DTIC FILE COPY

(1)

AD-A210 558

CMS Technical Summary Report #89-30

# SUBHARMONIC SOLUTIONS NEAR AN EQUILIBRIUM POINT FOR HAMILTONIAN SYSTEMS

Patricio L. Felmer

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

April 1989

(Received April 26, 1989)





Sponsored by

Air Force Office of Scientific Research
Washington, DC 20332

89 7 28 074

UNIVERSITY OF WISCONSIN-MADISON
CENTER FOR THE MATHEMATICAL SCIENCES

# SUBHARMONIC SOLUTIONS NEAR AN EQUILIBRIUM POINT FOR HAMILTONIAN SYSTEMS.

Patricio L. Felmer*

Technical Summary Report #89-30
April 1989

## ABSTRACT

In this work we study subharmonic solutions near an equilibrium point for the Hamiltonian system

$$\dot{z} = JA(t)z + J\hat{H}_z(z,t)$$

where $A(t)$ is a matrix, $\hat{H}_z(z,t) = o(|z|)$ and both $A$ and $\hat{H}$ are periodic in $t$.

On the linear part of the system we impose a condition expressed in terms of its symplectic invariants. The higher order term is assumed to be superquadratic near the equilibrium point, and we show that this condition can be reduced to the center manifold.

We transform the Hamiltonian system to a variational problem and we apply a minimax argument to find critical points.

AMS (MOS) Subject Classification: 34C20, 34C25, 49A40, 70H05, 70H15.

Key Words: Hamiltonian systems, canonical forms, Hamiltonian matrices, subharmonic solutions, minimax argument, center manifold.

*Mathematics Department, University of Wisconsin, Madison, WI 53706.
Departamento de Matemáticas, Universidad de Chile, Casilla 170, Correo-3, Santiago, Chile.

This research was sponsored by the Air Force Office of Scientific Research under Grant No. AFOSR-87-0202.

# SUBHARMONIC SOLUTIONS NEAR AN EQUILIBRIUM POINT FOR HAMILTONIAN SYSTEMS.

## §0 INTRODUCTION.

In this work we study the existence of subharmonic solutions near an equilibrium point for Hamiltonian systems

$$\dot{z} = JH_z(z,t) \tag{0.1}$$

where $z = (p,q) \in \mathbf{R}^{2n}$, $J$ denotes the standard symplectic structure in $\mathbf{R}^{2n}$ and $H_z$ denotes the partial derivative of $H$ with respect to $z$.

We assume that the Hamiltonian $H$ satisfies:

(H0) $H$ is $C^2$ near $z = 0$,

(H1) $H(0,t) = 0, \quad H_z(0,t) = 0 \quad \forall t \in \mathbf{R}$, and

(H2) $H$ is $T$–periodic in the t–variable.

Hypothesis (H0) and (H1) allows us to write the Hamiltonian as

$$H(z,t) = \frac{1}{2}(A(t)z,z) + \hat{H}(z,t) \tag{0.2}$$

where $A(t)$ denotes the Hessian matrix of $H$ at $z = 0$ and $\hat{H}(z,t) = o(|z|^2)$ represents the higher order terms of the Hamiltonian.

We then can rewrite system (0.1) as

$$\dot{z} = JA(t)z + J\hat{H}_z(z,t), \tag{HS}$$

and $z = 0$ represents an equilibrium point of (HS).

The problem of finding subharmonic solutions near $z = 0$ for (HS) has been the object of a considerable amount of work (See [2], [3], [7], [11], [12], and references therein). The first results are due to Birkhoff and Lewis in 1933, [3]. In 1968 Harris [7] gave a more general version of the results of Birkhoff and Lewis. And in 1977 Moser [11] presented an improved version of those mentioned above.

The work in [3] and [7] requires a strong nonresonance condition on the Floquet exponents of the linearized system. This allows reduction of the Hamiltonian to the Birkhoff normal form. With an additional assumption of nondegeneracy on the higher order term they prove the existence of subharmonics.

Moser in [11] restricted the nonresonance condition only to the Floquet exponents on the imaginary axis by reducing the problem to the center manifold. The nondegeneracy condition on the higher order terms was also imposed only on the center manifold. Moser presented his results working in the context of symplectic diffeomorphisms rather than directly with (HS). The differentiability hypothesis on the map was also reduced to the class $C^3$.

The work of Rabinowitz in [12] gave another direction in the methodology to attack the problem. By using minimax techniques of a global nature he proved the existence of subharmonics. He assumed that the linearized system has all its Floquet exponents on the imaginary axis, and different from each other, and that the higher order term is superquadratic in an appropriate way. He also obtain the result when the linearized system is trivial, i.e. $A(t) \equiv 0$.

Recently, Benci and Fortunato [2] gave weaker conditions on the Floquet exponents of the linearized system, keeping the higher order term $\hat{H}$ superquadratic.

Comparing the results of [3], [7], and [11] with those in [2] and [12], we see that in the latter the condition on the linear part is relaxed by assuming a stronger condition on the higher order term.

In this work we generalize results in [2] and [12]. In the first place we relax the conditions on the Floquet exponents. In the second place we reduce the system to the center manifold and we require the higher order terms of the reduced Hamiltonian to be superquadratic.

A more precise description of the results follows. Let us consider the linearized system

$$\dot{z} = JA(t)z. \tag{0.3}$$

After a coordinate transformation (0.3) can be written as

$$\dot{z} = JBz \tag{0.4}$$

where $B$ is a constant symmetric matrix. The matrix $JB$ can be further transformed into a canonical form where only symplectic changes of coordinates are allowed.

It is the nature of the canonical form of $JB$ that will determine the possibility of solving (HS) for subharmonics.

We do not expect to find subharmonic solutions in the case all Floquet exponents are outside the imaginary axis. See examples in [2] and [11]. Thus we will assume that there is at least one purely imaginary Floquet exponent. However , as we will see in Section 3, this is not sufficient.

In Section 2 we will define a Floquet exponent of positive, negative or indefinite type. This is a symplectic invariant related to a perturbed eigenvalue problem for the blocks of the canonical form of $JB$.

In terms of this notion, we assume

(H3) The linear system (0.3) has at least one purely imaginary Floquet exponent that is not of negative type.

For the higher order term $\hat{H}$ we will assume

(H4) There are constants $r > 0, c > 0, q + 2 > p > q > 2$ that

$$(\hat{H}_z(z,t), z) \geq q\hat{H}(z,t) \geq c|z|^p,$$

$$\forall z \in \mathbf{R}^{2n}, |z| \leq r \quad \text{and} \quad \forall t \in \mathbf{R}.$$

Then the following theorem holds.

**Theorem I.**

*Suppose (H0), (H1), (H2), (H3) and (H4) hold. Then there is an integer $N$ so that for every $k \geq N$ there exists a nontrivial solution $z_k$ of (HS) with period $kT$, and $\{z_k\}_{k=N}^{\infty}$ converges to zero as $k \to \infty$ uniformly in $C^1(\mathbf{R}, \mathbf{R}^{2n})$.*

**Remark 0.1.** Benci and Fortunato in [2] considered the following hypothesis on the linearized system (0.3).

(H3') Every symplectic matrix sufficiently close to $X(T)$ has at least one eigenvalue on the unit circle.

Here $X(t)$ represents the fundamental matrix of (0.3), i.e. $X(t)$ satisfies

$$X(0) = I_{2n} \tag{0.5}$$

$$\dot{X}(t) = JA(t)X(t). \tag{0.6}$$

We will show that (H3') implies (H3), but the converse is not true. See Section 3.3.

**Remark 0.2.** In [12] results similar to Theorem I were given in the case $A(t) = B$ is constant and either $B = 0$ or $JB$ has purely imaginary eigenvalues, all different. We will see that both cases satisfy hypothesis (H3). We note that $B = 0$ does not satisfy hypothesis (H3') of Benci and Fortunato.

**Remark 0.3.** In the election of the names Floquet exponent of positive, negative or indefinite type we want to reflect the possibility of finding subharmonics for (HS) with the higher order term $\hat{H}$ satisfying (H4) or the opposite:

(H4-) There are constants $r > 0, c > 0, q + 2 > p > q > 2$ such that

$$(\hat{H}_z(z,t), z) \leq -q\hat{H}(z,t) \leq -c|z|^p$$

$$\forall z \in \mathbf{R}^{2n}, |z| \leq r, \forall t \in \mathbf{R},$$

or both. See Remark 3.4.

**Remark 0.4** Our hypothesis (H4) is slightly more general that the corresponding one in [2] and [12]. They assume $p = q$. This will be needed in the results obtained on the center manifold. See Section 4.

**Remark 0.5.** If we further assume that 0 is not a Floquet exponent for (0.3) then using an argument given in [2], we can prove that the solutions given by Theorem I have minimal period $kT$, for $k$ large and prime. This hypothesis requires that the only $T$-periodic solution of (0.3) be the trivial one. It will be violated if (HS) is the variational equation about a periodic solution of an autonomous system.

Theorem I gives a result generalizing earlier works by relaxing conditions on the linearized system. The second goal of this work is to generalize condition (H4), that has to do with the nonlinear term $\hat{H}$.

By assuming extra differentiability

(H5) $\quad H$ is $C^3$ near $z = 0$,

we reduce system (HS) to the center manifold. Specifically we will obtain a Hamiltonian system

$$\dot{w} = JB(t)w + J\hat{K}_w(w,t), \tag{0.7}$$

where the linearized system

$$\dot{w} = JB(t)w \tag{0.8}$$

has all Floquet exponents on the imaginary axis.

We assume

(H6) There are constants $c > 0, r > 0, q + 2 > p > q > 2$ such that

$$(\hat{K}_w(w,t), w) \geq q\hat{K}(w,t) \geq c|w|^p$$

$$\forall w \in \mathbf{R}^{2m}, |w| \leq r, \forall t \in \mathbf{R}.$$

where $2m$ is the dimension of the center manifold of (HS).

An application of Theorem I will give

**Theorem II.**

*If (H5), (H1), (H2), (H3) and (H6) hold then there exists an integer $N$ so that for every $k \geq N$ there exists a nontrivial solution $z_k$ of (HS) with period $kT$ and $\{z_k\}_{k=1}^{\infty}$ converges to zero as $k \to \infty$ uniformly in $C^1(\mathbf{R}, \mathbf{R}^{2n})$.*

**Remark 0.6** Since hypothesis (H6) is in general hard to check we are interested in giving conditions on the original function $\hat{H}$ that imply (H6).

Roughly speaking, if the superquadratic condition in (H4) is satisfied for every $z$ on the center manifold then (H6) holds. A precise formulation of this condition is given in Section 4.4.

This work is organized in 4 Sections and 3 Appendices. In Section §1. we recall some basic results about Hamiltonian matrices. In particular we discuss the canonical form for Hamiltonian matrices. In Section §2 we define the concept of eigenvalue of positive, negative and indefinite type, and we study some consequences related to a perturbed eigenvalue problem for the canonical blocks. In Section §3 we study a generalized eigenvalue problem associated to (0.3) and we prove a proposition fundamental in the proof of Theorem I. Then we prove Theorem I. In section §4 we describe the reduction to the center manifold and we prove Theorem II.

In Appendix A we describe some basic properties of the logarithm of matrices. In Appendix B we go briefly through perturbation theory for eigenvalue problems with symmetric matrices and we apply it to give a proof to the results in Section §3. And in Appendix C we present the changes necessary to the proof of the Center Manifold Theorem in order to suit our needs.

I want to express my gratitude to my thesis advisor Professor Paul Rabinowitz for his guidance, encouragement and patience. Also I want to thank professor E. Zehnder who calls to my attention the paper by Laub and Meyer on canonical form for Hamiltonian matrices.

## TABLE OF CONTENT.

## §1.BACKGROUND ON HAMILTONIAN MATRICES

In this section we will describe some basic facts about Hamiltonian matrices. In particular we will present the canonical form of a Hamiltonian matrix; this canonical form show the symplectic invariants we will use in our study of subharmonics solutions for (HS).

A constant real matrix $C$ satisfying

$$JC^{\tau}J = C \tag{1.1}$$

is called Hamiltonian matrix. Here $J$, as usual, denotes the standard symplectic structure in $\mathbf{R}^{2n}$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad I = I_n, \tag{1.2}$$

and $\tau$ denotes the transpose of a matrix. We note that if $B$ is a symmetric matrix, then $JB$ is a Hamiltonian matrix, and conversely, if C is a Hamiltonian matrix then $B = JC$ is symmetric. Hamiltonian matrices correspond to the linear part of autonomous Hamiltonian systems.

If $C$ is the matrix of a linear system

$$\dot{z} = Cz \tag{1.3}$$

where $C$ is Hamiltonian, then we can transform (1.3) through a linear change of coordinates $Q$ without changing the Hamiltonian structure if $Q$ satisfies

$$Q^{\tau}JQ = J. \tag{1.4}$$

A matrix $Q$ satisfying (1.4) is called a symplectic matrix.

The transformation $Q^{-1}CQ$, with $Q$ a symplectic matrix, defines a similarity relation. In such a situation canonical forms are of great interest.

Williamson, in [15], obtained the basic results for canonical forms of Hamiltonian matrices . A more constructive presentation of canonical forms is proposed by Laub and Meyer [10]. We will take the canonical form as presented in the paper of Bryuno [4], there simpler canonical forms than in [10] are given.

Before discussing the canonical forms, we state some basic lemmas on Hamiltonian matrices. For the proof we refer the reader to [10].

**Lemma 1.1.**

*If $\lambda$ is an eigenvalue of the Hamiltonian matrix $C$ then $-\lambda, \bar{\lambda}$ and $-\bar{\lambda}$ are also eigenvalues of $C$ with the same multiplicity as $\lambda$. In particular 0 has even multiplicity if it is an eigenvalue of $C$.*

With respect to the eigenspace associated to an eigenvalue of a Hamiltonian matrix we have the following lemma.

Let us define

$$\eta_k(\lambda) = \ker(C - \lambda I)^k, \quad \text{and} \tag{1.5}$$

$$\eta(\lambda) = \bigcup_{k\geq 1} \eta_k(\lambda). \tag{1.6}$$

**Lemma 1.2.**

*If $C$ is a Hamiltonian matrix and $\lambda$ is an eigenvalue of $C$ then if $\mu$ is another eigenvalue of $C$, $\mu \neq -\lambda$ implies*

$$(z, Jw) = 0 \qquad \forall z \in \eta(\lambda), \quad \forall w \in \eta(\lambda).$$

**Corollary 1.1.**

*If $C$ is a Hamiltonian matrix and $Q$ is an invertible matrix so that*

$$\mathcal{J} = Q^{-1}CQ \tag{1.7}$$

*is the real Jordan canonical form of $C$, then*

$$Q^{\tau}JQ = \operatorname{diag}(q_1, q_2, ..., q_p) \tag{1.8}$$

*is a block diagonal matrix with one block $q_i$ for each pair $(\lambda, -\lambda)$ of eigenvalues of $C$.*

Even though the real Jordan canonical form of a Hamiltonian matrix already shows some special structure, it is not enough for our purposes. The next theorem gives the (real) canonical form of a Hamiltonian matrix when only symplectic changes of coordinates are allowed. The proof can be found in [4] and [15].

The following notation will be useful in this and the next sections.

$$d = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}, \tag{1.9}$$

$$\Delta = \operatorname{diag}(1, 0, ..., 0) \quad \text{and} \quad I = I_n. \tag{1.10}$$

The size of each matrix will be easily determined by the context.

