to formulate a boundary and interface problem for elliptic partial differential equations in $\Omega_o \cup \Omega_+$ that completely and directly characterizes the latter spatially dependent fields, to determine when this problem is well-posed in a reasonable sense, and subsequently to show how one can construct sequences of approximations converging in a useful manner to the solution (when the problem is well-posed). As a preview, we note at this point that the desire to account for the effect of driving fields such as $\mathbf{T}_o$ and $\mathbf{F}_o$ introduces into the correct formulation of the problem governing the steady-state behavior peculiar features that are not of concern when it is supposed that only a fluid-incident-field velocity potential Φ^{ι} is present to sustain the motion. For example, one would hope that the data of the correct steady-state problem comprise merely the limiting amplitudes $\mathbf{t}_o$, $\mathbf{f}_o$, and φ^{ι}, along with the angular frequency ω, the underlying geometry, and the material (constitutive) parameters, *i.e.*, that the limiting amplitudes $\mathbf{u}$ and φ can be characterized without knowledge of the transients $\mathbf{T}_o^{\tau}$, $\mathbf{F}_o^{\tau}$, and $\Phi^{\iota\tau}$ acting during the evolution from the quiescent state. Perhaps surprisingly, this is evidently not the case; in general, one must know $\mathbf{T}_o^{\tau}$ and $\mathbf{F}_o^{\tau}$, although $\Phi^{\iota\tau}$ is not required. We shall return to this matter shortly.

It is necessary to proceed through the examination of the original space-time formulation to identify the correct time-independent problem to be studied for the determination of the limiting amplitudes of the response, essentially because there is a possible loss of the unique-solution property in a too-naïve passage from the evolution problem to the time-harmonic setting. One must retain the unique-solution property in such a way as to force the solution of the steady-state problem to provide precisely the limiting amplitudes sought; when this is done, we naturally refer to the resultant formulation as a *limiting-amplitude problem*. But this difficulty is not of an unfamiliar sort. Indeed, the situation is analogous to that obtaining in the simpler case of acoustic radiation or scattering in the exterior of, say, a rigid body, wherein the Sommerfeld radiation condition (cf. (1.34), *infra*) is the appropriate requirement to be imposed for the purpose of picking out, from among all solutions of the Helmholtz equation in Ω_+ that possess the required Neumann (normal-derivative) data on Γ_+, precisely that function that represents the amplitude of the time-harmonic motion asymptotically approached from the quiescent state. Thus, the Sommerfeld condition can be referred to as the "(exterior) limiting-amplitude condition" for that case. One should consult in this regard the fundamental work of Wilcox [1] for the developments pertinent to that more familiar setting. The present problem is more complex, requiring an "interior limiting-amplitude condition," as well as an exterior one. Indeed, by merely setting down the necessary conditions on $\mathbf{u}$ and φ that evidently result from (1.10), (1.11), (1.13), and (1.14) in consequence of the (sufficiently regular) assumed forms, one arrives at the following collection of requirements (cf., also, Theorem 5.1 and the remarks following the proof of Theorem 5.2, *infra*):

$$\Delta\varphi + \kappa_+^2\varphi = 0, \qquad \text{in } \Omega_+, \tag{1.23}$$

$$\mathbf{\Delta}^{\star}_{\lambda,\mu}\mathbf{u} + \varrho_o\omega^2\mathbf{u} = -\mathbf{f}_o, \qquad \text{in } \Omega_o, \tag{1.24}$$

$$\left.\begin{aligned} \varphi_{,\mathbf{n}_+} - i\varrho_+\omega\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+ &= -\varphi^{\iota}_{,\mathbf{n}_+}, \\ \mathbf{T}^{\mathbf{n}_+}[\mathbf{u}] + (\gamma_+ - i\omega)\varphi|_{\Gamma_+}\mathbf{n}_+ &= \mathbf{t}_+ - (\gamma_+ - i\omega)\varphi^{\iota}|_{\Gamma_+}\mathbf{n}_+, \end{aligned}\right\} \qquad \text{on } \Gamma_+, \tag{1.25}$$

and

$$\mathbf{T}^{\mathbf{n}_-}[\mathbf{u}] = \mathbf{t}_-, \qquad \text{on } \Gamma_-. \tag{1.26}$$

However, one can already see that this problem need not be well posed, in particular, need not possess at most one solution. For example, suppose that there is a nontrivial field $\mathbf{u}_\omega : \Omega_o \to \mathbf{C}^3$ such that

$$\boldsymbol{\Delta}^\star_{\lambda,\mu}\mathbf{u}_\omega + \varrho_o\omega^2\mathbf{u}_\omega = \mathbf{0}, \qquad \text{in } \Omega_o, \tag{1.27}$$

with

$$\mathbf{u}_\omega|_{\Gamma_+} \cdot \mathbf{n}_+ = 0, \qquad \text{on } \Gamma_+, \tag{1.28}$$

and

$$\mathbf{T}^{\mathbf{n}_o}[\mathbf{u}_\omega] = \mathbf{0}, \qquad \text{on } \Gamma_o. \tag{1.29}$$

Then, obviously, the nontrivial pair $(\mathbf{u} = \mathbf{u}_\omega, \varphi = 0)$ satisfies (1.23) and the homogeneous versions of (1.24) through (1.26). In fact, for certain domains Ω_o and frequencies ω, there do exist nontrivial $\mathbf{u}_\omega$ satisfying (1.27) through (1.29). When Ω_o is a ball (so Ω_- is absent) or a spherical annulus (so Ω_- is a ball), such fields can be constructed by means of separation-of-variables; cf. Love [2] for the former case. The eigenfunctions $\mathbf{u}_\omega$ in question for either of these spherical geometries correspond to vibrations in which the particles of the body move on spheres concentric with the boundary. Some of these motions are "purely rotatory," with all particles oscillating along arcs of circles lying in planes normal to, and having centers on, a single axis passing through the center of the body; it is reasonable to anticipate that modes of this simple type can be supported by any body of revolution, although we have not yet worked through the analysis of the pertinent eigenvalue problem. For other classically familiar geometries in which separation-of-variables-type arguments are applicable, see Ref. 3, from which the appropriate computations can be constructed on the basis of a reformulation of the eigenvalue problem governing the purely rotatory modes for a certain class of shapes of revolution. For example, Ref. 3 provides some of the details necessary in the study of this problem for ellipsoids of revolution. It would be very useful to prove (as intuition suggests) that the class of bodies of revolution is precisely the collection of domains Ω_o for which such nontrivial eigenfunctions can exist, since the entire development that we shall give is much simplified when it is known that fields of this type cannot be supported by the particular Ω_o under consideration. In the absence of such a proof, and since, in any event, we wish to establish results that shall be valid for the important class of bodies of revolution, we shall proceed under the assumption that there may be nontrivial $\mathbf{u}_\omega$ satisfying (1.27) through (1.29) for the chosen Ω_o and ω. Such a field $\mathbf{u}_\omega$ can be termed the "complex amplitude of a nonradiating mode," since the corresponding elastic-displacement field $\mathbf{U}_\omega$ given by

$$\mathbf{U}_\omega(\cdot, t) := \mathbf{u}_\omega(\cdot)e^{-i\omega t} \qquad \text{in } \Omega_o \text{ for } t \in \mathbf{R} \tag{1.30}$$

then churns about in the elastic medium in Ω_o but has no effect on the inviscid fluid; it is completely uncoupled from the fluid at the interface Γ_+, and so, in particular, induces no outward radiation of acoustic energy. No measurement made in the fluid will serve to detect its presence in the elastic body.

Conceivably, there are more complex situations in which (1.23) through (1.26) do not suffice to determine at most one pair $(\mathbf{u}, \varphi)$. To see how such a situation might come about, let us suppose that Ω_o is, say, a Lipschitz domain with, for simplicity, Ω_- absent (so $\Gamma_o = \Gamma_+$), and the positive ω is such that the second fundamental boundary-value problem of steady-state elastic vibrations of angular frequency ω for Ω_o, λ, μ, and ϱ_o is uniquely solvable in a strong sense for any boundary data chosen from some linear space $\tilde{\mathbf{H}}(\Gamma_+)$ of $\mathbf{C}^3$-valued functions defined almost everywhere (a.e.) on Γ_+. Thus, we assume that whenever $\mathbf{t} \in \tilde{\mathbf{H}}(\Gamma_+)$, there exists a unique corresponding field $\mathbf{u}_\mathbf{t}$ in the Sobolev space $\mathbf{H}^2(\Omega_o) := H^2(\Omega_o)^3$ satisfying

$$\boldsymbol{\Delta}^\star_{\lambda,\mu}\mathbf{u}_\mathbf{t} + \varrho_o\omega^2\mathbf{u}_\mathbf{t} = \mathbf{0} \qquad \text{a.e. in } \Omega_o \tag{1.31}$$

and

$$\mathbf{T}^{\mathbf{n}_o}[\mathbf{u_t}] = \mathbf{t} \qquad \text{a.e. on } \Gamma_+, \tag{1.32}$$

the latter condition being fulfilled in the Sobolev-trace sense; for more on the notations $\mathbf{H}^2(\Omega_o)$, *etc.*, and terminology being used here, cf. §2, *infra*. Note that $\tilde{\mathbf{H}}(\Gamma_+)$ must lie in the fractional-order Sobolev space $\mathbf{H}^{\frac{1}{2}}(\Gamma_+) := H^{\frac{1}{2}}(\Gamma_+)^3$ associated with the boundary Γ_+. Parenthetically we note that, by interior elliptic-regularity results, each $\mathbf{u_t}$ must have real-analytic real and imaginary parts, and so (1.31) is actually satisfied at each point of Ω_o. The trace $\mathbf{u_t}|_{\Gamma_+}$ is then automatically determined as an element of $\mathbf{H}^{\frac{1}{2}}(\Gamma_+)$ by $\mathbf{t}$, so that there is defined, as a result of the assumed strong unique solvability, an injective linear operator $\mathcal{B}_\omega : \tilde{\mathbf{H}}(\Gamma_+) \to \mathbf{H}^{\frac{1}{2}}(\Gamma_+)$:

$$\mathcal{B}_\omega \mathbf{t} := \mathbf{u_t}|_{\Gamma_+} \quad (= \mathcal{B}_\omega \mathbf{T}^{\mathbf{n}_+}[\mathbf{u_t}]) \qquad \text{whenever } \mathbf{t} \in \tilde{\mathbf{H}}(\Gamma_+).$$

Now suppose that there exists a nontrivial function $\hat{\varphi} \in C^2(\Omega_+)$ that possesses, in some reasonable sense, a trace $\hat{\varphi}|_{\Gamma_+}$ and a normal derivative $\hat{\varphi}_{,\mathbf{n}_+}$ in the Lebesgue space $L_2(\Gamma_+)$ (*e.g.*, suppose that $\hat{\varphi}$ is in $H^2_{loc}(\overline{\Omega}_+)$, and interpret the trace and normal derivative in the Sobolev sense), with $\hat{\varphi}|_{\Gamma_+}\mathbf{n}_+ \in \tilde{\mathbf{H}}(\Gamma_+)$, and is a solution of a nonlinear and nonlocal eigenvalue problem for $-\Delta$ in Ω_+, specifically, that it satisfies

$$\Delta\hat{\varphi} + \kappa_+^2 \hat{\varphi} = 0, \qquad \text{in } \Omega_+,$$

along with the homogeneous (nonlocal) boundary condition

$$\hat{\varphi}_{,\mathbf{n}_+} + \varrho_+ c_+^2 \kappa_+^2 \mathbf{n}_+ \cdot \mathcal{B}_\omega(\hat{\varphi}|_{\Gamma_+}\mathbf{n}_+) = 0. \tag{1.33}$$

Then, taking $\hat{\mathbf{u}} := \mathbf{u}_{\hat{\mathbf{t}}}$, wherein $\hat{\mathbf{t}} := -(\gamma_+ - i\omega)\hat{\varphi}|_{\Gamma_+}\mathbf{n}_+$, we obtain a nontrivial pair $(\hat{\mathbf{u}}, \hat{\varphi})$ satisfying (1.23) and the homogeneous versions of (1.24) through (1.26), as can be checked. One would expect that this sort of eventuality could be ruled out with the additional imposition of the Sommerfeld condition. We remark that a generalized exterior Robin problem for (1.23) and a boundary condition of a form corresponding to the nonhomogeneous version of (1.33) is studied in Refs. 4 and 5 (Ref. 5 being a revised and somewhat amplified version of Ref. 4), under certain restrictions on the operator appearing in the boundary condition. In that case, it is found that the Sommerfeld condition does serve to prohibit the existence of a nontrivial eigenfunction of the sort that we have just described.

At any rate, it is evident that some conditions augmenting (1.23) through (1.26) are wanted before there is a chance that the resultant problem will be well posed. Keeping in mind the hypothesized origin of the fields being sought here, we should demand that these additional requirements also follow from the original initial-value problem, as conditions necessarily fulfilled by assumed-to-exist complex limiting amplitudes of the elastic and fluid perturbation fields; cf. the assumptions on the forms in (1.21) and (1.22). To see more specifically what is involved, let us formulate two requirements that we might consider imposing in addition to (1.23) through (1.26), *viz.*, for the exterior field, the Sommerfeld radiation condition,

$$\lim_{r\to\infty} r\{\hat{\mathbf{e}} \cdot \operatorname{grad}\varphi(r\hat{\mathbf{e}}) - i\kappa_+\varphi(r\hat{\mathbf{e}})\} = 0 \qquad \text{uniformly for } \hat{\mathbf{e}} \in S_1, \tag{1.34}$$

with S_1 denoting the boundary of the unit ball in $\mathbf{R}^3$, and, for the interior field, the orthogonality condition

$$\int_{\Omega_o} \mathbf{u} \cdot \overline{\mathbf{u}}_\omega \, d\lambda_3 = 0 \qquad \text{for every } \mathbf{u}_\omega \text{ satisfying (1.27) through (1.29)}, \tag{1.35}$$

an overbar signifying complex conjugation and λ_3 denoting the Lebesgue measure on $\mathbf{R}^3$; the precise sense(s) in which (1.27) through (1.29) are to be satisfied in (1.35) (and in (1.36), *infra*) shall be specified later. Now, one can show, as we shall in §2, that the problem of finding a pair $(\mathbf{u}, \varphi)$ satisfying (1.23) through (1.26), (1.34), and (1.35) can have at most one solution in a reasonably large class of pairs, each comprising an interior vector field and an exterior scalar field. But this is not sufficient to fulfill our stated objective; we must also decide whether (1.34) and (1.35) are ***limiting-amplitude conditions***, *i.e.*, whether they are necessarily satisfied by our hypothesized complex limiting amplitudes figuring in the time-harmonic states approached by fields evolving from a quiescent state under the influence of driving agents as in (1.15) through (1.17). In fact, (1.35) does n . in general meet this criterion. Rather, evidently one should require that

$$\int_{\Omega_o} \mathbf{u} \cdot \overline{\mathbf{u}}_\omega \, d\lambda_3 = \frac{i}{2\varrho_o \omega} \int_0^\infty \left\{ \int_{\Gamma_o} \mathbf{T}_o^\tau(\cdot, s) \cdot \overline{\mathbf{u}}_\omega|_{\Gamma_o} \, d\lambda_{\Gamma_o} + \int_{\Omega_o} \mathbf{F}_o^\tau(\cdot, s) \cdot \overline{\mathbf{u}}_\omega \, d\lambda_3 \right\} e^{i\omega s} \, ds$$

for every $\mathbf{u}_\omega$ satisfying (1.27) through (1.29). (1.36)

One of the goals of this report is to supply motivation for the claim that (1.34) and (1.36) are, respectively, the correct exterior and interior limiting-amplitude conditions to be imposed, by proving that they must obtain at least under sufficiently strong hypotheses concerning the transients in (1.15) through (1.17), (1.21), and (1.22). Once this has been established, (1.36) will imply that the interior limiting amplitude $\mathbf{u}$ is not independent of the particular manners by which asymptotic approaches to steady-state evolve under the influence of various $\mathbf{T}_o$ and $\mathbf{F}_o$ differing only in their transient parts. The reason for this circumstance resides in the fact that the transient parts of $\mathbf{T}_o$ and $\mathbf{F}_o$, while dying away, nevertheless may leave a vestige of their action by inducing a nonzero $L_2(\Omega_o)^3$-orthogonal projection of the resultant limiting amplitude $\mathbf{u}$ onto the subspace of solutions of (1.27) through (1.29), and different transient components may induce different projections. It is trivial to see that this is so for the case in which $\mathbf{T}_o = \mathbf{0}$ and $\mathbf{F}_o = \mathbf{F}_o^\tau \neq \mathbf{0}$. While the proof of the assertion in the case $\mathbf{T}_o = \mathbf{T}_o^\tau \neq \mathbf{0}$ and $\mathbf{F}_o = \mathbf{0}$ requires much more work, we choose not to provide the reasoning in the present report.

There is an additional consideration necessitated by the presence of the forcing tractions $\mathbf{T}_o$ and body force $\mathbf{F}_o$. We describe this by making some remarks that shall remain for the present in the nature of a plausible conjecture, since we shall not prove them here. Consider for a moment the driving of the elastic body in Ω_o, but *in vacuo*, from a quiescent state by means of (bounded!) applied tractions and body force as in (1.15) and (1.16): in this case, when ω has any one of an infinite, discrete set of positive values corresponding to the eigenvalues of the traction problem for $-\mathbf{\Delta}^*_{\lambda,\mu}$ in Ω_o, a "resonance" must be expected, *i.e.*, one or more of the pertinent eigenmodes will be excited and the displacement response of the body will not be bounded in $\Omega_o \times (0, \infty)$, so certainly a time-harmonic state will not be approached asymptotically. Now, when the elastic body is surrounded by fluid, evidently the situation with regard to possible resonances is significantly altered, for one would expect that the fluid acts as an energy sink into which the body suffers losses by working at the interface. Consequently, even when ω corresponds to one of the eigenvalues just specified, it can be anticipated that the tendency to resonate owing to excitement of a traction eigenmode will be suppressed to the extent that the response will remain bounded unless the body possesses such a mode that can exchange no energy with the fluid, ω corresponds to the eigenvalue for that mode, and $\mathbf{t}_o$ and $\mathbf{f}_o$ fail to satisfy a certain orthogonality condition relative to that mode. When, as in our case, the fluid is inviscid, the special eigenfunctions in question are just those (nontrivial) $\mathbf{u}_\omega$ satisfying (1.27) through (1.29) for some $\omega > 0$. Formal computations based on Laplace transformation indicate that a condition on $\mathbf{t}_o$ and $\mathbf{f}_o$ that is necessary and sufficient to ensure that no resonance be induced by excitement of any such (nonradiating) mode appears as

$$\int_{\Gamma_o} \mathbf{t}_o \cdot \overline{\mathbf{u}}_\omega|_{\Gamma_o} \, d\lambda_{\Gamma_o} + \int_{\Omega_o} \mathbf{f}_o \cdot \overline{\mathbf{u}}_\omega \, d\lambda_3 = 0 \qquad \text{for every } \mathbf{u}_\omega \text{ satisfying (1.27) through (1.29).} \qquad (1.37)$$

On the other hand, it is also easy to use certain integral identities to show that (1.37) is necessary for the existence of a solution to the problem generated by (1.23) through (1.26), (1.34), and (1.36), while the same condition will appear naturally in the development of the existence assertion of §4. Observe that (1.37) is automatically fulfilled (and (1.36) reduces to (1.35)) when the only forcing agent in the problem is the fluid incident field. While the conjectures in this discussion probably cannot be substantiated unless the domain Ω_o satisfies some additional requirements on its shape, they provide an heuristic basis for anticipating the need for (1.37).

As we proceed, it becomes clear that it would be of great assistance to have knowledge of conditions on the shape of Ω_o sufficient to guarantee that, for the given $\omega > 0$, there can exist no nontrivial fields of the sort that we have termed "complex amplitudes of nonradiating modes" for Ω_o and ω, those very special (eigen)functions $\mathbf{u}_\omega$ satisfying (1.27) through (1.29). For, when we are certain that such fields do exist or when we lack any separate assertion of their nonexistence, we must deal at every step of the analysis with their presence or, respectively, possible presence, an aspect significantly complicating matters. In particular, this is true in the numerical work, since it turns out that if one wants to approximate *both* the elastic field $\mathbf{u}$ and the acoustic field φ satisfying (1.23) through (1.26), (1.34), and (1.36) (when such a pair exists), then complete information concerning the solutions of the problem embodied in (1.27) through (1.29) is required. However, we also wish to show that if one is content with the approximate calculation of the fluid field alone, then one can make do with very meager information about the latter problem.

To sum up, our aim is threefold:

1. We wish to establish the unique solvability of the problem generated by (1.23) through (1.26), (1.34), (1.36), and (1.37), in a weak sense to be specified.
2. We intend to show how the Galerkin method can be used to construct a convergent sequence of approximations for the unique solution of the problem just cited above, provided that one "knows all about" both the exterior Neumann problem for the Helmholtz equation (1.23) with the Sommerfeld condition (1.34) and the collection of solutions of the problem (1.27) through (1.29). Also, we want to verify that the acoustic part of the solution can be approximated without explicit knowledge of the latter family of functions.
3. We wish to prove that, under sufficiently strong hypotheses concerning the assumed asymptotic approach to steady-state, (1.34) and (1.36) are properties of the complex amplitudes $\mathbf{u}$ and φ in the assumed forms (1.21) and (1.22) of the solution $(\mathbf{U}, \Phi)$ of (1.10) through (1.14), with forcing data having the forms appearing in (1.15) through (1.17). The results here are less than satisfactory, in that they provide only a "plausibility argument," on the basis of which we are motivated to suspect strongly that (1.34) and (1.36) are genuine limiting-amplitude conditions.

We shall carry out the analysis in this same order. Thus, in §2 we shall reduce the study of the original interior-exterior problem to the examination of a purely interior problem (in Ω_o), while giving strong and weak formulations for both the original and the purely interior problems. Subsequently, we shall restrict our attention to the weak problems, deferring to a future report a discussion of regularity results for the weak solutions, in particular, postponing the specification of conditions sufficient to guarantee that a weak solution is a strong, or even a classical, solution. The abstract results that we need for the analysis of the weak purely interior problem are presented in §3. There, we base our approach on developments of Hildebrandt and Wienholtz [6] generalizing the Lax-Milgram Lemma, that we either take over entirely or modify to suit our own purposes. In particular, an evidently new result is given in Theorem 3.3, identifying a simple sufficient condition under which the Galerkin procedure can be directly applied to a variational problem with an indefinite sesquilinear form appearing as a perturbation of a definite form by a compact one. This machinery is applied in §4 to accomplish the first two aims listed above. In §5 we undertake to provide the promised motivation for accepting (1.34) and (1.36) as limiting-amplitude conditions. Finally, the proofs of two auxiliary results used in §5 are given in the Appendix.

2. STRONG FORMULATION; DERIVATION OF THE PURELY INTERIOR PROBLEM AND WEAK FORMULATIONS

Following a description of the basic notations to be used throughout, we proceed directly in this section to the formulations and reformulations of the problem represented by (1.23) through (1.26), (1.34), and (1.36). The plan consists in the use of certain facts about the Neumann problem for the Helmholtz equation in Ω_+ to reduce the original interface problem to one involving only the elastic field $\mathbf{u}$ in Ω_o, with subsequent construction of the fluid field φ in Ω_+ from $\mathbf{u}$.

We adhere to the standard notations for the usual spaces of complex functions on subsets of $\mathbf{R}^N$ ($N \geq 2$). Thus, $C^0(F) \equiv C(F)$ indicates the complex-linear space of continuous $\mathbf{C}$-valued maps defined on $F \subset \mathbf{R}^N$. Let Ω be open in $\mathbf{R}^N$; for a positive integer k, $C^k(\Omega)$ is the linear manifold of all elements of $C(\Omega)$ having all partial derivatives of orders less than or equal to k also belonging to $C(\Omega)$. When $f \in C^k(\Omega)$, in addition to the notation $f_{,j}$, we may also indicate partial derivatives of f in an obvious manner as $f_{,\alpha}$, wherein $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an N-index of order $|\alpha| := \alpha_1 + \ldots + \alpha_N \leq k$. The elements of $C^k(\Omega)$ having supports compact and contained in Ω is the linear manifold denoted by $C_0^k(\Omega)$. The members of $C^k(\overline{\Omega})$ are those functions that lie in $C^k(\Omega)$ and are, along with all of their partial derivatives, bounded and uniformly continuous (and so possess, with all of their partial derivatives, continuous extensions to $\overline{\Omega}$); $C^k(\overline{\Omega})$ is a Banach space under the usual norm, given by

$$\|f\|_{C^k(\overline{\Omega})} := \max_{0\leq|\alpha|\leq k} \sup_{\mathbf{x}\in\Omega} |f_{,\alpha}(\mathbf{x})|.$$

In turn, for a number ν with $0 < \nu \leq 1$, $C^{k,\nu}(\overline{\Omega})$ is the linear manifold in $C^k(\overline{\Omega})$ comprising those functions that are, along with all of their partial derivatives, uniformly ν-Hölder continuous in Ω; provided with the norm given by

$$\|f\|_{C^{k,\nu}(\overline{\Omega})} := \|f\|_{C^k(\overline{\Omega})} + \max_{0\leq|\alpha|\leq k} \sup_{\substack{\mathbf{x},\mathbf{y}\in\Omega \\ \mathbf{x}\neq\mathbf{y}}} \frac{|f_{,\alpha}(\mathbf{x}) - f_{,\alpha}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\nu},$$

$C^{k,\nu}(\overline{\Omega})$ is a Banach space. The spaces $C^\infty(\Omega)$, $C_0^\infty(\Omega)$, and $C^\infty(\overline{\Omega})$ are defined as the intersections of, respectively, all $C^k(\Omega)$, $C_0^k(\Omega)$, and $C^k(\overline{\Omega})$, for $k = 1, 2, 3, \ldots$.

Whenever $\mathcal{X}$ is a measure space, with positive measure m, $L_2(\mathcal{X})$ is the collection of all (equivalence classes of) complex m-measurable functions f defined m-a.e. on $\mathcal{X}$ with $\int_{\mathcal{X}} |f|^2 \, d\mathrm{m} < \infty$; in this setting, $L_2(\mathcal{X})$ is equipped with the inner product $\langle \cdot, \cdot \rangle_{L_2(\mathcal{X})}$,

$$\langle f, g \rangle_{L_2(\mathcal{X})} := \int_{\mathcal{X}} f\overline{g} \, d\mathrm{m} \qquad \text{for } f, g \in L_2(\mathcal{X}),$$

under which it is a Hilbert space. The Lebesgue measure on $\mathbf{R}^N$ shall be denoted by λ_N.

Concerning regularity hypotheses for a bounded and connected open subset Ω of $\mathbf{R}^3$, we use the definitions of Nečas [7]. Corresponding to a positive a, let S_a denote the open square $(-a, a) \times (-a, a)$ in $\mathbf{R}^2$. By an *affine isometry of* $\mathbf{R}^3$, we mean a function $\mathbf{x} \mapsto \mathbf{L}\mathbf{x} = \mathbf{x}_o + \mathbf{L}_o\mathbf{x}$ on $\mathbf{R}^3$, with $\mathbf{x}_o \in \mathbf{R}^3$ and $\mathbf{L}_o : \mathbf{R}^3 \to \mathbf{R}^3$ a linear isometry. Now, let k be either a nonnegative integer or ∞; Ω is said to be *of type* $\mathfrak{R}^k$ provided there exist positive numbers a and b, a positive integer n, a collection $\{f_l\}_{l=1}^n$ of real-valued functions contained in $C^k(\overline{S}_a)$, and a family $\{\mathbf{L}_l\}_{l=1}^n$ of surjective affine isometries of $\mathbf{R}^3$ such that

(a) for $l = 1, \ldots, n$, $\mathbf{L}_l$ maps the graph $\mathcal{G}\{f_l\}$ of f_l into $\partial\Omega$, the set

$$\{(x_1, x_2, x_3) \mid (x_1, x_2) \in \overline{\mathcal{S}}_a, f_l(x_1, x_2) < x_3 < f_l(x_1, x_2) + b\}$$

into Ω, and the set

$$\{(x_1, x_2, x_3) \mid (x_1, x_2) \in \overline{\mathcal{S}}_a, f_l(x_1, x_2) - b < x_3 < f_l(x_1, x_2)\}$$

into $\mathbf{R}^3 \setminus \overline{\Omega}$,

and

(b) $\partial\Omega = \bigcup_{l=1}^{n} \mathbf{L}_l(\mathcal{G}\{f_l\})$.

Further, if these conditions are fulfilled and the family $\{f_l\}_{l=1}^{n}$ is in $C^{k,\nu}(\overline{\mathcal{S}}_a)$ for some number $\nu \in (0, 1]$, then Ω is said to be *of type* $\mathfrak{R}^{k,\nu}$. Bounded and connected open sets of type $\mathfrak{R}^{0,1}$ are also termed *Lipschitz domains*. For the definition of the Lebesgue measure (induced by λ_3) on the boundary $\partial\Omega$ of a Lipschitz domain Ω, cf. Ref. 7, §3.1.1 or Ref. 8, §1.1; we shall denote this measure by $\lambda_{\partial\Omega}$, and its restriction to the measurable subsets of a measurable subset Γ of $\partial\Omega$ by λ_Γ. For such Ω, the unit exterior normal field $\mathbf{n}$ exists $\lambda_{\partial\Omega}$-a.e. on $\partial\Omega$, with Cartesian components $\lambda_{\partial\Omega}$-measurable (and bounded); by "exterior," of course we mean here that, whenever a normal is defined at $\mathbf{x} \in \partial\Omega$, the inclusion $\mathbf{x} + s\mathbf{n}(\mathbf{x}) \in \mathbf{R}^3 \setminus \overline{\Omega}$ obtains for all sufficiently small positive s. Explicitly, suppose that $\mathbf{x} \in \partial\Omega$ is such that for some $l \in \{1, \ldots, n\}$ and $(\hat{x}_1, \hat{x}_2) \in \mathcal{S}_a$ we have $\mathbf{x} = \mathbf{L}_l(\hat{x}_1, \hat{x}_2, f_l(\hat{x}_1, \hat{x}_2))$: then, provided that $f_{l,1}(\hat{x}_1, \hat{x}_2)$ and $f_{l,2}(\hat{x}_1, \hat{x}_2)$ exist, $\mathbf{n}(\mathbf{x})$ exists and can be computed from

$$\mathbf{n}(\mathbf{x}) = \mathbf{L}_{lo}(\hat{\mathbf{n}}_l(\hat{x}_1, \hat{x}_2)),$$

wherein $\mathbf{L}_{lo}$ is the surjective linear isometry associated with $\mathbf{L}_l$, and (in Cartesian components)

$$\hat{\mathbf{n}}_l(\hat{x}_1, \hat{x}_2) := \frac{(f_{l,1}(\hat{x}_1, \hat{x}_2), f_{l,2}(\hat{x}_1, \hat{x}_2), -1)}{\{1 + [f_{l,1}(\hat{x}_1, \hat{x}_2)]^2 + [f_{l,2}(\hat{x}_1, \hat{x}_2)]^2\}^{1/2}}$$

is the appropriate unit normal to the graph of f_l at the point $(\hat{x}_1, \hat{x}_2, f_l(\hat{x}_1, \hat{x}_2))$.

For the definitions and properties of the various Sobolev spaces, we rely on Nečas [7]; cf., also, Adams [9], wherein certain of the results are to be found. Returning to the case in which Ω is open in $\mathbf{R}^N$, for a positive integer k the corresponding Sobolev space $H^k(\Omega)$ ($\equiv W_2^{(k)}(\Omega)$ in Ref. 7, $\equiv W^{k,2}(\Omega)$ in Ref. 9) is the set of all elements of $H^0(\Omega) := L_2(\Omega)$ for which all weak (distributional) derivatives of orders less than or equal to k exist and belong to $L_2(\Omega)$; we extend the notations $f_{,j}$, $f_{,\alpha}$ to indicate the weak derivatives of an element $f \in H^k(\Omega)$. Defining an inner product on $H^k(\Omega)$ by

$$\langle f, g\rangle_{H^k(\Omega)} := \sum_{0 \le |\alpha| \le k} \langle f_{,\alpha}, g_{,\alpha}\rangle_{L_2(\Omega)} \qquad \text{for } f, g \in H^k(\Omega),$$

$H^k(\Omega)$ becomes a Hilbert space. The definition of $H^k(\Omega)$ is extended to encompass nonintegral values of $k > 0$ as in Ref. 7, §2.3.8, in a well-known manner, by using the approach of S. L. Slobodetskii and N. Aronszajn; such spaces are needed here for the definition of the fractional-order Sobolev spaces on $\partial\Omega$, *infra*. Thus, when s is positive and nonintegral, with $[s]$ denoting the greatest integer not exceeding s, one sets

$$\|u\|_{H^s(\Omega)} := \left\{ \|u\|^2_{H^{[s]}(\Omega)} + \sum_{|\alpha| = [s]} \int_\Omega \int_\Omega \frac{|u_{,\alpha}(\mathbf{x}) - u_{,\alpha}(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{N + 2(s - [s])}} \, d\lambda_N(\mathbf{x}) \, d\lambda_N(\mathbf{y}) \right\}^{\frac{1}{2}}$$

for those $u \in H^{[s]}(\Omega)$ for which the integrals appearing on the right are all finite; the resultant complex-linear space is equipped with the obvious inner product generating $\|\cdot\|_{H^s(\Omega)}$, forming a Hilbert space denoted by $H^s(\Omega)$. Finally, when Ω is not bounded, by $H^s_{loc}(\overline{\Omega})$ we shall mean the collection of all complex measurable functions defined a.e. in Ω and having restrictions in $H^s(\Omega')$ for every bounded open subset $\Omega' \subset \Omega$; in the standard manner, $H^s_{loc}(\overline{\Omega})$ is made into a locally convex linear topological space through the introduction of the seminorms $u \mapsto \|u|\Omega'\|_{H^s(\Omega')}$.

In this paragraph, $\Omega \subset \mathbf{R}^3$ shall be (at least) a Lipschitz domain, whence a good deal more is known about the structure of $H^k(\Omega)$ than in the case of a general open set. We shall write $\Gamma := \partial\Omega$. Thus, $C^k(\overline{\Omega})$ is dense in $H^k(\Omega)$ (cf. Ref. 7, Théorème 2.3.1 or Ref. 9, Theorem 3.18), and there is defined the Sobolev trace operator $u \mapsto u|_\Gamma$ from $H^1(\Omega)$ into $L_2(\Gamma)$, as the unique bounded extension to $H^1(\Omega)$ of the "classical" trace map $f \mapsto f|_\Gamma := f|\Gamma$ for $f \in C^1(\overline{\Omega})$, regarded as densely defined in $H^1(\Omega)$ and mapping into $L_2(\Gamma)$ (Ref. 7, Théorème 2.4.2). Moreover, both the natural injection of $H^1(\Omega)$ into $L_2(\Omega)$ and the trace operator carrying $H^1(\Omega)$ into $L_2(\Gamma)$ are compact (Ref. 7, Théorème 2.6.1 and Théorème 2.6.2). At various points, we shall make reference to Sobolev spaces $H^s(\Gamma)$ associated with Γ, for certain real (nonintegral as well as integral) values of s. Following §2.4.3 and §2.5.2 of Ref. 7, let k be a positive integer and Ω be of type $\mathfrak{R}^{k-1,1}$: when $0 < s \leq k$, the corresponding $H^s(\Gamma)$ is defined to comprise those g in $L_2(\Gamma)$ for which the function g_l defined λ_2-a.e. on $\mathcal{S}_a$ by $g_l(x_1, x_2) := g\left(\mathbf{L}_l(x_1, x_2, f_l(x_1, x_2))\right)$ lies in $H^s(\mathcal{S}_a)$ for $l = 1, \ldots, n$ (cf. the notations employed in the definitions of the regularity classes of open sets, *supra*); when provided with the inner product ensuing from this construction,

$$\langle g, h\rangle_{H^s(\Gamma)} := \sum_{l=1}^{n} \langle g_l, h_l\rangle_{H^s(\mathcal{S}_a)},$$

$H^s(\Gamma)$ forms a Hilbert space. With this definition, the range of the trace operator on $H^1(\Omega)$ is characterized as precisely the collection of elements in $H^{\frac{1}{2}}(\Gamma)$, and the resultant map, when regarded on $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$, is bounded (Ref. 7, Théorème 2.5.5 and Théorème 2.5.7). Finally, provided that $s > 0$ and $H^s(\Gamma)$ is defined, $H^{-s}(\Gamma)$ denotes the anti-dual of $H^s(\Gamma)$ (the collection of all bounded conjugate-linear functionals on $H^s(\Gamma)$) and $(\cdot,\cdot)_s$ the resultant duality pairing on $H^{-s}(\Gamma) \times H^s(\Gamma)$. With the usual identification, we can regard $H^0(\Gamma) := L_2(\Gamma)$ as a linear manifold in $H^{-s}(\Gamma)$, and then can write

$$(f, g)_s = \int_\Gamma f\overline{g}\, d\lambda_\Gamma = \langle f, g\rangle_{L_2(\Gamma)} \qquad \text{for } f \in H^0(\Gamma) \text{ and } g \in H^s(\Gamma).$$

The extension of the formula for integration by parts ("formule de Green"), classically a corollary of the Divergence Theorem, to the Sobolev-space setting for a Lipschitz domain is effected in Théorème 3.1.1 of Ref. 7. Thus, we have

$$\int_\Omega u_{,j} v\, d\lambda_3 = \int_\Gamma u|_\Gamma v|_\Gamma \mathrm{n}_j\, d\lambda_\Gamma - \int_\Omega u v_{,j}\, d\lambda_3 \qquad \text{for } u, v \in H^1(\Omega), \quad j = 1, 2, 3,$$

$(\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3)$ denoting Cartesian components of the unit exterior normal field $\mathbf{n}$, defined λ_Γ-a.e. on Γ.