**Theorem 1.1.**

*If $C$ is a Hamiltonian matrix then there exists a symplectic matrix $Q$ such that*

$$Q^{-1}CQ = \begin{pmatrix} D_{11} & 0 & \dots & 0 & D_{21} & 0 & \dots & 0 \\ 0 & D_{12} & \dots & 0 & 0 & D_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{1s} & 0 & 0 & \dots & D_{2s} \\ D_{31} & 0 & \dots & 0 & D_{41} & 0 & \dots & 0 \\ 0 & D_{32} & \dots & 0 & 0 & D_{42} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{3s} & 0 & 0 & \dots & D_{4s} \end{pmatrix} \tag{1.11}$$

*where the matrix formed by combining blocks in the same relative position*

$$D_i = \begin{pmatrix} D_{1i} & D_{2i} \\ D_{3i} & D_{4i} \end{pmatrix} \tag{1.12}$$

*has a simple form analogous to the real Jordan canonical blocks. Every block is associated to one eigenvalue of $C$, but there can be more than one block for a specific eigenvalue. More precisely the matrices $D_i$, that we call generically $D$, can have one of the following forms:*

*1. Block of C1 type.*

*Block associated to an eigenvalue $\lambda = \alpha + i\beta, \quad \alpha > 0, \beta > 0$ of multiplicity $l$.*

$$D = \begin{pmatrix} \alpha I + d & -\beta I & 0 & 0 \\ \beta I & \alpha I + d & 0 & 0 \\ 0 & 0 & -\alpha I - d^\tau & -\beta I \\ 0 & 0 & \beta I & -\alpha I - d^\tau \end{pmatrix} \tag{1.13}$$

*Here $D$ is a $4l \times 4l$ matrix.*

*2. Block of C2 type.*

*Block associated to an eigenvalue $\lambda = \alpha, \alpha > 0$ of multiplicity $l$.*

$$D = \begin{pmatrix} \alpha I + d & 0 \\ 0 & -\alpha I - d^\tau \end{pmatrix} \tag{1.14}$$

*Here D is a $2l \times 2l$ matrix.*

*3. Block of C3 type.*

*Block associated to an eigenvalue $\lambda = i\beta, \beta > 0$ of multiplicity $l$.*

$$D = \sigma \begin{pmatrix} 0 & K(\beta I + d^\tau) \\ -K(\beta I + d) & 0 \end{pmatrix} \tag{1.15}$$

*Here D is a $2l \times 2l$ matrix. The number $\sigma$ can be 1 or $-1$, and it is an additional symplectic invariant.*

*4. Block of C4 type.*

*Block associated to the eigenvalue $\lambda = 0$ of multiplicity $l$ odd.*

$$D = \begin{pmatrix} d & 0 \\ 0 & -d^\tau \end{pmatrix} \tag{1.16}$$

*Here D is a $2l \times 2l$ matrix.*

*5. Block of C5 type.*

*Block associated to the eigenvalue $\lambda = 0$ of multiplicity $l$.*

$$D = \begin{pmatrix} d & \sigma\Delta \\ 0 & -d^\tau \end{pmatrix} \tag{1.17}$$

*Here $D$ is a $2l \times 2l$ matrix. The number $\sigma$ can be 1 or $-1$, and it is an additional invariant.*

**Remark 1.1.** If we compare with the real Jordan canonical blocks, we see that the canonical blocks described in the theorem are more complicated and cases C3 and C5 carry an extra invariant, namely $\sigma$.

In what follows we only be interested in blocks of C3, C4, and C5 type, i.e. in blocks corresponding to purely imaginary eigenvalues.

The following lemmas are rather obvicus, but we state them for later reference.

In the case of a block of C3 type we can write

$$D = -\sigma\beta V - \sigma W \tag{1.18}$$

where

$$V = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & -Kd^{\tau} \\ Kd & 0 \end{pmatrix}. \tag{1.19}$$

**Lemma 1.3.**

1. *$V$ and $W$ are Hamiltonian matrices.*
2. *$VJ$ and $WJ$ are symmetric matrices.*
3. *$V$ commutes with $W$ and $J$.*
4. *$V$ is invertible, and $W$ is nilpotent.*

**Proof.** Parts 1., 2. and 4. are direct and part 3. requires an easy computation. ◇

**Lemma 1.4.**

$$\exp(2\pi V) = I \tag{1.20}$$

**Proof.** Noting that $KK = I$, for $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} \exp(\alpha V) &= I + \frac{\alpha V}{1!} - \frac{\alpha^2 I}{2!} + \dots \\ &= \cos(\alpha)I + \sin(\alpha)V. \diamond \end{aligned}$$

## §2. PERTURBED EIGENVALUE PROBLEM FOR THE CANONICAL BLOCKS.

This section is devoted to the definition of eigenvalue of positive, negative or indefinite type, in the case of purely imaginary eigenvalues of Hamiltonian matrices.

The definitions we are going to make are motivated by the application to subharmonic solutions of Hamiltonian systems that we will present in next section. However, we can already see the meaning of these definitions in studying some perturbed eigenvalue problems for the canonical blocks . In Section §3 we will see how this is related to the existence of subharmonics for the Hamiltonian system (HS).

### §2.1. Definitions.

In the following definitions we assume we have a Hamiltonian matrix $C$, and we have obtained a symplectic transformation $Q$ that takes $C$ into its canonical form. $D$ will denote a generic block in the canonical form of $C$ as given by Theorem 1.1.

**Definition 2.1.** *We say that a $2l \times 2l$ block D is positive (negative) if one of the following is true:*

1. *D is a block of C3 type, $l$ is even and $\sigma = 1$ ($\sigma = -1$).*
2. *D is a block of C5 type and $\sigma = 1$ ($\sigma = -1$).*

**Definition 2.2.** *If $\lambda$ is a purely imaginary eigenvalue of the Hamiltonian matrix $C$ we say that $\lambda$ is of positive type (negative type) if all blocks associated to $\lambda$ are positive (negative). We say that $\lambda$ is of indefinite type if $\lambda$ is not of positive type nor of negative type.*

**Remark 2.1.** The concepts of positive, negative and indefinite type are symplectic invariants.

**Remark 2.2.** See Remark 3.6 for a perturbation result for blocks of positive type and of negative type.

### §2.2. Perturbed eigenvalue problems.

Given a block D of the canonical form of a Hamiltonian matrix, that is of C3, C4 or C5 type we will define an associated eigenvalue problem depending on a parameter $\epsilon$. Using perturbation theory we can study the behaviour of the eigenvalues of the problem as a function of $\epsilon$. In particular we are interested in the sign of the small eigenvalues for small values of the parameter $\epsilon$. The results are given in Propositions 2.1, 2.2 and 2.3. We delay the proof of these propositions to Appendix B.

**Case 1. Block of C3 type.** Using the notation given in Section §1 we can write a block of C3 type in the form

$$D = -\sigma\beta V - \sigma W \tag{2.1}$$

where V and W were defined in (1.19).

Let us consider the following eigenvalue problem

$$-(\epsilon V + \sigma W)v + \lambda J v = 0 \tag{2.2}$$

where $\epsilon \in \mathbf{R}$ and $v \in \mathbf{R}^{2l}$. We study the eigenvalue $\lambda$ as a function of $\epsilon$.

This problem is equivalent to

$$(\epsilon VJ + \sigma WJ)w + \lambda w = 0, \quad w = Jv. \tag{2.3}$$

From Lemma 1.3 we know that this is a symmetric eigenvalue problem, so that the behaviour of the eigenvalues of (2.3) near $\epsilon = 0$ can be determined by using the perturbation theory for such problems.

The simplest case occurs when the block $D$ in question is $2 \times 2$ , i.e. $l = 1$. Then we have

$$VJ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad WJ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously 0 is an eigenvalue of $WJ$ of multiplicity 2, and for $\epsilon \neq 0$, (2.3) has

$$\lambda(\epsilon) = -\epsilon \tag{2.4}$$

as a double eigenvalue.

The situation in the general case is more complicated. It is easy to see that the matrix $WJ$ has 0 as an eigenvalue of multiplicity 2. We want to determine what happens to this eigenvalue for small $\epsilon \neq 0$. From Theorem B.1 we know that the eigenvalues of (2.3) are analytic functions of $\epsilon$ near $\epsilon = 0$. Thus $\lambda = 0$ will be continued as

$$\lambda_j(\epsilon) = \lambda_j^1 \epsilon + \lambda_j^2 \epsilon^2 + \ldots \qquad j = 1, 2. \tag{2.5}$$

Since the matrix $V$ is invertible, $W$ is nilpotent and, $V$ and $W$ commute, the matrix $(\epsilon V + \sigma W)J$ is invertible. Consequently in the series (2.5) there is at least one nonzero coefficient. The following proposition determine the first nonzero coefficient.

**Proposition 2.1.**
*If the block $D$ is $2l \times 2l$ then the series* (2.5) *can be written as*

$$\lambda_j(\epsilon) = \sigma^{l-1}(-\epsilon)^l + \ldots \qquad j = 1, 2. \tag{2.6}$$

**Proof.** See Appendix B.◇

**Remark 2.3.** If $D$ is a positive block of C3 type then according to Definition 2.1 $l$ is even and $\sigma = 1$, this implies that

$$\lambda_j(\epsilon) = \epsilon^l + \ldots \qquad j = 1, 2$$

is positive for $\epsilon \neq 0$ in a neighborhood of 0. If $D$ is a negative block then $\lambda_j(\epsilon)$ is negative near zero. If $D$ is not positive nor of negative type then $\lambda_j(\epsilon)$ takes positive and negative values near 0.

**Case 2. Block of C4 type.** We recall that a block of C4 type, according to Theorem 1.1 has the form

$$D = \begin{pmatrix} d & 0 \\ 0 & -d^T \end{pmatrix} \tag{2.7}$$

here $D$ is a $2l \times 2l$ matrix, with $l$ being odd.

As in the case just treated we will consider an eigenvalue problem

$$(\epsilon i I + D)v + \lambda J v = 0, \tag{2.8}$$

with $v \in \mathbf{R}^{2l}$ and $\epsilon \in \mathbf{R}$, or equivalently

$$(-\epsilon i J - DJ)w + \lambda w = 0, \quad w = Jv. \tag{2.9}$$

The matrix $DJ$ is easily seen to be symmetric, and $iJ$ is hermitian. We can treat (2.9) using perturbation theory as before. The eigenvalues continuing $\lambda = 0$ have the form (2.5) and the next proposition determine the first nonzero coefficient in the series.

**Proposition 2.2.**

*If the block $D$ is $2l \times 2l$ of C4 type then the series (2.5) can be written as*

$$\lambda_j(\epsilon) = (-1)^j \epsilon^l + \dots \qquad j = 1, 2. \tag{2.10}$$

**Proof.** See Appendix B. $\Diamond$

**Remark 2.4.** We observe that for $\epsilon$ small $\lambda_j(\epsilon)$, $j = 1, 2$ takes positive and negative values. According to Definition 2.1 a block of C4 type is not positive nor negative.

**Case 3. Block of C5 type.** Now we turn to study blocks of the C5 type, where again the invariant $\sigma$ appears.

From Theorem 1.1, a block of this type is of the form

$$D = \begin{pmatrix} d & \sigma\Delta \\ 0 & -d^T \end{pmatrix}$$

with D a $2l \times 2l$ matrix and $\sigma = 1$ or $-1$.

We consider the eigenvalue problem

$$(\epsilon i I + D)v + \lambda J v = 0, \tag{2.11}$$

with $v \in \mathbf{R}^{2l}$ and $\epsilon \in \mathbf{R}$, or equivalently

$$(-\epsilon i J - DJ)w + \lambda w = 0, \qquad w = Jv. \tag{2.12}$$

$DJ$ is symmetric and $iJ$ is hermitian. We apply again perturbation theory.

**Proposition 2.3**

*If the block $D$ is $2l \times 2l$ of C5 type, the eigenvalue of* (2.12) *continuing* $\lambda = 0$, *can be written as*

$$\lambda(\epsilon) = \sigma\epsilon^{(2l)} + \ldots \tag{2.13}$$

**Proof.** The proof of this proposition goes through the same lines as as the proof of Proposition 2.1 and 2.2. We omit it.◊

**Remark 2.5.** By analyzing (2.13) in the case $D$ is a positive block we can see that $\lambda(\epsilon)$ is positive in a neighborhood of $\epsilon = 0$. Similarly if $D$ is negative then $\lambda(\epsilon)$ is negative near $\epsilon = 0$.

### §2.3 $i\pi$ as a special eigenvalue of a Hamiltonian matrix

If we want to study a Hamiltonian system like

$$\dot{z} = JA(t)z + o(|z|),$$

and we assume that $A(t) \equiv A$ is constant then $JA$ is a Hamiltonian matrix and we can apply the Definitions 2.1 and 2.2 to $JA$. If $A(t)$ is time dependent then, using Floquet theory we can transform the system to one with a constant linear term. However if the system has negative Floquet multipliers we need to deal with a new situation.

Hamiltonian matrices can be obtained from symplectic matrices by taking the logarithm. A special situation arises when the symplectic matrix has negative eigenvalues.

To begin we recall some results that are the symplectic analogues of Lemmas 1.1 and 1.2. The proofs can be found in [10].

**Lemma 2.1.**

*If $\lambda$ is an eigenvalue of a symplectic matrix $X$ then $1/\lambda$, $\bar{\lambda}$, $1/\bar{\lambda}$ are also eigenvalues of $X$ with the same multiplicity as $\lambda$. In particular* 1 *and* $-1$ *have even multiplicity if they are eigenvalues of $X$.*

Let us define

$$\xi_k(\lambda) = \ker(X - \lambda I)^k, \quad \text{and} \tag{2.14}$$

$$\xi(\lambda) = \bigcup_{k \geq 1} \xi_k(\lambda), \tag{2.15}$$

and similarly

$$\tilde{\xi}_k(\lambda) = \ker(X^{-1} - \lambda I)^k, \quad \text{and} \tag{2.16}$$

$$\tilde{\xi}(\lambda) = \bigcup_{k \geq 1} \tilde{\xi}_k. \tag{2.17}$$

Then we have

**Lemma 2.2.**

*If $X$ is a symplectic matrix and $\lambda$ is an eigenvalue of $X$ then*

1. $\xi(\lambda) = \bar{\xi}(1/\lambda)$
2. *If $\mu$ is another eigenvalue of $X$ then $\mu \neq 1/\lambda$ implies*

$$(z, Jw) = 0 \qquad \forall z \in \xi(\lambda), \quad \forall w \in \xi(\mu).$$

**Corollary 2.1.**

*If $X$ is a symplectic matrix and $Q$ is an invertible matrix so that*

$$\mathcal{J} = Q^{-1} X Q \tag{2.18}$$

*is the real Jordan canonical form of $X$, then*

$$Q^{\tau} J Q = \text{diag}(q_1, q_2, ..., q_p) \tag{2.19}$$

*is a block diagonal matrix with one block $q_i$ for each pair $(\lambda, 1/\lambda)$ of eigenvalues of $X$.*

For $X$ symplectic let us consider

$$C = \log(X) \tag{2.20}$$

where log denotes the logarithm defined in Appendix A. If $X$ does not have eigenvalues on the negative real axis then $C$ is a real Hamiltonian matrix (See Lemma A.2 and A.3). However, if $X$ has negative real eigenvalues $C$ is no longer real. The matrix $C$ is not Hamiltonian in the sense we defined before, but it satisfies

$$JC^*J = C \tag{2.21}$$

where $*$ denotes the conjugate transpose of $C$. For such a matrix $C$ we want to have 'real' canonical forms. We will reduce this question to Theorem 1.1.

Let us assume that $X$ has $2p$ negative eigenvalues, and let us reorder J as

$$\tilde{J} = \begin{pmatrix} J_p & 0 \\ 0 & J_q \end{pmatrix} \tag{2.22}$$

where $J_p$ and $J_q$ are the standard symplectic structures in $\mathbf{R}^{2p}$ and $\mathbf{R}^{2q}$ respectively. Here $q = n - p$. From Lemma 2.2 there is an invertible matrix $Q$ so that

$$Q^{\tau} J Q = \tilde{J} \tag{2.23}$$

and

$$Y = Q^{-1} X Q \tag{2.24}$$

is a block matrix

$$Y = \begin{pmatrix} Y_p & 0 \\ 0 & Y_q \end{pmatrix} \tag{2.25}$$

with

$$Y_j^\tau J_j Y_j = J_j, \qquad j = p, q \tag{2.26}$$

and $Y_p$ has only negative eigenvalues, while $Y_q$ does not have negative eigenvalues.

Let

$$S_j = \log(Y_j), \qquad j = p, q \tag{2.27}$$

then, by Lemma A.3, we have

$$J_j S_j^* J_j = S_j, \qquad j = p, q. \tag{2.28}$$

Lemma A.2 guarantees that $S_q$ is real, and Lemma A.4 gives that

$$S_p = i\pi I_{2p} + \bar{S}_p \tag{2.29}$$

where $\bar{S}_p = \log(-Y_p)$ is a real matrix, and $I_{2p}$ is the identity in $\mathbf{R}^{2p}$.

From (2.28) we see that $\bar{S}_p$ satisfies

$$J_p \bar{S}_p^\tau J_p = \bar{S}_p. \tag{2.30}$$

If we define

$$\bar{I} = \begin{pmatrix} I_{2p} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \bar{S}_p & 0 \\ 0 & S_q \end{pmatrix}$$

then we have

$$\log(Y) = i\pi \bar{I} + S \tag{2.31}$$

and the real matrix $S$ satisfies

$$\tilde{J} S^\tau \tilde{J} = S. \tag{2.32}$$

If we define

$$C_i = \pi Q \bar{I} Q^{-1} \quad \text{and} \quad C_r = QSQ^{-1} \tag{2.33}$$

then from (2.31) , (2.32) and the block structure of $\bar{I}$, $\tilde{J}$ and $S$ the following hold

$$C = iC_i + C_r, \tag{2.34}$$

$$JC_r^\tau J = C_r, \tag{2.35}$$

$$JC_i^\tau J = -C_i, \quad \text{and} \tag{2.36}$$

$$C_i C_r = C_r C_i. \tag{2.37}$$

If $-1$ is an eigenvalue of $X$ of multiplicity $2k$ then 0 is an eigenvalue of multiplicity $2k$ of $\bar{S}_p$ and from (2.30) $\bar{S}_p$ is a Hamiltonian matrix. We have the following extension of Definition 2.2.

**Definition 2.3.** *Let $X$ be a symplectic matrix, and $C = \log(X)$. If $-1$ is an eigenvalue of $X$ of multiplicity $2k$, then $i\pi$ is an eigenvalue of $C$ of multiplicity $2k$ and we say that $i\pi$ is of positive type (negative type or indefinite type) if and only if 0 is an eigenvalue of positive type (negative type or indefinite type) of the matrix $\bar{S}_p$.*

## §3. PROOF OF THEOREM I.

In this section we begin the study of the Hamiltonian system (HS)

$$\dot{z} = JA(t)z + J\hat{H}_z(z,t)$$

where the Hamiltonian function is

$$H(z,t) = \frac{1}{2}(A(t)z,z) + \hat{H}(z,t) \tag{3.1}$$

and we assume it satisfies the basic hypotheses:

(H0) $H$ is of class $C^2$ near $z = 0$,

(H1) $H(0,t) = 0$ and $H_z(0,t) = 0 \quad \forall t \in \mathbf{R}$,

(H2) $H$ is 1-periodic in the $t$-variable.

Assuming that $H$ is 1-periodic instead of $T$-periodic does not restrict the analysis. Under these hypotheses (HS) represents the Hamiltonian system

$$\dot{z} = JH_z(z,t), \tag{3.2}$$

$A(t)$ is the Hessian matrix of $H$ with respect to $z$ at $z = 0$, and $\hat{H}$ represents the higher order terms of $H$, $\hat{H}(z,t) = o(|z|^2)$ uniformly in $t$, for $z$ near 0.

Let us consider the linear system

$$\dot{z} = JA(t)z. \tag{3.3}$$

If $\mathcal{X}(t)$ is the fundamental matrix if system (3.3), i.e.

$$\mathcal{X}(0) = I_{2n} \tag{3.4}$$

$$\dot{\mathcal{X}}(t) = JA(t)\mathcal{X}(t) \tag{3.5}$$

we define $X = \mathcal{X}(1)$ and $C = \log(X)$, where the logarithm is defined in Appendix A.

The eigenvalues of $C$ are called the Floquet exponents of (3.3). We will assume

(H3) There is at least one Floquet exponent of (3.3) on the imaginary axis that is not of negative type.

In other words, (H3) says that $C$ has at least one purely imaginary eigenvalue that is not of negative type. Finally we assume

(H4) There are constants $c > 0, r > 0, q + 2 > p > q > 2$ such that

$$(\hat{H}_z(z,t), z) \geq q\hat{H}(z,t) \geq c|z|^p \tag{3.6}$$

$$\forall z \in \mathbf{R}^{2n}, \quad |z| \leq r, \quad \forall t \in \mathbf{R}.$$

In this section we will prove

**Theorem I**

*If (H0), (H1), (H2), (H3) and (H4) hold, then there is an integer $N$ so that for every $k \geq N$, there exist a nontrivial solution $z_k$ of (HS) with period $k$, and $\{z_k\}_{k \geq N}$ converges to zero as $k \to \infty$ uniformly in $C^1(\mathbf{R}, \mathbf{R}^{2n})$.*

**Remark 3.1.** In Section §4 we will see that adding some differentiability to $H$ we will be able to prove the result of Theorem I under less restrictive conditions than (H4), namely we need only to assume (3.6) holds on the center manifold of (HS).

The proof of Theorem I is presented in the next two sections. We begin in Section §3.1 by proving a result on a generalized eigenvalue problem associated to (3.3). For that purpose we transform (3.3) to a linear system with constant coefficients and then we analize the resulting system in terms of the canonical form associated to it. Next we study the nonlinear problem in Section §3.2 using variational methods.

## §3.1 The Linearized Problem.

The proposition we will prove now is the basis of the proof of Theorem I. It is here that hypothesis (H3) is crucial. We will first reduce system (3.3) to a system with constant coefficients via Floquet Theory. Because the matrix $X = \mathcal{X}(1)$ can have negative eigenvalues we will need to introduce complex matrices or change our periodicity condition on the solutions. We prefer the second, because in Section §4 this will be more convenient.

Let $R$ be a matrix so that $R^2 = I$.

**Definition 3.1.**

*Let $k \in \mathbf{N}$, a function $z : \mathbf{R} \longrightarrow \mathbf{R}^{2n}$ is said to be $Rk$-periodic if*

$$z(t + k) = R^k z(t) \quad \forall t \in \mathbf{R}. \tag{3.7}$$

We note that every $R$-periodic function is 2-periodic.

**Lemma 3.1.**

*There are matrices $C_r, R$ and $P(t)$ so that $C_r$ is Hamiltonian, $R^2 = I$, $P(t)$ symplectic and $P(t+1) = P(t)R \quad \forall t \in \mathbf{R}$ for which we have*
*$z(t)$ is a $k$-periodic solution of (3.3) if and only if*

$$\xi(t) = P(t)^{-1} z(t)$$

*is an $Rk$-periodic solution of*

$$\dot{\xi} = C_r \xi. \tag{3.8}$$

**Proof.** The proof is an application of Floquet theory. Let $\mathcal{X}(t)$ be the fundamental matrix of (3.3) as defined above. Since (3.3) is Hamiltonian $\mathcal{X}(t)$ satisfies

$$\mathcal{X}(t)^\tau J \mathcal{X}(t) = J \qquad \forall t \in \mathbf{R}. \tag{3.9}$$

To obtain (3.9) we differentiate the matrix $Z(t) = \mathcal{X}(t)^\tau J \mathcal{X}(t)$ and use (3.4), (3.5) and the fact that $A(t)$ is symmetric. Identity (3.9) says that $\mathcal{X}(t)$ is symplectic for every $t \in \mathbf{R}$, consequently $X = \mathcal{X}(1)$ is symplectic.

By Lemma A.3 there is a matrix $C$ so that

$$X = \exp(C), \quad \text{and} \tag{3.10}$$

$$C = JC^* J. \tag{3.11}$$

As we have mentioned before $C$ is not necessarily real because $X$ can have negative eigenvalues. However we can always decompose $C$ as $C = iC_i + C_r$ with $C_i$ and $C_r$ satisfying (2.35), (2.36) and (2.37).

Let us define

$$P(t) = \mathcal{X}(t) \exp(-C_r t) \tag{3.12}$$

then $P(t)$ is invertible and obviously

$$\mathcal{X}(t) = P(t) \exp(C_r t). \tag{3.13}$$

Since $\mathcal{X}(t)$ is symplectic and $C_r$ satisfies (2.35) $P(t)$ is also symplectic, i.e.

$$P(t)^\tau J P(t) = J. \tag{3.14}$$

On the other hand, since (3.3) is 1–periodic it is easy to see that

$$\mathcal{X}(t+1) = \mathcal{X}(t)\mathcal{X}(1).$$

Therefore

$$\begin{aligned} P(t+1) &= \mathcal{X}(t+1) \exp(-C_r(t+1)) \\ &= \mathcal{X}(t) X \exp(-C) \exp(-C_r t + iC_i) \\ &= \mathcal{X}(t) \exp(-C_r t) \exp(iC_i) \\ &= P(t) \exp(iC_i). \end{aligned} \tag{3.15}$$

where we used (2.37) and the definition of $P(t)$. If we define

$$R = \exp(iC_i) \tag{3.16}$$

then from the definition of $C_i$, (2.33),

$$R^2 = I \tag{3.17}$$

and what we showed in (3.15) becomes

$$P(t+1) = P(t)R. \tag{3.18}$$

Finally we have the following identity obtained from differentiating (3.14) and using (3.5)

$$\dot{P}(t) + P(t)C_r = JA(t)P(t). \tag{3.19}$$

Let $z(t)$ be a $k$–periodic solution of (3.3), and $\xi(t) = P(t)^{-1}z(t)$. Then by differentiating $z(t) = P(t)\xi(t)$ and using (3.3) we obtain

$$\dot{P}(t)\xi(t) + P(t)\dot{\xi}(t) = JA(t)P(t)\xi(t). \tag{3.20}$$

Using (3.19) in (3.20) and multiplying by $P(t)^{-1}$ we have

$$\dot{\xi} = C_r\xi.$$

Since $P(t)$ satisfies (3.18), and $z$ is $k$–periodic we have

$$\xi(t+k) = R^k\xi(t).$$

The reverse implication is also easily obtained.$\diamondsuit$

Next we prove a lemma that will be used in Section §4, which is a sort of inverse of Lemma 3.1 in the sense that finding $Rk$-periodic solutions of a system like (3.8) is equivalent to find $k$-periodic solutions of a system with variable coefficients. Note that this is not given by Lemma 3.1 because in this situation we do not have the fundamental matrix $\mathcal{X}(t)$ available.

We defined $R$ in (3.16) in terms of $C_i$. We recall that $C_i$ was defined in (2.33) as

$$C_i = \pi Q\bar{I}Q^{-1}$$

where $Q$ is a symplectic matrix, and $\bar{I}$ is given by

$$\bar{I} = \begin{pmatrix} I_{2p} & 0 \\ 0 & 0 \end{pmatrix}$$

and $I_{2p}$ is the identity in $\mathbf{R}^{2p}$.

We can write the matrix $R$ in a different way, and we do it now. If

$$\tilde{I} = \begin{pmatrix} \tilde{I}_{2p} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.21}$$

with

$$\tilde{I}_{2p} = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \tag{3.22}$$

then it is easy to see that for the matrix

$$\tilde{C}_i = \pi Q \tilde{I} Q^{-1} \tag{3.23}$$

we have

$$R = \exp(\tilde{C}_i). \tag{3.24}$$

**Lemma 3.2.**

*$\xi(t)$ is an $Rk$–periodic solution of*

$$\dot{\xi} = C_r \xi$$

*if and only if $w(t) = R(t)\xi(t)$ is a $k$-periodic solution of*

$$\dot{w} = (\dot{R}(t) + R(t)C_r)R^{-1}(t)w$$

*where $R(t) = \exp(\tilde{C}_i t)$.*

**Proof.** By direct calculations.◇

**Remark 3.2.** We note that in general the matrices $\tilde{C}_i$ and $C_r$ do not commute. That is the reason why we cannot use $C_r + \tilde{C}_i$ as the logarithm of $X$.

The following proposition is fundamental.

**Proposition 3.1.**

*Assume the linear system (3.3) satisfies (H3). Then there is a 1–periodic, continuous matrix function $U(t)$, that is symmetric and positive definite, such that for every $k \in \mathbb{N}$ there is a $k$–periodic function $z_k$, and a number $\lambda_k$, $\lambda_k > 0$ and $\lambda_k = O(1/k)$ as $k \to \infty$, satisfying*

$$\dot{z} = JA(t)z + \lambda JU(t)z. \tag{3.25}$$

*In other words the generalized eigenvalue problem (3.25) has at least one eigenvalue $\lambda_k > 0$ with $\lambda_k = O(1/k)$ as $k \to \infty$.*

**Proof.** From Lemma 3.1 we can transform (3.3) into

$$\dot{\xi} = C_r \xi$$

with $C_r$ a Hamiltonian matrix. Let $N$ be the canonical form of $C_r$ as in Theorem 1.1. Then there is a $Q$ so that

$$Q^{\tau} J Q = J \quad \text{and} \tag{3.26}$$

$$Q^{-1} C_r Q = N. \tag{3.27}$$

From hypothesis (H3) we know that $N$ has at least one eigenvalue $\lambda$ that is purely imaginary and not of negative type. For that eigenvalue there exists a $2l \times 2l$ block $D$ that is not negative.

Let $k \in \mathbf{N}$, and consider the following eigenvalue problem

$$\dot{x} = Dx + \lambda Jx, \quad x(0) = x(k) \tag{3.28}$$

where $x \in \mathbf{R}^{2l}, \lambda \in \mathbf{R}$ and $J$ is the standard symplectic structure in $\mathbf{R}^{2l}$. We do not distinguish the dimension; it will be clear from the context.

The block D can be of C3, C4, or C5 type. We analyze each case separately.

CASE 1. Block of C3 type.

In this case $\lambda = i\beta, \beta > 0$, and the matrix $D$ has the form given by (1.18)

$$D = -\sigma\beta V - \sigma W \tag{3.29}$$

Let $\alpha \in [-2\pi, 2\pi]$ so that $\exp(-i\alpha) = \exp(-i\sigma\beta k)$; we determine the sign of $\alpha$ later. The eigenvalue problem (3.28) is related to

$$-(\alpha V + k\sigma W)v + \lambda k J v = 0. \tag{3.30}$$

In fact, let us assume $(\lambda, v)$ satisfies (3.30) and let us define

$$x_k(t) = \exp\big((-\sigma\beta V - \sigma W + \lambda J)t\big)\, v \tag{3.31}$$

Certainly $x_k(t)$ satisfies the differential equation in (3.28), but it also satisfies the boundary condition

$$\begin{aligned} x_k(k) &= \exp\big(k(-\sigma\beta V - \sigma W + \lambda J)\big)\, v \\ &= \exp(-\sigma\beta k V) \exp(-\sigma k W + \lambda k J)\, v \\ &= \exp(-\alpha V) \exp(-\sigma k W + \lambda k J)\, v \\ &= \exp(-\alpha V - \sigma k W + \lambda k J)\, v \\ &= v, \end{aligned}$$

here we used Lemmas 1.3 and 1.4, the definition of $\alpha$ and (3.30). Thus it is enough to study (3.30).

Dividing (3.30) by $k$ we get

$$-(\frac{\alpha}{k} V + \sigma W)v + \lambda J v = 0 \tag{3.32}$$

that is exactly the eigenvalue problem (2.2), with $\epsilon = \alpha/k$. Because D is not negative, considering $\alpha$ with the adequate sign, from Proposition 2.1 we have that for every $k$ large enough there is a $\lambda_k > 0$, an eigenvalue of (3.32) and hence of (3.30). Moreover $\lambda_k = O(1/k)$.

Thus, for every $k \in \mathbf{N}$ large enough, (3.28) has an eigenvalue $\lambda_k > 0$, with $\lambda_k = O(1/k)$ and associated eigenfunction $x_k$ that is $k$-periodic.