For each of the spaces of complex functions defined in the preceding, we introduce the corresponding space of $\mathbf{C}^3$-valued functions, identified as the Cartesian product of three copies of the original collection, and denote the new space with the corresponding boldface symbol, $\mathbf{C}(F)$, $\mathbf{C}^k(\Omega)$, $\mathbf{L}_2(\mathcal{X})$, $\mathbf{H}^k(\Omega)$, *etc.* $\mathbf{L}_2(\mathcal{X})$ is provided with the Hilbert-space structure induced by the inner product given as

$$\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbf{L}_2(\mathcal{X})} := \int_\mathcal{X} \mathbf{f} \cdot \overline{\mathbf{g}}\, dm \qquad \text{for } \mathbf{f}, \mathbf{g} \in \mathbf{L}_2(\mathcal{X});$$

$\mathbf{H}^k(\Omega)$ is equipped with the inner product defined by

$$\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbf{H}^k(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \langle \mathbf{f}_{,\alpha}, \mathbf{g}_{,\alpha}\rangle_{\mathbf{L}_2(\Omega)} \qquad \text{for } \mathbf{f}, \mathbf{g} \in \mathbf{H}^k(\Omega),$$

under which it also becomes a Hilbert space. Similarly, the other products of Hilbert spaces are given their natural Hilbert-space structures.

Returning to the developments begun in §1, we retain the notations already introduced there. We now suppose that Ω_o is a Lipschitz domain, but we impose the restriction that (the fluid-elastic interface) Γ_+ be a manifold of class C^2, just so that we can use directly the results of Refs. 4 and 5, wherein this much smoothness was demanded. Still, the components of (the vacuum-elastic boundary) Γ_- are allowed to be nonsmooth, with corners and edges of a severity permitted within the Lipschitz-domain setting. More precisely, then, the (bounded and connected) open set $\overline{\Omega}_- \cup \Omega_o$ shall be taken throughout to be of type $\Re^2$. Alternately, the regularity requirements on Γ_+ can be phrased in terms of the notations already introduced: let $a > 0$, $b > 0$, the integer n, and the families $\{f_l\}_{l=1}^n$ and $\{\mathbf{L}_l\}_{l=1}^n$ be associated with Ω_o as a result of its being of type $\Re^{0,1}$. If Ω_- is absent, let $n_+ := n$; otherwise, we may suppose that matters are arranged so that $\{1,\ldots,n\}$ can be partitioned into $\{1,\ldots,n_+\}$ and $\{n_+ + 1,\ldots,n\}$, with $\Gamma_+ = \bigcup_{l=1}^{n_+} \mathbf{L}_l(\mathcal{G}\{f_l\})$ and $\Gamma_- = \bigcup_{l=n_++1}^{n} \mathbf{L}_l(\mathcal{G}\{f_l\})$. Then our restriction is expressed by the inclusion $\{f_l\}_{l=1}^{n_+} \subset C^2(\overline{\mathcal{S}}_a)$. Observe that the spaces $H^s(\Gamma_+)$ and $\mathbf{H}^s(\Gamma_+)$ are defined for $|s| \leq 2$ and can be described more explicitly with the help of a, b, n_+, and the f_l and $\mathbf{L}_l$ for $l = 1,\ldots,n_+$. For brevity, we write (with $\mathbf{H}^0(\Omega_o) := \mathbf{L}_2(\Omega_o)$) $\mathbf{H}^k := \mathbf{H}^k(\Omega_o)$ for $k = 0,1,2,\ldots$. If $\mathbf{u} \in \mathbf{H}^1$, by $\mathbf{u}|_{\Gamma_-}$ and $\mathbf{u}|_{\Gamma_+}$ we mean the restrictions to Γ_- and Γ_+, respectively, of the trace $\mathbf{u}|_{\Gamma_o}$ of $\mathbf{u}$ on Γ_o, *i.e.*,

$$\mathbf{u}|_{\Gamma_-} := (\mathbf{u}|_{\Gamma_o})|\Gamma_- \quad \text{and} \quad \mathbf{u}|_{\Gamma_+} := (\mathbf{u}|_{\Gamma_o})|\Gamma_+ \qquad \text{whenever } \mathbf{u} \in \mathbf{H}^1.$$

With this understanding, we want to make sure of some simple facts concerning the linear maps $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+}$ and $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ on $\mathbf{H}^1$, which will be of importance in the later developments.

Lemma 2.1. *Recall the regularity conditions imposed on* Ω_o.

(*i*) *The linear operations* $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+}$ *and* $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ *on* $\mathbf{H}^1$ *are compact when regarded as mapping into* $\mathbf{L}_2(\Gamma_+)$ *and* $L_2(\Gamma_+)$, *respectively.*

(*ii*) *The linear operations* $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+}$ *and* $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ *carry* $\mathbf{H}^1$ *onto* $\mathbf{H}^{\frac{1}{2}}(\Gamma_+)$ *and into* $H^{\frac{1}{2}}(\Gamma_+)$, *respectively, and are bounded when so regarded.*

Proof: The assertions in (*i*) follow immediately from the compactness of the trace map $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_o}$ on $\mathbf{H}^1$ into $\mathbf{L}_2(\Gamma_o)$ and the obvious inequalities $\|(\mathbf{g}|\Gamma_+)\cdot\mathbf{n}_+\|_{L_2(\Gamma_+)} \leq \|\mathbf{g}|\Gamma_+\|_{\mathbf{L}_2(\Gamma_+)} \leq \|\mathbf{g}\|_{\mathbf{L}_2(\Gamma_o)}$, holding for $\mathbf{g} \in \mathbf{L}_2(\Gamma_o)$. Turning to statement (*ii*), let us first show that $\mathbf{H}^{\frac{1}{2}}(\Gamma_+) = \{\, \mathbf{u}|_{\Gamma_+} \mid \mathbf{u} \in \mathbf{H}^1 \,\}$. The inclusion "$\supset$" here is easily seen to hold by virtue of the inclusion $\mathbf{u}|_{\Gamma_o} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_o)$ and the remarks made concerning the generation of $\mathbf{H}^{\frac{1}{2}}(\Gamma_+)$. The proof of the opposite inclusion requires not much more work: taking any $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_+)$, extend its definition (if necessary) to get an element $\tilde{\mathbf{g}}$ of $\mathbf{H}^{\frac{1}{2}}(\Gamma_o)$ by setting, say, $\tilde{\mathbf{g}}(\mathbf{x}) := \mathbf{0}$ for $\mathbf{x} \in \Gamma_-$; there exists some $\tilde{\mathbf{u}} \in \mathbf{H}^1$ such that $\tilde{\mathbf{u}}|_{\Gamma_o} = \tilde{\mathbf{g}}$, whence we have $\tilde{\mathbf{u}}|_{\Gamma_+} = \mathbf{g}$. Thus, $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+}$ carries $\mathbf{H}^1$ onto $\mathbf{H}^{\frac{1}{2}}(\Gamma_+)$; the boundedness of this "restricted-trace" map then follows from the boundedness of the trace operator from $\mathbf{H}^1$ onto $\mathbf{H}^{\frac{1}{2}}(\Gamma_o)$ and the evident inequality $\|\mathbf{g}|\Gamma_+\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_+)} \leq \|\mathbf{g}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_o)}$, holding for $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_o)$. Finally, let $\mathbf{u} \in \mathbf{H}^1$: we show that $\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ lies in $H^{\frac{1}{2}}(\Gamma_+)$ and that there exists a positive c_1, independent of $\mathbf{u}$, such that $\|\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+\|_{H^{\frac{1}{2}}(\Gamma_+)} \leq c_1 \|\mathbf{u}|_{\Gamma_+}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma_+)}$, which then effectively completes the proof of (*ii*). To this end, by using the notation introduced above, select any $l \in \{1,\ldots,n_+\}$ and consider the function $(\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+)_l$ defined for λ_2-almost all $(x_1,x_2) \in \mathcal{S}_a$ by

$$\begin{aligned}(\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+)_l(x_1,x_2) &:= (\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+)\big(\mathbf{L}_l(x_1,x_2,f_l(x_1,x_2))\big) \\ &= \mathbf{u}|_{\Gamma_+}\big(\mathbf{L}_l(x_1,x_2,f_l(x_1,x_2))\big) \cdot \mathbf{L}_{lo}(\hat{\mathbf{n}}_l(x_1,x_2)) = (\mathbf{u}|_{\Gamma_+})_l(x_1,x_2) \cdot \mathbf{L}_{lo}(\hat{\mathbf{n}}_l(x_1,x_2)).\end{aligned}$$

Clearly, the inclusion $\hat{\mathbf{n}}_l \in \mathbf{C}^1(\overline{\mathcal{S}}_a)$ holds by our restriction on Ω_o, so the function $\hat{\mathbf{n}}_l$ is certainly uniformly Lipschitz continuous on $\overline{\mathcal{S}}_a$. Consequently, we get

$$\begin{aligned}
&\int_{\mathcal{S}_a}\int_{\mathcal{S}_a} \frac{|(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)_l(x_1,x_2) - (\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)_l(y_1,y_2)|^2}{|(x_1,x_2)-(y_1,y_2)|^3}\,d\lambda_2(x_1,x_2)\,d\lambda_2(y_1,y_2)\\
&\quad\le 2\left\{\int_{\mathcal{S}_a}|(\mathbf{u}|_{\Gamma_+})_l(y_1,y_2)|^2\int_{\mathcal{S}_a}\frac{|\hat{\mathbf{n}}_l(x_1,x_2)-\hat{\mathbf{n}}_l(y_1,y_2)|^2}{|(x_1,x_2)-(y_1,y_2)|^3}\,d\lambda_2(x_1,x_2)\,d\lambda_2(y_1,y_2)\right.\\
&\qquad\left.+\int_{\mathcal{S}_a}\int_{\mathcal{S}_a}\frac{|(\mathbf{u}|_{\Gamma_+})_l(x_1,x_2)-(\mathbf{u}|_{\Gamma_+})_l(y_1,y_2)|^2}{|(x_1,x_2)-(y_1,y_2)|^3}\,d\lambda_2(x_1,x_2)\,d\lambda_2(y_1,y_2)\right\}\\
&\quad\le 2\left\{M_l\|(\mathbf{u}|_{\Gamma_+})_l\|^2_{\mathbf{H}^0(\mathcal{S}_a)}+\int_{\mathcal{S}_a}\int_{\mathcal{S}_a}\frac{|(\mathbf{u}|_{\Gamma_+})_l(x_1,x_2)-(\mathbf{u}|_{\Gamma_+})_l(y_1,y_2)|^2}{|(x_1,x_2)-(y_1,y_2)|^3}\,d\lambda_2(x_1,x_2)\,d\lambda_2(y_1,y_2)\right\}\\
&\quad\le M_l'\|(\mathbf{u}|_{\Gamma_+})_l\|^2_{\mathbf{H}^{\frac12}(\mathcal{S}_a)},
\end{aligned}$$

in which M_l and M_l' depend upon Ω_o and l alone. Moreover, obviously

$$\|(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)_l\|_{H^0(\mathcal{S}_a)} \le \|(\mathbf{u}|_{\Gamma_+})_l\|_{\mathbf{H}^0(\mathcal{S}_a)} \le \|(\mathbf{u}|_{\Gamma_+})_l\|_{\mathbf{H}^{\frac12}(\mathcal{S}_a)}.$$

From these facts, we conclude that $(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)_l$ lies in $H^{\frac12}(\mathcal{S}_a)$, with norm not greater than a positive multiple of $\|(\mathbf{u}|_{\Gamma_+})_l\|_{\mathbf{H}^{\frac12}(\mathcal{S}_a)}$, the multiple depending upon only Ω_o and l. In turn, this implies that $\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+$ belongs to $H^{\frac12}(\Gamma_+)$, with norm not exceeding a positive multiple of $\|\mathbf{u}|_{\Gamma_+}\|_{\mathbf{H}^{\frac12}(\Gamma_+)}$, the multiple depending on Ω_o alone. ∎

For the various formulations of the problem loosely described in §1, and for its reformulation as a purely interior problem in Ω_o, we need some preliminary results on the (purely exterior) Neumann problem for the Helmholtz equation and the Sommerfeld condition in Ω_+, with boundary data lying in $L_2(\Gamma_+)$. For this, it is most convenient to rely on the developments already in place in Refs. 4 and 5. There, a generalized Robin problem, subsuming the Neumann problem, is studied by using potential-theoretic methods, although the analysis is not carried out within the Sobolev-space setting, but completely in $L_2(\Gamma_+)$; the later article Ref. 10 is concerned with the connections between the two approaches. In particular, in Refs. 4 and 5, traces and normal derivatives on Γ_+ are taken in the *normal-L_2 sense*. To recall the pertinent definitions, let $\mathbf{N}_s : \Gamma_+ \to \mathbf{R}^3$ be given for any $s > 0$ by

$$\mathbf{N}_s(\mathbf{x}) := \mathbf{x} + s\mathbf{n}_+(\mathbf{x}) \qquad \text{for each } \mathbf{x}\in\Gamma_+;$$

because of the regularity imposed upon Ω_o, there is some $s_+ > 0$ such that $\mathbf{N}_s(\Gamma_+) \subset \Omega_+$ whenever $0 < s < s_+$. Thus, if ψ is any function defined in Ω_+, then $\psi \circ \mathbf{N}_s$ is defined on Γ_+ for all sufficiently small positive s. Now suppose, for simplicity, that $\varphi \in C^1(\Omega_+)$. Then we say that φ *has a trace on* Γ_+ *in the normal-L_2 sense* iff $\lim_{s\to0^+}\varphi\circ\mathbf{N}_s$ exists in the $L_2(\Gamma_+)$-sense, *i.e.*, iff

$$\lim_{s\to0^+}\|\varphi\circ\mathbf{N}_s - \varphi|^2_{\Gamma_+}\|_{L_2(\Gamma_+)} = 0$$

for some $\varphi|^2_{\Gamma_+} \in L_2(\Gamma_+)$, in which case the latter function is termed *the normal-L_2 trace of* φ *on* Γ_+. Similarly, we say that φ *has a normal derivative on* Γ_+ *in the normal-L_2 sense* iff $\lim_{s\to0^+}\mathbf{n}_+\cdot(\operatorname{grad}\varphi)\circ\mathbf{N}_s$ exists in the $L_2(\Gamma_+)$-sense, *i.e.*, iff

$$\lim_{s\to0^+}\|\mathbf{n}_+\cdot(\operatorname{grad}\varphi)\circ\mathbf{N}_s - \varphi,^2_{\mathbf{n}_+}\|_{L_2(\Gamma_+)} = 0$$

for some $\varphi_{,\mathbf{n}_+}^2 \in L_2(\Gamma_+)$, in which case the latter function is called *the normal-L_2 normal derivative of φ on* Γ_+. When both $\varphi|_{\Gamma_+}^2$ and $\varphi_{,\mathbf{n}_+}^2$ exist, we say that *φ is L_2-regular at* Γ_+. With the understanding that the trace and normal derivative of an appropriate function φ in Ω_+ are always to be interpreted in this sense, henceforth we omit the superscript "2" and write, respectively, simply $\varphi|_{\Gamma_+}$ and $\varphi_{,\mathbf{n}_+}$ for these elements of $L_2(\Gamma_+)$.

Now, we introduce the collection $W(\Omega_+;\kappa_+)$ of radiating solutions of the Helmholtz equation (1.23) in Ω_+ that are also L_2-regular at Γ_+:

$$W(\Omega_+;\kappa_+) := \{\, \varphi \in C^2(\Omega_+) \mid \text{(1.23) and (1.34) hold, and } \varphi \text{ is } L_2\text{-regular at } \Gamma_+ \,\}.$$

For $\varphi \in W(\Omega_+;\kappa_+)$, we refer to $\varphi|_{\Gamma_+}$ and $\varphi_{,\mathbf{n}_+}$, respectively, as *the Dirichlet data* and *the Neumann data of* φ.

The basic facts that we shall use concerning the Neumann problem for (1.23) and (1.34) in Ω_+ are collected in Theorem 2.1. To state the results, we need to introduce an "outgoing" fundamental solution for the operator $\Delta + \kappa_+^2$; we choose the particular one given by, for each $\mathbf{x} \in \mathbf{R}^3$,

$$E_{\mathbf{x}}^{\kappa_+}(\mathbf{y}) := -\frac{e^{i\kappa_+|\mathbf{y}-\mathbf{x}|}}{2\pi|\mathbf{y}-\mathbf{x}|} \qquad \text{for } \mathbf{y} \in \mathbf{R}^3 \setminus \{\mathbf{x}\}.$$

Whenever $\mathbf{x} \in \mathbf{R}^3$, we write $E_{\mathbf{x},\mathbf{n}_+}^{\kappa_+} := \mathbf{n}_+ \cdot (\operatorname{grad} E_{\mathbf{x}}^{\kappa_+})|(\Gamma_+ \setminus \{\mathbf{x}\})$ for the normal derivative of $E_{\mathbf{x}}^{\kappa_+}$ on $\Gamma_+ \setminus \{\mathbf{x}\}$.

Theorem 2.1. *Recall the regularity conditions imposed upon Ω_+ and the inequalities $\operatorname{Re}\kappa_+ > 0$ and $\operatorname{Im}\kappa_+ \geq 0$ fulfilled by κ_+.*

(*i*) *The linear map $\varphi \mapsto \varphi_{,\mathbf{n}_+}$ is a bijection of $W(\Omega_+;\kappa_+)$ onto $L_2(\Gamma_+)$.*

(*ii*) *By (i), the linear operator $A_{\kappa_+} : L_2(\Gamma_+) \to L_2(\Gamma_+)$ given by*

$$A_{\kappa_+} g := \varphi_g|_{\Gamma_+}, \quad \text{wherein } \varphi_g \in W(\Omega_+;\kappa_+) \text{ with } \varphi_{g,\mathbf{n}_+} = g, \qquad \text{for each } g \in L_2(\Gamma_+)$$

is well-defined. This operator is compact, injective, and has dense range in $L_2(\Gamma_+)$, while $-i\kappa_+^2 A_{\kappa_+}$ is "strictly dissipative," i.e.,

$$\operatorname{Im}\langle \kappa_+^2 A_{\kappa_+} f, f\rangle_{L_2(\Gamma_+)} < 0 \qquad \text{whenever } f \in L_2(\Gamma_+) \text{ and } f \neq 0. \tag{2.1}$$

(*iii*) *For any $g \in L_2(\Gamma_+)$, the corresponding unique $\varphi_g \in W(\Omega_+;\kappa_+)$ such that $\varphi_{g,\mathbf{n}_+} = g$, i.e., the corresponding unique solution in $W(\Omega_+;\kappa_+)$ of (1.23), (1.34) with Neumann data g, is given by*

$$\varphi_g(\mathbf{x}) = \frac{1}{2}\int_{\Gamma_+} \{E_{\mathbf{x}}^{\kappa_+} g - E_{\mathbf{x},\mathbf{n}_+}^{\kappa_+} A_{\kappa_+} g\}\, d\lambda_{\Gamma_+} \qquad \text{for } \mathbf{x} \in \Omega_+. \tag{2.2}$$

Proof: This is proven in Ref. 4, §6 and §7; cf., also, the remarks there in §2 following the definition in $(2.2)_1$. Actually, in Ref. 4 it is assumed that $\kappa_+ \neq 0$, with $\operatorname{Im}\kappa_+ \geq 0$, and $\kappa_+ > 0$ if $\operatorname{Im}\kappa_+ = 0$. It is shown there that the statements (*i*), (*ii*), and (*iii*) hold on that larger set of κ_+-values, with the exception of (2.1). Rather, there is a generalization of (2.1) asserting that, for any such κ_+, the operator $-i\zeta A_{\kappa_+}$ is strictly dissipative, *i.e.*,

$$\operatorname{Im}\langle \zeta A_{\kappa_+} f, f\rangle_{L_2(\Gamma_+)} < 0 \qquad \text{whenever } f \in L_2(\Gamma_+) \text{ and } f \neq 0,$$

for any ζ lying in the complex set Z_{κ_+} defined by

$$Z_{\kappa_+} := \begin{cases} \{\zeta \in \mathbf{R} \mid \zeta > 0\} = (0,\infty), & \text{if } \operatorname{Im}\kappa_+ = 0, \\ \{\zeta \in \mathbf{C} \mid \operatorname{Im}\zeta \geq 0,\ \operatorname{Im}(\zeta\overline{\kappa_+}^2) \leq 0,\ \text{and } [\operatorname{Im}\zeta]^2 + [\operatorname{Im}(\zeta\overline{\kappa_+}^2)]^2 > 0\}, & \text{if } \operatorname{Im}\kappa_+ > 0. \end{cases}$$

One can check that, for the larger set of κ_+-values specified, κ_+^2 lies in Z_{κ_+} iff $\operatorname{Re}\kappa_+ > 0$, whence (2.1) results. ∎

The operator A_{κ_+} of Theorem 2.1 can be referred to as *the* $(L_2(\Gamma_+)$-*) boundary-data operator for* $W(\Omega_+;\kappa_+)$, since it maps the Neumann data $\varphi_{,\mathbf{n}_+}$ to the corresponding Dirichlet data $\varphi|_{\Gamma_+}$ for each $\varphi \in W(\Omega_+;\kappa_+)$. This operator has a number of interesting properties in addition to those just cited in Theorem 2.1, some of which are given in Refs. 4, 5, and 10. Henceforth, we shall suppose that we have at our disposal some means for constructing the image $A_{\kappa_+}g$ for each $g \in L_2(\Gamma_+)$, which is essentially equivalent to assuming that we can construct the solution of the Neumann problem for (1.23) and (1.34) in Ω_+ for any boundary data in $L_2(\Gamma_+)$. Various schemes are available for accomplishing this. For example, an explicit representation of the operator A_{κ_+} can be obtained if one is willing to carry out the Gram-Schmidt orthonormalization procedure for an appropriately chosen complete family in $L_2(\Gamma_+)$; cf. Lemma 6.3 of Ref. 4. If κ_+^2 is not in the countably infinite collection of (positive) Dirichlet eigenvalues for the operator $-\Delta$ in the interior domain $\overline{\Omega}_- \cup \Omega_o$ $(= \mathbf{R}^3 \setminus \overline{\Omega}_+)$, then A_{κ_+} is given by $(I + D_{\kappa_+})^{-1}S_{\kappa_+}$, wherein S_{κ_+} and D_{κ_+} are the compact operators in $L_2(\Gamma_+)$ constructed from the "direct values" of, respectively, the single- and double-layer potentials based upon the fundamental solution E^{κ_+}: for $f \in L_2(\Gamma_+)$, $S_{\kappa_+}f$ and $D_{\kappa_+}f$ are the elements of $L_2(\Gamma_+)$ defined by setting, for λ_{Γ_+}-a.a. $\mathbf{x} \in \Gamma_+$,

$$S_{\kappa_+}f(\mathbf{x}) := \int_{\Gamma_+} E_{\mathbf{x}}^{\kappa_+} f\, d\lambda_{\Gamma_+}$$

and

$$D_{\kappa_+}f(\mathbf{x}) := \int_{\Gamma_+} E_{\mathbf{x},\mathbf{n}_+}^{\kappa_+} f\, d\lambda_{\Gamma_+}.$$

However, unless $\operatorname{Im}\kappa_+ > 0$, one does not usually know whether κ_+^2 is not amongst the eigenvalues just specified, *i.e.*, whether the operator $(I + D_{\kappa_+})$ is injective, and so, in general, a more difficult operator inversion must be effected to capture A_{κ_+}. For example, it is shown in Theorem 8.4 of Ref. 4 that one can always compute on the basis of the representation

$$A_{\kappa_+} = \{S_{\kappa_+} - \xi(I - D_{\kappa_+})\}\{I + \overline{D_{\kappa_+}^*} + \xi W_{\kappa_+,\mathbf{n}_+}\}^{-1},$$

wherein $\overline{D_{\kappa_+}^*}$ is the compact operator in $L_2(\Gamma_+)$ obtained by conjugating the kernel of the $(L_2(\Gamma_+)$-adjoint) integral operator $D_{\kappa_+}^*$; ξ is any conveniently chosen complex number with positive imaginary part (or, in the more general case identified in the proof of Theorem 2.1, with $\operatorname{Im}\xi \neq 0$ and $\operatorname{Im}\xi \operatorname{Re}\kappa_+ \geq 0$); and $W_{\kappa_+,\mathbf{n}_+}$ is the densely defined and unbounded operator in $L_2(\Gamma_+)$ induced by the normal-L_2 normal derivative of the double-layer potential in Ω_+ (or in Ω_-). That is, defining the (restriction to Ω_+ of the) double-layer potential $W_{\kappa_+}^+\{f\}$ with density $f \in L_2(\Gamma_+)$ by

$$W_{\kappa_+}^+\{f\}(\mathbf{x}) := \int_{\Gamma_+} E_{\mathbf{x},\mathbf{n}_+}^{\kappa_+} f\, d\lambda_{\Gamma_+} \qquad \text{for } \mathbf{x} \in \Omega_+,$$

then $W_{\kappa_+,\mathbf{n}_+}f$ is defined to be the normal-L_2 normal derivative $W_{\kappa_+}^+\{f\}_{,\mathbf{n}_+}$, on the linear manifold of all $f \in L_2(\Gamma_+)$ for which this normal derivative exists (which turns out to be precisely $\mathcal{R}(A_{\kappa_+})$, the range of the operator A_{κ_+}; cf. Ref. 4, Corollary 8.3). Another scheme for calculating the action of A_{κ_+} is implicit in the results of Ref. 11 and stated more explicitly in Theorem 8.1 of Ref. 4. Although it has not been formulated in this manner elsewhere, the problem of deriving representations for A_{κ_+} that are valid even when κ_+^2 happens to be one of the Dirichlet eigenvalues of $-\Delta$ in $\overline{\Omega}_- \cup \Omega_o$ has been the object of much study; cf., *e.g.*, Refs. 11, 12, and the references given therein.

By relying on properties of the single- and double-layer potentials that are proven by Kirsch [13], some connections are made in Ref. 10 between the L_2-type results just cited and those that can be established within the Sobolev-space setting. For example, with $\overline{\Omega}_- \cup \Omega_o$ of type $\Re^2$, as we are supposing here, it can be shown that $\mathcal{R}(A_{\kappa_+})$ lies in $H^1(\Gamma_+)$. For a given real number s, with more smoothness hypothesized for Γ_+ (depending upon s), it turns out that the restriction $A_{\kappa_+}|H^s(\Gamma_+)$ forms a bounded bijection of $H^s(\Gamma_+)$ onto $H^{s+1}(\Gamma_+)$ when $s \geq 0$, while A_{κ_+} can be extended to a bounded bijection of $H^s(\Gamma_+)$ onto $H^{s+1}(\Gamma_+)$ if $s < 0$. By using results of this sort, we find, for example, if $\overline{\Omega}_- \cup \Omega_o$ is of type $\Re^{3,\nu}$ and g is a member of $H^{\frac{1}{2}}(\Gamma_+)$, then the function $\varphi_g \in W(\Omega_+; \kappa_+)$ defined by (2.2) lies in $H^2_{loc}(\overline{\Omega}_+)$, and so possesses both a Sobolev trace and a Sobolev normal derivative on Γ_+, elements of $H^{\frac{3}{2}}(\Gamma_+)$ and $H^{\frac{1}{2}}(\Gamma_+)$, respectively; moreover, these coincide with its trace and normal derivative on Γ_+ in the normal-L_2 sense, given by $A_{\kappa_+}g$ and g, respectively (cf., also, Proposition 2.2, *infra*). Presently. we shall see that we need to construct the solution φ_g of the exterior Neumann problem (with (1.34)) here only when the boundary data $g \in H^{\frac{1}{2}}(\Gamma_+)$; the preceding remarks will then supply the Sobolev-space properties of that solution when Γ_+ has the additional smoothness noted.

To concisely formulate the conditions (1.35) and (1.36), it is convenient to introduce the linear space $\mathcal{N}^2_{\Omega_o,\omega}$ of *complex amplitudes of strong nonradiating modes for* Ω_o and ω:

$$\mathcal{N}^2_{\Omega_o,\omega} := \{\, \mathbf{u}_\omega \in \mathbf{H}^2 \mid (1.27) \text{ through } (1.29) \text{ hold} \,\}. \tag{2.3}$$

Here, we mean that (1.27) through (1.29) are to hold a.e. on the respective sets indicated, with respect to the appropriate measure (λ_3 or λ_{Γ_o}), the conditions in (1.28) and (1.29) being interpreted in the sense of the Sobolev trace operator on $\mathbf{H}^1$; cf. (1.4) through (1.6) for the calculation of the action of $\mathbf{T}^{\mathbf{n}_o}[\,\cdot\,]$ on an element of $\mathbf{H}^2$. Now, with $(\mathcal{N}^2_{\Omega_o,\omega})^{(\perp)_0}$ denoting the orthogonal complement of $\mathcal{N}^2_{\Omega_o,\omega}$ taken with respect to $\mathbf{H}^0$, we shall write $\mathcal{M}^2_{\Omega_o,\omega}$ for the linear manifold of elements of this orthogonal complement that also lie in $\mathbf{H}^2$:

$$\mathcal{M}^2_{\Omega_o,\omega} := (\mathcal{N}^2_{\Omega_o,\omega})^{(\perp)_0} \cap \mathbf{H}^2. \tag{2.4}$$

Since $(\mathcal{N}^2_{\Omega_o,\omega})^{(\perp)_0}$ is closed in $\mathbf{H}^0$, it is easy to see that $\mathcal{M}^2_{\Omega_o,\omega}$ is closed in $\mathbf{H}^2$. Eventually, we shall find that $\mathcal{N}^2_{\Omega_o,\omega}$ is finite-dimensional (a fact that is not surprising, since, when it is nontrivial, $\mathcal{N}^2_{\Omega_o,\omega}$ is a subspace of an eigenspace for the traction problem for $-\mathbf{\Delta}^\star_{\lambda,\mu}$ in Ω_o), and so also closed in each $\mathbf{H}^k$, $k = 0, 2$. In particular. once this is known, we shall have the orthogonal direct-sum decomposition

$$\mathbf{H}^0 = \mathcal{N}^2_{\Omega_o,\omega} \oplus (\mathcal{N}^2_{\Omega_o,\omega})^{(\perp)_0}, \tag{2.5}$$

whence it shall follow readily that

$$\mathbf{H}^2 = \mathcal{N}^2_{\Omega_o,\omega} + \mathcal{M}^2_{\Omega_o,\omega} \quad \text{with} \quad \mathcal{N}^2_{\Omega_o,\omega} \cap \mathcal{M}^2_{\Omega_o,\omega} = \{\mathbf{0}\}, \tag{2.6}$$

i.e., we will have the (nonorthogonal) direct-sum decomposition of $\mathbf{H}^2$ into the subspaces indicated. In this case, we shall reduce the problem corresponding to (1.23) through (1.26), (1.34), and (1.36) to that in which (1.36) is replaced by (1.35), essentially by seeking separately the projections of the solution onto the subspace of amplitudes of nonradiating modes and onto the $\mathbf{H}^0$-orthogonal complement of that subspace (and we shall effect this reduction in both the strong and the weak formulations). Since we must frequently refer to these two sets of requirements, it is convenient to introduce some abbreviated references.

Terminology 2.1. Let $\mathbf{f}_o \in \mathbf{H}^0$ and $\mathbf{t}_o \in \mathbf{L}_2(\Gamma_o)$. Let $\tilde{\Omega}^\iota$ be open in $\mathbf{R}^3$ and contain $\Omega_- \cup \overline{\Omega}_o$, and suppose that $\varphi^\iota \in C^2(\tilde{\Omega}^\iota)$ satisfies (1.18). Then by $(\mathbf{P}_0) \equiv \big(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota)\big)$ we refer to the problem of determining an appropriate pair of functions $(\mathbf{u}, \varphi)$ satisfying, in a sense that is always to be specified in the context, the conditions (1.23) through (1.26), (1.34), and (1.35). Suppose also that $\mathbf{T}^\tau_o$ on $\Gamma_o \times (0, \infty)$ and $\mathbf{F}^\tau_o$ on

$\Omega_o \times (0,\infty)$ are measurable $\mathbf{C}^3$-valued functions such that $\mathbf{T}_o^\tau(\cdot,t) \in \mathbf{L}_2(\Gamma_o)$ and $\mathbf{F}_o^\tau(\cdot,t) \in \mathbf{H}^0$ for each $t > 0$ and

$$\ell(\mathbf{u}_\omega) \equiv \ell(\mathbf{u}_\omega; \mathbf{T}_o^\tau, \mathbf{F}_o^\tau) := \frac{i}{2\varrho_o\omega}\int_0^\infty \left\{\int_{\Gamma_o} \mathbf{T}_o^\tau(\cdot,s)\cdot \overline{\mathbf{u}}_\omega|_{\Gamma_o}\, d\lambda_{\Gamma_o} + \int_{\Omega_o} \mathbf{F}_o^\tau(\cdot,s)\cdot \overline{\mathbf{u}}_\omega\, d\lambda_3\right\} e^{i\omega s}\, ds \tag{2.7}$$

exists whenever $\mathbf{u}_\omega \in \mathbf{H}^1$ satisfies (1.27) through (1.29) in a specific sense to be designated in the context; for example, for the existence of the integrals in (2.7), it is sufficient to know that the functions $t \mapsto \|\mathbf{T}_o^\tau(\cdot,t)\|_{\mathbf{L}_2(\Gamma_o)}$ and $t \mapsto \|\mathbf{F}_o^\tau(\cdot,t)\|_{\mathbf{H}^0}$ are in $L_1(0,\infty)$. Then, by $(\mathbf{P}) \equiv \big(\mathbf{P}(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota; \mathbf{F}_o^\tau, \mathbf{T}_o^\tau)\big)$ we shall refer to the problem of determining an appropriate pair of functions $(\mathbf{u},\varphi)$ satisfying, in a sense to be designated in the context, the conditions (1.23) through (1.26), (1.34), and (1.36).

Of course, it is not really necessary to retain the explicit appearance of the complex amplitude φ^ι in the conditions in (1.25), since we could just replace the inhomogeneities there with some given elements of $L_2(\Gamma_+)$ and $\mathbf{L}_2(\Gamma_+)$, respectively. However, we wish to maintain contact with the underlying physical setting, and so choose to continue writing the interface conditions in their original form (1.25).

We shall first define "strong solutions" of $(\mathbf{P}_0)$ and of $(\mathbf{P})$, since one has for such solutions a better feel for the sense in which the various requirements are fulfilled. Eventually, however, we restrict our attention in the present report to the study of weak formulations of these problems. Whatever else is demanded of them, all solutions of (1.24) (or (1.27)) that we consider shall be $\mathbf{H}^1$-functions $\mathbf{u}$ that are, in particular, solutions in the distributional sense, *i.e.*, such that

$$\int_{\Omega_o} \mathbf{u}\cdot\{\Delta^\star_{\lambda,\mu}\mathbf{v} + \varrho_o\omega^2\mathbf{v}\}\, d\lambda_3 = -\int_{\Omega_o} \mathbf{f}_o\cdot\mathbf{v}\, d\lambda_3 \qquad \text{for every } \mathbf{v}\in\mathbf{C}_0^\infty(\Omega_o),$$

whence their (local) interior regularity properties will be known from the well-developed theory of regularity for strongly elliptic systems. For example, Theorem 3.I of Ref. 14 implies that, if $\mathbf{f}_o \in \mathbf{H}^k$ for some integer $k \geq 0$, then any $\mathbf{u} \in \mathbf{H}^0$ satisfying (1.24) in the distributional sense will have restrictions in $\mathbf{H}^{k+2}(\Omega)$ for any open Ω with $\overline{\Omega} \subset \Omega_o$. The Sobolev Imbedding Theorems then show that, if $\mathbf{f}_o \in \mathbf{H}^2$, such a $\mathbf{u}$ will be in $\mathbf{C}^2(\Omega_o)$, and so also a classical solution of (1.24) (in fact, the same conclusion can be drawn even if it is known only that $\mathbf{f}_o$ lies in $\mathbf{C}^{0,\nu}(\Omega_o)$); if $\mathbf{f}_o \in \mathbf{C}^\infty(\Omega_o)$, then $\mathbf{u}$ will satisfy that inclusion as well (for these assertions, cf. Theorem 3.II of Ref. 14). If $\mathbf{f}_o$ has real-analytic real and imaginary parts, the same will be true of $\mathbf{u}$ (cf. Ref. 15). Thus, the elements of $\mathcal{N}^2_{\Omega_o,\omega}$ enjoy the latter property of real-analyticity (as do those of $\mathcal{N}^1_{\Omega_o,\omega}$, to be defined presently). All of our solutions of (1.23) will be classical, in $W(\Omega_+;\kappa_+)$, and consequently will also have real-analytic real and imaginary parts. Global regularity ("regularity up to the boundary") for the solutions of our boundary and interface problems is another matter, that we shall touch upon in this report only in Proposition 2.2, *infra*, and there we shall give a limited result only for the scalar part of a solution.

Definition 2.1. Recall Terminology 2.1. The pair $(\mathbf{u},\varphi)$ is *a strong solution of* $\big(\mathbf{P}_0(\mathbf{f}_o,\mathbf{t}_o,\varphi^\iota)\big)$ iff $\mathbf{u} \in \mathcal{M}^2_{\Omega_o,\omega}$, $\varphi \in W(\Omega_+;\kappa_+)$, (1.24) holds a.e. in Ω_o, (1.25) holds a.e. on Γ_+, and (1.26) is true a.e. on Γ_-, with $\mathbf{u}|_{\Gamma_+}$, $\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}]$, and $\mathbf{T}^{\mathbf{n}_-}[\mathbf{u}]$ interpreted in the sense of the Sobolev trace operator, and $\varphi|_{\Gamma_+}$ and $\varphi,_{\mathbf{n}_+}$ taken in the normal-L_2 sense on Γ_+. Suppose it is known that $\mathcal{N}^2_{\Omega_o,\omega}$ is finite-dimensional, with dimension $n_\omega \geq 0$; if $n_\omega \geq 1$, let $\{\mathbf{u}_\omega^{(j)}\}_{j=1}^{n_\omega}$ constitute an $\mathbf{H}^0$-orthonormal basis for $\mathcal{N}^2_{\Omega_o,\omega}$. Suppose that the integral in (2.7) exists for every $\mathbf{u}_\omega \in \mathcal{N}^2_{\Omega_o,\omega}$. Then the pair $(\mathbf{u},\varphi)$ is *a strong solution of* $\big(\mathbf{P}(\mathbf{f}_o,\mathbf{t}_o,\varphi^\iota;\mathbf{F}_o^\tau,\mathbf{T}_o^\tau)\big)$ iff

$$\mathbf{u} = \mathbf{u}_0 + \begin{cases} 0 & \text{if } n_\omega = 0 \\ \sum_{j=1}^{n_\omega} \ell(\mathbf{u}_\omega^{(j)})\mathbf{u}_\omega^{(j)} & \text{if } n_\omega \geq 1 \end{cases} \tag{2.8}$$

and $(\mathbf{u}_0,\varphi)$ is a strong solution of $\big(\mathbf{P}_0(\mathbf{f}_o,\mathbf{t}_o,\varphi^\iota)\big)$ (cf. (2.7) for the coefficients in (2.8) when $n_\omega \geq 1$).

The requirement that the vector part of a strong solution of $(\mathbf{P}_0)$ lie in $\mathcal{M}^2_{\Omega_o,\omega}$ is the strong form of the condition (1.35); the requirement that the vector part of a strong solution of $(\mathbf{P})$ be of the form (2.8) with (in particular) $\mathbf{u}_0 \in \mathcal{M}^2_{\Omega_o,\omega}$ is the strong form of condition (1.36), ensuring that the latter holds for every $\mathbf{u}_\omega \in \mathcal{N}^2_{\Omega_o,\omega}$. As a consequence of Definition 2.1, we essentially need consider only the study of the existence, uniqueness, and construction of strong solutions of the problems $(\mathbf{P}_0)$. We will find that, even in the best of circumstances, one must identify $\mathcal{N}^2_{\Omega_o,\omega}$ to approximate the elastic-field portion of a solution of the latter problem, so that there will be but little additional work involved in using (2.8) if the ultimate goal should in fact be the solution of a problem $(\mathbf{P})$.

Obviously, under Definition 2.1 $\left(\mathbf{P}(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota; \mathbf{F}^\tau_o, \mathbf{T}^\tau_o)\right)$ can have at most one strong solution if the problem $\left(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota)\right)$ enjoys that same property. To emphasize the rôle played by (1.34) and (1.35), let us prove that there can exist at most one strong solution of the boundary and interface problem $(\mathbf{P}_0)$. In preparation, we need a familiar integral relation (which shall be used later, as well), *viz.*,

$$\int_{\Omega_o} \{\mathbf{v} \cdot \mathbf{\Delta}^\star_{\lambda,\mu}\mathbf{u} + \mathcal{E}_{\lambda,\mu}(\mathbf{u},\mathbf{v})\}\, d\lambda_3 = \int_{\Gamma_o} \mathbf{v}|_{\Gamma_o} \cdot \mathbf{T}^{\mathbf{n}_o}[\mathbf{u}]\, d\lambda_{\Gamma_o} \qquad \text{for } \mathbf{u} \in \mathbf{H}^2 \text{ and } \mathbf{v} \in \mathbf{H}^1, \tag{2.9}$$

in which

$$\mathcal{E}_{\lambda,\mu}(\mathbf{u},\mathbf{v}) := \lambda\varepsilon_{jj}[\mathbf{u}]\varepsilon_{kk}[\mathbf{v}] + 2\mu\varepsilon_{jk}[\mathbf{u}]\varepsilon_{jk}[\mathbf{v}] = \sigma_{jk}[\mathbf{u}]\varepsilon_{jk}[\mathbf{v}] = \sigma_{jk}[\mathbf{v}]\varepsilon_{jk}[\mathbf{u}] \qquad \text{for } \mathbf{u},\mathbf{v} \in \mathbf{H}^1. \tag{2.10}$$

This identity is easily derived with an application of the formula for integration by parts, cited previously.

Theorem 2.2. *There can exist at most one strong solution of the problem* $\left(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota)\right)$, *in the sense of Definition 2.1.*

Proof: Let the pair $(\mathbf{u}, \varphi)$ be a strong solution of $\left(\mathbf{P}_0(0,0,0)\right)$, *i.e.*, with $\mathbf{f}_o = \mathbf{0}$, $\mathbf{t}_o = \mathbf{0}$, and $\varphi^\iota = 0$. Then, by using (1.25) and the mapping property of A_{κ_+} following from its definition in Theorem 2.1(*ii*), and noting that $\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}]$ is a normal field on Γ_+, we have the equalities

$$\begin{aligned}\overline{\varphi}_{,\mathbf{n}_+} A_{\kappa_+}\varphi_{,\mathbf{n}_+} = \overline{\varphi}_{,\mathbf{n}_+}\varphi|_{\Gamma_+} &= \left(-i\varrho_+\omega\overline{\mathbf{u}}|_{\Gamma_+}\cdot\mathbf{n}_+\right)\left(-\frac{1}{\gamma_+ - i\omega}\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}]\cdot\mathbf{n}_+\right)\\ &= \frac{i\varrho_+\omega}{\gamma_+ - i\omega}\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}]\cdot\overline{\mathbf{u}}|_{\Gamma_+}\end{aligned}$$

between elements of $L_1(\Gamma_+)$, whence a little rearrangement and integration over Γ_+ produces

$$\frac{c_+^2}{\omega^2}\langle\kappa_+^2 A_{\kappa_+}\varphi_{,\mathbf{n}_+}, \varphi_{,\mathbf{n}_+}\rangle_{L_2(\Gamma_+)} = -\varrho_+\int_{\Gamma_+}\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}]\cdot\overline{\mathbf{u}}|_{\Gamma_+}\, d\lambda_{\Gamma_+}. \tag{2.11}$$

On the other hand, from (2.9) and by taking into account (1.26) and (1.24), we get

$$\begin{aligned}\int_{\Gamma_+}\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}]\cdot\overline{\mathbf{u}}|_{\Gamma_+}\, d\lambda_{\Gamma_+} &= \int_{\Gamma_o}\mathbf{T}^{\mathbf{n}_o}[\mathbf{u}]\cdot\overline{\mathbf{u}}|_{\Gamma_o}\, d\lambda_{\Gamma_o}\\ &= \int_{\Omega_o}\{\overline{\mathbf{u}}\cdot\mathbf{\Delta}^\star_{\lambda,\mu}\mathbf{u} + \mathcal{E}_{\lambda,\mu}(\mathbf{u},\overline{\mathbf{u}})\}\, d\lambda_3 = \int_{\Omega_o}\{-\varrho_o\omega^2\mathbf{u}\cdot\overline{\mathbf{u}} + \mathcal{E}_{\lambda,\mu}(\mathbf{u},\overline{\mathbf{u}})\}\, d\lambda_3,\end{aligned}$$

clearly implying that the integral on the right in (2.11) is real. Thus, $\langle\kappa_+^2 A_{\kappa_+}\varphi_{,\mathbf{n}_+}, \varphi_{,\mathbf{n}_+}\rangle_{L_2(\Gamma_+)}$ is real, so that the vanishing of $\varphi_{,\mathbf{n}_+}$ follows from (2.1). Statement (*i*) of Theorem 2.1 then allows us to assert that φ vanishes in Ω_+. Now, (1.24) through (1.26) and the inclusion $\mathbf{u} \in \mathbf{H}^2$ show that $\mathbf{u} \in \mathcal{N}^2_{\Omega_o,\omega}$; since we also know that $\mathbf{u} \in (\mathcal{N}^2_{\Omega_o,\omega})^{(\perp)_0}$ (by Definition 2.1), it must be that $\mathbf{u} = \mathbf{0}$. This completes the proof. ∎

It is interesting to note that, without the condition that $\mathbf{u}$ lie in $\mathcal{M}^2_{\Omega_o,\omega}$, we could still have deduced that $\varphi = 0$ by reasoning as in the proof, but then would have been able to conclude only that $\mathbf{u} \in \mathcal{N}^2_{\Omega_o,\omega}$.

Now, we turn to the reduction of the strongly formulated boundary and interface problem $(\mathbf{P}_0)$ to a purely interior problem (in Ω_o) with a nonlocal boundary condition, basing the replacement on the properties of the operator A_{κ_+}. Suppose that $(\mathbf{u}, \varphi)$ is a strong solution (and so also the unique strong solution) of $(\mathbf{P}_0)$. Then, by the boundary-data-mapping property of A_{κ_+} and the first equality in (1.25), we have

$$\varphi|_{\Gamma_+} = A_{\kappa_+}\varphi_{,\mathbf{n}_+} = A_{\kappa_+}(i\varrho_+\omega\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+) - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+},$$

and so the second equality in (1.25) becomes

$$\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}] + (\gamma_+ - i\omega)\{i\varrho_+\omega A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+) - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+ = \mathbf{t}_+ - (\gamma_+ - i\omega)\varphi^\iota|_{\Gamma_+}\mathbf{n}_+,$$

or

$$\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}] + \varrho_+ c_+^2\kappa_+^2\{A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)\}\mathbf{n}_+ = \mathbf{t}_+ - (\gamma_+ - i\omega)\{\varphi^\iota|_{\Gamma_+} - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+, \tag{2.12}$$

a boundary condition on $\mathbf{u}$ alone. This motivates

Definition 2.2. Recall Terminology 2.1. The function $\mathbf{u}$ is *a strong solution of the purely interior problem corresponding to* $(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota))$ iff $\mathbf{u} \in \mathcal{M}^2_{\Omega_o,\omega}$, (1.24) holds a.e. in Ω_o, (2.12) is true a.e. on Γ_+, and (1.26) holds a.e. on Γ_-.

To find a strong solution of the original problem $(\mathbf{P}_0)$, it is easy to see that it suffices to generate a strong solution of the corresponding purely interior problem. For, if $\mathbf{u}$ fulfills all requirements of Definition 2.2, let us construct the unique $\varphi \in W(\Omega_+;\kappa_+)$ having the Neumann data given by

$$\varphi_{,\mathbf{n}_+} = i\varrho_+\omega\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+ - \varphi^\iota_{,\mathbf{n}_+}; \tag{2.13}$$

observe that the function on the right here lies in $H^{\frac{1}{2}}(\Gamma_+)$ (cf. our previous remark). Explicitly, from the representation (2.2) of Theorem 2.1, this φ is given by

$$\begin{aligned}\varphi(\mathbf{x}) = &-\frac{1}{2}\int_{\Gamma_+}\{E^{\kappa_+}_{\mathbf{x}}\varphi^\iota_{,\mathbf{n}_+} - E^{\kappa_+}_{\mathbf{x},\mathbf{n}_+}A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\,d\lambda_{\Gamma_+}\\ &+\frac{i\varrho_+\omega}{2}\int_{\Gamma_+}\{E^{\kappa_+}_{\mathbf{x}}\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+ - E^{\kappa_+}_{\mathbf{x},\mathbf{n}_+}A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)\}\,d\lambda_{\Gamma_+}\qquad \text{for each } \mathbf{x}\in\Omega_+.\end{aligned} \tag{2.14}$$

With (2.13), we have automatically ensured that the first equality in (1.25) is fulfilled a.e. on Γ_+, while the validity of the second requirement there follows in the same sense from (2.13) and (2.12):

$$\begin{aligned}\mathbf{T}^{\mathbf{n}_+}[\mathbf{u}] + (\gamma_+ - i\omega)\varphi|_{\Gamma_+}\mathbf{n}_+ &= \mathbf{T}^{\mathbf{n}_+}[\mathbf{u}] + (\gamma_+ - i\omega)(A_{\kappa_+}\varphi_{,\mathbf{n}_+})\mathbf{n}_+\\ &= -\varrho_+\omega(\omega + i\gamma_+)\{A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+)\}\mathbf{n}_+ + \mathbf{t}_+ - (\gamma_+ - i\omega)\{\varphi^\iota|_{\Gamma_+} - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+\\ &\qquad + (\gamma_+ - i\omega)\{i\varrho_+\omega A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+) - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+\\ &= \mathbf{t}_+ - (\gamma_+ - i\omega)\varphi^\iota|_{\Gamma_+}\mathbf{n}_+.\end{aligned}$$

Consequently, $(\mathbf{u}, \varphi)$ satisfies all of the requirements placed in Definition 2.1 on a strong solution of $(\mathbf{P}_0)$. It is important to note here that only the normal component $\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+$ of the trace of $\mathbf{u}$ restricted to Γ_+ is needed for the construction of φ by use of (2.14), *i.e.*, to calculate the fluid field once the strong solution of the purely interior problem for the elastic field has been obtained; the complete implication of this observation for the numerical work will emerge in §4. We might also remark that the representation displayed for φ in (2.14) has split naturally into a sum of two integrals taken over Γ_+. Of these, Theorem 2.1 makes it clear that the first is a representation for the unique element of $W(\Omega_+;\kappa_+)$ with Neumann data $-\varphi^\iota_{,\mathbf{n}_+}$, which is nothing else but the complex amplitude of the time-harmonic scattered-field velocity potential that would be produced by the interaction of the time-harmonic incident-field velocity potential with complex amplitude φ^{ι} with a *rigid* obstacle occupying $\Omega_- \cup \overline{\Omega}_o$, under the usual steady-state assumption. The second term on the right in (2.14) must then encompass all contributions from the elastic nature of the obstacle, the impressed tractions, and the imposed body force. By using Green's Theorem, the first term can be recast into the form $-(1/2)\int_{\Gamma_+} E^{\kappa_+}_{\mathbf{x}},_{\mathbf{n}_+}\{\varphi^{\iota}|\Gamma_+ - A_{\kappa_+}\varphi^{\iota},_{\mathbf{n}_+}\}\, d\lambda_{\Gamma_+}$, in which there appears the same field $\varphi^{\iota}|\Gamma_+ - A_{\kappa_+}\varphi^{\iota},_{\mathbf{n}_+}$ on Γ_+ as in the new boundary condition (2.12) of the purely interior problem; this is the sum of the incident and the scattered fields, or "total field," on Γ_+ in the rigid-acoustic steady-state scattering interaction just mentioned.

It is appropriate to include here an observation pertinent to the eventual numerical approximation of φ and its far-field pattern from numerical approximations generated for $\mathbf{u}$ by computations based on the interior problem. We recall that to each element ψ of $W(\Omega_+;\kappa_+)$ there is associated a unique complex function ψ_∞ on the unit sphere S_1 of $\mathbf{R}^3$ with the property that

$$\psi(r\hat{\mathbf{e}}) = \frac{e^{i\kappa_+ r}}{r}\psi_\infty(\hat{\mathbf{e}}) + O\left(\frac{1}{r^2}\right) \qquad \text{as } r\to\infty, \text{ for each } \hat{\mathbf{e}}\in S_1.$$

This function ψ_∞ is variously termed *the far-field pattern*, or *radiation pattern, of* ψ; cf., *e.g.*, Ref. 12. Since $\psi_\infty(\hat{\mathbf{e}}) = \lim_{r\to\infty}(e^{i\kappa_+ r}/r)^{-1}\psi(r\hat{\mathbf{e}})$ for each $\hat{\mathbf{e}}\in S_1$, by using the representation for elements of $W(\Omega_+;\kappa_+)$ that is obtained from (2.2), it is easy to show that

$$\psi_\infty(\hat{\mathbf{e}}) = -\frac{1}{4\pi}\int_{\Gamma_+}\{\mathbf{e}^{\kappa_+}_{\hat{\mathbf{e}}}\psi,_{\mathbf{n}_+} - \mathbf{e}^{\kappa_+}_{\hat{\mathbf{e}}},_{\mathbf{n}_+}A_{\kappa_+}\psi,_{\mathbf{n}_+}\}\, d\lambda_{\Gamma_+} \qquad \text{for each } \hat{\mathbf{e}}\in S_1, \tag{2.15}$$

whenever $\psi\in W(\Omega_+;\kappa_+)$, with $\mathbf{e}^{\kappa_+}_{\hat{\mathbf{e}}}(\mathbf{y}) := e^{-i\kappa_+\hat{\mathbf{e}}\cdot\mathbf{y}}$, for $\mathbf{y}\in\mathbf{R}^3$. Now, we can formulate the desired statement in the form of

Proposition 2.1. *Let* $\mathbf{u}$ *be an element of* $\mathbf{H}^1$, *and let* φ *be the unique element of* $W(\Omega_+;\kappa_+)$ *satisfying (2.13), so that* φ *has the representation displayed in (2.14); of course, if* $\mathbf{u}$ *happens to be a strong solution of the purely interior problem corresponding to* $(\mathbf{P}_0)$, *then* $(\mathbf{u},\varphi)$ *is the unique strong solution of the problem* $(\mathbf{P}_0)$. *Suppose that* $(\mathbf{u}_n)_{n=N}^\infty$ *is a sequence in* $\mathbf{H}^1$ *converging weakly to* $\mathbf{u}$ *in* $\mathbf{H}^1$. *Construct the corresponding sequence* $(\varphi_n)_{n=N}^\infty$ in $W(\Omega_+;\kappa_+)$ according to, for $n\geq N$,

$$\begin{aligned}\varphi_n(\mathbf{x}) := &-\frac{1}{2}\int_{\Gamma_+}\{E^{\kappa_+}_{\mathbf{x}}\varphi^{\iota},_{\mathbf{n}_+} - E^{\kappa_+}_{\mathbf{x}},_{\mathbf{n}_+}A_{\kappa_+}\varphi^{\iota},_{\mathbf{n}_+}\}\, d\lambda_{\Gamma_+}\\ &+\frac{i\varrho_+\omega}{2}\int_{\Gamma_+}\{E^{\kappa_+}_{\mathbf{x}}\mathbf{u}_n|\Gamma_+\cdot\mathbf{n}_+ - E^{\kappa_+}_{\mathbf{x}},_{\mathbf{n}_+}A_{\kappa_+}(\mathbf{u}_n|\Gamma_+\cdot\mathbf{n}_+)\}\, d\lambda_{\Gamma_+} \qquad \text{for each } \mathbf{x}\in\Omega_+.\end{aligned} \tag{2.16}$$

Then, for each 3-index α, *the sequence* $(\varphi_{n},_{\alpha})_{n=N}^\infty$ *converges to* $\varphi,_{\alpha}$ *uniformly on each subset of* Ω_+ *that is closed in* $\mathbf{R}^3$. *Further, the sequence* $(\varphi_{n\infty})_{n=N}^\infty$ *of far-field patterns converges to the far-field pattern* φ_∞ *of* φ *uniformly on* S_1.

Proof: We have, for each $\mathbf{x}\in\Omega_+$ and $n\geq N$,

$$\varphi(\mathbf{x}) - \varphi_n(\mathbf{x}) = \frac{i\varrho_+\omega}{2}\int_{\Gamma_+}\{E^{\kappa_+}_{\mathbf{x}}(\mathbf{u}-\mathbf{u}_n)|\Gamma_+\cdot\mathbf{n}_+ - E^{\kappa_+}_{\mathbf{x}},_{\mathbf{n}_+}A_{\kappa_+}((\mathbf{u}-\mathbf{u}_n)|\Gamma_+\cdot\mathbf{n}_+)\}\, d\lambda_{\Gamma_+}. \tag{2.17}$$

Now, choose any set F that is closed in $\mathbf{R}^3$ and contained in Ω_+. Since $\operatorname{Im}\kappa_+\geq 0$ and the distance between F and the compact set $\Gamma_+ := \partial\Omega_+$ is positive, it is clear that $E^{\kappa_+}_{\mathbf{x}}|\Gamma_+$ and $E^{\kappa_+}_{\mathbf{x}},_{\mathbf{n}_+}$ are bounded on Γ_+, uniformly for $\mathbf{x}\in F$. Thus, by applying the Cauchy-Schwarz inequality to (2.17) and using the boundedness of A_{κ_+} in $L_2(\Gamma_+)$, we find that

$$\sup_{\mathbf{x}\in F}|\varphi(\mathbf{x}) - \varphi_n(\mathbf{x})| \leq \frac{\varrho_+\omega}{2}\lambda_{\Gamma_+}(\Gamma_+)\{c_1 + c_2\|A_{\kappa_+}\|\}\|(\mathbf{u}-\mathbf{u}_n)|\Gamma_+\|_{\mathbf{L}_2(\Gamma_+)} \qquad \text{for } n\geq N,$$

wherein c_1 and c_2 depend upon only F and κ_+. The compactness of the Sobolev restricted-trace map $\mathbf{v} \mapsto \mathbf{v}|_{\Gamma_+}$ from $\mathbf{H}^1$ into $\mathbf{L}_2(\Gamma_+)$ and the weak convergence of $(\mathbf{u}_n)_{n=N}^{\infty}$ to $\mathbf{u}$ in $\mathbf{H}^1$ then imply that the first assertion of the Proposition is true for $\alpha = (0,0,0)$. The proof for any other 3-index α can obviously be carried through in an entirely analogous manner, since the corresponding partial derivatives can be computed by differentiation under the integral in (2.14) and (2.16). The final assertion, concerning the convergence of the far-field patterns, is established just as easily, by similar reasoning based on (2.15). ∎

In the applications of Proposition 2.1 that are pertinent to the present setting, we shall even have available the norm-convergence of $(\mathbf{u}_n)_{n=N}^{\infty}$ to $\mathbf{u}$ in $\mathbf{H}^1$ (cf. Theorem 4.1, *infra*), not merely weak convergence; assuming some additional regularity for Γ_+, we can sketch the proof of a correspondingly stronger result for the convergence of the sequence $(\varphi_n)_{n=N}^{\infty}$ to φ:

Proposition 2.2. *Retaining the setting and notation of Proposition 2.1, suppose now also that $\overline{\Omega}_- \cup \Omega_o$ is of type $\Re^{3,\nu}$, while $(\mathbf{u}_n)_{n=N}^{\infty}$ converges to $\mathbf{u}$ in the norm of $\mathbf{H}^1$. Then φ and the sequence $(\varphi_n)_{n=N}^{\infty}$ lie in $H^2_{loc}(\overline{\Omega}_+)$, with $\lim_{n\to\infty} \varphi_n = \varphi$ in the locally convex topology of that space. Moreover, φ and the sequence $(\varphi_n)_{n=N}^{\infty}$ lie in $C^{0,\frac{1}{2}}(\overline{\Omega}_+)$, and $\lim_{n\to\infty}\varphi_n = \varphi$ in that Banach space (so, in particular, the convergence is uniform on $\overline{\Omega}_+$).*

Proof: Now, we know, by Lemma 2.1, that $(\mathbf{u}_n|_{\Gamma_+} \cdot \mathbf{n}_+)_{n=N}^{\infty}$ converges to $\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ in the norm of $H^{\frac{1}{2}}(\Gamma_+)$. We recall the definition of the double-layer potential-operator (restricted to Ω_+) $f \mapsto W^+_{\kappa_+}\{f\}$ on $L_2(\Gamma_+)$ (*supra*), and now define the single-layer potential-operator (restricted to Ω_+) $f \mapsto V^+_{\kappa_+}\{f\}$ on $L_2(\Gamma_+)$ by

$$V^+_{\kappa_+}\{f\}(\mathbf{x}) := \int_{\Gamma_+} E^{\kappa_+}_{\mathbf{x}} f \, d\lambda_{\Gamma_+} \qquad \text{for } \mathbf{x} \in \Omega_+.$$

We have sufficient regularity of Γ_+ to ensure that, according to Ref. 10, $A_{\kappa_+}|H^{\frac{1}{2}}(\Gamma_+)$ maps $H^{\frac{1}{2}}(\Gamma_+)$ boundedly into $H^{\frac{3}{2}}(\Gamma_+)$, and, by Theorem 2.16 of Ref. 13, that the double-layer potential-operator is bounded from $H^{\frac{3}{2}}(\Gamma_+)$ into $H^2_{loc}(\overline{\Omega}_+)$, while the single-layer potential-operator is bounded from $H^{\frac{1}{2}}(\Gamma_+)$ into $H^2_{loc}(\overline{\Omega}_+)$. Combining all of these facts, in view of (2.14) and (2.16) it follows that φ and $(\varphi_n)_{n=N}^{\infty}$ are contained in $H^2_{loc}(\overline{\Omega}_+)$, with $\lim_{n\to\infty}\varphi_n = \varphi$ in that space (cf. (2.17)). Directly, the Sobolev Imbedding Theorem (Theorem 5.4, Part II, of Ref. 9) allows us to assert that, for any bounded open set $\Omega \subset \Omega_+$, $\varphi|\Omega$ and the $\varphi_n|\Omega$ are elements of $C^{0,\frac{1}{2}}(\overline{\Omega})$, with the convergence $\lim_{n\to\infty}\varphi_n|\Omega = \varphi|\Omega$ in the norm of the latter Banach space. Taking Ω to be the intersection of Ω_+ with, say, a ball of sufficiently large radius, and accounting for the results of Proposition 2.1 concerning the uniform convergence of the φ_n and their partial derivatives to φ and its partial derivatives in the exterior of the ball, we see that the φ_n and φ lie in the Banach space $C^{0,\frac{1}{2}}(\overline{\Omega}_+)$, and $\lim_{n\to\infty}\varphi_n = \varphi$ in the norm of that space. ∎

Returning to the general setting, we shall study the purely interior problem by setting up and analyzing a weak version of it; this will also lead back in a natural manner to weak formulations for the original problems $(\mathbf{P}_0)$ and $(\mathbf{P})$. Regularity results for solutions of the weakly formulated interior problem would then allow us subsequently to return to the strong setting. As usual, the weak formulation arises from (2.9), written in the form

$$\int_{\Omega_o} \{\mathbf{\Delta}^*_{\lambda,\mu}\mathbf{u} + \varrho_o\omega^2\mathbf{u} + \mathbf{f}_o\} \cdot \overline{\mathbf{v}}\, d\lambda_3 + \int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u},\overline{\mathbf{v}}) - \varrho_o\omega^2\mathbf{u}\cdot\overline{\mathbf{v}}\}\, d\lambda_3$$
$$= \int_{\Gamma_o} \mathbf{T}^{\mathbf{n}_o}[\mathbf{u}] \cdot \overline{\mathbf{v}}|_{\Gamma_o}\, d\lambda_{\Gamma_o} + \int_{\Omega_o} \mathbf{f}_o \cdot \overline{\mathbf{v}}\, d\lambda_3 \qquad \text{for } \mathbf{u} \in \mathbf{H}^2 \text{ and } \mathbf{v} \in \mathbf{H}^1 \qquad (2.18)$$

(with $\mathbf{f}_o \in \mathbf{H}^0$). Now, if $\mathbf{u}$ is a strong solution of the purely interior problem for $(\mathbf{P}_0)$, as in Definition 2.2, then we must have

$$\int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u}, \overline{\mathbf{v}}) - \varrho_o\omega^2\mathbf{u}\cdot\overline{\mathbf{v}}\}\, d\lambda_3 + \varrho_+ c_+^2 \kappa_+^2 \int_{\Gamma_+} (A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+))\overline{\mathbf{v}}|_{\Gamma_+}\cdot\mathbf{n}_+\, d\lambda_{\Gamma_+}$$
$$= \int_{\Gamma_+} \{\mathbf{t}_+ - (\gamma_+ - i\omega)\{\varphi^\iota|_{\Gamma_+} - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+\}\cdot\overline{\mathbf{v}}|_{\Gamma_+}\, d\lambda_{\Gamma_+} + \int_{\Gamma_-} \mathbf{t}_-\cdot\overline{\mathbf{v}}|_{\Gamma_-}\, d\lambda_{\Gamma_-}$$
$$+ \int_{\Omega_o} \mathbf{f}_o\cdot\overline{\mathbf{v}}\, d\lambda_3 \qquad \text{whenever } \mathbf{v}\in\mathbf{H}^1. \tag{2.19}$$

This leads us to define the sesquilinear form $\sigma_\omega(\cdot,\cdot)$ on $\mathbf{H}^1$ according to

$$\sigma_\omega(\mathbf{u},\mathbf{v}) := \int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u}, \overline{\mathbf{v}}) - \varrho_o\omega^2\mathbf{u}\cdot\overline{\mathbf{v}}\}\, d\lambda_3 + \varrho_+ c_+^2 \kappa_+^2 \int_{\Gamma_+} (A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+))\overline{\mathbf{v}}|_{\Gamma_+}\cdot\mathbf{n}_+\, d\lambda_{\Gamma_+},$$
$$\text{for } \mathbf{u},\mathbf{v}\in\mathbf{H}^1, \tag{2.20}$$

and, corresponding to selected functions $\mathbf{f}_o$, $\mathbf{t}_o$, and φ^ι as in Terminology 2.1, the conjugate-linear functional Λ_ω on $\mathbf{H}^1$ by

$$\Lambda_\omega\mathbf{v} := \int_{\Gamma_+} \{\mathbf{t}_+ - (\gamma_+ - i\omega)\{\varphi^\iota|_{\Gamma_+} - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+\}\cdot\overline{\mathbf{v}}|_{\Gamma_+}\, d\lambda_{\Gamma_+} + \int_{\Gamma_-} \mathbf{t}_-\cdot\overline{\mathbf{v}}|_{\Gamma_-}\, d\lambda_{\Gamma_-}$$
$$+ \int_{\Omega_o} \mathbf{f}_o\cdot\overline{\mathbf{v}}\, d\lambda_3, \qquad \text{for each } \mathbf{v}\in\mathbf{H}^1. \tag{2.21}$$

Then (2.19) is written concisely as $\sigma_\omega(\mathbf{u},\mathbf{v}) = \Lambda_\omega\mathbf{v}$ for $\mathbf{v}\in\mathbf{H}^1$; the weak formulation of the purely interior problem corresponding to $(\mathbf{P}_0)$ shall be set up on the basis of this equality.

It is also convenient to identify a linear manifold comprising complex amplitudes of *weak* nonradiating modes, proceeding from (2.18). If $\mathbf{u}_\omega \in \mathcal{N}^2_{\Omega_o,\omega}$, then (2.18) shows that

$$\int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u}_\omega, \overline{\mathbf{v}}) - \varrho_o\omega^2\mathbf{u}_\omega\cdot\overline{\mathbf{v}}\}\, d\lambda_3 = 0 \qquad \text{for every } \mathbf{v}\in\mathbf{H}^1, \tag{2.22}$$

while $\mathbf{u}_\omega$ also lies in the subspace $\mathbf{H}^1_{0\mathbf{n}_+}$ given by

$$\mathbf{H}^1_{0\mathbf{n}_+} := \{\,\mathbf{u}\in\mathbf{H}^1 \mid \mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+ = 0\,\} \tag{2.23}$$

(the closure in $\mathbf{H}^1$ of the collection of all $\mathbf{u}\in\mathbf{C}^1(\overline{\Omega}_o)$ such that $\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+ = 0$). With this, we define the linear manifold $\mathcal{N}^1_{\Omega_o,\omega}$ in $\mathbf{H}^1$ by setting

$$\mathcal{N}^1_{\Omega_o,\omega} := \{\,\mathbf{u}_\omega\in\mathbf{H}^1_{0\mathbf{n}_+} \mid (2.22) \text{ holds}\,\}. \tag{2.24}$$

Then $\mathcal{N}^2_{\Omega_o,\omega} \subset \mathcal{N}^1_{\Omega_o,\omega}$. At least under sufficiently strong hypotheses of smoothness for Γ_o, we shall also have $\mathcal{N}^1_{\Omega_o,\omega} \subset \mathbf{H}^2$, whence it shall follow that $\mathcal{N}^2_{\Omega_o,\omega} = \mathcal{N}^1_{\Omega_o,\omega}$. For example, in the extreme case in which Ω_o is of type $\Re^\infty$, we shall have $\mathcal{N}^1_{\Omega_o,\omega} \subset \mathbf{C}^\infty(\overline{\Omega}_o)$; cf. Ref. 14. Proceeding as we did for $\mathcal{N}^2_{\Omega_o,\omega}$, we denote by $(\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_0}$ the orthogonal complement of $\mathcal{N}^1_{\Omega_o,\omega}$ taken with respect to $\mathbf{H}^0$, and introduce $\mathcal{M}^1_{\Omega_o,\omega} \subset \mathbf{H}^1$ by

$$\mathcal{M}^1_{\Omega_o,\omega} := (\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_0} \cap \mathbf{H}^1; \tag{2.25}$$

$\mathcal{M}^1_{\Omega_o,\omega}$ is a closed linear manifold (subspace) in $\mathbf{H}^1$. We shall show later that $\mathcal{N}^1_{\Omega_o,\omega}$ is finite-dimensional, and so also closed in both $\mathbf{H}^0$ and $\mathbf{H}^1$. Once this is known, just as before (cf. (2.5) and (2.6)) it will follow that the direct-sum decompositions

$$\mathbf{H}^0 = \mathcal{N}^1_{\Omega_o,\omega} \oplus (\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_0} \tag{2.26}$$

and

$$\mathbf{H}^1 = \mathcal{N}^1_{\Omega_o,\omega} + \mathcal{M}^1_{\Omega_o,\omega} \quad \text{with} \quad \mathcal{N}^1_{\Omega_o,\omega} \cap \mathcal{M}^1_{\Omega_o,\omega} = \{0\} \tag{2.27}$$

are valid (of which the first is orthogonal). We still have the interior regularity result asserting that the elements of $\mathcal{N}^1_{\Omega_o,\omega}$ have real and imaginary parts that are real-analytic in Ω_o, so (1.27) holds in the classical sense in Ω_o for each $\mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}$.

As already tacitly noted in Proposition 2.1, for the construction of the unique $\varphi \in W(\Omega_+;\kappa_+)$ satisfying (2.13), we need know only that $\mathbf{u} \in \mathbf{H}^1$. This affords a simple means for providing weak formulations for both $(\mathbf{P}_0)$ and $(\mathbf{P})$, based upon that for the purely interior problem for $(\mathbf{P}_0)$. Precisely, the weak formulations are given in

Definition 2.3. Recall Terminology 2.1. The function $\mathbf{u}$ is *a weak solution of the purely interior problem corresponding to* $(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota))$ iff $\mathbf{u} \in \mathcal{M}^1_{\Omega_o,\omega}$ and

$$\sigma_\omega(\mathbf{u},\mathbf{v}) = \Lambda_\omega \mathbf{v} \qquad \text{for every } \mathbf{v} \in \mathbf{H}^1, \tag{2.28}$$

with the sesquilinear form σ_ω and the conjugate-linear functional Λ_ω defined by (2.20) and (2.21), respectively. If $\mathbf{u}$ is a weak solution of the purely interior problem corresponding to $(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota))$ and φ is the corresponding unique element of $W(\Omega_+;\kappa_+)$ such that (2.13) holds, then the pair $(\mathbf{u},\varphi)$ is termed *a weak solution of* $(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota))$. Suppose further that the integral in (2.7) exists for every $\mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}$, and it is known that $\mathcal{N}^1_{\Omega_o,\omega}$ has finite dimension $n_\omega \geq 0$; if $n_\omega \geq 1$, let $\{\mathbf{u}^{(j)}_\omega\}_{j=1}^{n_\omega}$ be an $\mathbf{H}^0$-orthonormal basis for $\mathcal{N}^1_{\Omega_o,\omega}$. Then $(\mathbf{u},\varphi)$ is termed *a weak solution of* $(\mathbf{P}(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota; \mathbf{F}^\tau_o, \mathbf{T}^\tau_o))$ iff $\mathbf{u}$ can be written as in (2.8) and $(\mathbf{u}_0,\varphi)$ is a weak solution of $(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota))$.

Observe here that, for $(\mathbf{P}_0)$, the requirement $\mathbf{u} \in \mathcal{M}^1_{\Omega_o,\omega}$ constitutes a weak form of (1.35), while, for $(\mathbf{P})$, the condition that $\mathbf{u}$ have the form in (2.8) provides a weak realization of (1.36).

According to Terminology 2.1, we have been supposing that the piece of data $\mathbf{t}_o$ is in $\mathbf{L}_2(\Gamma_o)$; one can relax this requirement by positing instead the inclusion $\mathbf{t}_o \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_o)$, provided that appropriate modification is made in the definition of Λ_ω in (2.21), *viz.*, that the sum of the terms involving $\mathbf{t}_+$ and $\mathbf{t}_-$ be replaced by the more general expression $(\mathbf{t}_o, \mathbf{v}|_{\Gamma_o})_{\frac{1}{2}}$, involving the duality pairing for $\mathbf{H}^{\frac{1}{2}}(\Gamma_o)$. The resultant conjugate-linear functional Λ_ω is again bounded on $\mathbf{H}^1$, and the analysis of §4 can be carried through, *mutatis mutandis*. For simplicity, we shall continue to suppose that $\mathbf{t}_o \in \mathbf{L}_2(\Gamma_o)$.