The remaining cases correspond to blocks of C4 and C5 type. These blocks have two possible origins, they can come from an eigenvalue 0 of $C$ or from the eigenvalue $i\pi$ of $C$. We distinguish between the two possibilities.

CASE 2. Block of C4 type ( $\lambda = 0$ is an eigenvalue of $C$) In this case the matrix D has the form

$$D = \begin{pmatrix} d & 0 \\ 0 & -d^{\tau}. \end{pmatrix}$$

Let $\alpha = 2\pi$ and consider the eigenvalue problem

$$(\alpha i I + kD)v + \lambda k J v = 0. \tag{3.33}$$

If $(\lambda, v)$ is a solution of (3.33) then the function

$$x_k(t) = \exp\big((D + \lambda J)t\big)\, v \tag{3.34}$$

satisfies equation (3.28). Using (3.33) we see that

$$\begin{aligned} x_k(k) &= \exp(kD + \lambda k J)\, v \\ &= \exp(2\pi i I + kD + \lambda k J)\, v \\ &= v \end{aligned}$$

so that $x_k$ is $k$-periodic. Thus it is enough to solve (3.33), but dividing (3.33) by $k$ we see that it corresponds exactly to the eigenvalue problem (2.8) with $\epsilon = 2\pi/k$

$$(\frac{2\pi}{k} i I + D)v + \lambda J v = 0$$

and Proposition 2.2 guarantees that for large $k$ there is a $\lambda_k > 0, \lambda_k = O(1/k)$, an eigenvalue of (3.33). Thus we have obtained the same conclusion as in Case 1.

CASE 3. Block of C4 type (corresponding to the eigenvalue $\lambda = i\pi$ of C). Here we proceed as in Case 2 but we will take

$$\alpha_k = \begin{cases} \pi & \text{if } k \text{ is odd} \\ 2\pi & \text{if } k \text{ is even} \end{cases}$$

The only difference with Case 2 is that the corresponding function $x_k$ will satisfies

$$x_k(k) = (-1)^k x_k(0).$$

CASES 4 and 5. D is a block of C5 type (corresponding to $\lambda = 0$ or $i\pi$ of $C$). This cases are similar to Cases 2 and 3 respectively. We note that D is of positive type so that from Proposition 2.3 we can guarantee that the eigenvalue $\lambda_k$ can be chosen positive and also of order $O(1/k)$.

Now we summarize what we have. From (2.33) and (3.16) we see that

$$R = QR_1Q^{-1}$$

where

$$R_1 = \begin{pmatrix} -I_{2p} & 0 \\ 0 & I_{2q} \end{pmatrix}$$

Then, what we have shown is that the eigenvalue problem

$$\dot{\xi} = N\xi + \lambda J\xi, \qquad \xi(k) = R_1^k\xi(0) \tag{3.35}$$

has an eigenvalue $\lambda_k > 0$, $\lambda_k = O(1/k)$ with eigenfunction $\xi_k$, for any $k \in \mathbb{N}$ large enough.

Changing coordinates, $w_k(t) = Q\xi_k(t)$ satisfies

$$\dot{w} = C_r w + \lambda QJQ^{-1}w, \qquad w(k) = R^k w(0). \tag{3.36}$$

Finally defining

$$z_k(t) = P(t)w_k(t) \tag{3.37}$$

and

$$U(t) = -JP(t)QJQ^{-1}P(t)^{-1}, \tag{3.38}$$

using the same argument of the proof of Lemma 3.1, we see that

$$\dot{z}_k = JA(t)z_k + \lambda_k JU(t)z_k, \quad \text{and}$$

$$z_k(k) = z_k(0).$$

It is only left to prove that $U(t)$ is symmetric and positive definite. Because $Q$ and $P(t)$ are symplectic we have

$$Q^\tau J = JQ^{-1} \quad \text{and} \quad P(t)^\tau J = JP(t)^{-1} \tag{3.39}$$

and then from (3.38) and (3.39) we obtain

$$U(t) = J^\tau P(t)QQ^\tau P(t)^\tau J, \tag{3.40}$$

from where $U(t)$ is clearly symmetric and positive definite. A simple computation using (2.33), (3.16) and (3.18) will show that $U(t)$ is also 1-periodic.$\diamond$

## §3.2 The Nonlinear Problem.

Now we give a proof of Theorem I. We start with some preliminaries.

Let us consider the eigenvalue problem

$$-J\dot{z} - A(t)z = \lambda U(t)z \tag{3.41}$$

in $L^2([0,k], \mathbf{R}^{2n}) \equiv L_k^2$ with periodic boundary conditions. Here $U$ is the matrix function defined in Proposition 3.1. In $L_k^2$ we will consider two inner products

$$(u,v)_2 = \int_0^k (u(t), v(t))\, dt, \quad \text{and} \tag{3.42}$$

$$< u, v >_2 = \int_0^k (U(t)u(t), v(t))\, dt. \tag{3.43}$$

Let $|\cdot|_2$ and $\|\cdot\|_2$ be the norms induced by $(\cdot,\cdot)_2$ and $<\cdot,\cdot>_2$ respectively.

Since the matrix function $U(t)$ is periodic, continuous and positive definite there is a constant $a > 0$ such that

$$\frac{1}{a} \leq | U(t) | \leq a \quad \forall t \in \mathbf{R}, \quad \text{and} \tag{3.44}$$

$$\frac{1}{a} \leq | U(t)^{-1} | \leq a \quad \forall t \in \mathbf{R}. \tag{3.45}$$

where $|\cdot|$ denotes the $L(\mathbf{R}^{2n}, \mathbf{R}^{2n})$ norm induced on matrices (using the usual norm in $\mathbf{R}^{2n}$).

**Lemma 3.3.**

*There are constants $c_1 > 0$ and $c_2 > 0$ independent of $k$ so that*

$$| u |_2 \leq c_1 \| u \|_2 \quad \text{and} \tag{3.46}$$

$$\| u \|_2 \leq c_2 | u |_2 \quad \forall u \in L_k^2. \tag{3.47}$$

**Proof.**

$$\begin{aligned} | u |_2^2 &= \int_0^k (u(t), u(t))\, dt \\ &= \int_0^k (U(t)U(t)^{-1}u(t), u(t))\, dt \\ &= < U(t)^{-1}u, u >_2 \\ &\leq \| U^{-1}u \|_2 \| u \|_2 \,. \end{aligned} \tag{3.48}$$

But from (3.45)

$$\begin{aligned} \| U^{-1}u \|_2^2 &= \int_0^k (u(t), U(t)^{-1}u(t))\,dt \\ &\le a \mid u \mid_2^2 . \end{aligned} \tag{3.49}$$

Thus, from (3.48) and (3.49) we obtain

$$\mid u \mid_2 \le \sqrt{a} \parallel u \parallel_2 .$$

On the other hand

$$\begin{aligned} \| u \|_2^2 &= \int_0^k (U(t)u(t), u(t))\,dt \\ &\le \mid Uu \mid_2 \mid u \mid_2 . \end{aligned} \tag{3.50}$$

But

$$\begin{aligned} \mid Uu \mid_2^2 &= \int_0^k (U(t)u(t), U(t)u(t))\,dt \\ &\le a^2 \mid u \mid_2^2, \end{aligned} \tag{3.51}$$

so, from (3.50) and (3.51) we obtain

$$\parallel u \parallel_2 \le \sqrt{a} \mid u \mid_2 .$$

Taking $c_1 = c_2 = \sqrt{a}$ we get (3.46) and (3.47).◇

For the eigenvalue problem (3.41) we have the following standard result.

**Lemma 3.4.**

*The eigenvalue problem (3.41) possesses a sequence of eigenvalues $\{\nu_j\}_{j\in\mathbf{Z}}$ that extends from $-\infty$ to $+\infty$. Each eigenvalue is isolated and has finite multiplicity. Associated to the eigenvalues there is a basis of $L^2_k$, composed of eigenvectors $\{v_j\}_{j\in\mathbf{Z}}$, orthonormal with respect to the inner product $<,>_2$.*

**Proof.** We only give the general idea of the proof. First, by making a transformation like that of Proposition 3.1, and taking in account the way $U$ was defined we can transform system (3.41) into a system with constant coefficients. Second, by taking an appropriate trigonometric basis in $L^2_k$ we transform this constant coefficient differential equation into a matrix one, where the eigenvalues can be calculated. The trigonometric basis gives rise to eigenvalues $\pm n$ with a constant correction coming from the constant term.◇

**Remark 3.3.** By Proposition 3.1 there is an eigenvalue $\lambda_k > 0$ of (3.41) with $\lambda_k = O(1/k)$ as $k \to \infty$.

In order to study the existence of subharmonics of (HS) we will study the critical points of a functional. Let $E$ be the Sobolev space $W_k^{1/2,2}(\mathbf{R}, \mathbf{R}^{2n})$. For details about the definition of $E$ and the operators we define next we refer the reader to [13]. For $z = (p, q) \in E$ and smooth we define

$$Q(z) = \int_0^k q\dot{p} - \frac{1}{2}(z, A(t)z)\,dt. \tag{3.52}$$

If $\zeta = (\varphi, \psi) \in E$ is also smooth we can define the bilinear form

$$\begin{aligned} B(z,\zeta) &= \int_0^k \psi\dot{p} + q\varphi - (z, A(t)\zeta)\,dt \\ &= \int_0^k (-J\dot{z} - A(t)z, \zeta)\,dt. \end{aligned} \tag{3.53}$$

Both $Q$ and $B$ can be extended continuously to all $E$, and

$$\frac{1}{2}B(z,z) = Q(z) \qquad \forall z \in E. \tag{3.54}$$

Using Lemma 3.4 we can define the following closed subspaces of $E$

$$\begin{aligned} E^+ &= \overline{\text{span}}\{v_i /\ \nu_i > 0\} \\ E^- &= \overline{\text{span}}\{v_i /\ \nu_i < 0\}, \quad \text{and} \\ E^0 &= \text{span}\{v_i /\ \nu_i = 0\}. \end{aligned}$$

It is easy to see that Q is positive in $E^+$, negative in $E^-$ and zero in $E^0$. Also for $z = z^+ + z^- + z^0, z^+ \in E^+, z^- \in E^-$ and $z^0 \in E^0$ we have

$$Q(z^+ + z^- + z^0) = Q(z^+) + Q(z^-). \tag{3.55}$$

Moreover, we have the following lemma.

**Lemma 3.5.**
*If we define* $\| z \|_Q^2 = Q(z^+) - Q(z^-) + \int_0^k | z^0 |^2 \, dt$ *for every* $z \in E$*, then there are constants* $c_3$ *and* $c_4$ *such that*

$$c_3 \| z \|_Q \leq \| z \|_E \leq c_4 \| z \|_Q \tag{3.56}$$

*i.e.* $\| \cdot \|_Q$ *is an equivalent norm in E.*

**Proof.** For proving this lemma we would proceed as in Lemma 3.4 passing to the equivalent definitions of $Q$ and $\| \cdot \|_Q$ for an equivalent system with constants coefficients, where the lemma is clear.◇

As mentioned above in order to find subharmonics solutions of (HS) we will use a variational approach. In particular we will use a version of the generalized Mountain

Pass Theorem due to Benci and Rabinowitz. We state this theorem here and we refer the reader to [13] for its proof.

**Theorem 3.1.**

*Let $E$ be a real Hilbert space with $E = E_1 \oplus E_2$ and $E_2 = E_1^{\perp}$. Suppose $I \in C^1(E.\mathbf{R})$, satisfies the Palais-Smale condition, and*

*$(I_1)\, I(u) = \frac{1}{2}(Lu, u) + b(u)$, where $Lu = L_1P_1u + L_2P_2u$ and $L_i : E_i \longrightarrow E_i$ is bounded and selfadjoint, where $P_i : E \longrightarrow E_i$ is the orthogonal projection, i=1,2,*

*$(I_2)$ $b'$ is compact, and*

*$(I_3)$ there exist a subspace $\tilde{E} \subset E$ and sets $S \subset E, Q \subset \tilde{E}$ and constants $\alpha > \omega$ such that*

*(i) $S \subset E_1$ and $I|_S \geq \alpha$,*

*(ii) $Q$ is bounded and $I|_{\partial Q} \leq \omega$,*

*(iii) $S$ and $\partial Q$ link.*

*Then $I$ possesses a critical value $c \geq \alpha$.*

**Proof of Theorem I.**

The proof follows the lines of the proof of Theorem 1.1 in [2] by Benci and Fortunato. We will first consider a globally defined Hamiltonian following a trick introduced by Rabinowitz.

Let $\psi \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\psi'(s) < 0$ for $s \in (r/3, r/2)$ and

$$\psi(s) = \begin{cases} 1 & \text{if } s \leq r/3 \\ 0 & \text{if } s \geq r/2 \end{cases}$$

where $r$ was introduced in hypothesis (H4). We define the globalized Hamiltonian, by extending the higher order term

$$\tilde{H}(z,t) = \psi(| z |)\hat{H}(z,t) + R(1 - \psi(| z |)) \mid z \mid^q . \tag{3.57}$$

By choosing $R \geq 1$ large enough and using hypothesis (H4) we can show that

$$(\tilde{H}_z(z,t), z) \geq q\tilde{H}(z,t) \quad \forall z \in \mathbf{R}^{2n}. \tag{3.58}$$

Also from hypothesis (H4), for a positive constant $c_5$

$$\tilde{H}(z,t) \geq c_5(\psi(| z |) \mid z \mid^p + R(1 - \psi(| z |)) \mid z \mid^q) \quad \forall z \in \mathbf{R}^{2n} \tag{3.59}$$

and from the definition of $\tilde{H}$ we see that there is an $M > 1$ so that

$$\tilde{H}(z,t) = R \mid z \mid^q, \qquad \forall z \in \mathbf{R}^{2n}, \quad | z | \geq M. \tag{3.60}$$

We define the functional

$$f_k(z) = Q(z) - \int_0^k \tilde{H}(z,t)\, dt \tag{3.61}$$

for all $z$ in $E$. It is well known, see [13], that under our hypothesis $f_k$ is well defined in $E$, of class $C^1$ and the critical points of $f_k$ are classical $k$-periodic solutions of

$$\dot{z} = JA(t)z + J\tilde{H}_z(z,t). \tag{3.62}$$

In order to prove the existence of critical points for $f_k$ we will use min-max arguments. Specifically we will use Theorem 3.1 stated above, which requires that the linking condition given in (i), (ii) and (iii) be satisfied.