It is clear that a weak solution of the purely interior problem corresponding to $(\mathbf{P}_0)$ that is also known to possess some additional regularity will be a strong solution of the purely interior problem for $(\mathbf{P}_0)$. For, if $\mathbf{u}$ satisfies the requirements placed on a weak solution of the purely interior problem for $(\mathbf{P}_0)$ in Definition 2.3 and also lies in $\mathbf{H}^2$, then $\mathbf{u} \in \mathcal{M}^2_{\Omega_o,\omega}$ (since the inclusion $\mathcal{N}^2_{\Omega_o,\omega} \subset \mathcal{N}^1_{\Omega_o,\omega}$ implies that $(\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_0} \subset (\mathcal{N}^2_{\Omega_o,\omega})^{(\perp)_0}$), while (2.18) and (2.28) (*i.e.*, (2.19)) together imply that

$$\begin{aligned} -\int_{\Omega_o} \{\boldsymbol{\Delta}^\star_{\lambda,\mu}\mathbf{u} + \varrho_o\omega^2\mathbf{u} - \mathbf{f}_o\} \cdot \overline{\mathbf{v}}\, d\lambda_3 + \int_{\Gamma_o} \mathbf{T}^{\mathbf{n}_o}[\mathbf{u}] \cdot \overline{\mathbf{v}}|_{\Gamma_o}\, d\lambda_{\Gamma_o} \\ = \int_{\Gamma_+} \{-\varrho_+ c_+^2 \kappa_+^2 (A_{\kappa_+}(\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+))\mathbf{n}_+ + \mathbf{t}_+ - (\gamma_+ - i\omega)\{\varphi^\iota|_{\Gamma_+} - A_{\kappa_+}\varphi^\iota_{,\mathbf{n}_+}\}\mathbf{n}_+\} \cdot \overline{\mathbf{v}}|_{\Gamma_+}\, d\lambda_{\Gamma_+} \\ + \int_{\Gamma_-} \mathbf{t}_- \cdot \overline{\mathbf{v}}|_{\Gamma_-}\, d\lambda_{\Gamma_-} \qquad \text{for every } \mathbf{v} \in \mathbf{H}^1. \end{aligned} \tag{2.29}$$

Since the equality in (2.29) must hold, in particular, for every $\mathbf{v} \in \mathbf{C}^\infty_0(\Omega_o)$, (1.24) follows (a.e. in Ω_o); with the simplified form then taken on by (2.29), we conclude that (2.12) and (1.26) are true (a.e. on Γ_+ and Γ_-, respectively). Thus, such a $\mathbf{u}$ has all of the properties demanded of a strong solution of the purely interior problem in Definition 2.2, and, moreover, leads directly to the strong solution for ($\mathbf{P}_0$) itself (as described in the remarks following Definition 2.2) and to the strong solution for ($\mathbf{P}$) (as in Definition 2.1). Observe that we do not need to know here whether $\mathcal{N}^1_{\Omega_o,\omega}$ and $\mathcal{N}^2_{\Omega_o,\omega}$ coincide.

3. VARIATIONAL PROBLEMS INVOLVING A DEFINITE FORM WITH A COMPACT PERTURBATION; THE GALERKIN METHOD

For the analysis of the weak formulation of the purely interior problem, we require some facts about problems posed "in variational form;" in particular, for the construction of a convergent approximation scheme, we shall use results of Hildebrandt and Wienholtz [6] concerning the Galerkin method.

Throughout this section, $(H, \langle\cdot,\cdot\rangle_H)$ denotes a separable complex Hilbert space, $\sigma(\cdot,\cdot)$ a bounded sesquilinear form on H, and Λ a bounded conjugate-linear functional on H. We are interested in studying the solvability of the problem

$$\textit{find } f \in H \textit{ satisfying } \quad \sigma(f,h) = \Lambda h \quad \textit{for all } h \in H, \tag{3.1}$$

and in establishing a viable and convergent scheme for the approximation of a solution (when at least one exists), under certain hypotheses on the form σ. Of course, we know that there are uniquely determined a bounded linear operator T_σ on H and an element g_Λ of H such that

$$\sigma(g,h) = \langle T_\sigma g, h\rangle_H \qquad \text{for all } g,h \in H \tag{3.2}$$

and

$$\Lambda h = \langle g_\Lambda, h\rangle_H \qquad \text{for all } h \in H, \tag{3.3}$$

so that (3.1) is equivalent to the operator problem

$$\textit{find } f \in H \textit{ satisfying } \quad T_\sigma f = g_\Lambda. \tag{3.4}$$

The latter equivalence is useful for examining the solvability of (3.1), but we prefer not to base the computations of a numerical scheme upon the availability of explicit analytical forms for T_σ and g_Λ since such formulae may be difficult to deduce.

By $\mathcal{N}(L)$ we mean the null space of the operator or functional L. $\mathcal{N}_1(\sigma)$ is the subspace of H defined by

$$\mathcal{N}_1(\sigma) := \{\, g \in H \mid \sigma(g,h) = 0 \quad \text{for all } h \in H \,\}$$

(throughout, a *subspace* of a Hilbert space is defined to be a linear manifold that is closed in that Hilbert space). The (bounded) sesquilinear form $\sigma^*(\cdot,\cdot)$ *adjoint to* σ is defined by

$$\sigma^*(g,h) := \overline{\sigma(h,g)} \qquad \text{for } g,h \in H;$$

σ^* is represented by T_σ^*, the (Hilbert-space) adjoint of T_σ, in the same way that σ is represented by T_σ. Corresponding to the adjoint form, we have

$$\mathcal{N}_1(\sigma^*) := \{\, g \in H \mid \sigma^*(g,h) = 0 \quad \text{for all } h \in H \,\} = \{\, g \in H \mid \sigma(h,g) = 0 \quad \text{for all } h \in H \,\}.$$

With (3.2), it is easy to check that $\mathcal{N}_1(\sigma) = \mathcal{N}(T_\sigma)$ and $\mathcal{N}_1(\sigma^*) = \mathcal{N}(T_\sigma^*)$.

Invariably, we assume that $\sigma(\cdot,\cdot)$ is of the form $\delta(\cdot,\cdot) + \kappa(\cdot,\cdot)$, in which the sesquilinear forms δ and κ are bounded on H, with δ definite and κ compact. By the requirement of definiteness for δ, we mean that there exists $c_\delta > 0$ such that

$$|\delta(g,g)| \geq c_\delta \|g\|_H^2 \qquad \text{for all } g \in H; \tag{3.5}$$

by the requirement of compactness for κ, we demand that whenever g and h are elements of H and $(g_n)_{n=0}^{\infty}$ and $(h_n)_{n=0}^{\infty}$ are sequences in H converging weakly to g and h, respectively, then $(\kappa(g_n, h_n))_{n=0}^{\infty}$ converges in $\mathbf{C}$ to $\kappa(g,h)$. Of course, a compact form is necessarily bounded. For the uniquely determined bounded linear operators T_δ and T_κ on H such that

$$\delta(g,h) = \langle T_\delta g, h\rangle_H \qquad \text{and} \qquad \kappa(g,h) = \langle T_\kappa g, h\rangle_H \qquad \text{for all } g,h \in H, \tag{3.6}$$

we know then that T_δ is definite ($|\langle T_\delta g, g\rangle_H| \geq c_\delta \|g\|_H^2$ for all $g \in H$) and T_κ is compact (since $\|T_\kappa f\|_H^2 = \kappa(f, T_\kappa f)$ for $f \in H$ and the boundedness of T_κ certainly implies that it maps weakly convergent sequences into weakly convergent sequences). Under these hypotheses, the solvability of the problem (3.1) is completely characterized by the Fredholm Theorems that are available for it in the form of the following statement (for clarity, it is desirable to provide a proof of these facts showing that their validity depends on only the form hypothesized for σ, and is independent of the considerations involved in establishing an approximation scheme; cf. Ref. 6).

Theorem 3.1. *Let the sesquilinear form σ on H be the sum $\delta + \kappa$ of bounded sesquilinear forms, with δ definite and κ compact.*

(*i*) *The dimensions of $\mathcal{N}_1(\sigma)$ and $\mathcal{N}_1(\sigma^*)$ are finite and equal.*

(*ii*) *If Λ is a bounded conjugate-linear functional on H, then there exists some $f_0 \in H$ such that*

$$\sigma(f_0, h) = \Lambda h \qquad \text{for all } h \in H \tag{3.7}$$

iff the ("solvability") condition

$$\mathcal{N}_1(\sigma^*) \subset \mathcal{N}(\Lambda) \tag{3.8}$$

obtains, in which case the collection of all solutions of (3.1) is given by $f_0 + \mathcal{N}_1(\sigma)$. Consequently, there exists a unique solution of (3.1) iff $\mathcal{N}_1(\sigma) = \{0\}$ (or, equivalently, iff $\mathcal{N}_1(\sigma^*) = \{0\}$).

(*iii*) *Let $\mathcal{M}$ be a subspace of H that is complementary to $\mathcal{N}_1(\sigma)$, i.e., such that*

$$\mathcal{N}_1(\sigma) \cap \mathcal{M} = \{0\} \quad \text{and} \quad H = \dot{\mathcal{N}_1}(\sigma) + \mathcal{M}. \tag{3.9}$$

Then, whenever Λ is a bounded conjugate-linear functional on H such that (3.8) holds, there exists precisely one corresponding $f_{\Lambda\mathcal{M}} \in \mathcal{M}$ for which

$$\sigma(f_{\Lambda\mathcal{M}}, h) = \Lambda h \qquad \text{for all } h \in H; \tag{3.10}$$

moreover, there exists $M > 0$ with the property that, for all such Λ

$$\|f_{\Lambda\mathcal{M}}\|_H \leq M\|\Lambda\| \tag{3.11}$$

(with $\|\Lambda\|$ denoting the norm of Λ).

Proof: Adhering to the notation already introduced, we have $T_\sigma = T_\delta + T_\kappa$, with T_δ definite and T_κ compact. It is easy to see that the bounded operator T_δ is bijective. Indeed, the inequality $|\langle T_\delta g, g\rangle_H| \geq c_\delta\|g\|_H^2$, holding for all $g \in H$, gives $\|T_\delta g\|_H \geq c_\delta\|g\|_H$ for all g, whence we conclude that T_δ is injective and has closed range $\mathcal{R}(T_\delta)$ (as well as bounded inverse on $\mathcal{R}(T_\delta)$ into H). Clearly, the same reasoning can be effected for T_δ^*. Thus, $\mathcal{R}(T_\delta) = \mathcal{N}(T_\delta^*)^\perp = H$ (and $\mathcal{R}(T_\delta^*) = \mathcal{N}(T_\delta)^\perp = H$), proving the claim. Now we can write, *e.g.*, $T_\sigma = T_\delta(I + T_\delta^{-1}T_\kappa)$ and argue in the well-known manner, from the validity of the Fredholm Theorems for a perturbation of the identity by a compact operator, to conclude that those same statements are valid for T_σ. Having secured this, all of the assertions of the theorem follow from (3.2) and the equivalence of (3.1) and (3.4). Thus, (*i*) holds, in view of the equalities $\mathcal{N}_1(\sigma) = \mathcal{N}(T_\sigma)$ and $\mathcal{N}_1(\sigma^*) = \mathcal{N}(T_\sigma^*)$. Further, there exists $f_0 \in H$ such that (3.7) is true iff $T_\sigma f_0 = g_\Lambda$ (cf. (3.3)), iff $g_\Lambda \in \mathcal{N}(T_\sigma^*)^\perp = \mathcal{N}_1(\sigma^*)^\perp$, iff

$\Lambda h = \langle g_\Lambda, h\rangle_H = 0$ for all $h \in \mathcal{N}_1(\sigma^*)$, iff (3.8) holds. This proves (ii), the final assertions concerning the collection of all solutions of (3.1) and the existence of a unique solution now being obviously true. Finally, for (iii), let $f_0 \in H$ satisfy (3.7), wherein Λ is a bounded conjugate-linear functional on H satisfying (3.8) (such an f_0 existing by (ii)). Writing $f_0 = f_1 + f_{\Lambda\mathcal{M}}$, with $f_1 \in \mathcal{N}_1(\sigma)$ and $f_{\Lambda\mathcal{M}} \in \mathcal{M}$, then (3.10) clearly holds, and $f_{\Lambda\mathcal{M}}$ is the unique element of $\mathcal{M}$ satisfying (3.10) since $\mathcal{M}\cap\mathcal{N}_1(\sigma) = \{0\}$. This reasoning implies that the bounded operator $T_\sigma|\mathcal{M}$ (the restriction of T_σ to $\mathcal{M}$) maps the subspace $\mathcal{M}$ bijectively onto the subspace $\mathcal{N}_1(\sigma^*)^\perp = \mathcal{N}(T_\sigma^*)^\perp$, and so has bounded inverse defined on the latter. With M denoting the norm of this inverse, since $T_\sigma f_{\Lambda\mathcal{M}} = g_\Lambda$, inequality (3.11) follows upon noting that $\|g_\Lambda\|_H = \|\Lambda\|$. ∎

For the construction of approximations to solutions of the problem (3.1), we first consider the special case in which $\mathcal{N}_1(\sigma)$ is the trivial subspace, then treat the general case in which $\mathcal{N}_1(\sigma)$ may be nontrivial, under an additional hypothesis on σ (that will in fact obtain in our subsequent application of the Galerkin method).

Theorem 3.2. *Let the sesquilinear form σ on H be the sum $\delta+\kappa$ of bounded sesquilinear forms, with δ definite and κ compact. Suppose further that $\mathcal{N}_1(\sigma) = \{0\}$. Let Λ be a bounded conjugate-linear functional on H, and denote the associated unique solution of (3.1) by f_Λ. Then f_Λ can be convergently approximated in the norm of H by the Galerkin method for (3.1), by using as coördinate functions any family that is linearly independent and complete in H. That is, suppose that $\{h_n\}_{n=1}^\infty$ is linearly independent and complete in H. Then there exists a positive integer N such that for $n \geq N$ the system*

$$\sum_{k=1}^{n} \sigma(h_k, h_l)\xi_k = \Lambda h_l, \qquad l = 1,\ldots,n, \tag{3.12}$$

possesses a unique solution $(\xi_k^{(n)})_{k=1}^n$ and the resultant sequence $(f_n := \sum_{k=1}^n \xi_k^{(n)} h_k)_{n=N}^\infty$ converges to f_Λ in the norm of H. In fact, there exists a positive c that is independent of Λ and such that

$$\|f_\Lambda - f_n\|_H \leq c \inf_{f\in H_n} \|f_\Lambda - f\|_H \qquad \text{for } n \geq N, \tag{3.13}$$

in which H_n is the linear span of $\{h_k\}_{k=1}^n$.

Proof: Retrace the reasoning adduced in the proofs of Theorem 1 and Remark 3° of Ref. 6, *mutatis mutandis*, replacing the "bilinear" form B and linear functional L of Ref. 6 with the sesquilinear form σ and conjugate-linear functional Λ, and accounting for the minor difference between the form of the variational problem given in Eq. (1) of Ref. 6 and that of (3.1). The error inequality (3.13) results from an inspection of the proof of Theorem 1 of Ref. 6. We have written "bilinear" here in referring to the form B of Ref. 6 because there seems to be some confusion in terminology, so one must take some care in working through the developments of Ref. 6. Evidently a complex Hilbert space is considered there, while the form B, hypothesized as "bilinear," seems to be treated as sesquilinear in all of the reasoning subsequent to the proof of Theorem 1 of Ref. 6. It is possible that the term "bilinearity" in Ref. 6 is actually used to indicate the property that is more properly designated "sesquilinearity"; if this is so, then there has been omitted a sign of complex conjugation in Eq. (3) of Ref. 6. With the latter change, one can read Ref. 6 with the term "bilinear" replaced by "sesquilinear" throughout. ∎

When $\mathcal{N}_1(\sigma)$ is nontrivial and the solvability condition (3.8) is fulfilled, Hildebrandt and Wienholtz [6] base their method for approximating a solution of (3.1) upon application of the Galerkin method to the "symmetrized" problem

$$\textit{find} \quad f' \in H \quad \textit{satisfying} \quad \tilde{\sigma}(f', h) = \Lambda h \quad \textit{for all } h \in H, \tag{3.14}$$

in which $\tilde{\sigma}$ is the (Hermitian-symmetric) form corresponding to $T_\sigma T_\sigma^*$, given by $\tilde{\sigma}(g,h) := \langle T_\sigma T_\sigma^* g, h\rangle_H = \sigma(T_\sigma^* g, h)$. They show that the Galerkin method can be applied to the problem (3.14) to produce approximations converging to the unique solution of this problem that also lies in the subspace $\mathcal{N}(T_\sigma^*)^\perp = \mathcal{N}(T_\sigma T_\sigma^*)^\perp$; if f'_Λ is this solution, then $f := T_\sigma^* f'_\Lambda$ is a solution of the original problem (3.1), unique to within an element of $\mathcal{N}(T_\sigma)$. Although this scheme may be the best one available in the general case, it has some unattractive features, *e.g.*, one must know how to determine the action of T_σ^* on the selected coördinate functions. Of course, a method based upon computation of values of the original form σ would be much preferred; in at least one special case, *viz.*, when $\mathcal{N}_1(\sigma) = \mathcal{N}_1(\sigma^*)$, such a method can be employed (note that this condition is automatically satisfied if σ is Hermitian symmetric). Still, for a complete resolution of the problem one can evidently not escape the necessity for determining $\mathcal{N}_1(\sigma)$, although some information can be salvaged without addressing the latter task; cf. the remarks following the proof of the next theorem.

As a preliminary, let us recall some terminology. Suppose that $\mathcal{M}$ is a subspace of H that is complementary to the subspace $\mathcal{N}_1(\sigma)$, *i.e.*, such that (3.9) holds. Then the linear operator $P_\mathcal{M} : H \to H$ defined by

$$P_\mathcal{M} h := h_2, \quad \text{wherein} \quad h = h_1 + h_2 \quad \text{with } h_1 \in \mathcal{N}_1(\sigma),\ h_2 \in \mathcal{M}, \qquad \text{for each } h \in H$$

is bounded and also a projection (*i.e.*, satisfies $P_\mathcal{M}^2 = P_\mathcal{M}$), called *the operator of projection onto $\mathcal{M}$ along $\mathcal{N}_1(\sigma)$*. Obviously, $\mathcal{N}(P_\mathcal{M}) = \mathcal{N}_1(\sigma)$, while $I - P_\mathcal{M}$ is also a bounded projection operator, carrying H onto $\mathcal{N}_1(\sigma)$ ("along $\mathcal{M}$"), with $\mathcal{N}(I - P_\mathcal{M}) = \mathcal{M}$. Of course, $P_\mathcal{M}$ is self-adjoint, *i.e.*, an operator of orthogonal projection, iff $\mathcal{M} = \mathcal{N}_1(\sigma)^\perp$.

Theorem 3.3. *Let the sesquilinear form σ on H be the sum $\delta + \kappa$ of bounded sesquilinear forms, with δ definite and κ compact, and suppose that the equality*

$$\mathcal{N}_1(\sigma) = \mathcal{N}_1(\sigma^*) \tag{3.15}$$

is known to hold. Let $\mathcal{M}$ be a subspace of H that is complementary to $\mathcal{N}_1(\sigma)$, as in (3.9), and denote by $P_\mathcal{M}$ the operator of projection onto $\mathcal{M}$ along $\mathcal{N}_1(\sigma)$. Suppose that Λ is a bounded conjugate-linear functional on H for which the solvability condition (3.8) is valid. Then the unique $f_{\Lambda\mathcal{M}} \in \mathcal{M}$ satisfying (3.10) can be convergently approximated in the norm of H by use of the Galerkin method applied to the form σ. More precisely, let $\{h_n\}_{n=1}^\infty$ be a family in H such that $\{P_\mathcal{M} h_n\}_{n=1}^\infty$ is linearly independent and complete in the Hilbert space $(\mathcal{M}, \langle\cdot,\cdot\rangle_H)$. Then there exists N such that for $n \geq N$ the system (3.12) possesses a unique solution $(\xi_k^{(n)})_{k=1}^n$, and the resultant sequence $(f_n := \sum_{k=1}^n \xi_k^{(n)} P_\mathcal{M} h_k)_{n=N}^\infty$ converges to $f_{\Lambda\mathcal{M}}$ in the norm of H. In fact, there exists a positive c independent of Λ and such that

$$\|f_{\Lambda\mathcal{M}} - f_n\|_H \leq c \inf_{f \in P_\mathcal{M} H_n} \|f_{\Lambda\mathcal{M}} - f\|_H \qquad \text{for } n \geq N, \tag{3.16}$$

with H_n denoting the linear span of $\{h_k\}_{k=1}^n$.

Proof: As noted in the statement of the Theorem, $(\mathcal{M}, \langle\cdot,\cdot\rangle_H)$ is a Hilbert space, since $\mathcal{M}$ is assumed to be closed in H. With $\sigma_\mathcal{M}$ and $\Lambda_\mathcal{M}$ denoting, respectively, the restrictions of σ to $\mathcal{M} \times \mathcal{M}$ and Λ to $\mathcal{M}$, consider

$$\mathcal{N}_1(\sigma_\mathcal{M}) := \{\, g \in \mathcal{M} \mid \sigma_\mathcal{M}(g,h) = 0 \quad \text{for all } h \in \mathcal{M} \,\}.$$

Suppose that $g \in \mathcal{N}_1(\sigma_\mathcal{M})$: then we compute

$$\sigma(g,h) = \sigma(g, P_\mathcal{M} h + (I - P_\mathcal{M})h) = \sigma(g, (I - P_\mathcal{M})h) = 0 \qquad \text{for all } h \in H,$$

since $(I - P_\mathcal{M})h \in \mathcal{N}_1(\sigma) = \mathcal{N}_1(\sigma^*)$, by (3.15). Thus, $g \in \mathcal{M} \cap \mathcal{N}_1(\sigma) = \{0\}$, and so $\mathcal{N}_1(\sigma_\mathcal{M})$ is the trivial subspace. Consequently, it is clear that all of the required hypotheses are fulfilled so that we can apply Theorem 3.1(*ii*) and Theorem 3.2, with H, σ, and Λ replaced there by $(\mathcal{M}, \langle\cdot,\cdot\rangle_H)$, $\sigma_{\mathcal{M}}$, and $\Lambda_{\mathcal{M}}$, respectively. We conclude first that there is a unique solution of the problem

$$\textit{find} \quad f \in \mathcal{M} \quad \textit{such that} \quad \sigma_{\mathcal{M}}(f,h) = \Lambda_{\mathcal{M}} h \quad \textit{for all } h \in \mathcal{M};$$

this unique solution clearly coincides with the unique element $f_{\Lambda\mathcal{M}} \in \mathcal{M}$ satisfying (3.10), since the latter holds, in particular, for every $h \in \mathcal{M}$. Further, according to Theorem 3.2 and the given properties of the family $\{h_n\}_{n=1}^{\infty}$, there exists N such that for $n \geq N$, the system

$$\sum_{k=1}^{n} \sigma_{\mathcal{M}}(P_{\mathcal{M}}h_k, P_{\mathcal{M}}h_l)\xi_k = \Lambda_{\mathcal{M}} P_{\mathcal{M}} h_l, \qquad l = 1,\ldots,n, \tag{3.17}$$

has a unique solution $(\xi_k^{(n)})_{k=1}^{n}$, and the resultant sequence $(f_n := \sum_{k=1}^{n} \xi_k^{(n)} P_{\mathcal{M}} h_k)_{n=N}^{\infty}$ converges to $f_{\Lambda\mathcal{M}}$ in the norm of $\mathcal{M}$, *i.e.*, in the norm of H. But, because of (3.15) and (3.8), we find that

$$\sigma_{\mathcal{M}}(P_{\mathcal{M}}h_k, P_{\mathcal{M}}h_l) = \sigma\big((I - P_{\mathcal{M}})h_k + P_{\mathcal{M}}h_k, (I - P_{\mathcal{M}})h_l + P_{\mathcal{M}}h_l\big) = \sigma(h_k, h_l), \qquad k,l = 1,2,3,\ldots,$$

and

$$\Lambda_{\mathcal{M}} P_{\mathcal{M}} h_l = \Lambda\big((I - P_{\mathcal{M}})h_l + P_{\mathcal{M}}h_l\big) = \Lambda h_l, \qquad l = 1,2,3,\ldots,$$

so the system (3.17) coincides with the system (3.12). Finally, the error estimate (3.16) follows from (3.13). ∎

We conclude this section with a remark that will be important for the application of Theorem 3.3 that we have in mind, if we should content ourselves in the original fluid-elastic interaction problem with the approximation of the fluid field alone. Maintaining the setting and notation of this Theorem, suppose that $\tilde{H}$ is some normed linear space, $L : H \to \tilde{H}$ is a bounded linear operator, and we are interested in approximating $Lf_{\Lambda\mathcal{M}}$ in the norm of $\tilde{H}$. If it should be the case that

$$\mathcal{N}_1(\sigma) \subset \mathcal{N}(L), \tag{3.18}$$

then obviously

$$Lf_n = \sum_{k=1}^{n} \xi_k^{(n)} L P_{\mathcal{M}} h_k = \sum_{k=1}^{n} \xi_k^{(n)} L h_k, \qquad \text{for } n \geq N, \tag{3.19}$$

while the sequence $(Lf_n)_{n=N}^{\infty}$ converges to $Lf_{\Lambda\mathcal{M}}$ in the norm of $\tilde{H}$. In our application, we shall have $H = \mathbf{H}^1$, $\sigma = \sigma_\omega$, $\tilde{H} = H^{\frac{1}{2}}(\Gamma_+)$, and L given by $\mathbf{u} \mapsto \mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$, for $\mathbf{u} \in \mathbf{H}^1$ (cf. Lemma 2.1), choices for which (3.15) and (3.18) shall be shown to hold. The point here is that when (3.15) and (3.18) are fulfilled, we can convergently approximate $Lf_{\Lambda\mathcal{M}}$ with very little explicit information about the subspace $\mathcal{N}_1(\sigma)$, clearly a desirable feature. Indeed, we need only ensure that our choice of coördinate family $\{h_n\}_{n=1}^{\infty}$ is such that $\{P_{\mathcal{M}}h_n\}_{n=1}^{\infty}$ is linearly independent (*i.e.*, has the property that the only linear combination of a finite number of elements of $\{h_n\}_{n=1}^{\infty}$ that belongs to $\mathcal{N}_1(\sigma)$ is the trivial linear combination, with all coefficients zero) and complete in $(\mathcal{M}, \langle\cdot,\cdot\rangle_H)$; in the later application, we will even know that the required completeness property follows once we have selected a family $\{h_n\}_{n=1}^{\infty}$ that is complete in H itself (cf. Lemma 4.3, *infra*). Then, according to (3.12) and (3.19), the computations necessary to construct approximations converging in $\tilde{H}$ to $Lf_{\Lambda\mathcal{M}}$ can be accomplished without further regard for the exact nature of $\mathcal{N}_1(\sigma)$. In general, however, for the computation of approximations converging in H to $f_{\Lambda\mathcal{M}}$ itself (which will correspond in our application to the approximation of the elastic-field portion of the solution of the interaction problem), evidently we cannot avoid confronting the difficult and expensive eigenvalue problem that must be solved to identify $\mathcal{N}_1(\sigma)$ and so determine how to compute the action of the projection operator $P_{\mathcal{M}}$. Of course, if we were to know by some other reasoning that $\mathcal{N}_1(\sigma) = \{0\}$ (which will correspond in the upcoming application to $\mathcal{N}^1_{\Omega_o,\omega} = \{\mathbf{0}\}$), then these difficulties disappear.

4. SOLVABILITY AND APPROXIMATION-OF-SOLUTION RESULTS FOR THE WEAK PROBLEMS

In §2, we specified how a solution of the weak version of (**P**) is to be obtained from a weak solution of the purely interior problem corresponding to ($\mathbf{P}_0$) with data derived from that of (**P**). Thus, we study exclusively the weak form of the interior problem corresponding to ($\mathbf{P}_0$). Our first goal in this section is a description of the state of affairs concerning the solvability of the latter problem and the applicability of the Galerkin method for the generation of a sequence converging to the unique solution when the problem is uniquely solvable. We are restricting our considerations here, as in §2, to the case in which $\mathbf{t}_o \in \mathbf{L}_2(\Gamma_o)$, although the statement and proof of the following Theorem 4.1 can be modified in an obvious fashion to permit the weaker hypothesis $\mathbf{t}_o \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_o)$.

Theorem 4.1. *Recall the regularity conditions placed on Ω_o, the conditions $\operatorname{Re}\kappa_+ > 0$ and $\operatorname{Im}\kappa_+ \geq 0$ fulfilled by κ_+, and the restrictions $\lambda \geq 0$, $\mu > 0$ imposed upon the Lamé parameters.*

(*i*) *The dimension of the linear manifold $\mathcal{N}^1_{\Omega_o,\omega}$ is finite. In particular, $\mathcal{N}^1_{\Omega_o,\omega}$ is closed in both $\mathbf{H}^0$ and $\mathbf{H}^1$, and we have the direct-sum decompositions given in (2.26) and (2.27). Thus, there is defined the (bounded) operator $P_{\mathcal{M}^1} : \mathbf{H}^1 \to \mathbf{H}^1$ of projection onto the subspace $\mathcal{M}^1_{\Omega_o,\omega}$ along $\mathcal{N}^1_{\Omega_o,\omega}$.*

(*ii*) *There exists a weak solution of the purely interior problem corresponding to $\big(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota)\big)$, as in Definition 2.3, iff the associated conjugate-linear functional Λ_ω on $\mathbf{H}^1$ vanishes on $\mathcal{N}^1_{\Omega_o,\omega}$, i.e., iff*

$$\int_{\Gamma_+} \big(\mathbf{t}_+ - (\mathbf{t}_+ \cdot \mathbf{n}_+)\mathbf{n}_+\big) \cdot \overline{\mathbf{u}}_\omega|_{\Gamma_+}\, d\lambda_{\Gamma_+} + \int_{\Gamma_-} \mathbf{t}_- \cdot \overline{\mathbf{u}}_\omega|_{\Gamma_-}\, d\lambda_{\Gamma_-} + \int_{\Omega_o} \mathbf{f}_o \cdot \overline{\mathbf{u}}_\omega \, d\lambda_3 = 0$$

$$\text{for every } \mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}. \qquad (4.1)$$

When this condition holds, there exists precisely one weak solution $\mathbf{u}$ ($\in \mathcal{M}^1_{\Omega_o,\omega}$) for that problem; the $\mathbf{H}^1$-norm of this solution has the bound given by

$$\|\mathbf{u}\|_{\mathbf{H}^1} \leq M\|\Lambda_\omega\| \leq M\left\{\|\mathbf{f}_o\|_{\mathbf{H}^0} + \|\mathbf{t}_o\|_{\mathbf{L}_2(\Gamma_o)} + |\gamma_+ - i\omega|\big(\|\varphi^\iota|_{\Gamma_+}\|_{L_2(\Gamma_+)} + \|A_{\kappa_+}\|\,\|\varphi^\iota_{,\mathbf{n}_+}\|_{L_2(\Gamma_+)}\big)\right\}, \qquad (4.2)$$

for some positive M that is independent of the particular set of data $\{\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota\}$ chosen as in Terminology 2.1 and satisfying (4.1); here, $\|\Lambda_\omega\|$ is the norm of Λ_ω as an element of the anti-dual of $\mathbf{H}^1$, while $\|A_{\kappa_+}\|$ is the norm of A_{κ_+} acting in $L_2(\Gamma_+)$.

(*iii*) *Suppose that the solvability condition (4.1) holds. Then the unique weak solution $\mathbf{u}$ of the purely interior problem corresponding to $\big(\mathbf{P}_0(\mathbf{f}_o, \mathbf{t}_o, \varphi^\iota)\big)$ can be convergently approximated in the norm of $\mathbf{H}^1$ by using the Galerkin method. More precisely, suppose that $\{\mathbf{w}_n\}_{n=1}^\infty$ is a family in $\mathbf{H}^1$ such that $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^\infty$ is linearly independent and complete in the subspace $\mathcal{M}^1_{\Omega_o,\omega}$ of $\mathbf{H}^1$; here, completeness of $\{\mathbf{w}_n\}_{n=1}^\infty$ in $\mathbf{H}^1$ will suffice to imply the completeness of $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^\infty$ in $\mathcal{M}^1_{\Omega_o,\omega}$. Then there exists a positive integer N such that for $n \geq N$, the system*

$$\sum_{k=1}^{n} \sigma_\omega(\mathbf{w}_k, \mathbf{w}_l)\xi_k = \Lambda_\omega \mathbf{w}_l, \qquad l = 1, \ldots, n, \qquad (4.3)$$

possesses a unique solution $(\xi_k^{(n)})_{k=1}^n$, and the resultant sequence $(\mathbf{u}_n := \sum_{k=1}^n \xi_k^{(n)} P_{\mathcal{M}^1}\mathbf{w}_k)_{n=N}^\infty$ converges to $\mathbf{u}$ in the norm of $\mathbf{H}^1$. Moreover, the sequence $(\mathbf{u}_n|_{\Gamma_+} \cdot \mathbf{n}_+)_{n=N}^\infty$ converges to $\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ in the norm of $H^{\frac{1}{2}}(\Gamma_+)$; since

$$\mathbf{u}_n|_{\Gamma_+} \cdot \mathbf{n}_+ = \sum_{k=1}^{n} \xi_k^{(n)} \mathbf{w}_k|_{\Gamma_+} \cdot \mathbf{n}_+ \qquad \text{for } n \geq N, \qquad (4.4)$$

it follows, in view of the form of the systems in (4.3), that this sequence can be constructed even in the absence of explicit information concerning the subspace $\mathcal{N}^1_{\Omega_o,\omega}$ and the operator $P_{\mathcal{M}^1}$. Finally, the following error bounds hold for $n \geq N$, c and c' denoting positive numbers that are independent of the particular set of data $\{\mathbf{f}_o, \mathbf{t}_o, \varphi^i\}$ chosen as in Terminology 2.1 and satisfying (4.1):

$$\|\mathbf{u} - \mathbf{u}_n\|_{\mathbf{H}^1} \leq c \inf_{\mathbf{v} \in P_{\mathcal{M}^1}\mathbf{W}_n} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1},$$

and

$$\|(\mathbf{u} - \mathbf{u}_n)|_{\Gamma_+} \cdot \mathbf{n}_+\|_{H^{\frac{1}{2}}(\Gamma_+)} \leq c' \inf_{\mathbf{v} \in P_{\mathcal{M}^1}\mathbf{W}_n} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{H}^1},$$

wherein $\mathbf{W}_n$ denotes the linear span of $\{\mathbf{w}_k\}_{k=1}^n$.

Concerning the operator $P_{\mathcal{M}^1}$, let us interject here an explanatory note: since $\mathcal{N}^1_{\Omega_o,\omega} \subset \mathbf{H}^1$, it is clear that the operator on $\mathbf{H}^0$ of orthogonal projection onto $(\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_0}$ carries the linear manifold of elements in $\mathbf{H}^1$ into itself; in view of (2.25) and (2.27), the restriction of this $\mathbf{H}^0$-orthogonal projection operator to the linear manifold of elements in $\mathbf{H}^1$, when regarded as acting in $\mathbf{H}^1$, is just the (nonorthogonal projection) operator that we have denoted by $P_{\mathcal{M}^1}$.

We shall make some remarks on the statements of Theorem 4.1, supposing that it has been proven. Assuming that the solvability condition (4.1) is fulfilled, it is asserted that there is a unique weak solution $\mathbf{u}$ of the purely interior problem for $(\mathbf{P}_0)$, and a method is provided for constructing a sequence converging to this solution in $\mathbf{H}^1$. To actually base a computation on this method, we must employ a family $\{\mathbf{w}_n\}_{n=1}^\infty$ for which the corresponding family of projections $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^\infty$ is linearly independent and complete in the subspace $\mathcal{M}^1_{\Omega_o,\omega}$ of $\mathbf{H}^1$. Once we have secured such a family $\{\mathbf{w}_n\}_{n=1}^\infty$, the numerical solution of the systems in (4.3) requires no further knowledge about the subspace $\mathcal{N}^1_{\Omega_o,\omega}$, but evidently we cannot avoid having to find the projections $P_{\mathcal{M}^1}\mathbf{w}_n$ for $n = 1, 2, 3, \ldots$, if we insist on approximating the weak solution $\mathbf{u}$ itself, *i.e.*, if we wish to compute the $\mathbf{u}_n$ for $n \geq N$. However, a further implication of the Theorem is that we shall be able to approximate convergently in $H^{\frac{1}{2}}(\Gamma_+)$ the normal component $\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ of the restricted trace of $\mathbf{u}$, *i.e.*, that we *can* compute the $\mathbf{u}_n|_{\Gamma_+} \cdot \mathbf{n}_+$, without troubling to find those projections. In turn, we shall be able to approximate the function $\varphi \in W(\Omega_+; \kappa_+)$ satisfying (2.13), the fluid-field portion of the weak solution of $(\mathbf{P})$ itself, by using Proposition 2.1 (cf. (2.16)). Consequently, we shall be able to calculate approximations to the fluid-field portion of the fluid-elastic interaction with little information about the possible complex amplitudes of nonradiating modes for Ω_o and ω. It suffices to ensure that our chosen family $\{\mathbf{w}_n\}_{n=1}^\infty$ is, say, complete in $\mathbf{H}^1$ and such that $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^\infty$ is linearly independent, *i.e.*, such that any linear combination of a finite number of elements chosen from $\{\mathbf{w}_n\}_{n=1}^\infty$ that lies in $\mathcal{N}^1_{\Omega_o,\omega}$ is necessarily the trivial combination with all coefficients equal to zero (cf. Lemma 4.4, *infra*). We shall remark later, in Proposition 4.1, on the selection of coördinate functions fulfilling this latter requirement.