From (3.58) and (3.59) and assuming without loss of generality that $r < 1$ there are positive constants $c_6$, $c_7$ and $c_8$ so that

$$\tilde{H}(z,t) \geq c_6 \mid z \mid^q - c_7, \quad \forall z \in \mathbb{R}^{2n} \tag{3.63}$$

$$\tilde{H}(z,t) \leq c_8 \mid z \mid^q, \quad \forall z \in \mathbb{R}^{2n}. \tag{3.64}$$

Now we can prove the linking conditions. Let $z \in E^+$, then from (3.64)

$$\begin{aligned} f_k(z) &= \| z \|_Q^2 - \int_0^k \tilde{H}(z,t)\,dt \\ &\geq \| z \|_Q^2 - c_8 \int_0^k \mid z \mid^q \; dt \\ &= \| z \|_Q^2 - c_8 \mid z \mid_q^q \; dt \end{aligned} \tag{3.65}$$

where $| \cdot |_q$ denotes the usual norm in the space $L^q([0,k], \mathbb{R}^{2n})$ that we will write as $L^q_k$. As is well known, see [13], for every $s > 1$ there is a constant $\alpha(s)$ so that

$$\mid v \mid_s \leq \alpha(s) \parallel v \parallel_E . \tag{3.66}$$

Then (3.65), (3.66) and Lemma 3.5 imply the existence of a constant $c_9$ such that

$$f_k(z) \leq \| z \|_Q^2 - c_9 \| z \|_Q^q . \tag{3.67}$$

Now (3.67) shows there are constants $\rho > 0$ and $\alpha > 0$ such that

$$f_k(z) \geq \alpha > 0, \quad \forall z \in E^+, \quad \| z \|_Q = \rho. \tag{3.68}$$

On the other hand, let $R_1$ and $R_2$ be two positive numbers to be determined later. Let $\phi_k$ be one eigenfunction of (3.41) associated to the eingenvalue $\lambda_k$ we found in Proposition 3.1. Assume $\| \phi_k \|_2 = 1$. Let

$$Q_k = \{ v + s\phi_k / \; v \in E^0 + E^-, \; \| v \|_Q \leq R_2 \text{ and } 0 \leq s \leq R_1 \}.$$

Then for $z = v^0 + v^- + s\phi_k$ we have

$$f_k(z) = Q(z) - \int_0^k \tilde{H}(z,t)\,dt$$
$$= - \parallel v^- \parallel_Q^2 + \frac{1}{2}s^2\lambda_k - \int_0^k \tilde{H}(v + s\phi_k, t)\,dt. \tag{3.69}$$

If $s = 0$, from (3.69) we have

$$f_k(z) \leq 0. \tag{3.70}$$

On the remaining part of $\partial Q_k$ we also have this inequality. Using (3.63) and Hölder inequality in (3.69) we have

$$f_k(v + s\phi_k) \leq - \parallel v^- \parallel_Q^2 + \frac{1}{2}s^2\lambda_k - c_6 k^{\{2-q/2\}} \mid v + s\phi_k \mid_2^q + c_7 k. \tag{3.71}$$

By Lemma 3.3, and the orthogonality of $E^+, E^-$ and $E^0$ with respect to $<,>_2$

$$\mid v + s\phi_k \mid_2^2 \geq \frac{1}{a} \parallel v + s\phi_k \parallel_2^2$$
$$\geq \frac{1}{a}(\parallel v^0 \parallel_2^2 + s^2)$$
$$\geq \frac{1}{a^2} \mid v^0 \mid_2^2 + \frac{1}{a}s^2 \tag{3.72}$$

Thus, for a constant $c_10 > 0$

$$\mid v + s\phi_k \mid_2^q \geq c_10(\mid v^0 \mid_2^q + s^q). \tag{3.73}$$

Using (3.71) in (3.73) we get

$$f_k(v + s\phi_k) \leq \frac{1}{2}s^2\lambda_k - \parallel v^- \parallel_Q^2 - c_{11}(\mid v^0 \mid_2^q + s^q) + c_7 k \tag{3.74}$$

From (3.74) we find $R_1$ and $R_2$ so that with (3.70) we obtain

$$f_k(z) \leq 0 \qquad \forall z \in \partial Q_k. \tag{3.75}$$

See details in [13]. Now (3.68) and (3.75) imply the linking condition is satisfied. The functional $f_k$ also satisfies the Palais Smale condition as can be shown by standard calculations. Thus $f_k$ possesses a critical point $z_k$, that according to the min-max characterization satisfies

$$0 < \alpha < f_k(z_k) \leq c_\infty = \mathrm{Sup}\{f_k(z)/\ z \in Q_k\}$$

Next we are interested in knowing the behaviour of the critical value $f_k(z_k)$ when $k$ is large. We will show that

$$0 < f_k(z_k) \leq \frac{C}{k^\gamma}, \tag{3.76}$$

$$\text{with} \quad \gamma = \mathrm{Min}\{2/(p-2), (q-p+2)/(q-2)\},$$

and $C$ independent of $k$. We note that because $q+2 > p > q > 2$, the exponent $\gamma$ is positive.

Let $z = v + s\phi_k, v \in E^- \oplus E^0, s \in [0, R_1]$, then by (3.59)

$$\begin{aligned} f_k(z) &\leq \frac{1}{2}\lambda_k s^2 - \int_0^k \tilde{H}(z,t)\,dt \\ &\leq \frac{1}{2}\lambda_k s^2 - c_5 \int_0^k \psi(\mid z \mid) \mid z \mid^p + R(1-\psi(\mid z \mid) \mid z \mid^q \; dt \\ &\leq \frac{1}{2}\lambda_k s^2 - c_5 \int_0^k \psi(\mid z \mid) \mid z \mid^p + (1-\psi(\mid z \mid) \mid z \mid^q \; dt. \end{aligned} \tag{3.77}$$

Let $\mathcal{A} = \{t \in [0,k] / \mid z(t) \mid \leq 1\}$ and $\mathcal{B} = \{t \in [0,k] / \mid z(t) \mid > 1\}$. Then since $r < 1$ and $p > q$

$$\int_0^k \psi(\mid z \mid) \mid z \mid^p + (1-\psi(\mid z \mid)) \mid z \mid^q \; dt \geq \int_0^k \chi_{\mathcal{A}} \mid z \mid^p + \chi_{\mathcal{B}} \mid z \mid^q \; dt \tag{3.78}$$

where $\chi_{\mathcal{A}}$ and $\chi_{\mathcal{B}}$ are the characteristic functions of $\mathcal{A}$ and $\mathcal{B}$ respectively. Using the Hölder inequality and $p > q$ we have

$$\begin{aligned} &\int_0^k \chi_{\mathcal{A}} \mid z \mid^p + \chi_{\mathcal{B}} \mid z \mid^q \; dt \geq \\ &\geq k^{\{2-p/2\}} (\int_0^k \mid \chi_{\mathcal{A}} z \mid^2 \; dt)^{p/2} + k^{\{2-q/2\}} (\int_0^k \mid \chi_{\mathcal{B}} z \mid^2 \; dt)^{q/2} \\ &\geq k^{\{2-p/2\}} ((\int_0^k \mid \chi_{\mathcal{A}} z \mid^2 \; dt)^{p/2} + (\int_0^k \mid \chi_{\mathcal{B}} z \mid^2 \; dt)^{q/2}). \end{aligned} \tag{3.79}$$

We will distinguish two cases, in order to obtain (3.76).

Case 1. $\int_0^k \chi_{\mathcal{A}} \mid z \mid^2 \; dt \geq \int_0^k \chi_{\mathcal{B}} \mid z \mid^2 \; dt$. In this situation we clearly have

$$\int_0^k \chi_{\mathcal{A}} \mid z \mid^p + \chi_{\mathcal{B}} \mid z \mid^q \; dt \geq \frac{k^{\{2-p/2\}}}{2^{p/2}} \mid z \mid_2^p . \tag{3.80}$$

Then from (3.77), (3.78), (3.79) and (3.80) we obtain

$$f_k(z) \leq \frac{1}{2}\lambda_k s^2 - c_{12} k^{\{2-p/2\}} \mid z \mid_2^p . \tag{3.81}$$

using (3.73), and recalling that $c_10$ does not depend on $k$, we have

$$f_k(z) \leq \lambda_k s^2 - c_{13} k^{\{2-p/2\}} s^p.$$

Finally since $\lambda_k = O(1/k)$ we conclude that

$$f_k(z) \leq c_{13}(c_{14}\frac{s^2}{k} - \frac{s^p}{k^{\{p-2/2\}}}). \tag{3.82}$$

Maximizing the right hand side of (3.82), we obtain the existence of a constant $c_{15}$ such that

$$f_k(z) \leq \frac{c_{14}}{k^{\{2/p-2\}}}. \tag{3.83}$$

Case 2. $\int_0^k \chi_{\mathcal{A}} \mid z \mid^2 \, dt < \int_0^k \chi_{\mathcal{B}} \mid z \mid^2 \, dt$. By the same kind of analysis as Case 1, we obtain that there exist a constant $c_{16}$ such that

$$f_k(z) \leq \frac{c_{16}}{k^{\{q-p+2/q-2\}}}. \tag{3.84}$$

Putting together (3.83) and (3.84) we obtain (3.76).

The final step in the proof is to show that the sequence of solutions we have converges to zero. We show that $z_k \to 0$ uniformly in $C^1(\mathbb{R}, \mathbb{R}^{2n})$. This will finish the proof because for $\mid z \mid < r/3$ the higher order terms $\tilde{H}$ and $\hat{H}$ coincide. To begin since $z_k$ solves (3.62) we see

$$-Q(z_k) + \frac{1}{2}\int_0^k (\tilde{H}_z(z_k, t), z_k)\, dt = 0. \tag{3.85}$$

Then, from the definition of $f_k$, (3.85), and using (3.58) and (3.59) there is a constant $c_{17}$ such that

$$\begin{aligned} f_k(z_k) &\geq (\frac{1}{2} - \frac{1}{q}) \int_0^k (\tilde{H}_z(z_k, t), z_k)\, dt \\ &\geq q(\frac{1}{2} - \frac{1}{q}) \int_0^k \tilde{H}(z_k, t))\, dt \\ &\geq c_{17} \int_0^k \psi(\mid z_k \mid) \mid z_k \mid^p + R(1 - \psi(\mid z_k \mid)) \mid z_k \mid^q \, dt. \end{aligned} \tag{3.86}$$

Thus from (3.76) and (3.86) we see that

$$\lim_{k \to \infty} \int_0^k \psi(\mid z_k \mid) \mid z_k \mid^p + R(1 - \psi(\mid z_k \mid)) \mid z_k \mid^q \, dt = 0. \tag{3.87}$$

Let us show next that $\mid z_k \mid_\infty$ is bounded, where $\mid \cdot \mid_\infty$ denotes the norm in $L^\infty(\mathbb{R}, \mathbb{R}^{2n})$. Let $I_k$ be an interval contained in $[0, k]$ such that for any $t \in I_k = [\gamma_1, \gamma_2]$

$$\mid z_k(t) \mid \geq M, \quad \text{and} \quad \mid z_k(\gamma_1) \mid = M. \tag{3.88}$$

For those $k$ such that no such an interval exists we have $| z_k |_\infty \leq M$. For any $k$ large enough and such that $I_k$ is not empty we have by (3.87) that $[0,k] \setminus I_k \neq \phi$. Since $z_k$ solves (3.62), for any $t \in I_k$ we have

$$\begin{aligned} | z_k(t) | &\leq | z_k(\gamma_1) | + \int_{\gamma_1}^{t} | \dot{z}_k(s) | \, ds \\ &\leq M + \int_{\gamma_1}^{t} | A(s) || z_k(s) | \, ds + \int_{\gamma_1}^{t} | \tilde{H}_z(z_k, s) | \, ds \\ &\leq M + c_{18}( \int_{\gamma_1}^{t} | z_k(s) | + | \tilde{H}_z(z_k, s) | \, ds \end{aligned} \tag{3.89}$$

Using (3.60) and (3.89)

$$\begin{aligned} | z_k(t) | &\leq M + c_{18} \int_{\gamma_1}^{t} | z_k(s) |^q \, ds \\ &\leq M + c_{19} \int_0^k \psi(| z_k |) | z_k |^p + R(1 - \psi(| z_k |) | z_k |^q \, ds \end{aligned}$$

Then from (3.87) it follows that $| z_k |_\infty$ is bounded. Since $z_k$ solves (3.62) we conclude that $| \dot{z}_k |_\infty$ is bounded. We finally prove that this together with (3.87) implies that actually $| z_k |_\infty \to 0$. Let $0 < s < r/3$, and $m_k(s)$ denotes the measure of the set $\{t \in [0,k] / | z_k(t) | \geq s\}$, then from (3.86) we have

$$\begin{aligned} f_k(z_k) &\geq c_{17} \int_0^k \psi(| z_k |) | z_k |^p + R(1 - \psi(| z_k |)) | z_k |^q \, dt \\ &\geq c_{17} \mathrm{Min}_{|\xi|=s} \{ \psi(| \xi |) | \xi |^p + R(1 - \psi(| \xi |)) | \xi |^q \} m_k(s) \\ &\geq c_{17} | s |^p \, m_k(s). \end{aligned} \tag{3.90}$$

From (3.76) and (3.90) we see that $m_k(s) \to 0$ as $k \to \infty$. Let $0 < \epsilon < r/3$, $k \in \mathbf{N}$ and $t \in \mathbf{R}$, then either

$$| z_k(t) | \leq \epsilon \tag{3.91}$$

or there is $\sigma \in \mathbf{R}$ so that

$$| z_k(\sigma) | \leq \epsilon \quad \text{and} \quad | t - \sigma | \leq m_k(\epsilon). \tag{3.92}$$

Then, if $K$ is such that $| \dot{z}_k |_\infty \leq K$, from (3.92)

$$\begin{aligned} | z_k(t) | &\leq | z_k(\sigma) | + \int_\sigma^t | \dot{z}_k | \, ds \\ &\leq \epsilon + K m_k(\epsilon), \end{aligned}$$

from where $\limsup_{k\to\infty} \mid z_k \mid_\infty \leq \epsilon$ and since $\epsilon$ is arbitrary we have $\mid z_k \mid_\infty \to 0$, and then since $z_k$ solves (3.62) also $\mid \dot{z}_k \mid_\infty \to 0$ as $k \to \infty$.◇

**Remark 3.4.** If the linear system has one Floquet exponent of indefinite type, or it has one Floquet exponent of positive type and one of negative type then the system

$$\dot{z} = JA(t)z + J\hat{H}_z(z,t)$$

admits subharmonics in case $\hat{H}$ satisfies (H4) or (H4-). This is not true in general for system with only positive or only negative Floquet exponents as we will se in an example in the next Section.

### §3.3 Benci-Fortunato's condition and a counterexample.

In this section we will present a counterexample to show that hypothesis (H3) is necessary. We also show that (H3'), the related hypothesis assumed in [2], is stronger than ours. In the process we obtain a result on perturbation of definite matrices (positive or negative).

Let us start with the counterexample. Let $D$ be the negative block of dimension $4 \times 4$ given by

$$D = \begin{pmatrix} 0 & 0 & 0 & -\nu \\ 0 & 0 & -\nu & -1 \\ 1 & \nu & 0 & 0 \\ \nu & 0 & 0 & 0 \end{pmatrix}, \tag{3.93}$$

where $\nu > 0$; we note that $D$ is a block of C3 type with $\sigma = -1$. Consider the Hamiltonian

$$\hat{H}(z,t) = f(t)(z_1^4 + z_2^4 + z_3^4 + z_4^4) \tag{3.94}$$

with $f(t) \geq \alpha > 0$, a continuous 1-periodic function and $z = (z_1, z_2, z_3, z_4)$. Let us assume $\{z^k\}$ is a sequence of solutions of the system

$$\dot{z} = Dz + JH_z(z,t), \tag{3.95}$$

$$z(0) = z(k) \tag{3.96}$$

and $z^k \to 0$ uniformly in $C^1(\mathbf{R}, \mathbf{R}^{2n})$ as $k \to \infty$. We will show that $z^k \equiv 0, \quad \forall k \geq N$ where $N$ is a certain number in $\mathbf{N}$. Since $\hat{H}$ certainly satisfies (H1), (H2), and (H4), this example shows that (H3) is necessary.