The proof of Theorem 4.1 will be carried out by checking that we can apply to the present setting the abstract developments of the preceding §3; this will be facilitated by the establishment or citation of certain preliminary results, to which we proceed. The first statement is termed "the coercivity of strains" by Nečas and Hlaváček [8], but is more commonly known as "Korn's second inequality" (cf., *e.g.*, Fichera [14]).

Lemma 4.1. *Recall that Ω_o is assumed to be a Lipschitz domain. There exists a positive number c_o, depending upon only Ω_o, such that*

$$\int_{\Omega_o} \{\varepsilon_{jk}[\mathbf{v}]\varepsilon_{jk}[\overline{\mathbf{v}}] + \mathbf{v} \cdot \overline{\mathbf{v}}\}\, d\lambda_3 \geq c_o \|\mathbf{v}\|^2_{\mathbf{H}^1} \qquad \text{for every } \mathbf{v} \in \mathbf{H}^1. \tag{4.5}$$

Proof: The statement is proven in Theorem 3.4 of Ref. 8 for the case in which the underlying Sobolev space comprises $\mathbf{R}^3$-valued functions. But it is clear that this also suffices to cover the present situation, in which the elements of $\mathbf{H}^1$ are $\mathbf{C}^3$-valued, since we have, for $\mathbf{v} \in \mathbf{H}^1$, the equalities $\varepsilon_{jk}[\mathbf{v}]\varepsilon_{jk}[\overline{\mathbf{v}}] = \varepsilon_{jk}[\mathrm{Re}\,\mathbf{v}]\varepsilon_{jk}[\mathrm{Re}\,\mathbf{v}] +$

$\varepsilon_{jk}[\mathrm{Im}\,\mathbf{v}]\varepsilon_{jk}[\mathrm{Im}\,\mathbf{v}]$ and $\|\mathbf{v}\|^2_{\mathbf{H}^k} = \|\mathrm{Re}\,\mathbf{v}\|^2_{\mathbf{H}^k} + \|\mathrm{Im}\,\mathbf{v}\|^2_{\mathbf{H}^k}$ for $k = 0,1$. Alternately, one can appeal to the reasoning in Ref. 14, carried out under the assumption that the domain Ω_o satisfies merely a "restricted cone hypothesis" (and again for $\mathbf{R}^3$-valued functions). ∎

For the sesquilinear form σ_ω defined on $\mathbf{H}^1$ by (2.20), the associated "linear manifolds of degeneracy" $\mathcal{N}_1(\sigma_\omega)$ and $\mathcal{N}_1(\sigma^*_\omega)$ in $\mathbf{H}^1$ are defined as in §3. Recall the definition (2.24) of the linear manifold $\mathcal{N}^1_{\Omega_o,\omega}$ of complex amplitudes of weak nonradiating modes in $\mathbf{H}^1$.

Lemma 4.2. $\mathcal{N}^1_{\Omega_o,\omega} = \mathcal{N}_1(\sigma_\omega) = \mathcal{N}_1(\sigma^*_\omega)$.

Proof: Let $\mathbf{u}_\omega \in \mathcal{N}_1(\sigma_\omega)$, so that $\mathbf{u}_\omega \in \mathbf{H}^1$ and $\sigma_\omega(\mathbf{u}_\omega, \mathbf{v}) = 0$ for every $\mathbf{v} \in \mathbf{H}^1$. In particular, $\sigma_\omega(\mathbf{u}_\omega, \mathbf{u}_\omega) = 0$, and this implies, in turn, from (2.20), that

$$\mathrm{Im}\,\langle \kappa_+^2 A_{\kappa_+}(\mathbf{u}_\omega|_{\Gamma_+} \cdot \mathbf{n}_+), \mathbf{u}_\omega|_{\Gamma_+} \cdot \mathbf{n}_+ \rangle_{L_2(\Gamma_+)} = 0. \tag{4.6}$$

By (2.1), we must have $\mathbf{u}_\omega|_{\Gamma_+} \cdot \mathbf{n}_+ = 0$, so that $\mathbf{u}_\omega$ belongs to $\mathbf{H}^1_{0\mathbf{n}_+}$ (cf. (2.23)). But then (2.22) is true, as well (cf. the form of σ_ω in (2.20)), whence the inclusion $\mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}$ results. Thus, $\mathcal{N}_1(\sigma_\omega) \subset \mathcal{N}^1_{\Omega_o,\omega}$. The reversed inclusion obviously holds, so $\mathcal{N}_1(\sigma_\omega) = \mathcal{N}^1_{\Omega_o,\omega}$. Similarly, the assumption $\mathbf{u}_\omega \in \mathcal{N}_1(\sigma^*_\omega)$, *i.e.*, $\mathbf{u}_\omega \in \mathbf{H}^1$ and $\sigma_\omega(\mathbf{v}, \mathbf{u}_\omega) = 0$ for every $\mathbf{v} \in \mathbf{H}^1$, leads again to (4.6), from which the reasoning can be carried through as before, with trivial modifications, to arrive at the equality $\mathcal{N}_1(\sigma^*_\omega) = \mathcal{N}^1_{\Omega_o,\omega}$, in view of the fact that the first integral appearing in the definition of σ_ω generates an Hermitian-symmetric form on $\mathbf{H}^1$. ∎

Lemma 4.3. *If $\{\mathbf{w}_n\}_{n=1}^\infty$ is a family complete in $\mathbf{H}^1$, then the corresponding collection $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^\infty$ is complete in the Hilbert space $(\mathcal{M}^1_{\Omega_o,\omega}, \langle\cdot,\cdot\rangle_{\mathbf{H}^1})$, wherein $P_{\mathcal{M}^1} : \mathbf{H}^1 \to \mathbf{H}^1$ denotes the operator of projection onto $\mathcal{M}^1_{\Omega_o,\omega}$ along $\mathcal{N}^1_{\Omega_o,\omega}$.*

The statement of Lemma 4.3 anticipates the fact that $\mathbf{H}^1$ has the direct-sum decomposition given in (2.27). Therefore, we choose to defer the proof of this Lemma until after we have established that $\mathcal{N}^1_{\Omega_o,\omega}$ is finite-dimensional in the course of the proof of Theorem 4.1 (without using Lemma 4.3!).

Proof of Theorem 4.1: At the outset, we observe that the sesquilinear form σ_ω and the conjugate-linear functional Λ_ω are obviously bounded on $\mathbf{H}^1$. We wish to show that σ_ω can be written as the sum of a definite form and a compact form, *i.e.*, that it satisfies the fundamental hypothesis required in the developments of §3. To this end, let us define the sesquilinear forms $\delta(\cdot,\cdot)$ and $\kappa_\omega(\cdot,\cdot)$ on $\mathbf{H}^1$ by setting

$$\left.\begin{aligned} \delta(\mathbf{u},\mathbf{v}) &:= \int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u},\overline{\mathbf{v}}) + 2\mu\mathbf{u}\cdot\overline{\mathbf{v}}\}\, d\lambda_3 \\ \kappa_\omega(\mathbf{u},\mathbf{v}) &:= -(2\mu + \varrho_o\omega^2)\int_{\Omega_o} \mathbf{u}\cdot\overline{\mathbf{v}}\, d\lambda_3 + \varrho_+ c_+^2 \kappa_+^2 \int_{\Gamma_+} (A_{\kappa_+}(\mathbf{u}|_{\Gamma_+}\cdot\mathbf{n}_+))\overline{\mathbf{v}}|_{\Gamma_+}\cdot\mathbf{n}_+\, d\lambda_{\Gamma_+} \end{aligned}\right\} \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathbf{H}^1.$$

Clearly, $\sigma_\omega = \delta + \kappa_\omega$. The definiteness of (the Hermitian-symmetric form) δ follows from Lemma 4.1, upon recalling that $\lambda \geq 0$ and $\mu > 0$, for, we can write

$$\begin{aligned} \delta(\mathbf{v},\mathbf{v}) &= \int_{\Omega_o} \{\lambda|\varepsilon_{kk}[\mathbf{v}]|^2 + 2\mu\varepsilon_{jk}[\mathbf{v}]\varepsilon_{jk}[\overline{\mathbf{v}}] + 2\mu\mathbf{v}\cdot\overline{\mathbf{v}}\}\, d\lambda_3 \geq 2\mu \int_{\Omega_o} \{\varepsilon_{jk}[\mathbf{v}]\varepsilon_{jk}[\overline{\mathbf{v}}] + \mathbf{v}\cdot\overline{\mathbf{v}}\}\, d\lambda_3 \\ &\geq 2\mu c_o\|\mathbf{v}\|^2_{\mathbf{H}^1} \qquad \text{for } \mathbf{v} \in \mathbf{H}^1. \end{aligned} \tag{4.7}$$

The compactness of κ_ω is a simple consequence of the compactness of both the natural injection map carrying $\mathbf{H}^1$ into $\mathbf{H}^0$ and the operator $\mathbf{v} \mapsto \mathbf{v}|_{\Gamma_+} \cdot \mathbf{n}_+$ taking $\mathbf{H}^1$ into $L_2(\Gamma_+)$ (for the latter, cf. Lemma 2.1(*i*)), coupled with the boundedness of A_{κ_+} on $L_2(\Gamma_+)$. We remark that the compactness of A_{κ_+} is not needed here, but could have been used to produce the same conclusion if we knew only that $\mathbf{v} \mapsto \mathbf{v}|_{\Gamma_+} \cdot \mathbf{n}_+$ is bounded from

$\mathbf{H}^1$ into $L_2(\Gamma_+)$. Thus, we can apply Theorem 3.1. By doing so, we find first, from statement (i) of that Theorem, that

$$\dim \mathcal{N}^1_{\Omega_o,\omega} = \dim \mathcal{N}_1(\sigma_\omega) = \dim \mathcal{N}_1(\sigma^*_\omega) < \infty, \tag{4.8}$$

having also used Lemma 4.2 to get the first equality here. Moreover, by Theorem 3.1(ii), there exists some $\mathbf{u} \in \mathbf{H}^1$ satisfying (2.28) iff $\mathcal{N}_1(\sigma^*_\omega) \subset \mathcal{N}(\Lambda_\omega)$, or, again by Lemma 4.2, iff $\mathcal{N}^1_{\Omega_o,\omega} \subset \mathcal{N}(\Lambda_\omega)$, an inclusion that is equivalent to the explicit condition given in (4.1), in view of the definitions in (2.21), (2.23), and (2.24). Now we know, in particular, that $\mathcal{N}^1_{\Omega_o,\omega}$ is closed in $\mathbf{H}^0$ (and in $\mathbf{H}^1$), so (2.26) holds, whence (2.27) follows: $\mathcal{M}^1_{\Omega_o,\omega}$ is a subspace of $\mathbf{H}^1$ that is complementary to $\mathcal{N}_1(\sigma_\omega)$ $(=\mathcal{N}^1_{\Omega_o,\omega})$. By the direct-sum decomposition of (2.27), there is induced the (bounded) projection operator $P_{\mathcal{M}^1}$ of $\mathbf{H}^1$ onto $\mathcal{M}^1_{\Omega_o,\omega}$ along $\mathcal{N}^1_{\Omega_o,\omega}$ $(= \mathcal{N}_1(\sigma_\omega))$. Taking $\mathcal{M} = \mathcal{M}^1_{\Omega_o,\omega}$ in Theorem 3.1(iii), we can assert, when (4.1) obtains, that there exists precisely one $\mathbf{u} \in \mathcal{M}^1_{\Omega_o,\omega}$ that fulfills the requirement (2.28), and so provides the unique weak solution of the purely interior problem corresponding to $(\mathbf{P}_0)$; (3.11) implies that the $\mathbf{H}^1$-norm of this element has the bound given in (4.2) (with M independent of the data generating Λ_ω), since it is easy to see from (2.21) that $\|\Lambda_\omega\|$, the norm of Λ_ω as an element of the anti-dual of $\mathbf{H}^1$, is less than the number in brackets on the right in (4.2). This completes the proof of (i) and (ii).

To establish (iii), in which we are given that (4.1) holds, we denote by $\mathbf{u} \in \mathcal{M}^1_{\Omega_o,\omega}$ the unique weak solution of the purely interior problem for $(\mathbf{P}_0)$. We wish to invoke Theorem 3.3, making the obvious identifications $H = \mathbf{H}^1$, $\sigma = \sigma_\omega$, $\mathcal{M} = \mathcal{M}^1_{\Omega_o,\omega}$, *etc*; this is permissible by the facts discovered in the proof of (i) and (ii) and by Lemma 4.2 (giving (3.15)). With the additional use of Lemma 4.3, all of the conclusions of (iii) now follow immediately. For example, the convergence of the sequence $(\mathbf{u}_n|_{\Gamma_+} \cdot \mathbf{n}_+)_{n=N}^\infty$ to $\mathbf{u}|_{\Gamma_+} \cdot \mathbf{n}_+$ in the norm of $H^{\frac{1}{2}}(\Gamma_+)$ results simply from the convergence of $(\mathbf{u}_n)_{n=N}^\infty$ to $\mathbf{u}$ in the norm of $\mathbf{H}^1$ and the boundedness of the operation $\mathbf{v} \mapsto \mathbf{v}|_{\Gamma_+} \cdot \mathbf{n}_+$ on $\mathbf{H}^1$ into $H^{\frac{1}{2}}(\Gamma_+)$ (Lemma 2.1(ii)). The equalities of (4.4) are true because $(I - P_{\mathcal{M}^1})\mathbf{w}_k$ lies in $\mathcal{N}^1_{\Omega_o,\omega}$ for each k, so that $\mathbf{w}_k|_{\Gamma_+} \cdot \mathbf{n}_+ = (P_{\mathcal{M}^1}\mathbf{w}_k)|_{\Gamma_+} \cdot \mathbf{n}_+$. Concerning the error estimates displayed, the first is a direct result of (3.16), and the second follows from the first (by Lemma 2.1(ii)). ∎

Finally, we give the

Proof of Lemma 4.3: Recall the Hermitian-symmetric sesquilinear form δ defined on $\mathbf{H}^1$ in the proof of Theorem 4.1. Inequality (4.7) shows first that $\delta(\cdot,\cdot)$ is an inner product on $\mathbf{H}^1$, and then, when combined with the boundedness of δ, that the norms induced by $\delta(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle_{\mathbf{H}^1}$ on the complex linear space of elements in $\mathbf{H}^1$ are in fact equivalent:

$$2\mu c_o \|\mathbf{v}\|^2_{\mathbf{H}^1} \le \delta(\mathbf{v},\mathbf{v}) \le c'_o \|\mathbf{v}\|^2_{\mathbf{H}^1} \qquad \text{for } \mathbf{v} \in \mathbf{H}^1,$$

for some positive c'_o. By $\mathbf{H}^1_\delta$, we denote the collection of elements in $\mathbf{H}^1$ equipped with the inner product $\delta(\cdot,\cdot)$. Then $\mathbf{H}^1_\delta$ is complete along with $\mathbf{H}^1$, and the two Hilbert spaces are isomorphic *qua* Banach spaces; a family is complete in $\mathbf{H}^1$ iff it is complete in $\mathbf{H}^1_\delta$. The linear manifold $\mathcal{M}^1_{\Omega_o,\omega}$ is closed in $\mathbf{H}^1_\delta$, since we already know that it is closed in $\mathbf{H}^1$; a family is complete in the Hilbert space $(\mathcal{M}^1_{\Omega_o,\omega}, \langle\cdot,\cdot\rangle_{\mathbf{H}^1})$ iff that family is complete in the Hilbert space $(\mathcal{M}^1_{\Omega_o,\omega}, \delta(\cdot,\cdot))$.

Let us suppose, for the moment, that the inclusion

$$\mathcal{M}^1_{\Omega_o,\omega} \subset (\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_\delta} \tag{4.9}$$

has been shown to hold, wherein $(\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_\delta}$ indicates the orthogonal complement of $\mathcal{N}^1_{\Omega_o,\omega}$ in $\mathbf{H}^1_\delta$, and check that the claim of Lemma 4.3 follows therefrom. Thus, suppose that $\{\mathbf{w}_n\}_{n=1}^\infty$ is a family complete in $\mathbf{H}^1$; then it is also complete in $\mathbf{H}^1_\delta$. Let $\mathbf{u}$ be an element of $\mathcal{M}^1_{\Omega_o,\omega}$ that is δ-orthogonal to the family $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^\infty$ of projections:

$$\delta(\mathbf{u}, P_{\mathcal{M}^1}\mathbf{w}_n) = 0 \qquad \text{for } n = 1,2,3,\ldots\ .$$

Then, since $(I - P_{\mathcal{M}^1})\mathbf{w}_n \in \mathcal{N}^1_{\Omega_o,\omega}$ for $n \geq 1$, with (4.9) we get

$$\delta(\mathbf{u}, \mathbf{w}_n) = 0 \qquad \text{for } n = 1, 2, 3, \ldots,$$

and so conclude that $\mathbf{u} = \mathbf{0}$. This shows that the collection $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$ is complete in $(\mathcal{M}^1_{\Omega_o,\omega}, \delta(\cdot,\cdot))$, whence it is also complete in $(\mathcal{M}^1_{\Omega_o,\omega}, \langle\cdot,\cdot\rangle_{\mathbf{H}^1})$.

Therefore, for the completion of the proof, it is sufficient to verify (4.9). Then, let $\mathbf{u} \in \mathcal{M}^1_{\Omega_o,\omega}$, *i.e.*, by (2.25), let $\mathbf{u} \in \mathbf{H}^1$ and $\int_{\Omega_o} \mathbf{u} \cdot \overline{\mathbf{u}}_\omega \, d\lambda_3 = 0$ whenever $\mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}$. As a consequence, according to the definition in (2.24), we must have

$$\int_{\Omega_o} \mathcal{E}_{\lambda,\mu}(\mathbf{u}_\omega, \overline{\mathbf{u}}) \, d\lambda_3 = \int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u}_\omega, \overline{\mathbf{u}}) - \varrho_o\omega^2\mathbf{u}_\omega \cdot \overline{\mathbf{u}}\} \, d\lambda_3 = 0 \qquad \text{for every } \mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}. \tag{4.10}$$

Clearly, then, $\delta(\mathbf{u}, \mathbf{u}_\omega) = 0$ whenever $\mathbf{u}_\omega$ is in $\mathcal{N}^1_{\Omega_o,\omega}$, implying that (4.9) is correct.

Before leaving the proof, we note parenthetically that equality actually obtains in (4.9):

$$\mathcal{M}^1_{\Omega_o,\omega} = (\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_\delta}.$$

Indeed, if we suppose that $\mathbf{u} \in (\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_\delta}$, *i.e.*, that $\mathbf{u} \in \mathbf{H}^1$ and

$$\int_{\Omega_o} \{\mathcal{E}_{\lambda,\mu}(\mathbf{u}_\omega, \overline{\mathbf{u}}) + 2\mu\mathbf{u}_\omega \cdot \overline{\mathbf{u}}\} \, d\lambda_3 = 0 \qquad \text{for every } \mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega},$$

then the latter equality together with the second equality in (4.10) show that $\int_{\Omega_o} \mathbf{u}_\omega \cdot \overline{\mathbf{u}} \, d\lambda_3 = 0$ for every $\mathbf{u}_\omega \in \mathcal{N}^1_{\Omega_o,\omega}$, so $\mathbf{u} \in (\mathcal{N}^1_{\Omega_o,\omega})^{(\perp)_0} \cap \mathbf{H}^1 = \mathcal{M}^1_{\Omega_o,\omega}$. ∎

Let us return to the linear-independence question: we wish to know useful sufficient conditions on a family $\{\mathbf{w}_n\}_{n=1}^{\infty} \subset \mathbf{H}^1$ under which we shall be certain that $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$ is linearly independent. It is quite simple to prove that this linear independence obtains for any such family that is itself linearly independent and comprises piecewise-polynomials, as in the usual applications of the finite-element methods. We begin by recording an obvious necessary and sufficient criterion for the linear independence of a collection of projections $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$.

Lemma 4.4. *Let $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be a family in $\mathbf{H}^1$ and $P_{\mathcal{M}^1}$ the operator in $\mathbf{H}^1$ of projection onto $\mathcal{M}^1_{\Omega_o,\omega}$ along $\mathcal{N}^1_{\Omega_o,\omega}$. Then $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$ is linearly independent iff for each positive integer N, the inclusion $\sum_{n=1}^{N} a_n\mathbf{w}_n \in \mathcal{N}^1_{\Omega_o,\omega}$ for some $\{a_n\}_{n=1}^{N} \subset \mathbf{C}$ implies that $a_1 = \cdots = a_N = 0$. In particular, the linear independence of $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is necessary for the linear independence of $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$.*

Proof: For the necessity, suppose that $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$ is linearly independent, let N be a positive integer, and let $\{a_n\}_{n=1}^{N}$ be a complex set with $\sum_{n=1}^{N} a_n\mathbf{w}_n \in \mathcal{N}^1_{\Omega_o,\omega}$. Then $\sum_{n=1}^{N} a_n P_{\mathcal{M}^1}\mathbf{w}_n = \mathbf{0}$, whence $a_1 = \cdots = a_N = 0$, by the linear independence of the family of projections. In particular, the vanishing of $\sum_{n=1}^{N} a_n\mathbf{w}_n$ gives $a_1 = \cdots = a_N = 0$, so $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is linearly independent. For the sufficiency, let the condition hold, and suppose that $\sum_{n=1}^{N} a_n P_{\mathcal{M}^1}\mathbf{w}_n = \mathbf{0}$, for some positive integer N and complex set $\{a_n\}_{n=1}^{N}$. Then $\sum_{n=1}^{N} a_n\mathbf{w}_n \in \mathcal{N}^1_{\Omega_o,\omega}$, so we must have $a_1 = \cdots = a_N = 0$, implying that $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$ is linearly independent. ∎

As usual, by *a* ($\mathbf{C}^3$*-valued) polynomial in* $\mathbf{R}^3$ we mean a function on $\mathbf{R}^3$ of the form $\mathbf{x} \mapsto \sum_{0 \leq |\alpha| \leq K} \mathbf{c}_\alpha \mathbf{x}^\alpha$, for some nonnegative integer K and collection $\{\mathbf{c}_\alpha\}_{0 \leq |\alpha| \leq K}$ lying in $\mathbf{C}^3$; the standard notation $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ is used here, for a 3-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and an $\mathbf{x} \in \mathbf{R}^3$. A $\mathbf{C}^3$-valued function $\mathbf{w}$ defined λ_3-a.e. in Ω_o is said to be *a piecewise-polynomial in* Ω_o iff there exists a family $\{\mathcal{O}_j\}_{j=1}^{J}$ of pairwise-disjoint open subsets of Ω_o such that the complement $\Omega_o \setminus \bigcup_{j=1}^{J} \mathcal{O}_j$ has λ_3-measure zero and each of the restrictions $\mathbf{w}|\mathcal{O}_j$, $j = 1, \ldots, J$, coincides in $\mathcal{O}_j$ with some polynomial (perhaps depending upon j).

Proposition 4.1. *Let the family* $\{\mathbf{w}_n\}_{n=1}^{\infty}$ *lie in* $\mathbf{H}^1$, *and suppose that each* $\mathbf{w}_n$ *is a piecewise-polynomial in* Ω_o. *Then* $\{P_{\mathcal{M}^1}\mathbf{w}_n\}_{n=1}^{\infty}$ *is linearly independent iff* $\{\mathbf{w}_n\}_{n=1}^{\infty}$ *is linearly independent.*

Proof: By Lemma 4.4, we need only demonstrate the sufficiency. Then, let $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be linearly independent. Let N be a positive integer, $\{a_n\}_{n=1}^{N} \subset \mathbf{C}$, and suppose that $\sum_{n=1}^{N} a_n\mathbf{w}_n \in \mathcal{N}^1_{\Omega_o,\omega}$; according to Lemma 4.4, we must show that $a_1 = \cdots = a_N = 0$. Now, as already pointed out in §2, each element of $\mathcal{N}^1_{\Omega_o,\omega}$ has real-analytic real and imaginary parts, and is a classical solution of (1.27), so $\sum_{n=1}^{N} a_n\mathbf{w}_n$ has these properties. Since each $\mathbf{w}_n$ is a piecewise-polynomial in Ω_o, it is easy to see that we can find an open subset of Ω_o in which $\sum_{n=1}^{N} a_n\mathbf{w}_n$ coincides with a polynomial. Therefore, by the uniqueness theorem for real-analytic functions (cf., *e.g.*, Ref. 16), $\sum_{n=1}^{N} a_n\mathbf{w}_n$ must coincide with that polynomial throughout Ω_o. Consequently, the sum must vanish in Ω_o, since (1.27) has no nontrivial polynomial solutions (because $\varrho_o\omega^2 > 0$). Now the equality $\sum_{n=1}^{N} a_n\mathbf{w}_n = \mathbf{0}$, coupled with the linear independence of $\{\mathbf{w}_n\}_{n=1}^{\infty}$, implies the desired conclusion $a_1 = \cdots = a_N = 0$. ∎

Lemma 4.3 and Proposition 4.1 imply that the computation of approximations to the fluid-field portion of the unique weak solution of the fluid-elastic interaction problem (**P**) (corresponding to data fulfilling the requisite solvability condition) can be carried out in complete ignorance of the dimension of $\mathcal{N}^1_{\Omega_o,\omega}$ if we select the family $\{\mathbf{w}_n\}_{n=1}^{\infty}$ of coördinate functions to comprise piecewise-polynomials in Ω_o, taking care to ensure only that the particular such family chosen is complete in $\mathbf{H}^1$ and linearly independent. But the denseness of $\mathbf{H}^2$ in $\mathbf{H}^1$ and standard results from the theoretical foundations of finite-element methods concerning the interpolation of elements in $\mathbf{H}^2$ by appropriate piecewise-polynomials in $\mathbf{H}^1$ (cf., *e.g.*, Theorem 8.2.2 of Ref. 17) show how to construct coördinate families of piecewise-polynomials in Ω_o that are also complete in $\mathbf{H}^1$. Of course, none of this helps in overcoming the difficulties involved in computing approximations to the elastic part of the weak solution of the problem when either it is known that $\mathcal{N}^1_{\Omega_o,\omega}$ is nontrivial or there is no *a priori* assurance that $\mathcal{N}^1_{\Omega_o,\omega} = \{0\}$.

5. ON THE DERIVATION OF THE INTERIOR AND EXTERIOR LIMITING-AMPLITUDE CONDITIONS

This section is devoted to the promised motivation for accepting (1.34) and (1.36) as limiting-amplitude conditions, *i.e.*, under sufficiently stringent assumptions concerning the asymptotic approach to a time-harmonic state, as conditions satisfied by the assumed-to-exist complex limiting amplitudes $\mathbf{u}$ and φ of the (assumed-to-exist) solution $(\mathbf{U}, \Phi)$ of the fundamental initial-value problem governing the time-dependent fluid-elastic interaction, as described in §1. Actually, the interior and exterior aspects can be treated essentially separately; specifically, we shall provide the desired assertions in the form of two theorems, the first concerning solutions of the damped-wave equation (1.10) (and providing a basis for taking the Sommerfeld condition (1.34) as the appropriate limiting-amplitude condition for any boundary-value problem for (1.23), with κ_+ as specified), the second involving solutions of the Navier equations (1.11).

Consistently, we shall denote by $d_+(\mathbf{x})$ the distance between a point $\mathbf{x} \in \mathbf{R}^3$ and Γ_+:

$$d_+(\mathbf{x}) := \mathrm{dist}(\mathbf{x}, \Gamma_+) := \min_{\mathbf{y}\in\Gamma_+} |\mathbf{y} - \mathbf{x}|.$$

For brevity, in this section we shall write c and γ in place of c_+ and γ_+, respectively.

Theorem 5.1. *Let $\Omega_+ \subset \mathbf{R}^3$ be as specified in §1 and §2. Thus, Ω_+ is connected and the complement of the closure of a bounded and connected open set of type $\Re^2$; as before, set $\Gamma_+ := \partial\Omega_+$. Let $c > 0$ and $\gamma \in \mathbf{C}$, with either $\gamma = 0$ or $\operatorname{Re}\gamma > 0$. Let $\Phi \in C^2(\Omega_+ \times \mathbf{R})$ satisfy*

$$-\Delta\Phi + \frac{\gamma}{c^2}\Phi_{,4} + \frac{1}{c^2}\Phi_{,44} = 0, \qquad \text{in } \Omega_+ \times \mathbf{R}, \tag{5.1}$$

and vanish in $\Omega_+ \times (-\infty, 0)$:

$$\Phi(\cdot, t) = 0 \qquad \text{in } \Omega_+ \text{ for } t < 0. \tag{5.2}$$

Suppose that, for some real positive ω, Φ can be written in the form

$$\Phi(\cdot, t) = \Phi^\tau(\cdot, t) + \varphi(\cdot)e^{-i\omega t}, \qquad \text{in } \Omega_+ \text{ for each } t \in \mathbf{R}, \tag{5.3}$$

wherein $\varphi \in C^1(\Omega_+)$ and $\Phi^\tau \in C^1(\Omega_+ \times \mathbf{R})$, with the latter fulfilling the following conditions of transience and boundedness:

$$\lim_{s\to\infty} \Phi^\tau(\mathbf{y}, s) = 0 \qquad \text{whenever } \mathbf{y} \in \Omega_+ \tag{5.4$_1$}$$

and, for certain positive numbers δ_o and M,

$$\lim_{s\to\infty} \Phi^\tau_{,4}(\mathbf{y}, s) = 0 \quad \text{and} \quad \lim_{s\to\infty} \operatorname{grad}\Phi^\tau(\mathbf{y}, s) = \mathbf{0} \qquad \text{if } \mathbf{y} \in \Omega_+ \text{ and } 0 < d_+(\mathbf{y}) < \delta_o, \tag{5.4$_2$}$$

and

$$|\Phi^\tau(\mathbf{y}, s)| \le M, \quad |\Phi^\tau_{,4}(\mathbf{y}, s)| \le M, \quad \text{and} \quad |\operatorname{grad}\Phi^\tau(\mathbf{y}, s)| \le M$$

$$\text{for } \mathbf{y} \in \Omega_+ \text{ with } 0 < d_+(\mathbf{y}) < \delta_o \text{ and } s > 0. \tag{5.5}$$

Then φ has real and imaginary parts that are real-analytic in Ω_+ and satisfies both (1.23) and (1.34), wherein the complex $\kappa_+ = \kappa_+(\omega; c, \gamma)$ is computed as in (1.19) and (1.20).

We remark that (5.3) obviously implies that the hypothesized Φ^τ must have more regularity than is indicated by its inclusion in $C^1(\Omega_+ \times \mathbf{R})$. Theorem 5.1 is easy to prove because the hypotheses have been adjusted to make it so; our only interest here lies in showing that (1.34) must hold when a solution of (1.10) evolves in a "reasonable" manner from a quiescent state to approach a time-harmonic form with the complex amplitude φ. A far more difficult question concerns the formulation of conditions on the shape of Γ_+ and the forcing terms in the problem under which this approach to a time-harmonic state is assured. Similar remarks are pertinent to the next theorem.

It is very convenient to couch the statement and proof of Theorem 5.2 in terms of Hilbert-space-valued functions on $\mathbf{R}$; for the basic definitions and results concerning such maps, one can consult, *e.g.*, Refs. 18 and 19. With a separable Hilbert space H, we associate the collection $C(\mathbf{R}; H)$ of all functions on $\mathbf{R}$ into H that are strongly continuous, *i.e.*, that are continuous when H is equipped with the topology induced by its inner-product structure. For a positive integer m, by $C^m(\mathbf{R}; H)$ we denote the family of all elements $\mathbf{U} \in C(\mathbf{R}; H)$ possessing m strong derivatives $\mathbf{U}^{(1)} \equiv \mathbf{U}', \ldots, \mathbf{U}^{(m)}$ that are also in $C(\mathbf{R}; H)$, "strong" here signifying that the derivatives are defined with respect to the inner-product structure of H. For integration of such functions over subsets of $\mathbf{R}$, we use the Bochner theory (cf. Refs. 18 and 9), although the strong Riemann integral (cf. Ref. 19) will suffice, in view of the continuity hypotheses that we pose in the upcoming assertions (in the interest of simplicity). If we were to weaken these hypotheses to allow for the existence of derivatives in a generalized sense, then use of the Bochner theory would be essential.

Theorem 5.2. *With setting and notation as set forth in §1 and §2, let the functions* $\mathbf{F}_o : \mathbf{R} \to \mathbf{H}^0$, $\mathbf{T}_o : \mathbf{R} \to \mathbf{L}_2(\Gamma_o)$, *and* $\Psi_+ : \mathbf{R} \to H^{\frac{1}{2}}(\Gamma_+)$ *vanish in* $(-\infty, 0)$. *Let* $\mathbf{F}_o$ *belong to* $C(\mathbf{R}; \mathbf{H}^0)$ *and* $\mathbf{T}_o$ *be contained in* $C(\mathbf{R}; \mathbf{L}_2(\Gamma_o))$. *Suppose further that, for some* $\omega > 0$,

$$\left.\begin{aligned} \mathbf{F}_o(t) &= \mathbf{F}_o^\tau(t) + \mathbf{f}_o e^{-i\omega t} \\ \mathbf{T}_o(t) &= \mathbf{T}_o^\tau(t) + \mathbf{t}_o e^{-i\omega t} \end{aligned}\right\} \quad \textit{for each } t \in \mathbf{R}, \tag{5.6}$$

wherein $\mathbf{f}_o \in \mathbf{H}^0$ *and* $\mathbf{t}_o \in \mathbf{L}_2(\Gamma_o)$, *while*

$$\lim_{t\to\infty} \|\mathbf{F}_o^\tau(t)\|_{\mathbf{H}^0} = \lim_{t\to\infty} \|\mathbf{T}_+^\tau(t) - \mathbf{T}_+^\tau(t)\cdot\mathbf{n}_+\mathbf{n}_+\|_{\mathbf{L}_2(\Gamma_+)} = \lim_{t\to\infty} \|\mathbf{T}_-^\tau(t)\|_{\mathbf{L}_2(\Gamma_-)} = 0. \tag{5.7}$$

Let $\mathbf{U} : \mathbf{R} \to \mathbf{H}^2$ *vanish in* $(-\infty, 0)$, *belong to* $C^2(\mathbf{R}; \mathbf{H}^0)$ *when regarded as taking values in* $\mathbf{H}^0$, *and satisfy*

$$\varrho_o \mathbf{U}''(t) - \boldsymbol{\Delta}^*_{\lambda,\mu}\mathbf{U}(t) = \mathbf{F}_o(t) \qquad \textit{for each } t \in \mathbf{R}, \tag{5.8}$$

along with

$$\left.\begin{aligned} \mathbf{T}^{\mathbf{n}_+}[\mathbf{U}(t)] &= \mathbf{T}_+(t) + \Psi_+(t)\mathbf{n}_+ \\ \mathbf{T}^{\mathbf{n}_-}[\mathbf{U}(t)] &= \mathbf{T}_-(t) \end{aligned}\right\} \quad \textit{for each } t \in \mathbf{R}. \tag{5.9}$$

Suppose that $\mathbf{U}$ *has the form*

$$\mathbf{U}(t) = \mathbf{U}^\tau(t) + \mathbf{u}e^{-i\omega t}, \qquad t \in \mathbf{R}, \tag{5.10}$$

for some $\mathbf{u} \in \mathbf{H}^0$ *and* $\mathbf{U}^\tau \in C^2(\mathbf{R}; \mathbf{H}^0)$, *the latter satisfying the conditions of transience*

$$\lim_{t\to\infty} \|\mathbf{U}^{\tau(j)}(t)\|_{\mathbf{H}^0} = 0 \qquad \textit{for } j = 0, 1, \textit{ and } 2. \tag{5.11}$$

Then, for every $\mathbf{u}_\omega \in \mathcal{N}^2_{\Omega_o,\omega}$,

$$\int_{\Gamma_o} \mathbf{t}_o \cdot \overline{\mathbf{u}}_\omega|_{\Gamma_o}\, d\lambda_{\Gamma_o} + \int_{\Omega_o} \mathbf{f}_o \cdot \overline{\mathbf{u}}_\omega\, d\lambda_3 = 0, \tag{5.12}$$

and

$$\int_{\Omega_o} \mathbf{u}\cdot\overline{\mathbf{u}}_\omega\, d\lambda_3 = \frac{i}{2\varrho_o\omega}\int_0^\infty \left\{\int_{\Gamma_o} \mathbf{T}_o^\tau(s)\cdot\overline{\mathbf{u}}_\omega|_{\Gamma_o}\, d\lambda_{\Gamma_o} + \int_{\Omega_o} \mathbf{F}_o^\tau(s)\cdot\overline{\mathbf{u}}_\omega\, d\lambda_3\right\} e^{i\omega s}\, ds. \tag{5.13}$$

The proof of Theorem 5.1 can be carried out in a straightforward manner with the use of an extension of the classical Kirchhoff representation for certain solutions of the usual wave equation ((5.1) with $\gamma = 0$; cf., *e.g.*, Ref. 20) to cover the more general case of the damped-wave equation (5.1). This extension can be established with the aid of a causal fundamental solution for the hyperbolic operator in question (cf., *e.g.*, Ref. 21). Alternately, one can proceed as in the proof of the representation result that we provide in the form of Lemma A.1 of the Appendix; it is convenient to introduce here some additional notation in preparation for the use of that Lemma. When $\mathbf{x} \in \mathbf{R}^3$, let the corresponding distance-function $r_\mathbf{x}$ on $\mathbf{R}^3$ be defined by

$$r_\mathbf{x}(\mathbf{y}) := |\mathbf{y} - \mathbf{x}| \qquad \text{for } \mathbf{y} \in \mathbf{R}^3.$$

Suppose that Ψ is a complex function defined on a cylinder $F \times \mathbf{R}$, for some $F \subset \mathbf{R}^3$. With Ψ, $c > 0$, and any chosen $\mathbf{x} \in \mathbf{R}^3$ and $t \in \mathbf{R}$, we associate a function defined on F, denoted by $[\Psi]_{(\mathbf{x},t)} : F \to \mathbf{C}$ and termed *the (c-)retardation of* Ψ *with respect to* $(\mathbf{x}, t)$, according to the rule

$$[\Psi]_{(\mathbf{x},t)}(\mathbf{y}) := \Psi\left(\mathbf{y}, t - \frac{1}{c}r_\mathbf{x}(\mathbf{y})\right) \qquad \text{for each } \mathbf{y} \in F.$$

Recall our convention of §1: for an appropriate complex function Ψ defined in $\Omega_+ \times \mathbf{R}$, by $\Psi_{,\mathbf{n}_+}$ we mean the function on $\Gamma_+ \times \mathbf{R}$ obtained by taking the spatial normal derivative at fixed values of the fourth argument. That is, $\Psi_{,\mathbf{n}_+}(\cdot,t) := \Psi(\cdot,t)_{,\mathbf{n}_+}$ on Γ_+ for $t \in \mathbf{R}$.