Expressing equation (3.95) in component form yields:

$$\dot{z}_1 = -\nu z_4 + \frac{\partial \hat{H}}{\partial z_3} \tag{3.97}$$

$$\dot{z}_2 = -\nu z_3 - z_4 + \frac{\partial \hat{H}}{\partial z_4} \tag{3.98}$$

$$\dot{z}_3 = \nu z_2 + z_1 - \frac{\partial \hat{H}}{\partial z_1} \tag{3.99}$$

$$\dot{z}_4 = \nu z_1 - \frac{\partial \hat{H}}{\partial z_2}. \tag{3.100}$$

Let $z = z_k$ be a solution of (3.95) and (3.96). Multiplying (3.97) by $z_3$ and (3.99) by $z_1$, integrating between 0 and $k$, and using (3.96) we obtain

$$\int_0^k (\nu z_4 z_3 - \frac{\partial \hat{H}}{\partial z_3} z_3)\, dt = \int_0^k (\nu z_2 z_1 + z_1^2 - \frac{\partial \hat{H}}{\partial z_1} z_1)\, dt. \tag{3.101}$$

Multiplying (3.98) by $z_4$, (3.100) by $z_2$, integrating between 0 and $k$ and using (3.96) we obtain

$$\int_0^k (\nu z_3 z_4 + z_4^2 - \frac{\partial \hat{H}}{\partial z_4} z_4)\, dt = \int_0^k (\nu z_1 z_2 - \frac{\partial \hat{H}}{\partial z_2} z_2)\, dt. \tag{3.102}$$

Subtracting (3.101) from (3.102)

$$\int_0^k z_4^2 - \frac{\partial \hat{H}}{\partial z_4} z_4 + z_1^2 - \frac{\partial \hat{H}}{\partial z_1} z_1\, dt = - \int_0^k \frac{\partial \hat{H}}{\partial z_2} z_2 + \frac{\partial \hat{H}}{\partial z_3} z_3\, dt,$$

by (3.94) this implies

$$\int_0^k z_4^2 + z_1^2\, dt \le \int_0^k 4f(t)(z_4^4 + z_1^4)\, dt \le c \int_0^k (z_4^2 + z_1^2)^2\, dt$$

where $c = 4\,\mathrm{Sup} f(t)$. If $z_1 \neq 0$ or $z_4 \neq 0$ then there exist $\tau \in [0, k]$ so that

$$| z_1^2(\tau) + z_4^2(\tau) | \ge \frac{1}{c}.$$

Since c is independent of $k$ we obtain that $z_1 \equiv 0$ and $z_4 \equiv 0$ for $k \ge N$, where $N$ is such that $| z^k(t) | \le 1/\sqrt{c} \quad \forall k \ge N \quad \forall t \in \mathbf{R}$. Since $z_1$ and $z_4$ are zero, equations (3.97) and (3.100) imply that also $z_2$ and $z_3$ are zero. Thus $z^k \equiv 0$ for all $k \ge N$.

**Remark 3.5** In [2] a counterexample is given to show that (H3'), the hypothesis related to (H3), is necessary. As we show later this condition is stronger than ours, so that the same counterexample shows the necessity of (H3). However the counterexample given here is finer because the linear term has all Floquet exponents on the imaginary axis.

Let us consider now a block $D$ of the canonical form of a Hamiltonian matrix $C$. Then we have the following result.

**Lemma 3.6.**

*If any Hamiltonian perturbation of D has at least one eigenvalue on the imaginary axis, then D is of indefinite type.*

**Proof.** Consider the eigenvalue problem

$$\dot{x} = Dx + \lambda Jx, \tag{3.103}$$

$$x(0) = x(k), \tag{3.104}$$

Since $D$ satisfies the hypothesis of Lemma 2.1 in [2], there is an $N$ so that $\forall\, k \geq N$, (3.103) and (3.104) has a solution $(x^k, \lambda_k)$ with

$$\frac{\pi}{k} \leq \lambda_k \leq \frac{3\pi}{k}. \tag{3.105}$$

Let $v = x^k(0)$. Then by (3.104),

$$v = \exp((D + \lambda_k J)k)\, v. \tag{3.106}$$

The block $D$ certainly can not be of C1 or C2 type since in this case the eigenvalue corresponding to the block is outside the imaginary axis, a small perturbation of D will have all its eigenvalues outside the imaginary axis too. Let us assume that D is of C3 type. Then we can write the matrix $D$ as

$$D = -\sigma\beta V - \sigma W$$

If we choose $\alpha \in [0, 2\pi]$ so that $\exp(-i\alpha) = \exp(-i\sigma\beta k)$, then from (3.106) and the properties of $V$ we have

$$v = \exp(-\alpha V - \sigma k W + \lambda_k k J)\, v. \tag{3.107}$$

By Lemma A.4, taking the logarithm of the matrix in (3.107) gives

$$(-(\alpha + m_k)V - \sigma k W + \lambda_k k J)\, v = 0 \tag{3.108}$$

where $m_k \in \mathbb{Z}$. Dividing (3.108) by $k$, we obtain that $\lambda_k$ is an eigenvalue of the problem

$$((\frac{-\alpha - m_k}{k})V - \sigma W + \lambda_k J)v = 0. \tag{3.109}$$

Now (3.109) has the form of the eigenvalue problem (2.2):

$$-(\epsilon V + \sigma W)v + \lambda J v = 0. \tag{3.110}$$

As we mentioned in section §2, the eigenvalues of (3.110) are analytic near $\epsilon = 0$. Thus they have the form

$$\lambda_j(\epsilon) = \lambda_j^0 + \lambda_j^1 \epsilon + \lambda_j^2 \epsilon^2 + \dots. \tag{3.111}$$

Noting that the matrix $(\epsilon I + \sigma V^{-1}W)$ has only large eigenvalues for $\epsilon$ large, (3.109) implies $\mu_k = \{-\alpha - m_k\}/k$ is bounded. Suppose that $\mu_k$ is bounded away from zero. Since $\epsilon I + \sigma V^{-1}W$ is invertible $\forall \epsilon \neq 0$ , for $\epsilon \geq \epsilon_0$ its eigenvalues will be bounded away from zero contradicting (3.105). By repeating this argument for any subsequence of $\{\mu_k\}$ we can conclude that $\mu_k \to 0$ as $k \to 0$. Then for some $j$, a subsequence of $\{\mu_k\}$, and $k$ large enough

$$\lambda_k = \lambda_j(\mu_k) \tag{3.112}$$

If $\lambda_j^s$ is the first nonzero coefficient in (3.111), then (3.105) implies

$$\lambda_j^s \mu_k^s > 0$$

and then the block $D$ can not be negative.

If we look at the proof of Lemma 2.1 in [2] we realize that there is an eigenvalue $\nu_k$ of (3.103) and (3.104) such that

$$-\frac{3\pi}{k} \leq \nu_k \leq -\frac{\pi}{k}.$$

Then we can repeat the argument given above to obtain that $D$ is not positive. This finishes the proof for a block of C3 type. For a block of C5 type we would proceed in the same way. We would actually prove that this implies a contradiction because a C5 block is either positive or negative. ◇

**Remark 3.6** As a consequence if $D$ is a positive (or negative) block then there are arbitrarily small Hamiltonian perturbations of $D$ not having any eigenvalue on the imaginary axis. This is an interesting result we have proved in a very indirect way.

We now compare hypothesis (H3') of Benci and Fortunato with our hypothesis (H3).

(H3') Every Hamiltonian perturbation of C has at least one eigenvalue on the imaginary axis.

**Proposition 3.2**

*If a Hamiltonian matrix C satisfies (H3') then C has at least one purely imaginary eigenvalue that is not of positive type, and at least one that is not of negative type.*

**Proof.** We prove it by contradiction. If $C$ does not have eigenvalues on the imaginary axis then clearly (H3') is false. Let assume now that all eigenvalues of $C$ on the imaginary axis are of positive type. Then by definition, all blocks of $C$ either correspond to eigenvalues outside the imaginary axis or they are positive. Lemma 3.6 applied to each of the latter gives a perturbation that takes all eigenvalues outside the imaginary axis, i.e. (H3') is false. We could do the same argument if all eigenvalues on the imaginary axis are of negative type. ◇

**Remark 3.7.** This proposition shows that our hypothesis (H3) is weaker than (H3').

## §4. REDUCTION TO THE CENTER MANIFOLD AND PROOF OF THEOREM II.

In Section §3 we proved Theorem I which guarantees the existence of nontrivial subharmonics converging to zero for the Hamiltonian system (HS), under hypotheses (H0), (H1), (H2), (H3) and (H4).

In this section we will prove a similar theorem but we will impose the superquadratic condition (H4) for $\hat{H}$ only on the center manifold.

In order to obtain our result we will need some extra differentiability on the Hamiltonian $H$:

(H5) $H$ is of class $C^3$ near $z = 0$.

Under hypotheses (H1), (H2) and (H5) we will reduce system (HS) to a Hamiltonian system

$$\dot{w} = JB(t)w + J\hat{K}_w(w,t) \tag{4.1}$$

on the center manifold.

We will replace (H4) by:

(H6) There are constants $c > 0, r > 0, q + 2 > p > q > 2$ such that

$$(\hat{K}_w(w,t), w) \geq q\hat{K}(w,t) \geq c \mid w \mid^q \tag{4.2}$$

$$\forall w \in \mathbf{R}^{2m}, \quad \mid w \mid \leq r, \quad t \in \mathbf{R}.$$

Here m is the dimension of the center manifold. A direct application of Theorem I leads to

**Theorem II.**

*If* (H5), (H1), (H2), (H3) *and* (H6) *hold, then there exists an integer $N$ so that for every $k \geq N$, there exists a nontrivial solution $z_k$ of* (HS) *with period $kT$ and $\{z_k\}_{k=1}^{\infty}$ converges to zero as $k \to \infty$ uniformly in $C^1(\mathbf{R}, \mathbf{R}^{2n})$.*

The reduction to the center manifold allows us to prove the theorem under condition (H6) that in principle is weaker than (H4).

As we will see later in this section the reduced Hamiltonian $\hat{K}$ is not known explicitly, so that it is interesting to have a condition on the original Hamiltonian $\hat{H}$ that implies (H6). If we assume some hypothesis on the behaviour of the second derivatives of $\hat{H}$ near 0 then we can prove that (H4) imply (H6). See Proposition 4.3. Actually we only need to impose (H4) on the center manifold. We will be more precise about this point in §4.4 where the adequate notation will be available.

### §4.1. Reduction to a system with a constant linear term.

With hypotheses (H5), (H1) and (H2) we can write the Hamiltonian system (0.1) as:

$$\dot{z} = JA(t)z + J\hat{H}_z(z,t) \tag{4.3}$$

where $\hat{H}(z,t) = o(|z|^2)$. System (4.3) is refered as (HS). We have the analog of Lemma 3.1. Using the same notation considered there, we have

**Lemma 4.1.**

*There are matrices $\bar{C}_r$, $\bar{R}$ and $\bar{P}(t)$ so that $\bar{C}_r$ is Hamiltonian, $\bar{R}^2 = I$, and $\bar{P}(t)$ is symplectic satisfying $\bar{P}(t+1) = \bar{P}(t)\bar{R}$, for which we have:*

*$z(t)$ is a $k$-periodic solution of*

$$\dot{z} = JA(t)z + J\hat{H}_z(z,t)$$

*if and only if $\xi(t) = \bar{P}(t)^{-1}z(t)$ is an $\bar{R}k$-periodic solution of*

$$\dot{\xi} = \bar{C}_r\xi + J\hat{H}_{1\xi}(\xi,t) \tag{4.4}$$

*where*

$$\hat{H}_1(\xi,t) = \hat{H}(\bar{P}(t)z,t). \tag{4.5}$$

**Proof.** Can take $\bar{C}_r = C_r$, $\bar{R} = R$ and $\bar{P}(t) = P(t)$ as in Lemma 3.1, and do the proof similarly.◇

**Remark 4.1.** The new Hamiltonian is not periodic in general, but it satisfies

$$\hat{H}_1(\xi,t+1) = \hat{H}_1(\bar{R}\xi,t) \tag{4.6}$$

$\forall t \in \mathbf{R}$, $\forall \xi \in \mathbf{R}^{2n}$. We will say that $\hat{H}_1$ is $\bar{R}$-periodic.

**Remark 4.2.** In Lemma 4.1 the matrices $\bar{C}_r$, $\bar{R}$ and $\bar{P}$ are not unique, they can be transformed by symplectic transformations. They can be chosen so to fit with the setting of the Center Manifold Theorem presented in Appendix C.

By definition of $C_r$ and $C_i$ we have a symplectic transformation $Q$ so that

$$Q^{-1}C_rQ = S$$

and

$$Q^{-1}C_iQ = \pi\bar{I}$$

see (2.33) and before for the definition of $S$ and $\bar{I}$. The matrix $S$ is a block matrix

$$S = \begin{pmatrix} \bar{S}_p & 0 \\ 0 & S_q \end{pmatrix}$$

where $\bar{S}_p$ and $S_q$ are Hamiltonian matrices. Let $Q_p$ and $Q_q$ be symplectic matrices transforming $\bar{S}_p$ and $S_q$ to their corresponding canonical forms (See Theorem 1.1). Then if

$$Q_1 = \begin{pmatrix} Q_p & 0 \\ 0 & Q_q \end{pmatrix}$$

and

$$Q_2 = QQ_1$$

the matrix

$$\bar{C}_r = Q_2^{-1} C_r Q_2 \tag{4.7}$$

is the canonical form of $C_r$, and due to the block structure of $S$ and $\bar{I}$

$$\pi \bar{I} = Q_2^{-1} C_i Q_2. \tag{4.8}$$

We define

$$\bar{R} = e^{i\pi\bar{I}} \tag{4.9}$$

and

$$\bar{P}(t) = Q_2^{-1} P(t) Q_2. \tag{4.10}$$

For the matrices $\bar{C}_r$, $\bar{R}$ and $\bar{P}(t)$ Lemma 4.1 holds. By the block structure of the canonical form matrix, we can reorder $\bar{C}_r$ so that

$$\bar{C}_r = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

and

$$\bar{R} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

with $C_1$ having eigenvalues only on the imaginary axis, $C_2$ having eigenvalues only outside the imaginary axis, and

$$R_1 = \begin{pmatrix} -I_{n_1} & 0 \\ 0 & I_{n_1'} \end{pmatrix}.$$

Moreover $C_2$ and $R_2$ have the block structure

$$C_2 = \begin{pmatrix} C_{21} & 0 \\ 0 & C_{22} \end{pmatrix}$$

and

$$R_2 = \begin{pmatrix} R_{21} & 0 \\ 0 & R_{22} \end{pmatrix}$$

$C_{21}$ has eigenvalues with positive real part, $C_{22}$ has eigenvalues with negative real part and the matrices $R_{21}$ and $R_{22}$ have the form

$$R_{2i} = \begin{pmatrix} -I_{n_{2i}} & 0 \\ 0 & I_{n_{2i}'} \end{pmatrix}$$

for $i = 2, 3$. The following commutativity property is consequence of the block structure and the special form of $R$:

$$C_{21}R_{21} = R_{21}C_{21}, \quad C_{22}R_{22} = R_{22}C_{22}, \quad \text{and} \quad C_1R_1 = R_1C_1. \tag{4.11}$$

Remark 4.1 holds with this new matrices.

**Remark 4.3.** In the canonical form we considered before we can take the 'off diagonal' elements to be any number $\gamma \neq 0$. See [4].

We can define symmetric matrices $A_1$ and $A_2$ so that

$$J_jA_j = C_j \quad j = 1, 2. \tag{4.12}$$

Then, after the transformation we can write (4.3) as

$$\dot{\xi}_1 = J_1A_1\xi_1 + \hat{H}_{1\xi_1}(\xi_1, \xi_2, t) \tag{4.13}$$

$$\dot{\xi}_2 = J_2A_2\xi_2 + \hat{H}_{1\xi_2}(\xi_1, \xi_2, t). \tag{4.14}$$

In Section §4.2 we drop the index 1 in $H$ and the bar in $R$.

## §4.2 Reduction of (4.13) and (4.14) to the center manifold.

We want to reduce (4.13) and (4.14) to the center manifold. Because this system is not periodic but $R$-periodic we will need a special version of the Center Manifold Theorem. In Appendix C we provide the changes necessary to modify the original existence proof in order to suit our needs. Our basic references for this theorem are the original paper by Kelley [9] and the book by Carr [5].