Proof of Theorem 5.1: We shall first assume that $\mathbf{R}^3\backslash\overline{\Omega}_+$ is of type $\Re^1$, $\Phi \in C^2(\overline{\Omega}_+ \times \mathbf{R})$, $\Phi^\tau \in C^1(\overline{\Omega}_+ \times \mathbf{R})$, $\varphi \in C^1(\overline{\Omega}_+)$, $(5.4)_1$ holds for every $\mathbf{y} \in \overline{\Omega}_+$, and $(5.4)_2$ and (5.5) are valid for $\mathbf{y} \in \Gamma_+$ instead of for the points $\mathbf{y}$ specified there. Notice that then (5.3), as well as the corresponding equalities between the extended first partial derivatives, must hold in all of $\overline{\Omega}_+$ for each $t \in \mathbf{R}$. Under these conditions, we shall show that φ must be given in Ω_+ by

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Gamma_+}\left\{\left(\frac{e^{i\kappa_+ r_\mathbf{x}}}{r_\mathbf{x}}\right)\varphi_{,\mathbf{n}_+} - \left(\frac{e^{i\kappa_+ r_\mathbf{x}}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+}\varphi\right\} d\lambda_{\Gamma_+}, \qquad \text{for } \mathbf{x} \in \Omega_+. \tag{5.14}$$

Then, returning to the original hypotheses, and setting, for $\epsilon > 0$,

$$(\Omega_+)_\epsilon := \{\, \mathbf{y} \in \Omega_+ \mid d_+(\mathbf{y}) > \epsilon \,\},$$

we shall have shown that (5.14) holds with Ω_+ replaced by $(\Omega_+)_\epsilon$ (and, of course, Γ_+ replaced by $\partial(\Omega_+)_\epsilon$), for any sufficiently small positive ϵ. From this, clearly it shall follow that φ has real-analytic real and imaginary parts in all of Ω_+, that (1.23) holds, and that therefore (1.34) is true, the latter obtaining because $\operatorname{Im}\kappa_+ \geq 0$ (as pointed out in §1). Thus, it suffices to prove (5.14) under the modified hypotheses listed; in particular, it is to be understood for the remainder of the proof that (5.4) and (5.5) are taken to hold for each $\mathbf{y} \in \Gamma_+$.

With this agreement in force, we have the representation for Φ available by appropriately specializing in the statement of Lemma A.1. Specifically, taking $\gamma_0 = 0$ and $\gamma_1 = \gamma$ there, we get

$$\begin{aligned}\Phi(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\Gamma_+}\Bigg\{&\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right)[\Phi_{,\mathbf{n}_+}]_{(\mathbf{x},t)} - \left(\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+} + \frac{\gamma}{4c}(e^{-\gamma r_\mathbf{x}/2c})_{,\mathbf{n}_+}\right)[\Phi]_{(\mathbf{x},t)} \\ &+\frac{1}{c}\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right) r_{\mathbf{x},\mathbf{n}_+}[\Phi_{,4}]_{(\mathbf{x},t)} \\ &+\frac{\gamma}{2c}\int_0^{t-\frac{1}{c}r_\mathbf{x}} e^{-\gamma(t-\tau)/2}\Bigg\{g_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\Phi_{,\mathbf{n}_+}(\cdot,\tau) \\ &+\frac{1}{c^2}(r_\mathbf{x}^2)_{,\mathbf{n}_+}g'_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\Phi(\cdot,\tau)\Bigg\}\,d\tau\Bigg\}\,d\lambda_{\Gamma_+}\end{aligned}$$

$$\text{for } \mathbf{x} \in \Omega_+ \text{ and } t \in \mathbf{R}. \tag{5.15}$$

Here, with I_ν denoting the modified Bessel function of the first kind and order ν, the function $g_\gamma : \mathbf{R}\backslash\{0\} \to \mathbf{C}$ is defined to vanish for negative values of its argument and to be given by

$$g_\gamma(s) := \frac{I_1(\gamma s^{\frac{1}{2}}/2)}{s^{\frac{1}{2}}} \qquad \text{for } s > 0; \tag{5.16}$$

then $g_\gamma|(0,\infty) \in C^\infty([0,\infty))$. In particular, $g_0 = 0$, so that the terms in (5.15) that contain g_γ vanish for the ordinary wave equation, and this gives the classical Kirchhoff representation.

Now, let $\mathbf{x}$ be a point chosen in Ω_+; since $\varphi(\mathbf{x}) = \lim_{t\to\infty} e^{i\omega t}\Phi(\mathbf{x},t)$, we examine the product $e^{i\omega t}\Phi(\mathbf{x},t)$ for $t \in \mathbf{R}$. Employing (5.15) and observing that, by (5.3) and the additional regularity assumptions noted,

$[\Phi]_{(\mathbf{x},t)}(\mathbf{y}) = [\Phi^\tau]_{(\mathbf{x},t)}(\mathbf{y}) + \varphi(\mathbf{y})e^{-i\omega t}e^{i\omega r_\mathbf{x}(\mathbf{y})/c}$, $[\Phi_{,4}]_{(\mathbf{x},t)}(\mathbf{y}) = [\Phi^\tau_{,4}]_{(\mathbf{x},t)}(\mathbf{y}) - i\omega\varphi(\mathbf{y})e^{-i\omega t}e^{i\omega r_\mathbf{x}(\mathbf{y})/c}$, and $[\Phi,_{\mathbf{n}_+}]_{(\mathbf{x},t)}(\mathbf{y}) = [\Phi^\tau_{,\mathbf{n}_+}]_{(\mathbf{x},t)}(\mathbf{y}) + \varphi,_{\mathbf{n}_+}(\mathbf{y})e^{-i\omega t}e^{i\omega r_\mathbf{x}(\mathbf{y})/c}$ hold for $\mathbf{y} \in \Gamma_+$, we find

$$\begin{aligned} e^{i\omega t}\Phi(\mathbf{x},t) = &-\frac{e^{i\omega t}}{4\pi}\int_{\Gamma_+}\Bigg\{\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right)[\Phi^\tau_{,\mathbf{n}_+}]_{(\mathbf{x},t)} - \left(\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+} + \frac{\gamma}{4c}(e^{-\gamma r_\mathbf{x}/2c})_{,\mathbf{n}_+}\right)[\Phi^\tau]_{(\mathbf{x},t)} \\ &\qquad + \frac{1}{c}\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right) r_{\mathbf{x},\mathbf{n}_+}[\Phi^\tau_{,4}]_{(\mathbf{x},t)} \\ &\qquad + \frac{\gamma}{2c}\int_0^{t-\frac{1}{c}r_\mathbf{x}} e^{-\gamma(t-\tau)/2}\Big\{g_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\Phi^\tau_{,\mathbf{n}_+}(\cdot,\tau) \\ &\qquad\qquad + \frac{1}{c^2}(r_\mathbf{x}^2)_{,\mathbf{n}_+}g'_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\Phi^\tau(\cdot,\tau)\Big\}\,d\tau\Bigg\}\,d\lambda_{\Gamma_+} \\ &-\frac{1}{4\pi}\int_{\Gamma_+}\Bigg\{\left(\frac{e^{-\alpha r_\mathbf{x}/c}}{r_\mathbf{x}}\right)\varphi_{,\mathbf{n}_+} - \left(\left(\frac{e^{-\alpha r_\mathbf{x}/c}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+} - \frac{\gamma^2}{8c^2}(e^{-\alpha r_\mathbf{x}/c})r_{\mathbf{x},\mathbf{n}_+}\right)\varphi \\ &\qquad + \frac{\gamma}{2c}\int_0^{t-\frac{1}{c}r_\mathbf{x}} e^{-\alpha(t-\tau)}\Big\{g_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\varphi_{,\mathbf{n}_+} \\ &\qquad\qquad + \frac{1}{c^2}(r_\mathbf{x}^2)_{,\mathbf{n}_+}g'_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\varphi\Big\}\,d\tau\Bigg\}\,d\lambda_{\Gamma_+} \end{aligned}$$

$$\text{for } t \in \mathbf{R}, \tag{5.17}$$

having set

$$\alpha \equiv \alpha(\omega,\gamma) := \frac{\gamma}{2} - i\omega. \tag{5.18}$$

In the second integral over Γ_+ on the right in (5.17), a simplification has resulted from the combination

$$\left(\frac{e^{-\gamma r_\mathbf{x}/2c}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+}e^{i\omega r_\mathbf{x}/c} + \frac{i\omega}{c}\left(\frac{e^{-\alpha r_\mathbf{x}/c}}{r_\mathbf{x}}\right)r_{\mathbf{x},\mathbf{n}_+} = \left(\frac{e^{-\alpha r_\mathbf{x}/c}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+}, \qquad \text{on } \Gamma_+.$$

We write (5.17) in the form

$$e^{i\omega t}\Phi(\mathbf{x},t) = -\frac{e^{i\omega t}}{4\pi}\mathcal{I}_1(\mathbf{x},t) - \frac{1}{4\pi}\mathcal{I}_2(\mathbf{x},t),$$

$\mathcal{I}_1(\mathbf{x},t)$ denoting the first integral over Γ_+ appearing on the right, $\mathcal{I}_2(\mathbf{x},t)$ denoting the second. We intend to show that

$$\lim_{t\to\infty}\mathcal{I}_1(\mathbf{x},t) = 0 \tag{5.19}$$

and

$$\lim_{t\to\infty}\mathcal{I}_2(\mathbf{x},t) = \int_{\Gamma_+}\left\{\left(\frac{e^{i\kappa_+ r_\mathbf{x}}}{r_\mathbf{x}}\right)\varphi_{,\mathbf{n}_+} - \left(\frac{e^{i\kappa_+ r_\mathbf{x}}}{r_\mathbf{x}}\right)_{,\mathbf{n}_+}\varphi\right\}d\lambda_{\Gamma_+}, \tag{5.20}$$

whence (5.14) shall follow, and the proof will be effectively complete, as outlined. The equalities (5.19) and (5.20) are established by considering in turn the various terms appearing in $\mathcal{I}_1(\mathbf{x},t)$ and $\mathcal{I}_2(\mathbf{x},t)$, identifying for each integrand its pointwise limit as $t\to\infty$ and an estimate valid for all sufficiently large t, in such a way that the Dominated-Convergence Theorem of Lebesgue will be applicable to give the limiting value of that term. Introducing the number

$$d^+(\mathbf{x}) := \max_{\mathbf{y}\in\Gamma_+}|\mathbf{y}-\mathbf{x}|,$$

henceforth, unless otherwise specified, we suppose (at least) that $t > d^+(\mathbf{x})/c$, so that $t - r_\mathbf{x}(\mathbf{y})/c$ is positive for all $\mathbf{y}\in\Gamma_+$.

With this plan in mind, we examine first the integral

$$\mathcal{I}_{11}(\mathbf{x},t) := \int_{\Gamma_+} \left(\frac{e^{-\gamma r_{\mathbf{x}}/2c}}{r_{\mathbf{x}}}\right) [\Phi^{\tau}_{;\mathbf{n}_+}]_{(\mathbf{x},t)}\, d\lambda_{\Gamma_+}.$$

Since $[\Phi^{\tau}_{;\mathbf{n}_+}]_{(\mathbf{x},t)}(\mathbf{y}) = \Phi^{\tau}_{;\mathbf{n}_+}(\mathbf{y}, t - r_{\mathbf{x}}(\mathbf{y})/c)$ for every $\mathbf{y} \in \Gamma_+$, it is clear from (the modified form of) $(5.4)_2$ that the integrand here converges pointwise on Γ_+ to zero as $t \to \infty$. Moreover, it is just as clear that the integrand is majorized in modulus by

$$\frac{1}{d_+(\mathbf{x})} \max_{\mathbf{y}\in\Gamma_+} |\Phi^{\tau}_{;\mathbf{n}_+}\left(\mathbf{y}, t - \frac{1}{c} r_{\mathbf{x}}(\mathbf{y})\right)| \le \frac{M}{d_+(\mathbf{x})}$$

for all $t > d^+(\mathbf{x})/c$, the inequality here following from (the modified form of) (5.5). Consequently, the Dominated-Convergence Theorem allows the assertion that $\lim_{t\to\infty} \mathcal{I}_{11}(\mathbf{x},t) = 0$. The integrals corresponding to the second and third terms in the integrand of $\mathcal{I}_1(\mathbf{x},t)$ can be shown also to have limit zero, in a similar manner, by using the other (modified) hypotheses in (5.4) and (5.5). Now (5.19) has been verified if $\gamma = 0$, so we assume in the further reasoning concerning $\mathcal{I}_1(\mathbf{x},t)$ that $\operatorname{Re}\gamma > 0$.

Next, let

$$\mathcal{I}_{14}(\mathbf{x},t) := \int_{\Gamma_+} \int_0^{t-\frac{1}{c}r_{\mathbf{x}}} e^{-\gamma(t-\tau)/2} g_\gamma\left((t-\tau)^2 - \frac{1}{c^2} r_{\mathbf{x}}^2\right) \Phi^{\tau}_{;\mathbf{n}_+}(\cdot,\tau)\, d\tau\, d\lambda_{\Gamma_+},$$

which can be written

$$\mathcal{I}_{14}(\mathbf{x},t) = \int_{\Gamma_+} \int_0^{\infty} e^{-\gamma\tau/2} \frac{I_1(\gamma(\tau^2 - r_{\mathbf{x}}^2/c^2)^{\frac{1}{2}}/2)}{(\tau^2 - r_{\mathbf{x}}^2/c^2)^{\frac{1}{2}}} \Phi^{\tau}_{;\mathbf{n}_+}(\cdot, t-\tau) h_0\left(\tau - \frac{1}{c} r_{\mathbf{x}}\right) h_0(t-\tau)\, d\tau\, d\lambda_{\Gamma_+}, \qquad (5.21)$$

wherein h_0 denotes the Heaviside function:

$$h_0(s) := \begin{cases} 0 & \text{if } s < 0, \\ 1 & \text{if } s > 0. \end{cases}$$

Once again, with $(5.4)_2$ it is evident that the integrand in (5.21) converges to zero as $t \to \infty$, pointwise on $\Gamma_+ \times (0,\infty)$. To derive an acceptable uniform estimate for the modulus of that integrand, we need some information about the behavior of I_1 when its argument lies in the right half-plane and has large modulus. A standard asymptotic result for $I_\nu(z)$ when the principal argument $\arg z$ is restricted by $|\arg z| \le (\pi/2) - \delta$, with δ fixed in $(0, \pi/2)$ (cf., *e.g.*, Eq. (5.11.10) of Ref. 22), coupled with the fact that now $\operatorname{Re}\gamma > 0$, shows that there exist positive numbers $M_\nu(\gamma)$ and $s_\nu(\gamma)$ such that

$$|I_\nu(\gamma s^{\frac{1}{2}}/2)| \le \frac{e^{s^{\frac{1}{2}}\operatorname{Re}\gamma/2}}{(\pi|\gamma|)^{\frac{1}{2}} s^{\frac{1}{4}}} \left\{1 + \frac{M_\nu(\gamma)}{s^{\frac{1}{2}}}\right\} \qquad \text{for } s > s_\nu(\gamma). \qquad (5.22)$$

Thus, for $\mathbf{y} \in \Gamma_+$, $\tau > \sqrt{(d^+(\mathbf{x})/c)^2 + s_1(\gamma)^2}$, and any t, the modulus of the integrand in (5.21) at $(\mathbf{y},\tau;t)$ is bounded by

$$e^{-\tau\operatorname{Re}\gamma/2} \frac{e^{(\tau^2 - (r_{\mathbf{x}}(\mathbf{y})/c)^2)^{\frac{1}{2}}\operatorname{Re}\gamma/2}}{(\pi|\gamma|)^{\frac{1}{2}}(\tau^2 - (r_{\mathbf{x}}(\mathbf{y})/c)^2)^{\frac{3}{4}}} |\Phi^{\tau}_{;\mathbf{n}_+}(\mathbf{y}, t-\tau) h_0(t-\tau)| \le \frac{M}{(\pi|\gamma|)^{\frac{1}{2}}(\tau^2 - (d^+(\mathbf{x})/c)^2)^{\frac{3}{4}}}.$$

Meanwhile, in the remaining (bounded) portion $\Gamma_+ \times (0, \sqrt{(d^+(\mathbf{x})/c)^2 + s_1(\gamma)^2}\,]$ of $\Gamma_+ \times (0,\infty)$, and for, say, $t > 0$, it is easy to see that the integrand in (5.21) is uniformly bounded in modulus, in view of the boundedness of g_γ on bounded subsets of $(0,\infty)$ and of $\Phi^{\tau}_{;\mathbf{n}_+}$ on $\Gamma_+ \times (0,\infty)$. Consequently, there is a nonnegative element of $L_1(\Gamma_+ \times (0,\infty))$ dominating the modulus of the integrand in (5.21) for all sufficiently large t, whence $\lim_{t\to\infty} \mathcal{I}_{14}(\mathbf{x},t) = 0$.

Denoting by $\mathcal{I}_{15}(\mathbf{x},t)$ the final term of $\mathcal{I}_1(\mathbf{x},t)$, an integration by parts produces

$$\begin{aligned}\mathcal{I}_{15}(\mathbf{x},t) :&= \frac{\gamma}{2c^3}\int_{\Gamma_+}\int_0^{t-\frac{1}{c}r_\mathbf{x}} (r_\mathbf{x}^2)_{,\mathbf{n}_+} e^{-\gamma(t-\tau)/2} g'_\gamma\left((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2\right)\Phi^\tau(\cdot,\tau)\,d\tau\,d\lambda_{\Gamma_+}\\ &= \frac{\gamma}{4c^3}\int_{\Gamma_+}\Bigg\{-2c(e^{-\gamma r_\mathbf{x}/2c})r_{\mathbf{x},\mathbf{n}_+}[\Phi^\tau]_{(\mathbf{x},t)}g_\gamma(0^+) + (r_\mathbf{x}^2)_{,\mathbf{n}_+}\frac{e^{-\gamma t/2}}{t}\Phi^\tau(\cdot,0)g_\gamma\left(t^2-\frac{1}{c^2}r_\mathbf{x}^2\right)\\ &\quad + (r_\mathbf{x}^2)_{,\mathbf{n}_+}\int_0^{t-\frac{1}{c}r_\mathbf{x}}\left(\Phi^\tau_{,4}(\cdot,\tau) + \left(\frac{\gamma}{2}+\frac{1}{(t-\tau)}\right)\Phi^\tau(\cdot,\tau)\right)\\ &\qquad \left(\frac{e^{-\gamma(t-\tau)/2}}{(t-\tau)}\right)g_\gamma\left((t-\tau)^2-\frac{1}{c^2}r_\mathbf{x}^2\right)d\tau\Bigg\}\,d\lambda_{\Gamma_+},\end{aligned} \tag{5.23}$$

with $g_\gamma(0^+) := \lim_{s\to 0^+} g_\gamma(s) = \gamma/4$. Obviously, the integral corresponding to the first term in the integrand on the right in (5.23) can be treated by essentially the same reasoning applied to $\mathcal{I}_{11}(\mathbf{x},\cdot)$ and so vanishes in the limit as $t \to \infty$. For the second term in the integrand, with the aid of (5.22) we derive the estimates

$$\begin{aligned}&\left|(r_\mathbf{x}^2)_{,\mathbf{n}_+}(\mathbf{y})\frac{e^{-\gamma t/2}}{t}\Phi^\tau(\mathbf{y},0)g_\gamma\left(t^2-\frac{1}{c^2}r_\mathbf{x}^2(\mathbf{y})\right)\right|\\ &\le 2d^+(\mathbf{x})\frac{e^{-t\operatorname{Re}\gamma/2}}{t}\frac{e^{\operatorname{Re}\gamma(t^2-(r_\mathbf{x}(\mathbf{y})/c)^2)^{\frac{1}{2}}/2}}{(\pi|\gamma|)^{\frac{1}{2}}(t^2-(r_\mathbf{x}(\mathbf{y})/c)^2)^{\frac{3}{4}}}\left\{1+\frac{M_\nu(\gamma)}{(t^2-(r_\mathbf{x}(\mathbf{y})/c)^2)^{\frac{1}{2}}}\right\}|\Phi^\tau(\mathbf{y},0)|\\ &\le \frac{4d^+(\mathbf{x})}{(\pi|\gamma|)^{\frac{1}{2}}t(t^2-(d^+(\mathbf{x})/c)^2)^{\frac{3}{4}}}\max_{\mathbf{z}\in\Gamma_+}|\Phi^\tau(\mathbf{z},0)|\\ &\qquad\qquad \text{for all } \mathbf{y}\in\Gamma_+ \text{ and all sufficiently large } t > d^+(\mathbf{x})/c,\end{aligned} \tag{5.24}$$

showing at once that the corresponding integral of this term over Γ_+ has limit zero as $t \to \infty$. The same assertion can be proven for the final (iterated-integral) term on the right in (5.23) by proceeding along the lines of the analysis carried out for $\mathcal{I}_{14}(\mathbf{x},\cdot)$; only straightforward modifications are required, so we omit the details. The limit equality in (5.19) can now be regarded as established.

Turning to $\mathcal{I}_2(\mathbf{x},\cdot)$, it is first of all obvious that (5.20) is true if $\gamma = 0$, since $g_0 = 0$ and $-\alpha(\omega,0)/c = i\omega/c = \kappa_+(\omega;c,0)$. Thus, we suppose in the remainder of the proof that $\operatorname{Re}\gamma > 0$. At the outset, we perform integrations by parts in the inner integral appearing in $\mathcal{I}_2(\mathbf{x},t)$, once for the term involving g_γ and twice for the term containing g'_γ. In the former, we base the computation on the observation

$$\frac{\gamma}{2}g_\gamma((t-\tau)^2 - r_\mathbf{x}^2(\mathbf{y})/c^2) = -\frac{G'_\gamma(\tau;\mathbf{y},t)}{(t-\tau)} \qquad \text{for } 0<\tau<t-r_\mathbf{x}(\mathbf{y})/c, \text{ with } \mathbf{y}\in\Gamma_+ \text{ and } t>d^+(\mathbf{x})/c, \tag{5.25}$$

wherein, for such τ, $\mathbf{y}$, and t,

$$G_\gamma(\tau;\mathbf{y},t) := I_0\left(\frac{\gamma}{2}((t-\tau)^2 - \frac{1}{c^2}r_\mathbf{x}^2(\mathbf{y}))^{\frac{1}{2}}\right);$$

in the term involving g'_γ, a first integration by parts results in the appearance of g_γ, whereupon (5.25) is used once again, to effect a second integration by parts. With K_ν denoting Macdonald's function of order ν (cf. Ref. 22), sometimes also called the modified Bessel function of the third kind and order ν (cf. Ref. 23), we use the equality

$$K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}, \qquad \text{for } z \neq 0 \text{ and } |\arg z| < \pi, \tag{5.26}$$

to replace the appearance of the exponential function in favor of $K_{\frac{1}{2}}$, subsequently applying the well-known formula asserting that the derivative of the function $z \mapsto z^{-\nu}K_\nu(z)$ is simply the function $z \mapsto -z^{-\nu}K_{\nu+1}(z)$. In this manner, we find

$$\frac{\gamma}{2c}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})} e^{-\alpha(t-\tau)}g_\gamma\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)d\tau$$

$$= -\frac{e^{-\alpha r_{\mathbf{x}}(\mathbf{y})/c}}{r_{\mathbf{x}}(\mathbf{y})}+\frac{e^{-\alpha t}}{ct}I_0\left(\frac{\gamma}{2}(t^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y}))^{\frac{1}{2}}\right)$$

$$+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\alpha^{\frac{3}{2}}}{c}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})}\frac{K_{\frac{3}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{1}{2}}}I_0\left(\frac{\gamma}{2}\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right)d\tau$$

and

$$\frac{\gamma}{2c^3}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})} e^{-\alpha(t-\tau)}g'_\gamma\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)d\tau$$

$$= -\frac{\gamma^2}{16c^2}\frac{e^{-\alpha r_{\mathbf{x}}(\mathbf{y})/c}}{r_{\mathbf{x}}(\mathbf{y})}+\frac{\gamma}{4c^3}\frac{e^{-\alpha t}}{t}g_\gamma\left(t^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)$$

$$-\frac{\alpha^3}{2c^3}\left\{\frac{c^3}{(\alpha r_{\mathbf{x}}(\mathbf{y}))^3}+\frac{c^2}{(\alpha r_{\mathbf{x}}(\mathbf{y}))^2}\right\}e^{-\alpha r_{\mathbf{x}}(\mathbf{y})/c}$$

$$+\frac{\alpha^3}{2c^3}\left\{\frac{1}{(\alpha t)^3}+\frac{1}{(\alpha t)^2}\right\}e^{-\alpha t}I_0\left(\frac{\gamma}{2}\left(t^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right)$$

$$+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\alpha^{\frac{5}{2}}}{2c^3}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})}\frac{K_{\frac{5}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{3}{2}}}I_0\left(\frac{\gamma}{2}\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right)d\tau,$$

each holding for all $\mathbf{y}\in\Gamma_+$ and $t > d^+(\mathbf{x})/c$. Inserting these results into the expression for $\mathcal{I}_2(\mathbf{x},t)$ and performing some rearrangements and simplifications (under which all terms involving the exponential $e^{-\alpha r_{\mathbf{x}}/c}$ disappear), we come to

$$\begin{aligned}\mathcal{I}_2(\mathbf{x},t) = \int_{\Gamma_+}\Bigg\{&\frac{e^{-\alpha t}}{t}I_0\left(\frac{\gamma}{2}\left(t^2-\frac{1}{c^2}r_{\mathbf{x}}^2\right)^{\frac{1}{2}}\right)\varphi_{,\mathbf{n}_+}\\ &+\frac{1}{4c^3}\Bigg(\gamma\frac{e^{-\alpha t}}{t}\frac{I_1(\gamma(t^2-r_{\mathbf{x}}^2/c^2)^{\frac{1}{2}}/2)}{(t^2-r_{\mathbf{x}}^2/c^2)^{\frac{1}{2}}}\\ &\qquad+2\left(\alpha+\frac{1}{t}\right)\frac{e^{-\alpha t}}{t^2}I_0\left(\frac{\gamma}{2}\left(t^2-\frac{1}{c^2}r_{\mathbf{x}}^2\right)^{\frac{1}{2}}\right)\Bigg)(r_{\mathbf{x}}^2)_{,\mathbf{n}_+}\varphi\Bigg\}\,d\lambda_{\Gamma_+}\\ &+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\alpha^{\frac{3}{2}}}{c}\int_{\Gamma_+}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}}\frac{K_{\frac{3}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{1}{2}}}I_0\left(\frac{\gamma}{2}\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2\right)^{\frac{1}{2}}\right)\varphi_{,\mathbf{n}_+}\,d\tau\,d\lambda_{\Gamma_+}\\ &+\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\alpha^{\frac{5}{2}}}{2c^3}\int_{\Gamma_+}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}}\frac{K_{\frac{5}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{3}{2}}}I_0\left(\frac{\gamma}{2}\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2\right)^{\frac{1}{2}}\right)(r_{\mathbf{x}}^2)_{,\mathbf{n}_+}\varphi\,d\tau\,d\lambda_{\Gamma_+}\end{aligned}$$

$$\text{for } t > d^+(\mathbf{x})/c. \qquad (5.27)$$

There is no difficulty in showing that the first integral on the right in (5.27) approaches zero as $t\to\infty$, since we have available for each of the terms in the integrand an estimate of the sort appearing in (5.24) (used to analyze the second term in the integral over Γ_+ on the right in (5.23)). One should note here that $\operatorname{Re}\alpha = \operatorname{Re}(\gamma/2)$, while both $\varphi|_{\Gamma_+}$ and $\varphi_{,\mathbf{n}_+}$ are now in $C(\Gamma_+)$, and so are bounded.

Finally, we must deal with the second and third integrals taken over Γ_+ on the right in (5.27). To evaluate the pointwise limits of the integrands, *i.e.*, of the inner integrals, we first write

$$\begin{aligned}\mathcal{I}_{22}(\mathbf{y},t;\mathbf{x}) :&= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\alpha^{\frac{3}{2}}}{c} \int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})} \frac{K_{\frac{3}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{1}{2}}} I_0\left(\frac{\gamma}{2}\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right) d\tau \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\alpha^{\frac{3}{2}}}{c} \int_0^{\sqrt{t^2-r_{\mathbf{x}}^2(\mathbf{y})/c^2}} \frac{K_{\frac{3}{2}}\left(\alpha\left(\tau^2 + r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{1}{2}}\right)}{\left(\tau^2 + r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{3}{4}}} I_0\left(\frac{\gamma}{2}\tau\right) \tau\, d\tau \end{aligned} \tag{5.28}$$

and

$$\begin{aligned}\mathcal{I}_{23}(\mathbf{y},t;\mathbf{x}) :&= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\alpha^{\frac{5}{2}}}{2c^3}(r_{\mathbf{x}}^2)_{,\mathbf{n}_+}(\mathbf{y}) \int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})} \frac{K_{\frac{5}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{3}{2}}} I_0\left(\frac{\gamma}{2}\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right) d\tau \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\alpha^{\frac{5}{2}}}{2c^3}(r_{\mathbf{x}}^2)_{,\mathbf{n}_+}(\mathbf{y}) \int_0^{\sqrt{t^2-r_{\mathbf{x}}^2(\mathbf{y})/c^2}} \frac{K_{\frac{5}{2}}\left(\alpha\left(\tau^2 + r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{1}{2}}\right)}{\left(\tau^2 + r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{5}{4}}} I_0\left(\frac{\gamma}{2}\tau\right) \tau\, d\tau \end{aligned} \tag{5.29}$$

for $\mathbf{y} \in \Gamma_+$ and $t > d^+(\mathbf{x})/c$.

Now, for the evaluation of the limits as $t \to \infty$ of the integrals on the right in (5.28) and (5.29), we have prepared Lemma A.2 of the Appendix. In the statement of that Lemma, we wish to take $a = r_{\mathbf{x}}(\mathbf{y})/c$ (with $\mathbf{y}$ chosen in Γ_+), $\beta = i\gamma/2 = -(\operatorname{Im}\gamma/2) + i(\operatorname{Re}\gamma/2)$, and $z = \alpha = (\operatorname{Re}\gamma/2) + i((\operatorname{Im}\gamma/2) - \omega)$; in particular, we shall then have

$$J_0(\beta\tau) = J_0\left(i\frac{\gamma}{2}\tau\right) = I_0\left(\frac{\gamma}{2}\tau\right) \qquad \text{for } \tau > 0$$

(with J_0 denoting, of course, the Bessel function of the first kind and order zero) and $\operatorname{Re} z = \operatorname{Im}\beta = \operatorname{Re}\gamma/2 > 0$, and consequently must use equality (A.19) for the evaluation of the limits in question. With these identifications,

$$z^2 + \beta^2 = -\omega(\omega + i\gamma) = |\omega(\omega + i\gamma)|e^{i(\vartheta - \pi)} \neq 0,$$

in which we have denoted the principal argument of $\omega(\omega + i\gamma)$ by ϑ. Since $\operatorname{Re}\gamma > 0$ (and $\omega > 0$), ϑ lies in $(0,\pi)$, whence $\vartheta - \pi$ lies in $(-\pi, 0)$, and we compute, recalling (1.19),

$$\sqrt{z^2 + \beta^2} = \sqrt{-\omega(\omega + i\gamma)} = |\omega(\omega + i\gamma)|e^{i\vartheta/2}e^{-i\pi/2} = -i\sqrt{\omega(\omega + i\gamma)} = -ic\kappa_+.$$

Provided that $\operatorname{Re}\xi > 1$, (A.19) then gives

$$\begin{aligned}\int_0^\infty \frac{K_\xi\left(\alpha\left(\tau^2 + r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{1}{2}}\right)}{\left(\tau^2 + r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{\xi}{2}}} I_0\left(\frac{\gamma}{2}\tau\right) \tau\, d\tau &= \frac{1}{z^\xi}\left(\frac{\sqrt{z^2+\beta^2}}{r_{\mathbf{x}}(\mathbf{y})/c}\right)^{\xi-1} K_{\xi-1}\left(r_{\mathbf{x}}(\mathbf{y})\sqrt{z^2+\beta^2}/c\right) \\ &= \frac{1}{\alpha^\xi}\left(\frac{-ic^2\kappa_+}{r_{\mathbf{x}}(\mathbf{y})}\right)^{\xi-1} K_{\xi-1}\left(-i\kappa_+ r_{\mathbf{x}}(\mathbf{y})\right). \end{aligned} \tag{5.30}$$

By first taking $\xi = 3/2$ in (5.30) and recalling (5.26), it is easy to check that

$$\lim_{t\to\infty} \mathcal{I}_{22}(\mathbf{y},t;\mathbf{x}) = \frac{e^{i\kappa_+ r_{\mathbf{x}}(\mathbf{y})}}{r_{\mathbf{x}}(\mathbf{y})} \qquad \text{for each } \mathbf{y} \in \Gamma_+; \tag{5.31}$$

by setting $\xi = 5/2$ in (5.30) and using (5.26) with the recursion relation for K_ξ to compute

$$K_{\frac{3}{2}}(z) = \left(1 + \frac{1}{z}\right) K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{1}{z^{\frac{3}{2}}} + \frac{1}{z^{\frac{1}{2}}}\right) e^{-z},$$

one also finds

$$\begin{aligned}\lim_{t\to\infty} \mathcal{I}_{23}(\mathbf{y}, t; \mathbf{x}) &= r_{\mathbf{x}}(\mathbf{y}) r_{\mathbf{x},\mathbf{n}_+}(\mathbf{y}) \left(\frac{-i\kappa_+}{r_{\mathbf{x}}(\mathbf{y})}\right)^{\frac{3}{2}} \left\{ \frac{1}{(-i\kappa_+ r_{\mathbf{x}}(\mathbf{y}))^{\frac{3}{2}}} + \frac{1}{(-i\kappa_+ r_{\mathbf{x}}(\mathbf{y}))^{\frac{1}{2}}} \right\} e^{i\kappa_+ r_{\mathbf{x}}(\mathbf{y})} \\ &= -\left(\frac{e^{i\kappa_+ r_{\mathbf{x}}}}{r_{\mathbf{x}}}\right)_{,\mathbf{n}_+}(\mathbf{y}) \qquad \text{for each } \mathbf{y} \in \Gamma_+. \end{aligned} \tag{5.32}$$