**Theorem 4.2. (Center Manifold Theorem)**
*If $H$ satisfies (H5) then there exists a $C^2$ function*

$$h : N \times \mathbf{R} \longrightarrow \mathbf{R}^{2n_2}$$

*where $N$ is a neighborhood of the origin in $\mathbf{R}^{2n_1}$, such that*

$$h(R_1\xi_1, t + 1) = R_2h(\xi_1, t) \tag{4.15}$$

$$h(0, t) = 0, \quad h'(0, t) = 0 \tag{4.16}$$

*and the manifold*

$$M = \{(\xi_1, h(\xi_1, t), t)/ \quad \xi_1 \in N, t \in \mathbf{R}\}$$

*is an invariant manifold for (4.13) and (4.14). $M$ is called the center manifold.*

**Proof.** See Appendix C.◇

System (4.13) and (4.14) can then be reduced to

$$\dot{\xi}_1 = J_1 A_1 \xi_1 + J_1 \hat{H}_{\xi_1}(\xi_1, h(\xi_1, t), t) \tag{4.17}$$

or equivalently

$$\dot{\xi}_1 = J_1 H_{\xi_1}(\xi_1, h(\xi_1, t), t) \tag{4.18}$$

where $H$ is given by

$$H(\xi_1, \xi_2, t) = \frac{1}{2}(A_1\xi_1, \xi_1) + \frac{1}{2}(A_2\xi_2, \xi_2) + \hat{H}(\xi_1, \xi_2, t).$$

System (4.18) is a Hamiltonian system if it is interpreted properly; namely we have to consider in $\mathbf{R}^{2n_1}$ a nonconstant symplectic form. In order to apply the results of Section §3, particularly Theorem I, we need to transform (4.18) to put it as a Hamiltonian system in $\mathbf{R}^{2n_1}$ with the standard symplectic form. Let us define the reduced Hamiltonian

$$H_2(\xi_1, t) = H(\xi_1, h(\xi_1, t), t). \tag{4.19}$$

Using (4.6), (4.11) and (4.15) we see that

$$\begin{aligned} H_2(\xi_1, t+1) &= \frac{1}{2}(A_1\xi_1, \xi_1) + \frac{1}{2}(A_2 h(\xi_1, t+1), h(\xi_1, t+1)) \\ &\quad + \hat{H}(\xi_1, h(\xi_1, t+1), t+1) \\ &= \frac{1}{2}(A_1 R_1\xi_1, R_1\xi_1) + \frac{1}{2}(A_2 h(R_1\xi_1, t), h(R_1\xi_1, t)) \\ &\quad + \hat{H}(R_1\xi_1, h(R_1\xi_1, t), t) \\ &= H_2(R_1\xi_1, t). \end{aligned}$$

in other words we have proved that

$$H_2(\xi_1, t+1) = H_2(R_1\xi_1, t) \quad \forall t \in \mathbf{R}, \quad \xi_1 \in N. \tag{4.20}$$

The derivative of $H_2$ is given by

$$H_{2\xi_1}(\xi_1, t) = H_{\xi_1}(\xi_1, h(\xi_1, t), t) + (h'(\xi_1, t))^T H_{\xi_2}(\xi_1, h(\xi_1, t), t) \tag{4.21}$$

where $'$ denotes derivarive with respect to $\xi_1$.

On the other hand, $M$ being an invariant manifold, the right hand side of (4.13) and (4.14) should be tangent to $M$, that is

$$\begin{pmatrix} J_1 H_{\xi_1}(\xi_1, h(\xi_1, t), t) \\ J_2 H_{\xi_2}(\xi_1, h(\xi_1, t), t) \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \dot{h}(\xi_1, t)\mu + h'(\xi_1, t)\lambda \\ \mu \end{pmatrix} \tag{4.22}$$

where $\cdot$ denotes derivative with respect to $t$, $\lambda \in \mathbf{R}^{2n_1}$ and $\mu \in \mathbf{R}$. From (4.22) we obtain that

$$J_2 H_{\xi_2}(\xi_1, h(\xi_1,t), t) = h'(\xi_1,t) J_1 H_{\xi_1}(\xi_1, h(\xi_1,t), t) + \dot{h}(\xi_1,t). \tag{4.23}$$

Inserting (4.23) in (4.21) we obtain

$$H_{2\xi_1}(\xi_1,t) = (I - h'^{\tau} J_2 h' J_1) H_{\xi_1}(\xi_1, h(\xi_1,t), t) - h'^{\tau} J_2 \dot{h}, \tag{4.24}$$

and then system (4.18) can be written as

$$-(J_1 + h'^{\tau} J_2 h')\dot{\xi}_1 = H_{2\xi_1}(\xi_1,t) + h'^{\tau} J_2 \dot{h}. \tag{4.25}$$

This system is Hamiltonian if we consider $\mathbf{R}^{2n_1}$ with the symplectic form given by $J_1 + h'^{\tau} J_2 h'$. We will perform another transformation so that (4.25) is a Hamiltonian system in $\mathbf{R}^{2n_1}$ with the standard symplectic form.

Before doing so let us collect some properties of the terms involved in (4.25). Let us define

$$\omega_1(\xi_1,t) = h'^{\tau}(\xi_1,t) J_2 h'(\xi_1,t) \quad \text{and} \tag{4.26}$$

$$\omega_2(\xi_1,t) = h'^{\tau}(\xi_1,t) J_2 \dot{h}(\xi_1,t). \tag{4.27}$$

From (4.15) we see that

$$h(\xi_1, t+1) = R_2 h(R_1\xi_1, t) \tag{4.28}$$

and then

$$h'(\xi_1, t+1) = R_2 h'(R_1\xi_1, t) R_1. \tag{4.29}$$

Using the simple form of $R_1$, (4.26), (4.27) and (4.29) we obtain

$$\omega_1(\xi_1, t+1) = R_1 \omega_1(R_1\xi_1, t) R_1 \quad \text{and} \tag{4.30}$$

$$\omega_2(\xi_1, t+1) = R_1 \omega_2(R_1\xi_1, t). \tag{4.31}$$

For notational convenience in the next section we will drop the index 1 from $J_1, \xi_1, R_1$ and $n_1$, and the index 2 from $H_2$.

## §4.3 Transformation of (4.25) via the Darboux Theorem.

In this section we will find a transformation that will change (4.25) into a Hamiltonian system with constant symplectic form. Before we go to the construction let us assume we have

$$\begin{aligned} F :& V \times \mathbf{R} \longrightarrow V' \times \mathbf{R} \\ & (w,t) \longmapsto (\xi, \theta) \end{aligned}$$

$$\begin{aligned} g :& V \times \mathbf{R} \longrightarrow \mathbf{R} \\ & (w,t) \longmapsto g(w,t), \end{aligned}$$

where F is a diffeomorphism of class $C^1$, g is a function of class $C^1$, and $V$ and $V'$ are neighborhoods of the origin. Let us assume $F$ and $g$ satisfy:

(F0) $F(w,t) = (G(w,t),t)$

(F1) $G(0,t) = 0$, $G'(0,t) = 0$, $g(0,t) = 0$ and $g_w(0,t) = 0$,

(F2) $G(w,t+1) = RG(Rw,t)$ and $g(w,t+1) = g(Rw,t)$,

(F3) $G'^{\tau}(w,t)(J + \omega_1(F(w,t)))G'(w,t) = J$ and

$G'^{\tau}(w,t)(J + \omega_1(F(w,t)))\dot{G}(w,t) + G'^{\tau}(w,t)\omega_2(F(w,t)) = g_w(w,t)$

where here and in what follows $'$ and $\cdot$ denote derivatives with respect to $w$ and $t$ respectively.

**Lemma 4.2.**

*Let $F$ and $g$ as above. Then $\xi(t)$ is an $Rk$-periodic solution of (4.25) if and only if $w(t)$, defined implicitly by*

$$F(w(t),t) = (\xi(t),t), \tag{4.32}$$

*is an $Rk$-periodic solution of*

$$\dot{w} = JH_{3w}(w,t) \tag{4.33}$$

*where*

$$H_3(w,t) = H(F(w,t)) + g(w,t). \tag{4.34}$$

**Proof.** Since $\xi(t)$ is $Rk$-periodic it satisfies $\xi(t+k) = R^k\xi(t)$. Then from (F2)

$$R^k\xi(t) = G(w(t+k),t+k) = R^kG(R^kw(t+k),t)$$

so that

$$\xi(t) = G(R^kw(t+k),t), \tag{4.35}$$

and this implies $w(t+k) = R^kw(t)$.

Let us assume now that $\xi(t)$ is a solution of (4.25). Then differentiating (4.32) and multiplying by $-(J+\omega_1)$ we obtain

$$-(J+\omega_1)\dot{\xi} = -(J+\omega_1)G'(w,t)\dot{w} - (J+\omega_1)\dot{G}(w,t). \tag{4.36}$$

From (4.34)

$$H_{3w}(w,t) = G'^{\tau}H_{\xi}(\xi,t) + g_w(w,t). \tag{4.37}$$

Multiplying (4.36) by $G'^{\tau}$, using (4.25) and (4.37) we obtain

$$-G'^{\tau}(J+\omega_1)G'\dot{w} = H_{3w}(w,t) + G'^{\tau}\omega_2 + G'^{\tau}(J+\omega_1)\dot{G} - g_w(w,t). \tag{4.38}$$

From (F3) we see that (4.38) is exactly

$$-J\dot{w} = H_{3w}(w,t)$$

completing the proof.◇

**Remark 4.4.** From (4.20), (F2) and (4.34) we see that

$$H_3(w,t+1) = H_3(Rw,t), \tag{4.39}$$

and from (F1) we have $H_3(0,t) = 0$ and $H_{3w}(0,t) = 0$.

It remains to prove the existence of the diffeomorphism $F$ and the function $g$. This is done via the Darboux Theorem. We present here an elementary proof of the Darboux Theorem. For a proof using the language of differential forms see, for example, [1].

We will introduce an extra variable and find a function

$$\begin{aligned} \mathcal{F} :& V \times \mathbf{R} \times \mathbf{R} \longrightarrow V' \times \mathbf{R} \times \mathbf{R} \\ & (w,t,u) \longmapsto \mathcal{F}(w,t,u) = (\xi,\theta,\mu). \end{aligned}$$

Taking appropriate projections of $\mathcal{F}$ we will define $F$ and $g$. We call $z = (w,t,u)$ and $\eta = (\xi,\theta,\mu)$.

Let us consider the matrices

$$\Omega_0 = \begin{pmatrix} J & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \tag{4.40}$$

$$\Omega(\eta) = \begin{pmatrix} \omega_1(\xi,\theta) & \omega_2(\xi,\theta) & 0 \\ -\omega_2^T(\xi,\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.41}$$

If we supplement $R$ with two extra entries

$$\tilde{R} = \begin{pmatrix} R & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then from (4.30) and (4.31) we have

$$\Omega(\xi,\theta+1,\mu) = \tilde{R}\Omega(R\xi,\theta,\mu)\tilde{R}. \tag{4.42}$$

**Lemma 4.3.**

$$\frac{\partial \Omega_{ij}}{\partial \eta_k} - \frac{\partial \Omega_{kj}}{\partial \eta_i} + \frac{\partial \Omega_{ki}}{\partial \eta_j} = 0, \quad 1 \le i,j,k \le 2n+2 \tag{4.43}$$

**Proof.** By direct calculation.◇

**Remark 4.5** Lemma 4.3 says the the 2-form defined by $\Omega$ is closed.

Because $\Omega$ satisfies (4.43) we have the following lemma.

**Lemma 4.4. (Poincaré Lemma)**
*There is a function* $\alpha : N \times \mathbf{R}^2 \longrightarrow \mathbf{R}^{2n+2}$ *such that*

$$\Omega_{ij}(\eta) = \frac{\partial \alpha_i}{\partial \eta_j}(\eta) - \frac{\partial \alpha_j}{\partial \eta_i}(\eta) \quad 1 \le i, j \le 2n+2. \tag{4.44}$$

*Moreover the function* $\alpha$ *satisfies*

$$\alpha(\xi, \theta + 1, \mu) = \tilde{R}\alpha(\xi, \theta, \mu). \tag{4.45}$$

**Proof.** We define $\alpha$ by the following integrals

$$\alpha_i(\eta) = \int_0^1 \sum_{k=1}^{2n} \Omega_{ik}(\tau\xi, \theta, \mu)\tau\xi_k \, d\tau, \quad 1 \le i \le 2n \quad \text{and} \tag{4.46}$$

$$\alpha_i(\eta) = \int_0^1 \sum_{k=1}^{2n} \Omega_{ik}(\tau\xi, \theta, \mu)\xi_k \, d\tau, \quad i = 2n+1, 2n+2. \tag{4.47}$$

First let us assume $1 \le i, j \le 2n$. Then differentiating (4.46) we have

$$\frac{\partial \alpha_i}{\partial \eta_j}(\eta) = \int_0^1 \Omega_{ij}(\tau\xi, \theta, \mu)\tau + \sum_{k=1}^{2n} \frac{\partial \Omega_{ik}}{\partial \xi_j}(\tau\xi, \theta, \mu)\tau^2\xi_k \, d\tau \tag{4.48}$$

and

$$\frac{\partial \alpha_j}{\partial \eta_i}(\eta) = \int_0^1 \Omega_{ji}(\tau\xi, \theta, \mu)\tau + \sum_{k=1}^{2n} \frac{\partial \Omega_{jk}}{\partial \xi_i}(\tau\xi, \theta, \mu)\tau^2\xi_k \, d\tau. \tag{4.49}$$

Noting that $\Omega_{ij} = -\Omega_{ji}$, subtracting (4.49) from (4.48), and using (4.43) we obtain

$$\begin{aligned} \frac{\partial \alpha_i}{\partial \eta_j}(\eta) - \frac{\partial \alpha_j}{\partial \eta_i}(\eta) &= \int_0^1 2\Omega_{ij}(\tau\xi, \theta, \mu)\tau + \sum_{k=1}^{2n}(\frac{\partial \Omega_{ik}}{\partial \eta_j} - \frac{\partial \Omega_{jk}}{\partial \eta_i})\tau^2\eta_k \, d\tau \\ &= \int_0^1 2\Omega_{ij}(\tau\xi, \theta, \mu)\tau + \sum_{k=1}^{2n}(\frac{\partial \Omega_{ij}}{\partial \eta_k}(\tau\xi, \theta, \mu)\tau^2\eta_k \, d\tau. \end{aligned} \tag{4.50}$$

In this last expression we identify the integrand as

$$\frac{d}{d\tau}\Omega_{ij}(\tau\xi, \theta, \mu)\tau^2, \tag{4.51}$$

so that

$$\frac{\partial \alpha_i}{\partial \eta_j}(\eta) - \frac{\partial \alpha_j}{\partial \eta_i}(\eta) = \Omega_{ij}(\eta).$$

The case $i = 2n+1, j = 2n+2$ is trivial because $\alpha_j = 0$ , $\alpha_i$ does not depend on $u$ and $\Omega_{ij} = 0$. The other cases are obtained in a similar way.