Now, to prove that the pointwise limits in (5.31) and (5.32) imply (5.20), we have only to produce appropriate estimates for the integrands of the second and third integrals taken over Γ_+ in (5.27). To this end, we cite the bound

$$|K_\xi(\alpha s)| \le \left(\frac{\pi}{2|\alpha|}\right)^{\frac{1}{2}} \frac{e^{-s \operatorname{Re} \alpha}}{s^{\frac{1}{2}}} \left\{1 + \frac{\tilde{M}_\xi(\alpha)}{s}\right\} \qquad \text{for } s > \tilde{s}_\xi(\alpha), \tag{5.33}$$

wherein $\tilde{M}_\xi(\alpha)$ and $\tilde{s}_\xi(\alpha)$ are certain positive numbers depending upon only the indicated parameters, following from an asymptotic formula for $K_\xi(z)$ that is valid when $|\arg z| \le \pi - \delta$, with δ fixed in $(0, \pi)$ (cf. Eq. (5.11.9) of Ref. 22). With this, we find that the integrands in question are uniformly bounded for $\mathbf{y} \in \Gamma_+$ and all sufficiently large t. For example, consider the second term on the right in (5.27): if $\mathbf{y} \in \Gamma_+$ and $t > t_o > d^+(\mathbf{x})/c$,

$$\begin{aligned} &\left| \varphi_{,\mathbf{n}_+}(\mathbf{y}) \int_0^{t - \frac{1}{c} r_{\mathbf{x}}(\mathbf{y})} \frac{K_{\frac{3}{2}}(\alpha(t-\tau))}{(t-\tau)^{\frac{1}{2}}} I_0\left(\frac{\gamma}{2}\left((t-\tau)^2 - \frac{1}{c^2} r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right) d\tau \right| \\ &\qquad \le \max_{\mathbf{z}\in\Gamma_+} |\varphi_{,\mathbf{n}_+}(\mathbf{z})| \left\{ \int_{\frac{1}{c} r_{\mathbf{x}}(\mathbf{y})}^{t_o} \left| \frac{K_{\frac{3}{2}}(\alpha\tau)}{\tau^{\frac{1}{2}}} I_0\left(\frac{\gamma}{2}\left(\tau^2 - \frac{1}{c^2} r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right)\right| d\tau \right. \\ &\qquad\qquad \left. + \int_{t_o}^{t} \left| \frac{K_{\frac{3}{2}}(\alpha\tau)}{\tau^{\frac{1}{2}}} I_0\left(\frac{\gamma}{2}\left(\tau^2 - \frac{1}{c^2} r_{\mathbf{x}}^2(\mathbf{y})\right)^{\frac{1}{2}}\right)\right| d\tau \right\}. \end{aligned}$$

By using (5.33) and recalling (5.22), the second integral on the right here is, for a fixed, sufficiently large t_o,

$$\le \frac{4}{(2|\alpha\gamma|)^{\frac{1}{2}}} \int_{t_o}^{t} \frac{e^{-(\operatorname{Re}\alpha)\tau}}{\tau} \frac{e^{(\operatorname{Re}\gamma/2)\left(\tau^2 - r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{1}{2}}}}{\left(\tau^2 - r_{\mathbf{x}}^2(\mathbf{y})/c^2\right)^{\frac{1}{4}}} d\tau \le \frac{4}{(2|\alpha\gamma|)^{\frac{1}{2}}} \int_{t_o}^{\infty} \frac{1}{\tau\left(\tau^2 - (d^+(\mathbf{x})/c)^2\right)^{\frac{1}{4}}} d\tau,$$

and, for that t_o, the first integral on the right is

$$\le \max_{0 \le s \le t_o} \left| I_0\left(\frac{\gamma}{2} s\right)\right| \int_{\frac{1}{c} d_+(\mathbf{x})}^{t_o} \frac{|K_{\frac{3}{2}}(\alpha\tau)|}{\tau^{\frac{1}{2}}} d\tau,$$

inequalities providing uniform bounds as claimed. The third term on the right in (5.27) can be handled in a similar manner, whence we conclude by (5.31), (5.32), and the Dominated-Convergence Theorem that (5.20) is indeed correct. This completes the proof of Theorem 5.1. ∎

Proof of Theorem 5.2: Fix $\mathbf{u}_\omega \in \mathcal{N}^2_{\Omega_o,\omega}$; corresponding to this choice, set

$$\mathbf{U}_\omega(t) := \mathbf{u}_\omega e^{-i\omega t} \qquad \text{for each } t \in \mathbf{R},$$

to get an element $\mathbf{U}_\omega$ of $C^\infty(\mathbf{R};\mathbf{H}^2)$ with the properties $\mathbf{\Delta}^\star_{\lambda,\mu}\mathbf{U}_\omega(t)+\varrho_o\omega^2\mathbf{U}_\omega(t)=\mathbf{0}$, $\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}_\omega(t)]=\mathbf{0}$, and $\mathbf{U}_\omega(t)|_{\Gamma_+}\cdot\mathbf{n}_+=0$ for each $t\in\mathbf{R}$; obviously, any one of the derivatives of $\mathbf{U}_\omega$, say, $\mathbf{U}'_\omega$, possesses the same properties. Consequently, the equality

$$\int_{\Omega_o}\{\overline{\mathbf{U}}'_\omega(s)\cdot\mathbf{\Delta}^\star_{\lambda,\mu}\mathbf{U}(s)-\mathbf{U}(s)\cdot\mathbf{\Delta}^\star_{\lambda,\mu}\overline{\mathbf{U}}'_\omega(s)\}\,d\lambda_3$$
$$=\int_{\Gamma_o}\{\overline{\mathbf{U}}'_\omega(s)|_{\Gamma_o}\cdot\mathbf{T}^{\mathbf{n}_o}[\mathbf{U}(s)]-\mathbf{U}(s)|_{\Gamma_o}\cdot\mathbf{T}^{\mathbf{n}_o}[\overline{\mathbf{U}}'_\omega(s)]\}\,d\lambda_{\Gamma_o},$$

holding for each $s\in\mathbf{R}$ and resulting from two applications of (2.9), leads to

$$\varrho_o\int_{\Omega_o}\{\mathbf{U}''(s)\cdot\overline{\mathbf{U}}'_\omega(s)+\omega^2\mathbf{U}(s)\cdot\overline{\mathbf{U}}'_\omega(s)\}\,d\lambda_3$$
$$=i\omega e^{i\omega s}\left\{\int_{\Gamma_o}\mathbf{T}_o(s)\cdot\overline{\mathbf{u}}_\omega|_{\Gamma_o}\,d\lambda_{\Gamma_o}+\int_{\Omega_o}\mathbf{F}_o(s)\cdot\overline{\mathbf{u}}_\omega\,d\lambda_3\right\}\qquad\text{for each } s\in\mathbf{R},\tag{5.34}$$

in view of (5.8) and (5.9). We note here that the function Ψ_+ need be considered no further. By using (5.6) and (5.10), in particular, by computing $\mathbf{U}''(s)$ from the latter equality as $\mathbf{U}^{\tau\prime\prime}(s)-\omega^2\mathbf{u}e^{-i\omega s}$ for each $s\in\mathbf{R}$, a first application of (5.34) yields

$$\varrho_o\int_{\Omega_o}\{\mathbf{U}^{\tau\prime\prime}(s)+\omega^2\mathbf{U}^\tau(s)\}\cdot\overline{\mathbf{U}}'_\omega(s)\,d\lambda_3$$
$$=i\omega e^{i\omega s}\left\{\int_{\Gamma_o}\mathbf{T}^\tau_o(s)\cdot\overline{\mathbf{u}}_\omega|_{\Gamma_o}\,d\lambda_{\Gamma_o}+\int_{\Omega_o}\mathbf{F}^\tau_o(s)\cdot\overline{\mathbf{u}}_\omega\,d\lambda_3\right\}$$
$$+i\omega\left\{\int_{\Gamma_o}\mathbf{t}_o\cdot\overline{\mathbf{u}}_\omega|_{\Gamma_o}\,d\lambda_{\Gamma_o}+\int_{\Omega_o}\mathbf{f}_o(s)\cdot\overline{\mathbf{u}}_\omega\,d\lambda_3\right\}\qquad\text{for each } s\in\mathbf{R}.$$

Directly from the latter equality, (5.12) results by employing the conditions of transience (5.7) and (5.11) (for $j=0$ and 2), the vanishing of $\mathbf{u}_\omega|_{\Gamma_+}\cdot\mathbf{n}_+$, and the Cauchy-Schwarz inequality.

For the proof of (5.13), we return to (5.34). By noting that

$$\mathbf{U}''\cdot\overline{\mathbf{U}}'_\omega+\omega^2\mathbf{U}\cdot\overline{\mathbf{U}}'_\omega=(\mathbf{U}'\cdot\overline{\mathbf{U}}'_\omega)'+\omega^2(\mathbf{U}\cdot\overline{\mathbf{U}}_\omega)'$$

holds by virtue of the relation $\mathbf{U}''_\omega=-\omega^2\mathbf{U}_\omega$ and again using (5.6), but now taking into account (5.12), we are led this time from (5.34) to

$$\varrho_o\int_{\Omega_o}(\mathbf{U}'\cdot\overline{\mathbf{U}}'_\omega+\omega^2\mathbf{U}\cdot\overline{\mathbf{U}}_\omega)'(s)\,d\lambda_3=i\omega e^{i\omega s}\left\{\int_{\Gamma_o}\mathbf{T}^\tau_o(s)\cdot\overline{\mathbf{u}}_\omega|_{\Gamma_o}\,d\lambda_{\Gamma_o}+\int_{\Omega_o}\mathbf{F}^\tau_o(s)\cdot\overline{\mathbf{u}}_\omega\,d\lambda_3\right\}$$
$$\text{for each } s\in\mathbf{R}.$$

Choose any $t>0$ and integrate this equality over $(0,t)$; applications of the Cauchy-Schwarz inequality and accounting for the continuity of the functions $s\mapsto\|\mathbf{U}^{(j)}(s)\|_{\mathbf{H}^0}$ serve to verify that Fubini's Theorem can be invoked to justify reversing the order of integration in the resultant left-hand side. By recalling that $\mathbf{U}(0)=\mathbf{U}'(0)=\mathbf{0}$ and using (5.10) to replace the appearance of $\mathbf{U}$ and $\mathbf{U}'$, we come to

$$2\varrho_o\omega^2\int_{\Omega_o}\mathbf{u}\cdot\overline{\mathbf{u}}_\omega\,d\lambda_3+\varrho_o\int_{\Omega_o}\{\mathbf{U}^{\tau\prime}(t)\cdot\overline{\mathbf{U}}'_\omega(t)+\omega^2\mathbf{U}^\tau(t)\cdot\overline{\mathbf{U}}_\omega(t)\}\,d\lambda_3$$
$$=i\omega\int_0^t e^{i\omega s}\left\{\int_{\Gamma_o}\mathbf{T}^\tau_o(s)\cdot\overline{\mathbf{u}}_\omega|_{\Gamma_o}\,d\lambda_{\Gamma_o}+\int_{\Omega_o}\mathbf{F}^\tau_o(s)\cdot\overline{\mathbf{u}}_\omega\,d\lambda_3\right\}ds,$$

holding then for each $t > 0$. Appealing to (5.11), this time for $j = 0$ and 1, we see that the second term on the left here has the limit 0 as $t \to \infty$. Thus, the limit of the integral on the right exists, and (5.13) results upon letting $t \to \infty$. ∎

Finally, supposing that the hypotheses of Theorem 5.2 remain in force, we remark that it is easy to identify additional conditions on $\mathbf{U}^\tau$ under which $\mathbf{u}$ satisfies (1.24). For example, suppose that $\mathbf{U}^\tau(t) \in \mathbf{H}^2$ for each $t \in \mathbf{R}$ and $\lim_{t\to\infty} \|\mathbf{U}^\tau(t)\|_{\mathbf{H}^2} = 0$. Then $\mathbf{u}$ must lie in $\mathbf{H}^2$ and (5.8) will give, with the limits being taken in the norm of $\mathbf{H}^0$,

$$\lim_{t\to\infty} e^{i\omega t}\{-\mathbf{\Delta}^\star_{\lambda,\mu}\mathbf{U}^\tau(t) + \varrho_o \mathbf{U}^{\tau\prime\prime}(t)\} - \mathbf{\Delta}^\star_{\lambda,\mu}\mathbf{u} - \varrho_o\omega^2\mathbf{u} = \lim_{t\to\infty} e^{i\omega t}\mathbf{F}^\tau_o(t) + \mathbf{f}_o,$$

whence (1.24) follows. Similarly, one can formulate reasonable conditions of this sort on the functions $\mathbf{U}$, $\mathbf{\Phi}$, $\mathbf{T}_o$, $\mathbf{F}_o$, $\mathbf{\Phi}'$, and on the summands in their forms assumed in §1, under which (1.18), (1.25), and (1.26) are consequences of (1.7), (1.13), and (1.14), respectively.

6. REFERENCES

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Appendix

AUXILIARY RESULTS

We have collected here the formulations and fairly detailed proofs of two statements that are used in the proof of Theorem 5.1.

The first result provides an integral representation for sufficiently regular and (for simplicity) initially quiescent solutions of the so-called damped-wave equation (1.10) or (5.1) in the exterior of a cylinder in space-time. The representation coincides with that resulting from the use of Green's Theorem in conjunction with a causal fundamental solution (in the distributional sense) for the hyperbolic operator involved (cf., *e.g.*, Ref. A.1), and reduces to the classical Kirchhoff expression (cf., *e.g.*, Ref. A.2) in the case of the ordinary wave equation. A full and precise statement of the representation does not seem to be readily available, and so we have chosen to outline here the main points in a derivation; the present proof does not employ the theory of distributions but is more in the spirit of "elliptic" reasoning, essentially consisting in an extension of the development of the Kirchhoff result as presented in Ref. A.2. Actually, we shall consider a more general hyperbolic operator, since but little additional work is thereby required.

It is convenient to introduce some notation, to prepare for the statement and proof of the first Lemma. When $\mathbf{x} \in \mathbf{R}^3$, let the corresponding distance-function $r_{\mathbf{x}}$ on $\mathbf{R}^3$ be defined by

$$r_{\mathbf{x}}(\mathbf{y}) := |\mathbf{y} - \mathbf{x}| \qquad \text{for } \mathbf{y} \in \mathbf{R}^3.$$

Suppose that Ψ is a complex function defined on a cylinder $F \times \mathbf{R}$, for some $F \subset \mathbf{R}^3$. With Ψ, $c > 0$, and any chosen $\mathbf{x} \in \mathbf{R}^3$ and $t \in \mathbf{R}$, we associate a function defined on F, denoted by $[\Psi]_{(\mathbf{x},t)} : F \to \mathbf{C}$ and termed *the (c-)retardation of* Ψ *with respect to* $(\mathbf{x}, t)$, according to the rule

$$[\Psi]_{(\mathbf{x},t)}(\mathbf{y}) := \Psi\left(\mathbf{y}, t - \frac{1}{c} r_{\mathbf{x}}(\mathbf{y})\right) \qquad \text{for each } \mathbf{y} \in F.$$

Recall our convention of §1: for an appropriate complex function Ψ defined in $\Omega_+ \times \mathbf{R}$, by $\Psi_{,\mathbf{n}_+}$ we mean the function on $\Gamma_+ \times \mathbf{R}$ obtained by taking the spatial normal derivative at fixed values of the fourth argument. That is, $\Psi_{,\mathbf{n}_+}(\cdot, t) := \Psi(\cdot, t)_{,\mathbf{n}_+}$ on Γ_+ for $t \in \mathbf{R}$. We shall write $B_a(\mathbf{x}) := \{\, \mathbf{y} \in \mathbf{R}^3 \mid |\mathbf{y} - \mathbf{x}| < a \,\}$ for the open ball in $\mathbf{R}^3$ of radius $a > 0$ and centered at $\mathbf{x} \in \mathbf{R}^3$. Finally, we use the standard notation I_ν for the modified Bessel function of the first kind and order ν; cf. the definition in Eq. (5.7.1) of Ref. A.3.

Lemma A.1. *Let the open set* Ω_+ *in* $\mathbf{R}^3$ *be connected and the complement of the closure of a bounded connected Lipschitz domain; write* $\Gamma_+ := \partial\Omega_+$, *and let* $\mathbf{n}_+$ *denote the unit-normal field that is defined at all appropriate points of* Γ_+ *and "directed into* Ω_+*." Suppose that* c *is a real positive number, while* γ_0 *and* γ_1 *are complex. Let* $\Phi \in C^2(\overline{\Omega}_+ \times \mathbf{R})$ *satisfy*

$$-\Delta\Phi + \frac{\gamma_0}{c^2}\Phi + \frac{\gamma_1}{c^2}\Phi_{,4} + \frac{1}{c^2}\Phi_{,44} = 0, \qquad \textit{in } \Omega_+ \times \mathbf{R}, \tag{A.1}$$

and vanish in $\Omega_+ \times (-\infty, 0)$:

$$\Phi(\cdot, t) = 0 \qquad \textit{in } \Omega_+ \text{ for } t < 0. \tag{A.2}$$

Then Φ has the representation given by

$$\Phi(\mathbf{x},t) = -\frac{1}{4\pi}\int_{\Gamma_+}\Bigg\{\left(\frac{e^{-\gamma_1 r_{\mathbf{x}}/2c}}{r_{\mathbf{x}}}\right)[\Phi_{,\mathbf{n}_+}]_{(\mathbf{x},t)} - \left(\left(\frac{e^{-\gamma_1 r_{\mathbf{x}}/2c}}{r_{\mathbf{x}}}\right)_{,\mathbf{n}_+} - \frac{\xi}{8c^2}(e^{-\gamma_1 r_{\mathbf{x}}/2c})r_{\mathbf{x},\mathbf{n}_+}\right)[\Phi]_{(\mathbf{x},t)}$$
$$+\frac{1}{c}\left(\frac{e^{-\gamma_1 r_{\mathbf{x}}/2c}}{r_{\mathbf{x}}}\right) r_{\mathbf{x},\mathbf{n}_+}[\Phi_{,4}]_{(\mathbf{x},t)}$$
$$+\frac{\sqrt{\xi}'}{2c}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}} e^{-\gamma_1(t-\tau)/2}\Big\{f_\xi\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2\right)\Phi_{,\mathbf{n}_+}(\cdot,\tau)$$
$$+\frac{1}{c^2}(r_{\mathbf{x}}^2)_{,\mathbf{n}_+}f_\xi'\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2\right)\Phi(\cdot,\tau)\Big\}\,d\tau\Bigg\}\,d\lambda_{\Gamma_+},$$

$$\text{for } \mathbf{x}\in\Omega_+ \text{ and } t\in\mathbf{R}, \qquad \text{(A.3)}$$

wherein

$$\xi := \gamma_1^2 - 4\gamma_0, \qquad \text{(A.4)}$$

and $\sqrt{\xi}'$ denotes a selected square root of ξ, corresponding to which the function $f_\xi : \mathbf{R}\setminus\{0\}\to\mathbf{C}$ is defined by

$$f_\xi(x) := \begin{cases} 0 & \text{if } x<0 \\ I_1(\sqrt{\xi}' x^{\frac{1}{2}}/2)/x^{\frac{1}{2}} & \text{if } x>0 \end{cases} \qquad \text{(A.5)}$$

(so that f_0 is the zero-function).

Proof: A preliminary transformation serves to simplify the form of the underlying partial differential equation: specifically, setting

$$\Psi(\mathbf{y},s) := \Phi(\mathbf{y},s)e^{\gamma_1 s/2} \qquad \text{for } \mathbf{y}\in\Omega_+ \text{ and } s\in\mathbf{R}, \qquad \text{(A.6)}$$

we get a function $\Psi\in C^2(\overline{\Omega}_+\times\mathbf{R})$, vanishing in $\Omega_+\times(-\infty,0)$ and satisfying, with ξ defined as in (A.4),

$$-\Delta\Psi - \frac{\xi}{4c^2}\Psi + \frac{1}{c^2}\Psi_{,44} = 0, \qquad \text{in } \Omega_+\times\mathbf{R}, \qquad \text{(A.7)}$$

as one can readily check. In the remainder of the proof, let $\mathbf{x}$ be fixed in Ω_+; until further notice, let t be fixed and positive. Then $[\Psi]_{(\mathbf{x},t)}$ and $[\Psi_{,4}]_{(\mathbf{x},t)}$ are defined in Ω_+ and lie in $C(\overline{\Omega}_+)$, while the supports of $[\Psi]_{(\mathbf{x},t)}$ and $[\Psi_{,4}]_{(\mathbf{x},t)}$ are compact in $\overline{\Omega}_+$, in fact, are contained in the closure of the set $\Omega_+\cap B_{ct}(\mathbf{x})$, since $[\Psi]_{(\mathbf{x},t)}(\mathbf{y})$ and $[\Psi_{,4}]_{(\mathbf{x},t)}(\mathbf{y})$ vanish when $t-(1/c)r_{\mathbf{x}}(\mathbf{y})<0$. Further, from the chain-rule computations yielding

$$[\Psi]_{(\mathbf{x},t),j}(\mathbf{y}) = [\Psi_{,j}]_{(\mathbf{x},t)}(\mathbf{y}) - \frac{1}{c}r_{\mathbf{x},j}(\mathbf{y})[\Psi_{,4}]_{(\mathbf{x},t)}(\mathbf{y}) \qquad \text{for } \mathbf{y}\in\Omega_+\setminus\{\mathbf{x}\},\ j=1,2,3,$$

and the further expressions giving $[\Psi]_{(\mathbf{x},t),jk}$ and $[\Psi_{,4}]_{(\mathbf{x},t),j}$ in $\Omega_+\setminus\{\mathbf{x}\}$ for $j,k=1,2,3$, it is apparent that $[\Psi]_{(\mathbf{x},t)}$ and $[\Psi_{,4}]_{(\mathbf{x},t)}$ may fail to have partial derivatives at $\mathbf{x}$, but their restrictions to $\Omega_+\setminus\{\mathbf{x}\}$ do lie in $C^2(\overline{\Omega}_+\setminus\{\mathbf{x}\})$ and $C^1(\overline{\Omega}_+\setminus\{\mathbf{x}\})$, respectively, with their first partial derivatives remaining bounded in that set. A bit of computation, along with an appeal to the obvious fact that the retardation with respect to $(\mathbf{x},t)$ of the function appearing on the left in (A.7) vanishes throughout Ω_+, will verify that the Laplacian of $[\Psi]_{(\mathbf{x},t)}$ satisfies (cf. Ref. A.2, for the case $\xi=0$)

$$\frac{1}{r_{\mathbf{x}}}[\Psi]_{(\mathbf{x},t),jj} = -\frac{2}{c}\left\{\frac{1}{r_{\mathbf{x}}}r_{\mathbf{x},j}[\Psi_{,4}]_{(\mathbf{x},t)}\right\}_{,j} - \left(\frac{\xi}{4c^2}\right)\frac{1}{r_{\mathbf{x}}}[\Psi]_{(\mathbf{x},t)}, \qquad \text{in } \Omega_+\setminus\{\mathbf{x}\}. \qquad \text{(A.8)}$$

For the moment, let f denote a complex function defined in $\mathbf{R}\setminus\{0\}$, with $f(s)=0$ for $s<0$ and the restriction $f|(0,\infty)\in C^2([0,\infty))$; we shall presently choose f judiciously. With f, $\mathbf{x}$, t, and any $\tilde\Psi\in C(\Omega_+\times\mathbf{R})$, we associate the function $\{f\star\tilde\Psi\}_{(\mathbf{x},t)}:\Omega_+\to\mathbf{C}$ according to

$$\{f\star\tilde\Psi\}_{(\mathbf{x},t)}(\mathbf{y}) := \int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})} f\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)\tilde\Psi(\mathbf{y},\tau)\,d\tau \qquad \text{for } \mathbf{y}\in\Omega_+.$$

The argument of f appearing here is

$$(t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y}) = \left\{\left(t - \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right) - \tau\right\}\left\{\left(t + \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right) - \tau\right\},$$

which is positive when $0 < \tau < t - r_{\mathbf{x}}(\mathbf{y})/c$ and negative when $t - r_{\mathbf{x}}(\mathbf{y})/c < \tau < 0$. In particular, $\{f \star \tilde{\Psi}\}_{(\mathbf{x},t)}$ vanishes outside the set $\Omega_+ \cap B_{ct}(\mathbf{x})$, since $f(s) = 0$ for $s < 0$; actually, the values of f for negative values of its argument will be irrelevant for us, since we shall be interested exclusively in the cases when $\tilde{\Psi}$ is either Ψ or $\Psi_{,j}$, each of which vanishes in $\Omega_+ \times (-\infty, 0)$. The regularity properties of $\{f \star \tilde{\Psi}\}_{(\mathbf{x},t)}$ can be discerned a bit more easily by making a change of variable: setting, for definiteness, $f(0) := f(0^+) := \lim_{s\to 0^+} f(s)$, we can write

$$\{f \star \tilde{\Psi}\}_{(\mathbf{x},t)}(\mathbf{y})$$

$$= \left(t - \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right)\int_0^1 f\left(\left(t - \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right)(1-s)\left(\left(t + \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right) - \left(t - \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right)s\right)\right)\tilde{\Psi}\left(\mathbf{y}, \left(t - \frac{1}{c}r_{\mathbf{x}}(\mathbf{y})\right)s\right)ds$$

whenever $\mathbf{y} \in \Omega_+$ (by first supposing that $t - r_{\mathbf{x}}(\mathbf{y})/c \neq 0$, and then noting that the result is true also when $t - r_{\mathbf{x}}(\mathbf{y})/c = 0$). From the latter form, it is a simple matter to check that $\{f \star \tilde{\Psi}\}_{(\mathbf{x},t)}$ is in $C(\overline{\Omega}_+)$, and, moreover, that the functions $\{f\star\Psi\}_{(\mathbf{x},t)}$ and $\{f\star(\Psi_{,j})\}_{(\mathbf{x},t)}$ of particular interest are at least in $C^2(\overline{\Omega}_+ \setminus \{\mathbf{x}\})$ and $C^1(\overline{\Omega}_+ \setminus \{\mathbf{x}\})$, respectively, with their first partial derivatives bounded in $\Omega_+ \setminus \{\mathbf{x}\}$. The only difficulty at all lies in examining the behavior of these functions near $\mathbf{x}$ and Γ_+ and on the set $\Omega_+ \cap \partial B_{ct}(\mathbf{x})$. Consider, for example, $\{f\star\Psi\}_{(\mathbf{x},t)}$: from the second form, it is easily seen that the first partial derivatives of $\{f\star\Psi\}_{(\mathbf{x},t)}$ exist and equal zero at any $\mathbf{y} \in \Omega_+$ with $t - r_{\mathbf{x}}(\mathbf{y})/c = 0$, owing to the regularity hypothesized for f and Ψ and the vanishing of Ψ in $\Omega_+ \times (-\infty, 0)$. At a point $\mathbf{y} \in \Omega_+$, $\mathbf{y} \neq \mathbf{x}$, and contained in the ball $B_{ct}(\mathbf{x})$, the usual formula for differentiation of a parameter-dependent integral can be applied to compute the first partial derivatives of $\{f \star \Psi\}_{(\mathbf{x},t)}$. Upon doing so, an inspection shows that the resultant expression vanishes outside $B_{ct}(\mathbf{x})$, and so is valid in all of $\Omega_+ \setminus \{\mathbf{x}\}$; that is, we have

$$\begin{aligned}\{f \star \Psi\}_{(\mathbf{x},t),j}(\mathbf{y}) &= -\frac{1}{c}f(0^+)r_{\mathbf{x},j}(\mathbf{y})[\Psi]_{(\mathbf{x},t)}(\mathbf{y}) \\ &\quad + \int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})}\left\{f\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)\Psi_{,j}(\mathbf{y},\tau)\right. \\ &\qquad \left. - \frac{1}{c^2}(r_{\mathbf{x}}^2)_{,j}(\mathbf{y})f'\left((t-\tau)^2 - \frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\right)\Psi(\mathbf{y},\tau)\right\}d\tau,\end{aligned}$$

$$\text{for } \mathbf{y} \in \Omega_+ \setminus \{\mathbf{x}\}. \qquad \text{(A.9)}$$

Directly from (A.9), it is obvious that the first partial derivatives of $\{f\star\Psi\}_{(\mathbf{x},t)}$ can be extended continuously to all points of Γ_+ and are bounded in $\Omega_+ \setminus \{\mathbf{x}\}$. The second partial derivatives of $\{f \star \Psi\}_{(\mathbf{x},t)}$ can be shown to be in $C(\overline{\Omega}_+ \setminus \{\mathbf{x}\})$ (but not necessarily bounded there) in a similar manner, starting from (A.9). The study of the first derivatives of $\{f \star (\Psi_{,j})\}_{(\mathbf{x},t)}$ is entirely analogous.

Now, by using (A.7) and the vanishing of Ψ in $\Omega_+ \times (-\infty, 0)$, an elementary, albeit rather tedious, computation reveals that the Laplacian of $\{f \star \Psi\}_{(\mathbf{x},t)}$ in $\Omega_+ \setminus \{\mathbf{x}\}$ satisfies

$$\{f\star\Psi\}_{(\mathbf{x},t)},_{jj}(\mathbf{y}) - 2\{f\star(\Psi,_j)\}_{(\mathbf{x},t)},_j(\mathbf{y})$$

$$= -\frac{2}{c}f(0^+)\frac{1}{r_{\mathbf{x}}(\mathbf{y})}[\Psi]_{(\mathbf{x},t)}(\mathbf{y})$$

$$-\frac{4}{c^2}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}(\mathbf{y})}\Big\{\Big((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\Big)f''\Big((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\Big)$$
$$+2f'\Big((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\Big)-\frac{\xi}{16}f\Big((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2(\mathbf{y})\Big)\Big\}\Psi(\mathbf{y},\tau)\,d\tau,$$

$$\text{for } \mathbf{y}\in\Omega_+\setminus\{\mathbf{x}\}. \qquad \text{(A.10)}$$

Evidently, we should select $f|(0,\infty)$ in such a way that this restriction lies in $C^2([0,\infty))$ and the bracketed factor in the integrand on the right in (A.10) vanishes for $0<\tau<t-r_{\mathbf{x}}(\mathbf{y})/c$, for each $\mathbf{y}\in\Omega_+\cap B_{ct}(\mathbf{x})$; since $(t-\tau)^2-r_{\mathbf{x}}^2(\mathbf{y})/c^2$ is positive for all such τ and $\mathbf{y}$, we demand that

$$sf''(s)+2f'(s)-\frac{\xi/4}{4}f(s)=0 \qquad \text{for all } s>0.$$

It follows that we must take $f|(0,\infty)$ to be given by a multiple of f_ξ, wherein

$$f_\xi(s) := \frac{I_1\left(\sqrt{\xi}'s^{\frac{1}{2}}/2\right)}{s^{\frac{1}{2}}} \qquad \text{for } s>0, \qquad \text{(A.11)}$$

with $\sqrt{\xi}'$ indicating a fixed choice of square root of ξ (in which case we shall even have $f|(0,\infty)\in C^\infty([0,\infty))$). For a reason that will become apparent shortly, we choose the multiple so that $f(0^+)=\xi/8c$; since we find $f_\xi(0^+)=\sqrt{\xi}'/4$ from (A.11), this requires that we take

$$f(s) := \frac{\sqrt{\xi}'}{2c}f_\xi(s) \qquad \text{for } s>0.$$

Now, with f so specified, (A.10) gives

$$\{f\star\Psi\}_{(\mathbf{x},t)},_{jj} = 2\{f\star(\Psi,_j)\}_{(\mathbf{x},t)},_j - \left(\frac{\xi}{4c^2}\right)\frac{1}{r_{\mathbf{x}}}[\Psi]_{(\mathbf{x},t)}, \qquad \text{in } \Omega_+\setminus\{\mathbf{x}\};$$

combining this with (A.8), we come to

$$\frac{1}{r_{\mathbf{x}}}[\Psi]_{(\mathbf{x},t)},_{jj} - \{f\star\Psi\}_{(\mathbf{x},t)},_{jj} = -\frac{2}{c}\left\{\frac{1}{r_{\mathbf{x}}}r_{\mathbf{x}},_j[\Psi,_4]_{(\mathbf{x},t)}\right\},_j - 2\{f\star(\Psi,_j)\}_{(\mathbf{x},t)},_j, \qquad \text{in } \Omega_+\setminus\{\mathbf{x}\}. \qquad \text{(A.12)}$$

Now let the positive number R be greater than ct and so large that the ball $B_R(\mathbf{x})$ contains $\mathbf{R}^3\setminus\Omega_+$; for all sufficiently small positive ϵ, denote by $\Omega_+^{R,\epsilon}$ the bounded open set $\{\Omega_+\cap B_R(\mathbf{x})\}\setminus\overline{B_\epsilon(\mathbf{x})}$, with boundary $\partial\Omega_+^{R,\epsilon}=\Gamma_+\cup\partial B_R(\mathbf{x})\cup\partial B_\epsilon(\mathbf{x})$. For any such ϵ, according to the preparation laid down in the discussions of the regularity properties of $[\Psi]_{(\mathbf{x},t)}$, $[\Psi,_4]_{(\mathbf{x},t)}$, $\{f\star\Psi\}_{(\mathbf{x},t)}$, and $\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}$, we can apply Green's Theorem to each of the terms in the integral of the left-hand side of (A.12) over $\Omega_+^{R,\epsilon}$ (recognizing that the Laplacians of both the function $1/r_{\mathbf{x}}$ and the function having everywhere the value 1 vanish in $\Omega_+^{R,\epsilon}$), while simultaneously using the Divergence Theorem to transform each of the terms in the integral of the right-hand side over the same open set. By proceeding in this manner and using the support properties of the integrands along with the condition $t-R/c<0$ to conclude that all resultant integrals taken over the portion $\partial B_R(\mathbf{x})$ of $\partial\Omega_+^{R,\epsilon}$ must vanish, we get (with n_{+j}, $j=1,2,3$, denoting the Cartesian components of $\mathbf{n}_+$)

$$-\int_{\Gamma_+}\left\{\left(\frac{1}{r_{\mathbf{x}}}\right)[\Psi]_{(\mathbf{x},t)},_{\mathbf{n}_+}-\left(\frac{1}{r_{\mathbf{x}}}\right),_{\mathbf{n}_+}[\Psi]_{(\mathbf{x},t)}-\{f\star\Psi\}_{(\mathbf{x},t)},_{\mathbf{n}_+}\right\}d\lambda_{\Gamma_+}$$
$$-\int_{\partial B_\epsilon(\mathbf{x})}r_{\mathbf{x}},_j\left\{\left(\frac{1}{r_{\mathbf{x}}}\right)[\Psi]_{(\mathbf{x},t)},_j-\left(\frac{1}{r_{\mathbf{x}}}\right),_j[\Psi]_{(\mathbf{x},t)}-\{f\star\Psi\}_{(\mathbf{x},t)},_j\right\}d\lambda_{\partial B_\epsilon(\mathbf{x})}$$

$$=\int_{\Omega_+^{R,\epsilon}}\left\{\left(\frac{1}{r_{\mathbf{x}}}\right)[\Psi]_{(\mathbf{x},t)},_{jj}-\{f\star\Psi\}_{(\mathbf{x},t)},_{jj}\right\}d\lambda_3$$
$$=\int_{\Omega_+^{R,\epsilon}}\left\{-\frac{2}{c}\left(\frac{1}{r_{\mathbf{x}}}r_{\mathbf{x}},_j[\Psi,_4]_{(\mathbf{x},t)}\right)-2\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}\right\},_j d\lambda_3$$

$$=2\int_{\Gamma_+}\left\{\left(\frac{1}{cr_{\mathbf{x}}}\right)r_{\mathbf{x}},_{\mathbf{n}_+}[\Psi,_4]_{(\mathbf{x},t)}+\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}\mathrm{n}_{+j}\right\}d\lambda_{\Gamma_+}$$
$$+2\int_{\partial B_\epsilon(\mathbf{x})}\left\{\left(\frac{1}{cr_{\mathbf{x}}}\right)[\Psi,_4]_{(\mathbf{x},t)}+\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}r_{\mathbf{x}},_j\right\}d\lambda_{\partial B_\epsilon(\mathbf{x})},$$

so, noting that

$$[\Psi,_{\mathbf{n}_+}]_{(\mathbf{x},t)}=[\Psi]_{(\mathbf{x},t)},_{\mathbf{n}_+}+\frac{1}{c}r_{\mathbf{x}},_{\mathbf{n}_+}[\Psi,_4]_{(\mathbf{x},t)}|_{\Gamma_+},\qquad \text{on } \Gamma_+,$$

we have

$$\int_{\partial B_\epsilon(\mathbf{x})}\left\{\left(\frac{1}{r_{\mathbf{x}}}\right)[\Psi]_{(\mathbf{x},t)},_j r_{\mathbf{x}},_j+\left(\frac{1}{r_{\mathbf{x}}^2}\right)[\Psi]_{(\mathbf{x},t)}-\{f\star\Psi\}_{(\mathbf{x},t)},_j r_{\mathbf{x}},_j\right.$$
$$\left.+\left(\frac{2}{cr_{\mathbf{x}}}\right)[\Psi,_4]_{(\mathbf{x},t)}+2\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}r_{\mathbf{x}},_j\right\}d\lambda_{\partial B_\epsilon(\mathbf{x})}$$
$$=-\int_{\Gamma_+}\left\{\left(\frac{1}{r_{\mathbf{x}}}\right)[\Psi,_{\mathbf{n}_+}]_{(\mathbf{x},t)}-\left(\frac{1}{r_{\mathbf{x}}}\right),_{\mathbf{n}_+}[\Psi]_{(\mathbf{x},t)}+\left(\frac{1}{cr_{\mathbf{x}}}\right)r_{\mathbf{x}},_{\mathbf{n}_+}[\Psi,_4]_{(\mathbf{x},t)}\right.$$
$$\left.+2\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}\mathrm{n}_{+j}-\{f\star\Psi\}_{(\mathbf{x},t)},_{\mathbf{n}_+}\right\}d\lambda_{\Gamma_+}. \qquad \text{(A.13)}$$

The expression on the left in (A.13) is consequently independent of the sufficiently small positive ϵ; its limit for $\epsilon \to 0^+$ is easily proven to be $4\pi\Psi(\mathbf{x},t)$, by noting that the first partial derivatives of $r_{\mathbf{x}}$, $[\Psi]_{(\mathbf{x},t)}$, and $\{f\star\Psi\}_{(\mathbf{x},t)}$ are bounded in $\Omega_+\setminus\{\mathbf{x}\}$, appealing to the continuity of $[\Psi]_{(\mathbf{x},t)}$, $[\Psi,_4]_{(\mathbf{x},t)}$, and $\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}$ in Ω_+, and observing that $[\Psi]_{(\mathbf{x},t)}(\mathbf{x})=\Psi(\mathbf{x},t)$. Thus, by inserting the definition of $\{f\star(\Psi,_j)\}_{(\mathbf{x},t)}$ and using (A.9), from (A.13) we derive the equality

$$\Psi(\mathbf{x},t)=-\frac{1}{4\pi}\int_{\Gamma_+}\left\{\left(\frac{1}{r_{\mathbf{x}}}\right)[\Psi,_{\mathbf{n}_+}]_{(\mathbf{x},t)}-\left(\left(\frac{1}{r_{\mathbf{x}}}\right),_{\mathbf{n}_+}-\frac{\xi}{8c^2}r_{\mathbf{x}},_{\mathbf{n}_+}\right)[\Psi]_{(\mathbf{x},t)}\right.$$
$$+\left(\frac{1}{cr_{\mathbf{x}}}\right)r_{\mathbf{x}},_{\mathbf{n}_+}[\Psi,_4]_{(\mathbf{x},t)}$$
$$+\frac{\sqrt{\xi}'}{2c}\int_0^{t-\frac{1}{c}r_{\mathbf{x}}}\left\{f_\xi\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2\right)\Psi,_{\mathbf{n}_+}(\cdot,\tau)\right.$$
$$\left.\left.+\frac{1}{c^2}(r_{\mathbf{x}}^2),_{\mathbf{n}_+}f'_\xi\left((t-\tau)^2-\frac{1}{c^2}r_{\mathbf{x}}^2\right)\Psi(\cdot,\tau)\right\}d\tau\right\}d\lambda_{\Gamma_+}. \qquad \text{(A.14)}$$

Upon using (A.6) to replace each appearance of Ψ in (A.14) by the appropriate expression involving Φ, it is a simple matter to verify that (A.3) results after some rearrangement. Finally, we have assumed here that

$t > 0$, but it is obvious that (A.3) holds for $t \leq 0$, as well, since the expression on the right in that equality vanishes for any such t. ∎

Next, we turn to the explicit evaluation of an integral of Sonine-Gegenbauer type, for which it is first necessary to interject some remarks about complex algebra. With $\operatorname{Log}(\cdot)$ denoting the principal branch of the logarithm-function, given as usual by

$$\operatorname{Log}\zeta := \ln|\zeta| + i\arg\zeta \qquad \text{for } \zeta \neq 0,$$

wherein $\ln x$ is the natural logarithm of the positive number x and $\arg\zeta \in (-\pi, \pi]$ indicates the principal argument of the complex number $\zeta \neq 0$, throughout $\zeta \mapsto \zeta^\alpha$ signifies the principal branch of the general power function:

$$\zeta^\alpha := \exp(\alpha\operatorname{Log}\zeta) = \exp(\alpha\ln|\zeta| + i\alpha\arg\zeta) \qquad \text{for } \zeta \in \mathbf{C},\ \zeta \neq 0, \text{ and } \alpha \in \mathbf{C}.$$

When $\alpha = 1/2$, we may write $\sqrt{\zeta}$ in place of ζ^α. Then we have the following rules of calculation:

$$\zeta^{\alpha_1}\zeta^{\alpha_2} = \zeta^{\alpha_1+\alpha_2} \qquad \text{and} \qquad \zeta^{-\alpha} = 1/\zeta^\alpha,$$

but, if $\alpha \neq 0$,

$$(\zeta_1\zeta_2)^\alpha = \zeta_1^\alpha\zeta_2^\alpha \quad \text{and} \quad (\zeta_1/\zeta_2)^\alpha = \zeta_1^\alpha/\zeta_2^\alpha \qquad \text{iff} \quad -\pi < \arg\zeta_1 + \arg\zeta_2 \leq \pi;$$

moreover, in general we have only

$$(\zeta^\alpha)^\beta = \zeta^{\alpha\beta}e^{2\pi i n(\zeta,\alpha)\beta},$$

in which $n(\zeta, \alpha)$ denotes the unique integer such that

$$\arg\left(e^{i\operatorname{Im}(\alpha\operatorname{Log}\zeta)}\right) = \operatorname{Im}(\alpha\operatorname{Log}\zeta) + 2n(\zeta,\alpha)\pi.$$

Consequently, if $\beta \neq 0$, then

$$(\zeta^\alpha)^\beta = \zeta^{\alpha\beta} \qquad \text{iff} \quad -\pi < \operatorname{Im}\alpha\ln|\zeta| + \operatorname{Re}\alpha\arg\zeta \leq \pi.$$

For example, the equality $(\zeta^\alpha)^\beta = \zeta^{\alpha\beta}$ holds if both ζ and α are real or if only α is real and $-1 < \alpha \leq 1$. Later, we shall employ these facts without explicit mention.

Now, with K_ξ denoting Macdonald's function of order ξ, J_ν the Bessel function of the first kind and order ν, and z, a, and β complex numbers, let us formally set

$$\mathcal{I}(z\,|\,a,\beta;\xi,\nu) := \int_0^\infty \frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x)x^{\nu+1}\,dx. \tag{A.15}$$

The standard works provide an explicit evaluation of $\mathcal{I}(z\,|\,a,\beta;\xi,\nu)$ for z and β real and positive, a complex with its principal argument $\arg a$ restricted by $|\arg a| < \pi/2$, and ξ and ν complex, with $\operatorname{Re}\nu > -1$; under these conditions,

$$\mathcal{I}(z\,|\,a,\beta;\xi,\nu) = \frac{\beta^\nu}{z^\xi}\left(\frac{\sqrt{z^2+\beta^2}}{a}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(a\sqrt{z^2+\beta^2}), \tag{A.16}$$

as shown, *e.g.*, by Watson [A.4, p. 416, Eq. 13.47(2)]. However, our interest necessarily lies in evaluating $\mathcal{I}(z\,|\,a,\beta;\xi,\nu)$ for certain nonreal values of z and β, although only for a real and positive. Watson (*loc. cit.*) notes merely that (A.16) is valid for complex z and β "with certain limitations," while Erdélyi *et al.* [A.5, p. 94, Eq. 7.14.2(46)] provide the formula with no indication of the restrictions on z and β. Evidently, the evaluation has not been carried out for values of the parameters pertinent to our setting, and so we extend the reasoning of Refs. A.3 or A.4 to develop the required result in

Lemma A.2. *Let ξ and ν be complex, with $\operatorname{Re}\nu > -1$. Let a be real and nonzero, and suppose that β is a nonzero complex number with $\arg\beta \neq \pi$. Introduce the half-plane $\mathbf{C}_\beta^+ \subset \mathbf{C}$ by*

$$\mathbf{C}_\beta^+ := \{\, z \in \mathbf{C} \mid \operatorname{Re} z > |\operatorname{Im}\beta|\,\}.$$

(*i*) *The integral appearing on the right in (A.15) exists for each $z \in \mathbf{C}_\beta^+$; the resultant function $\mathcal{I}(\,\cdot\mid a,\beta;\xi,\nu)$ is analytic in $\mathbf{C}_\beta^+$ and is given by*

$$\mathcal{I}(z\mid a,\beta;\xi,\nu) = \frac{\beta^\nu}{z^\xi}\left(\frac{\sqrt{z^2+\beta^2}}{|a|}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(|a|\sqrt{z^2+\beta^2}) \qquad \text{for } \operatorname{Re} z > |\operatorname{Im}\beta|. \tag{A.17}$$

(*ii*) *Suppose it also true that $\operatorname{Re}(\xi-\nu-1) > 0$. Then the integral on the right in (A.15) exists for each nonzero z lying on the boundary of $\mathbf{C}_\beta^+$, i.e., for $z \neq 0$ and $\operatorname{Re} z = |\operatorname{Im}\beta|$ (which is all of the boundary of $\mathbf{C}_\beta^+$ unless $\operatorname{Im}\beta = 0$); moreover, in this case, the resultant values provide a continuous extension of the function $\mathcal{I}(\,\cdot\mid a,\beta;\xi,\nu)$ to such boundary points, so that*

$$\int_0^\infty \frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x)x^{\nu+1}\,dx = \lim_{\substack{\zeta\to z\\ \zeta\in\mathbf{C}_\beta^+}} \frac{\beta^\nu}{\zeta^\xi}\left(\frac{\sqrt{\zeta^2+\beta^2}}{|a|}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(|a|\sqrt{\zeta^2+\beta^2})$$

$$\text{for } \operatorname{Re} z = |\operatorname{Im}\beta| \text{ and } z \neq 0. \tag{A.18}$$

Explicitly, (A.18) appears as

$$\int_0^\infty \frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x)x^{\nu+1}\,dx = \begin{cases} \dfrac{\beta^\nu}{z^\xi}\left(\dfrac{\sqrt{z^2+\beta^2}}{|a|}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(|a|\sqrt{z^2+\beta^2}) & \text{if } z^2+\beta^2 \neq 0 \\[2ex] \dfrac{\beta^\nu 2^{\xi-\nu-2}}{z^\xi a^{2\xi-2\nu-2}}\Gamma(\xi-\nu-1) & \text{if } z^2+\beta^2 = 0 \end{cases}$$

$$\text{for } \operatorname{Im}\beta \neq 0 \text{ and } \operatorname{Re} z = |\operatorname{Im}\beta|, \tag{A.19}$$

and

$$\int_0^\infty \frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x)x^{\nu+1}\,dx = \begin{cases} \dfrac{\beta^\nu}{(ib)^\xi}\left(\dfrac{\sqrt{\beta^2-b^2}}{|a|}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(|a|\sqrt{\beta^2-b^2}) & \text{if } |b| < \beta \\[2ex] \dfrac{\beta^\nu}{(ib)^\xi}\left(\dfrac{\pm i\sqrt{b^2-\beta^2}}{|a|}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(\pm i|a|\sqrt{b^2-\beta^2}) & \\ & \text{if } |b| > \beta \text{ and } b \gtrless 0 \\[2ex] \dfrac{2^{\xi-\nu-2}}{(\pm i)^\xi \beta^{\xi-\nu} a^{2\xi-2\nu-2}}\Gamma(\xi-\nu-1) & \text{if } b = \pm\beta \end{cases}$$

$$\text{for } \beta \text{ real and positive, and } z = ib \text{ with } b \text{ real and nonzero.} \tag{A.20}$$

Before giving the proof, we remark that we need only the result given in (A.19), for $\operatorname{Im}\beta \neq 0$, in the proof of Theorem 5.1. The form of (A.18) for the case in which β is real and positive and $z = ib$, with b real and nonzero, appearing here as (A.20), has been presented only for the sake of completeness. If desired, (A.20) can be recast into a form involving the Hankel functions of the first and second kinds rather than the Macdonald functions, by using the familiar relations between these functions given in Ref. A.3, Eqs. (5.7.5) and (5.7.6); in this manner, one can derive, *e.g.*, the formula to be found in Ref. A.5, p. 94, Eq. 7.14.2(48).

Proof: We begin by studying the z-dependence of the convergence properties of the integral appearing in (A.15), with the remaining parameters fixed and restricted as in the hypotheses. Throughout, it is to be understood that x takes only real values, while c_1, c_2, *etc.*, indicate positive numbers that may change from estimate to estimate, and may depend upon the secondary parameters in a given setting, but shall be constant with respect to the primary variable(s); we may indicate the explicit dependence of one or more of these numbers upon certain of the secondary parameters.

First, the restriction $\operatorname{Re}\nu > -1$ ensures that the integrand in (A.15) is sufficiently well-behaved in the neighborhood of the lower limit. For, given any positive M, from the expansion of $J_\nu(\zeta)$ in ascending powers of ζ we have an inequality

$$|J_\nu(\zeta)| \leq c_1(M)|\zeta|^{\operatorname{Re}\nu} \qquad \text{for } 0 < |\zeta| \leq |\beta|M \text{ and } |\arg\zeta| < \pi,$$

so that, if also $0 < \delta_1 < \delta_2$, there exists a $c_2(M, \delta_1, \delta_2)$ such that

$$\left|\frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x)x^{\nu+1}\right| \leq c_2(M, \delta_1, \delta_2)x^{2\operatorname{Re}\nu+1}$$

$$\text{for } 0 < x \leq M,\ 0 < \delta_1 \leq |z| \leq \delta_2, \text{ and } |\arg z| < \pi, \qquad \text{(A.21)}$$

since it is easy to check that K_ξ is bounded in any slit annulus $\{\, z \in \mathbf{C} \mid 0 < \delta_3 \leq |z| \leq \delta_4,\ |\arg\zeta| < \pi \,\}$.

Turning to the examination of the integrand in (A.15) for large values of x, we recall the asymptotic formulas

$$K_\xi(\zeta) = \sqrt{\frac{\pi}{2\zeta}}e^{-\zeta}\left\{1 + O\left(\frac{1}{|\zeta|}\right)\right\} \qquad \text{as } |\zeta| \to \infty \text{ with } |\arg\zeta| \leq \pi - \delta_0 \qquad \text{(A.22)}$$

(Ref. A.3, Eq. (5.11.9)) and

$$J_\nu(\zeta) = \sqrt{\frac{2}{\pi\zeta}}\left\{\cos\left(\zeta - \frac{(2\nu+1)\pi}{4}\right) + e^{|\operatorname{Im}\zeta|}O\left(\frac{1}{|\zeta|}\right)\right\} \qquad \text{as } |\zeta| \to \infty \text{ with } |\arg\zeta| \leq \pi - \delta_0 \qquad \text{(A.23)}$$

(following from Ref. A.3, Eq. (5.11.6)), wherein δ_0 is fixed in $(0, \pi)$, from which it is clear that, given any positive δ_1, there can be found corresponding positive numbers c_1, c_2, and $M'(\delta_1)$ such that

$$|K_\xi(z\sqrt{x^2+a^2})| \leq c_1\frac{e^{-(\operatorname{Re}z)\sqrt{x^2+a^2}}}{(x^2+a^2)^{\frac{1}{4}}} \qquad \text{for } x \geq M'(\delta_1) \text{ and } 0 < \delta_1 \leq |z| \text{ with } |\arg z| \leq \pi - \delta_0 \qquad \text{(A.24)}$$

and

$$|J_\nu(\beta x)| \leq c_2\frac{e^{|\operatorname{Im}\beta|x}}{x^{\frac{1}{2}}} \qquad \text{for } x \geq M'(\delta_1), \qquad \text{(A.25)}$$

the latter inequality following since $\cos\zeta$ and $\sin\zeta$ are always majorized in modulus by $e^{|\operatorname{Im}\zeta|}$. Consequently, for x and z as in (A.24),

$$\left|\frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x)x^{\nu+1}\right| \leq c_3\frac{e^{-(\operatorname{Re}z)\sqrt{x^2+a^2}}}{(x^2+a^2)^{\frac{1}{4}}}\frac{e^{|\operatorname{Im}\beta|x}}{x^{\frac{1}{2}}}\frac{x^{\operatorname{Re}\nu+1}}{(x^2+a^2)^{\frac{\operatorname{Re}\xi}{2}}}$$

$$\leq c_3 x^{-\operatorname{Re}(\xi-\nu)}e^{-\operatorname{Re}z(\sqrt{x^2+a^2}-x)}e^{-(\operatorname{Re}z-|\operatorname{Im}\beta|)x},$$

which gives an estimate companion to (A.21),

$$\left| \frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x) x^{\nu+1} \right| \le c_3(\delta_1) x^{-\mathrm{Re}(\xi-\nu)} e^{-(\mathrm{Re}\, z - |\mathrm{Im}\, \beta|)x}$$

$$\text{for } x \ge M'(\delta_1) \text{ and } 0 < \delta_1 \le |z| \text{ with } \mathrm{Re}\, z \ge 0. \qquad \text{(A.26)}$$

Now suppose that $z \in \mathbf{C}_\beta^+$, *i.e.*, that $\mathrm{Re}\, z > |\mathrm{Im}\, \beta|$: then, recalling that $\mathrm{Re}\, \nu > -1$, so $2\mathrm{Re}\, \nu + 1 > -1$, inequalities (A.21) and (A.26) show first that the integrand appearing in (A.15) is in $L_1(0, \infty)$, whence $\mathcal{I}(z \,|\, a, \beta; \xi, \nu)$ is defined. Selecting any $z_0 \in \mathbf{C}_\beta^+$ and applying (A.21) and (A.26) a second time, it is clear that the modulus of the integrand in (A.15) is majorized by a nonnegative function in $L_1(0, \infty)$, uniformly for all z lying in a closed disc contained in $\mathbf{C}_\beta^+$ and centered at z_0; since the function $z \mapsto K_\xi(z\sqrt{x^2+a^2})$ is continuous (even analytic) in $\mathbf{C}_\beta^+$ for fixed x, Lebesgue's Dominated-Convergence Theorem implies that $\mathcal{I}(\,\cdot\,|\, a, \beta; \xi, \nu)$ is continuous at z_0, and so also in all of $\mathbf{C}_\beta^+$. Moreover, the same estimates imply that, whenever $\mathcal{C}$ is a closed rectifiable path lying in $\mathbf{C}_\beta^+$, so that $|z|$ and $\mathrm{Re}\, z - |\mathrm{Im}\, \beta|$ are uniformly bounded below by a positive number for all z on $\mathcal{C}$, Fubini's Theorem can be applied to reverse the order of integration in $\int_{\mathcal{C}} \mathcal{I}(z \,|\, a, \beta; \xi, \nu)\, dz$, whence Cauchy's Theorem shows that the integral vanishes, and so $\mathcal{I}(\,\cdot\,|\, a, \beta; \xi, \nu)$ is analytic in $\mathbf{C}_\beta^+$, by Morera's Theorem. This proves the first two assertions of (i).

In this paragraph, let $\mathrm{Re}\,(\xi - \nu) > 1$ hold (as well as $\mathrm{Re}\, \nu > -1$), and suppose that $\mathrm{Re}\, z = |\mathrm{Im}\, \beta|$. Returning to (A.21) and (A.26), it is evident that the integrand in (A.15) is in this case still in $L_1(0, \infty)$ when $\mathrm{Im}\, \beta \ne 0$ (so that $z \ne 0$), but when $\mathrm{Im}\, \beta = 0$ we must stipulate that $z \ne 0$ to obtain the same inclusion. Thus, $\mathcal{I}(\,\cdot\,|\, a, \beta; \xi, \nu)$ is now defined by (A.15) at all nonzero boundary points of $\mathbf{C}_\beta^+$. Let us show at this point that $\mathcal{I}(\,\cdot\,|\, a, \beta; \xi, \nu)$ is continuous in $\overline{\mathbf{C}_\beta^+} \setminus \{0\}$, in particular, that the same strategy employed to prove the continuity in $\mathbf{C}_\beta^+$ also serves to verify that

$$\int_0^\infty \frac{K_\xi(z\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x) x^{\nu+1}\, dx = \lim_{\substack{\zeta \to z \\ \zeta \in \overline{\mathbf{C}_\beta^+} \setminus \{0\}}} \int_0^\infty \frac{K_\xi(\zeta\sqrt{x^2+a^2})}{(\sqrt{x^2+a^2})^\xi} J_\nu(\beta x) x^{\nu+1}\, dx$$

$$\text{for } \mathrm{Re}\, z = |\mathrm{Im}\, \beta| \text{ and } z \ne 0 \text{ when } \mathrm{Re}\,(\xi - \nu - 1) > 0. \qquad \text{(A.27)}$$

For this, let the nonzero z be fixed, with $\mathrm{Re}\, z = |\mathrm{Im}\, \beta|$: obviously, as ζ approaches z while remaining in $\overline{\mathbf{C}_\beta^+} \setminus \{0\}$, the integrand on the right in (A.27) approaches that appearing on the left, for each positive x. Further, we again appeal to (A.21) and (A.26), this time to conclude that there exists a nonnegative element of $L_1(0, \infty)$ that majorizes the modulus of the integrand on the right in (A.27), uniformly for all ζ lying in the intersection of $\overline{\mathbf{C}_\beta^+}$ with a closed disc centered at z and of radius sufficiently small to exclude the origin. Thus, the Dominated-Convergence Theorem can be applied again, establishing (A.27). Now the second sentence of (ii) has been proven except for the equality in (A.18), which shall follow immediately once (A.17) has been shown to be correct.

Returning to the general case, as a preliminary to proving (A.17) (as well as (A.19) and (A.20)), let us review the properties of the function defined by the expression appearing on the right in that equality. Let D_β be the subset of $\mathbf{C}$ defined by

$$\mathrm{D}_\beta := \{\, z \in \mathbf{C} \mid \arg(z^2 + \beta^2) = \pi \,\} \cup \{i\beta, -i\beta\} = \{\, z \in \mathbf{C} \mid \mathrm{Re}\,(z^2 + \beta^2) \le 0 \text{ and } \mathrm{Im}\,(z^2 + \beta^2) = 0 \,\};$$

then D_β is closed, and $z \mapsto \sqrt{z^2 + \beta^2}$ is analytic in $\mathbf{C} \setminus \mathrm{D}_\beta$ (recall that $\sqrt{(\cdot)}$ denotes the principal branch of the square-root function, analytic in the plane cut by removal of the nonpositive points on the real axis).

Now, the set D_β is found as the collection of all $z = x + iy$ (x, $y \in \mathbf{R}$) lying in the closed set described by the inequality

$$x^2 - y^2 \le (\operatorname{Im}\beta)^2 - (\operatorname{Re}\beta)^2$$

and on the hyperbola determined by the condition

$$xy = -\operatorname{Re}\beta \operatorname{Im}\beta$$

(which degenerates to coincide with the real and imaginary axes when $\operatorname{Re}\beta\operatorname{Im}\beta = 0$). Thus, when $\operatorname{Im}\beta = 0$, so that β is real and positive, D_β lies on the imaginary axis (just the boundary of $\mathbf{C}_\beta^+$ in that case), comprising those points ib with $|b| \ge \beta$; if $\operatorname{Im}\beta \ne 0$, D_β is contained within the closed strip $\{ z \mid |\operatorname{Re} z| \le |\operatorname{Im}\beta| \}$ and meets the boundary of that strip only at the two points $\pm i\beta$, while touching the boundary of $\mathbf{C}_\beta^+$ at a single point (*viz.*, $-i\beta$ if $\operatorname{Im}\beta > 0$, $i\beta$ if $\operatorname{Im}\beta < 0$). In either case, we find that $\mathbf{C}_\beta^+ \subset \mathbf{C} \setminus D_\beta$. Consequently, the restriction to $\mathbf{C}_\beta^+$ of the function $z \mapsto \sqrt{z^2+\beta^2}$ is analytic throughout $\mathbf{C}_\beta^+$, and it is easy to see that the restriction can be extended by continuity to all of $\overline{\mathbf{C}_\beta^+}$; in fact, if $\operatorname{Im}\beta \ne 0$, the values of the continuous extension are again given simply by $\sqrt{(\cdot)^2+\beta^2}$, while for $\operatorname{Im}\beta = 0$ these values are found to be, recalling that then β is assumed positive, and writing $z = ib$, with b real,

$$\begin{cases} \sqrt{\beta^2 - b^2} & \text{if } |b| \le \beta \\ i\sqrt{b^2-\beta^2} & \text{if } b > \beta \\ -i\sqrt{b^2-\beta^2} & \text{if } b < -\beta. \end{cases} \tag{A.28}$$

Next, since $K_{\xi-\nu-1}$ is analytic in the open set obtained by deleting from $\mathbf{C}$ the nonpositive portion of the real axis, while $\sqrt{z^2+\beta^2} \ne 0$ and $|\arg\sqrt{z^2+\beta^2}| < \pi/2$ for $z \in \mathbf{C} \setminus D_\beta$, by setting

$$\mathcal{K}(z) \equiv \mathcal{K}(z \mid a, \beta; \xi, \nu) := \left(\frac{\sqrt{z^2+\beta^2}}{|a|}\right)^{\xi-\nu-1} K_{\xi-\nu-1}(|a|\sqrt{z^2+\beta^2}) \qquad \text{for } z \in \mathbf{C} \setminus D_\beta,$$

we obtain a function analytic in $\mathbf{C} \setminus D_\beta$. Moreover, from what has been said, the restriction $\mathcal{K}_+ := \mathcal{K}|\mathbf{C}_\beta^+$ is analytic in $\mathbf{C}_\beta^+$ and can be extended continuously to all points of the boundary of $\mathbf{C}_\beta^+$ with the possible exception of those z on the boundary at which $z^2+\beta^2 = 0$ (there being one such point if $\operatorname{Im}\beta \ne 0$, two if $\operatorname{Im}\beta = 0$). The value of the continuous extension of $\mathcal{K}_+$ at a point z on the boundary with $z^2 + \beta^2 \ne 0$ is just $\mathcal{K}(z)$ when $\operatorname{Im}\beta \ne 0$, and in the contrary case when $\operatorname{Im}\beta = 0$ can be easily written down by use of the values supplied in (A.28) for the continuous extension of the restriction of $\zeta \mapsto \sqrt{\zeta^2+\beta^2}$ to $\mathbf{C}_\beta^+$. Now, if $\operatorname{Re}\chi > 0$ it is easy to check, by using the definition and series expansions for K_χ (cf., *e.g.*, Ref. A.3, §5.7) that

$$\lim_{\substack{\zeta\to 0 \\ |\arg\zeta|<\pi}} \zeta^\chi K_\chi(\zeta) = 2^{\chi-1}\Gamma(\chi),$$

and so

$$\lim_{\substack{z\to\pm i\beta \\ z\in\mathbf{C}\setminus D_\beta}} \mathcal{K}(z \mid a,\beta;\xi,\nu) = \frac{2^{\xi-\nu-2}}{a^{2\xi-2\nu-2}}\Gamma(\xi-\nu-1) \qquad \text{when } \operatorname{Re}(\xi-\nu-1) > 0 \tag{A.29}$$

(of course, with Γ denoting the Gamma function). Thus, when $\operatorname{Re}(\xi-\nu-1) > 0$ we conclude that $\mathcal{K}_+$ can be extended by continuity to *all* of $\overline{\mathbf{C}_\beta^+}$, the value of the extension at the point (if $\operatorname{Im}\beta \ne 0$) or points (if $\operatorname{Im}\beta = 0$) z of the boundary at which $z^2+\beta^2$ vanishes being found from (A.29). Finally, the modifications in these statements necessary to describe the properties of the function $z \mapsto (\beta^\nu/z^\xi)\mathcal{K}_+(z \mid a,\beta;\xi,\nu)$, $z \in \mathbf{C}_\beta^+$, are obvious; in particular, the origin must be excluded from the boundary points of extended-continuity of the latter function if $\operatorname{Im}\beta = 0$ and $\operatorname{Re}\xi > 0$. In fact, when $\operatorname{Re}(\xi-\nu-1) > 0$, we do have $\operatorname{Re}\xi > 0$ (since

$\mathrm{Re}\,\nu > -1$), and we find that the values of the continuous extension of this function to the nonzero points of the boundary of $\mathbf{C}_\beta^+$ are given by the expressions appearing on the right-hand sides of (A.19) and (A.20), in the respective cases $\mathrm{Im}\,\beta \neq 0$ and $\mathrm{Im}\,\beta = 0$.

According to what has been shown, to establish (A.17) it is sufficient to derive the equality for, say, z real and greater than $|\mathrm{Im}\,\beta|$; actually, we shall suppose that z lies in the larger set $\mathbf{C}_\beta^{++} \subset \mathbf{C}_\beta^+$ specified by

$$\mathbf{C}_\beta^{++} := \{\,\zeta \mid \mathrm{Re}\,(\zeta^2) > (\mathrm{Im}\,\beta)^2 \text{ and } \mathrm{Re}\,\zeta > 0\,\}$$

and prove that $\mathcal{I}(z\,|\,a,\beta;\xi,\nu)$ is then given by $(\beta^\nu/z^\xi)\mathcal{K}(z\,|\,a,\beta;\xi,\nu)$. For this, we require the evaluation of two auxiliary integrals. First, from Ref. A.6, p. 146, Eq. (29), we have

$$\int_0^\infty e^{-\zeta_1\tau-(\zeta_2/(4\tau))}\tau^{\tilde{\xi}-1}\,d\tau = 2\left(\frac{\zeta_2}{4\zeta_1}\right)^{\tilde{\xi}/2} K_{\tilde{\xi}}(\zeta_1^{1/2}\zeta_2^{1/2}) \qquad \text{for } \mathrm{Re}\,\zeta_1 > 0,\ \mathrm{Re}\,\zeta_2 > 0, \text{ and } \tilde{\xi} \in \mathbf{C}. \tag{A.30}$$

Also, in Ref. A.4, p. 394, Eq. 13.3(4), we find

$$\int_0^\infty J_\nu(\zeta_1\tau)e^{-\zeta_2\tau^2}\tau^{\nu+1}\,d\tau = \frac{\zeta_1^\nu}{(2\zeta_2)^{\nu+1}}e^{-\zeta_1^2/(4\zeta_2)} \qquad \text{for } \zeta_1 \neq 0,\ \mathrm{Re}\,\zeta_2 > 0, \text{ and } \mathrm{Re}\,\nu > -1. \tag{A.31}$$

Select and fix any $z \in \mathbf{C}_\beta^{++}$. Then we can apply (A.30) with $\zeta_1 = 1$, $\zeta_2 = z^2(x^2+a^2)$ (for $x \in \mathbf{R}$), and $\tilde{\xi} = -\xi$ to get (by recalling that $K_{-\xi} = K_\xi$ and noting that $\sqrt{z^2} = z$, $\big(z^2(x^2+a^2)\big)^{1/2} = z\sqrt{x^2+a^2}$, and $\big(z^2(x^2+a^2)/4\big)^{-\xi/2} = 2^\xi/(z^\xi(\sqrt{x^2+a^2})^\xi)$ hold because $\mathrm{Re}\,z > 0$)

$$K_\xi(z\sqrt{x^2+a^2}) = \frac{z^\xi(\sqrt{x^2+a^2})^\xi}{2^{\xi+1}}\int_0^\infty e^{-\tau-\left(z^2(x^2+a^2)/(4\tau)\right)}\tau^{-\xi-1}\,d\tau,$$

and so

$$\begin{aligned}\mathcal{I}(z\,|\,a,\beta;\xi,\nu) &= \frac{z^\xi}{2^{\xi+1}}\int_0^\infty\left\{\int_0^\infty e^{-\tau-\left(z^2(x^2+a^2)/(4\tau)\right)}\tau^{-\xi-1}\,d\tau\right\}J_\nu(\beta x)x^{\nu+1}\,dx\\ &= \frac{z^\xi}{2^{\xi+1}}\int_0^\infty\left\{\int_0^\infty e^{-z^2x^2/(4\tau)}J_\nu(\beta x)x^{\nu+1}\,dx\right\}e^{-\tau-\left(z^2a^2/(4\tau)\right)}\tau^{-\xi-1}\,d\tau,\end{aligned} \tag{A.32}$$

in which we have used Fubini's Theorem to reverse the order of integration, a step that we shall justify presently. We apply (A.31) for the evaluation of the inner integral in (A.32) for each positive τ (a computation that is permissible because here $\mathrm{Re}\,(z^2) > 0$ and $\mathrm{Re}\,\nu > -1$) and obtain $(2^{\nu+1}\beta^\nu/z^{2\nu+2})\tau^{\nu+1}e^{-\beta^2\tau/z^2}$, whence

$$\mathcal{I}(z\,|\,a,\beta;\xi,\nu) = 2^{\nu-\xi}\beta^\nu z^{\xi-2\nu-2}\int_0^\infty e^{-((z^2+\beta^2)/z^2)\tau-\left(z^2a^2/(4\tau)\right)}\tau^{\nu-\xi}\,d\tau. \tag{A.33}$$

Now we observe that the inclusion $z \in \mathbf{C}_\beta^{++}$ ensures not only that z and z^2 have positive real parts but also that the same is true of $z^2+\beta^2$ and $(z^2+\beta^2)/z^2$ (and so each number has principal argument in $(-\pi/2, \pi/2)$). Indeed, we find

$$\mathrm{Re}\,(z^2+\beta^2) = \mathrm{Re}\,(z^2) - (\mathrm{Im}\,\beta)^2 + (\mathrm{Re}\,\beta)^2 > 0,$$

and

$$\mathrm{Re}\left(\frac{z^2+\beta^2}{z^2}\right) = \frac{1}{|z|^4}\left\{\{\mathrm{Re}\,(z^2) + (\mathrm{Re}\,\beta)^2\}\{\mathrm{Re}\,(z^2) - (\mathrm{Im}\,\beta)^2\} + \{\mathrm{Im}\,(z^2) + \mathrm{Re}\,\beta\,\mathrm{Im}\,\beta\}^2\right\} > 0$$

(note that the second of these inequalities provides an independent check on the existence of the integral in (A.33)). This shows, firstly, that to evaluate the integral on the right-hand side of (A.33) it is permissible to apply (A.30) a second time by taking $\zeta_1 = (z^2+\beta^2)/z^2$, $\zeta_2 = z^2a^2$, and $\tilde{\xi} = \nu - \xi + 1$, and, secondly, that the following computations are correct: $\zeta_1^{1/2} = \sqrt{z^2+\beta^2}/z$, $\zeta_2^{1/2} = |a|z$, and

$$\left(\frac{\zeta_2}{4\zeta_1}\right)^{(\nu-\xi+1)/2} = \left(\frac{a^2z^2}{4}\frac{z^2}{z^2+\beta^2}\right)^{(\nu-\xi+1)/2} = \frac{2^{\xi-\nu-1}}{z^{2\xi-2\nu-2}}\left(\frac{\sqrt{z^2+\beta^2}}{|a|}\right)^{\xi-\nu-1}$$

In this manner, the equality in (A.17) clearly results from (A.33) and (A.30), and so the proof of (A.17) is complete, *modulo* the demonstration that the second equality in (A.32) is correct. Taking up this latter question, we first write

$$\begin{aligned}\int_0^\infty\int_0^\infty & \left|e^{-\tau-\left(z^2(x^2+a^2)/(4\tau)\right)}\tau^{-\xi-1}J_\nu(\beta x)x^{\nu+1}\right| d\tau\, dx \\ &= \int_0^\infty\left\{\int_0^\infty e^{-\tau-\left(\mathrm{Re}\,(z^2)(x^2+a^2)/(4\tau)\right)}\tau^{-\mathrm{Re}\,\xi-1}\,d\tau\right\}|J_\nu(\beta x)|x^{\mathrm{Re}\,\nu+1}\,dx \\ &= 2^{\mathrm{Re}\,\xi+1}\int_0^\infty \frac{K_{\mathrm{Re}\,\xi}\left(\sqrt{\mathrm{Re}\,(z^2)(x^2+a^2)}\right)}{\left(\sqrt{\mathrm{Re}\,(z^2)(x^2+a^2)}\right)^{\mathrm{Re}\,\xi}}|J_\nu(\beta x)|x^{\mathrm{Re}\,\nu+1}\,dx, \end{aligned} \tag{A.34}$$

having once more applied (A.30) to obtain the second equality here. To show that the nonnegative integrand appearing in the final integral in (A.34) is in $L_1(0,\infty)$, we return to (A.21) and (A.26), replacing there ξ by $\mathrm{Re}\,\xi$ and z by $\sqrt{\mathrm{Re}\,(z^2)}$, to conclude that for any positive M there can be found $c_2(M)$ such that

$$\frac{K_{\mathrm{Re}\,\xi}\left(\sqrt{\mathrm{Re}\,(z^2)(x^2+a^2)}\right)}{\left(\sqrt{\mathrm{Re}\,(z^2)(x^2+a^2)}\right)^{\mathrm{Re}\,\xi}}|J_\nu(\beta x)|x^{\mathrm{Re}\,\nu+1} \le c_2(M)x^{2\mathrm{Re}\,\nu+1} \qquad \text{for } 0 < x \le M, \tag{A.35}$$

and that there exists a sufficiently large positive M' for which we have an estimate of the form

$$\frac{K_{\mathrm{Re}\,\xi}\left(\sqrt{\mathrm{Re}\,(z^2)(x^2+a^2)}\right)}{\left(\sqrt{\mathrm{Re}\,(z^2)(x^2+a^2)}\right)^{\mathrm{Re}\,\xi}}|J_\nu(\beta x)|x^{\mathrm{Re}\,\nu+1} \le c_3x^{-\mathrm{Re}\,(\xi-\nu)}e^{-\left(\sqrt{\mathrm{Re}\,(z^2)}-|\mathrm{Im}\,\beta|\right)x} \qquad \text{for } x \ge M'. \tag{A.36}$$

Since $z \in \mathbf{C}_\beta^{++}$, we have $\sqrt{\mathrm{Re}\,(z^2)} - |\mathrm{Im}\beta| > 0$, whence it follows from (A.35) and (A.36) that the final integral in (A.34) is finite, and this justifies the application of Fubini's Theorem upon which the second equality in (A.32) is based. This completes the proof of (A.17), and so also of (i).

Considering once more the case in which $\mathrm{Re}\,(\xi-\nu-1) > 0$, since the limiting relation (A.18) is an immediate consequence of (A.17) and (A.27), now the explicit forms (A.19) and (A.20) of (A.18) in the respective cases $\mathrm{Im}\,\beta \ne 0$ and $\mathrm{Im}\,\beta = 0$ follow from our discussion of the properties of the function $z \mapsto (\beta^\nu/z^\xi)\mathcal{K}_+(z\,|\,a,\beta;\xi,\nu)$, $z \in \mathbf{C}_\beta^+$, in which we supplied the details necessary for checking that the expressions on the right in (A.19) and (A.20) give the values of the continuous extension of this function to the nonzero points of the boundary of $\mathbf{C}_\beta^+$. ∎

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