Condition (4.45) is easily obtained by using (4.42) and the definition of $\alpha$. ◇

**Lemma 4.5. (Darboux Theorem)**

*There is a diffeomorphism* $\mathcal{F} : V \times \mathbf{R}^2 \longrightarrow V' \times \mathbf{R}^2$ *such that*

$$\mathcal{F}(z)'^{\tau}(\Omega_0 + \Omega(\mathcal{F}(z)))\mathcal{F}'(z) = \Omega_0. \tag{4.52}$$

**Proof.** The function $\mathcal{F}$ will be defined by a differential equation. For $s \in \mathbf{R}$, let

$$\tilde{\Omega}_s(\eta) = \Omega_0 + s\Omega(\eta). \tag{4.53}$$

Then, since $\Omega_0$ is invertible and $\Omega(0, \theta, \mu) = 0$, there is $\epsilon > 0$ so that $\tilde{\Omega}_s(\eta)$ is invertible for every $\eta \in B_\epsilon(0) \times \mathbf{R}^2$. Then we can define

$$X(\eta, s) = -\tilde{\Omega}_s^{-1}(\eta)\alpha(\eta), \tag{4.54}$$

and consider the differential equation

$$\begin{aligned} \frac{d\eta}{ds} &= X(\eta, s) \\ \eta(0) &= z. \end{aligned} \tag{4.55}$$

Since $\alpha(0, \theta, \mu) = 0 \quad \forall(\theta, \mu) \in \mathbf{R}^2$, equation (4.55) with the initial condition

$$\eta(0) = (0, t, u)$$

has the solution

$$\eta(s) = (0, t, u) \quad \forall s \in \mathbf{R}.$$

Then, by the general theory of ordinary differential equations we can define

$$\mathcal{H} : B_{\epsilon_1}(0) \times \mathbf{R}^2 \times [0, 2] \longrightarrow \mathbf{R}^{2n} \times \mathbf{R}^2$$

where $\mathcal{H}(z, s)$ is the solution of (4.55), and $0 < \epsilon_1 < \epsilon$. $\mathcal{H}(z, 1)$ is a diffeomorphism of class $C^1$, see [6]. We define $\mathcal{F}(z) = \mathcal{H}(z, 1)$. Let us check that the function $\mathcal{F}$ satisfies (4.52). By definition we have

$$\tilde{\Omega}(\eta, s) = \Omega_0 + s\Omega(\eta).$$

Then we have the following derivatives:

$$\frac{\partial\tilde{\Omega}_{ij}}{\partial z_k}(\mathcal{H}(z,s),s) = s\frac{\partial\Omega_{ij}}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_k} \tag{4.56}$$

$$\frac{\partial\tilde{\Omega}_{ij}}{\partial s}(\mathcal{H}(z,s),s) = s\frac{\partial\Omega_{ij}}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial s} + \Omega_{ij}. \tag{4.57}$$

Here, and in what follows we will use Einstein's convention for sums. Let us consider the differential equation (4.55) componentwise:

$$\tilde{\Omega}_{ij}(\mathcal{H}(z,s),s)\frac{d\mathcal{H}_j}{ds}(z,s) = -\alpha_i(\mathcal{H}(z,s))$$

and differentiate it with respect to $z_k$. Using (4.56) we obtain

$$s\frac{\partial\Omega_{ij}}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_k}\frac{d\mathcal{H}_j}{ds}(z,s) + \tilde{\Omega}_{ij}\frac{d}{ds}\frac{\partial\mathcal{H}_j}{\partial z_k}(z,s) = -\frac{\partial\alpha_i}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_k}. \tag{4.58}$$

Our purpose is to show that

$$\mathcal{H}'^{\tau}(z,1)\tilde{\Omega}(\mathcal{H}(z,1),1)\mathcal{H}'(z,1) = \Omega_0. \tag{4.59}$$

Since we obviously have

$$\mathcal{H}'^{\tau}(z,0)\tilde{\Omega}(\mathcal{H}(z,0),0)\mathcal{H}'(z,0) = \Omega_0$$

it is enough to show that

$$\frac{d}{ds}\mathcal{H}'^{\tau}(z,s)\tilde{\Omega}(\mathcal{H}(z,s),s)\mathcal{H}'(z,s) = 0.$$

We do this componentwise. Differentiating and using (4.57) and (4.58)

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{\partial\mathcal{H}_i}{\partial z_m}(z,s)\tilde{\Omega}_{ij}(\mathcal{H}(z,s),s)\frac{\partial\mathcal{H}_j}{\partial z_k}(z,s)\right) = \\
&-(\tilde{\Omega}_{ji}\frac{d}{ds}\frac{\partial\mathcal{H}_i}{\partial z_m})\frac{\partial\mathcal{H}_j}{\partial z_k} + (\tilde{\Omega}_{ij}\frac{d}{ds}\frac{\partial\mathcal{H}_j}{\partial z_k})\frac{\partial\mathcal{H}_i}{\partial z_m} + \frac{\partial\mathcal{H}_i}{\partial z_m}\frac{d\tilde{\Omega}_{ij}}{ds}\frac{\partial\mathcal{H}_j}{\partial z_k} \\
&= \frac{\partial\alpha_j}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial z_k} - \frac{\partial\alpha_i}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_k}\frac{\partial\mathcal{H}_i}{\partial z_m} + s\frac{\partial\Omega_{ji}}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial z_k}\frac{\partial\mathcal{H}_i}{\partial s} \\
&- s\frac{\partial\Omega_{ij}}{\partial\eta_l}\frac{\partial\mathcal{H}_l}{\partial z_k}\frac{\partial\mathcal{H}_i}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial s} + s\frac{\partial\Omega_{ij}}{\partial\eta_l}\frac{\partial\mathcal{H}_i}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial z_k}\frac{\partial\mathcal{H}_l}{\partial s} + \Omega_{ij}\frac{\partial\mathcal{H}_i}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial z_k}.
\end{aligned} \tag{4.60}$$

Changing some of the indices in (4.60) and reordering the terms we obtain

$$\begin{aligned}&\frac{d}{ds}\left(\frac{\partial\mathcal{H}_i}{\partial z_m}(z,s)\tilde{\Omega}_{ij}(\mathcal{H}(z,s),s)\frac{\partial\mathcal{H}_j}{\partial z_k}(z,s)\right) = \\ &= \left(\frac{\partial\alpha_j}{\partial\eta_i}-\frac{\partial\alpha_i}{\partial\eta_j}+\Omega_{ij}\right)\frac{\partial\mathcal{H}_i}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial z_k}+s\left(\frac{\partial\Omega_{ji}}{\partial\eta_l}-\frac{\partial\Omega_{li}}{\partial\eta_j}+\frac{\partial\Omega_{lj}}{\partial\eta_i}\right)\frac{\partial\mathcal{H}_l}{\partial z_m}\frac{\partial\mathcal{H}_j}{\partial z_k}\frac{\partial\mathcal{H}_i}{\partial s}.\end{aligned} \tag{4.61}$$

Now (4.61) is zero by Lemmas 4.3 and 4.4.◇

Now we are prepared to prove the existence of $F$ and $g$.

**Proposition 4.1**
*There is a diffeomorphism $F$ and a function $g$ satisfying (F0), (F1) ,(F2) and (F3).*

**Proof.** Let us consider the diffeomorphism $\mathcal{F}$ defined in Lemma 4.5, and let us write it as

$$\mathcal{F}(z) = (G(z), T(z), u + g(z)). \tag{4.62}$$

We recall that $z = (w,t,u)$. From (4.41), (4.46) and (4.47) we see that $X$ is independent of $\eta_{2n+2} = \mu$, hence $G, T$ and $g$ are functions of $(w,t)$ only. From (4.41) and (4.47) we see that $\alpha_{2n+2} = 0$, and from the block structure of $\Omega_s$ we have that

$$\tilde{\Omega}_s^{-1} = \begin{pmatrix} (J+s\omega_1)^{-1} & 0 & s(J+s\omega_1)^{-1}\omega_2 \\ 0 & 0 & -1 \\ -s\omega_2^{\tau}(J+s\omega_1)^{-1} & 1 & 0 \end{pmatrix} \tag{4.63}$$

therefore $X_{2n+1}(\eta,s) = 0$, and this implies that

$$T(z) = t. \tag{4.64}$$

From (4.42), (4.45) and (4.54) we see that

$$X(\xi,\theta+1,\mu,s) = \tilde{R}X(\tilde{R}(\xi,\theta,\mu),s). \tag{4.65}$$

The function $\mathcal{H}(Rw,t,u,s)$ satisfies equation (4.55) with the initial condition

$$\mathcal{H}(Rw,t,u,0) = (Rw,t,u), \tag{4.66}$$

according to the proof of Lemma 4.4. On the other hand, the function

$$\mathcal{H}_*(w,t,u,s) = (RG(w,t+1,s),t,u+g(w,t+1,s),s) \tag{4.67}$$

satisfies

$$\mathcal{H}_*(w,t,u,0) = (Rw,t,u), \tag{4.68}$$

and using (4.65)

$$\begin{aligned}
\frac{d\mathcal{H}_*}{ds}(w,t,u,s) &= \frac{d}{ds}(RG(w,t+1,s),t+1,u+g(w,t+1,s)) \\
&= \frac{d}{ds}\bar{R}\mathcal{H}(w,t+1,u,s) \\
&= \bar{R}X(G(w,t+1,s),t+1,u+g(w,t+1,s),s) \\
&= X(RG(w,t+1,s,t,u+g(w,t+1,s),s) \\
&= X(\mathcal{H}_*(w,t,u,s),s).
\end{aligned}$$

Then by the uniqueness of solutions of ordinary differential equations

$$\mathcal{H}(Rw,t,u,1) = \mathcal{H}_*(w,t,u,1)$$

and hence

$$G(Rw,t,u) = RG(w,t+1,u) \quad \text{and} \tag{4.69}$$

$$g(Rw,t,u) = g(w,t+1,u). \tag{4.70}$$

Define $F(w,t) = (G(w,t),t)$. Because F is the projection of $\mathcal{F}$, and (4.64) holds together with the fact given after (4.62), it follows that F is a diffeomorphism. It certainly satisfies (F0). By (4.16) and the form of $X$, $F$ and $g$ satisfies (F1). From (4.69) and (4.70) it satisfies (F2). Lastly (F3) is a consequence of Lemma 4.5.◇

## §4.4 Analysis of Hypothesis (H6).

Our reduction process led us to the Hamiltonian system

$$\dot{w} = JH_{3w}(w,t).$$

By Remark 4.4 we have that $H_3$ satisfies (H1) then we can write this system as

$$\dot{w} = JA_1w + J\hat{H}_{3w}(w,t). \tag{4.71}$$

with $\hat{H}_3 = o(\mid w \mid^2)$. A further transformation of (4.71) in order to take it into a periodic system will give

$$\dot{w} = JB(t)w + J\hat{K}_w(w,t), \tag{4.72}$$

and we can apply Theorem I to it if we assume (H6). See Section §4.5.

In this section we will study condition (H6) for $\hat{H}_3$, and we will derive some sufficient conditions for (H6) to hold in terms of the Hamiltonian $\hat{H}$ or rather $\hat{H}_1$ from Section §4.1. For convenience we will use the notation of Section §4.1, and we will refer to $\hat{H}_1$ as the original Hamiltonian. We assume

(H7) There are constants $c > 0, r > 0, q+2 > p > q > 2$ such that

$$(\hat{H}_{1\xi_1}(\xi_1, 0, t), \xi_1) \geq q\hat{H}_1(\xi_1, 0, t) \geq c \mid \xi_1 \mid^p \tag{4.73}$$

$$\forall \xi_1 \in \mathbf{R}^{n_1}, \quad \mid \xi_1 \mid \leq r, \quad \forall t \in \mathbf{R},$$

and a condition on the cross derivatives

(H8) $\hat{H}_{1\xi_1\xi_2}(\xi_1, 0, t) = O(\mid \xi_1 \mid^{p-2})$ and $\hat{H}_{1t\xi_2}(\xi_1, 0, t) = O(\mid \xi_1 \mid^{p-1})$ near $\xi_1 = 0$.

**Remark 4.6.** Clearly (H4) implies (H7).

**Remark 4.7.** Due to the method we are using it seems necessary to have a condition like (H8) in order to deduce (H7) from (H4).

**Remark 4.8.** Under hypothesis (H8) we see from the Center Manifold Theorem (Appendix C) that

$$h'(\xi_1, t) = O(\mid \xi_1 \mid^{p-2}) \quad \text{and} \quad \dot{h}(\xi_1, t) = O(\mid \xi_1 \mid^{p-1}). \tag{4.74}$$

After integrating (4.74) and using that $h(0, t) \equiv 0$ we also have

$$h(\xi_1, t) = O(\mid \xi_1 \mid^{p-1}). \tag{4.75}$$

The following lemma describes the behaviour of $G$ and $g$ near $w = 0$. Let us write $G$ as $G(w, t) = w + G_1(w, t)$ and consider $\beta(p) = \text{Min}\{2p-4, p-1\}$.

**Lemma 4.6.**

*Under hypothesis (H8)*

$$G_1'(w, t) = O(\mid w \mid^{\beta(p)}) \tag{4.76}$$

*and*

$$g_w(w, t) = o(\mid w \mid^{p-1}) \tag{4.77}$$

**Proof.** By the same argument given in Proposition 4.1 we can show that $\mathcal{H}$ has the form

$$\mathcal{H}(w, t, u, s) = (\tilde{G}(w, t, s), t, u + \tilde{g}(w, t, s)) \tag{4.78}$$

where naturally $\tilde{G}(w, t, 1) = G(w, t)$ and $\tilde{g}(w, t, 1) = g(w, t)$. To show the lemma it will be necessary to analyze the vector field $X$ given by (4.54). Let $X_1$ represents the first $2n$ components of $X$. Recalling that $X$ does not depend on $\mu$, from (4.55) we see that $\tilde{G}$ satisfies

$$\frac{d}{ds}\tilde{G} = X_1(\tilde{G}, t, s), \qquad \tilde{G}(w, t, 0) = w. \tag{4.79}$$

If $\bar{\alpha}$ denotes the first $2n$ components of $\alpha$ from (4.54) we have

$$X_1(\xi, \theta, s) = -(J + s\omega_1)^{-1}\bar{\alpha}. \tag{4.80}$$

We have

$$-(J + s\omega_1)^{-1} = J + sJ\omega_1 J + s^2(J\omega_1)^2 J + \dots \tag{4.81}$$

Then from the definition of $\omega_1$ and (4.74)

$$-(J + s\omega_1)^{-1} = J + O(|\ \xi\ |^{2p-4}). \tag{4.82}$$

Differentiating (4.81) with respect to $w$ and using (4.74) again

$$(-(J + s\omega_1)^{-1})' = O(|\ \xi\ |^{p-2}). \tag{4.83}$$

On the other hand, from the definition of $\alpha$ in (4.46)

$$\bar{\alpha} = O(|\ \xi\ |^{2p-3}) \tag{4.84}$$

and differentiating $\bar{\alpha}$ with respect to $\xi$

$$\bar{\alpha}' = O(|\ \xi\ |^{\beta(p)}) \tag{4.85}$$

Then from (4.80), (4.82)-(4.85)

$$X_1 = O(|\ \xi\ |^{2p-3}) \tag{4.86}$$

and

$$X_1' = O(|\ \xi\ |^{\beta(p)}). \tag{4.87}$$

Differentiating (4.79) we find that $\tilde{G}'$ satisfies the differential equation

$$\frac{d}{ds}\tilde{G}' = X_1'\tilde{G}' \quad \text{and} \quad \tilde{G}'(w,t,0) = I. \tag{4.88}$$

Then, integrating (4.88) and using (4.87) we have

$$|\ G'(w,t) - I\ | \leq \left|\ \int_0^1 X_1'\tilde{G}'ds\ \right| \leq c\int_0^1 |\ X_1'\ |\ ds \leq c_1 \int_0^1 |\ \tilde{G}(w,t,s)\ |^{\beta(p)}\ ds. \tag{4.89}$$

By (4.87) we see that $X_1'(0,t,s) \equiv 0$, then noting that $\tilde{G}(0,t,s) \equiv 0$ we find that $\tilde{G}'(0,t,s) = I$, from where

$$\tilde{G}(w,t,s) = w + o(|\ w\ |). \tag{4.90}$$

Thus from (4.89) and (4.90)

$$|\ \tilde{G}'(w,t) - I\ | \leq c_2\ |\ w\ |^{\beta(p)}\ . \tag{4.91}$$

To show (4.77) we consider the $2n+2$ component of (4.55), then we have that $\tilde{g}$ satisfies

$$\frac{d}{ds}\tilde{g}(w,t,s) = X_{2n+2}(\tilde{G}(w,t,s),t,s). \tag{4.92}$$

But from the definition of $X$ and $\Omega$

$$X_{2n+2} = (-s\omega_2^{\tau}(J+s\omega_1)^{-1},1,0)\alpha. \tag{4.93}$$

Here $\omega_2 = h'^{\tau}J\dot{h}$ so that

$$\omega_2 = O(|\ \xi\ |^{2p-3}). \tag{4.94}$$

Since $\dot{h}'(0,t) = 0$ we have $\dot{h}'(\xi,t) = o(1)$ then, using (4.74)

$$\omega_2' = o(|\ \xi\ |^{p-2}). \tag{4.95}$$

Now, using the definition of $\alpha$, for the $2n+1$ component we have

$$\alpha_{2n+1}' = o(|\ \xi\ |^{p-1}) \tag{4.96}$$

and then, using (4.81)-(4.85), (4.95) and (4.96), from (4.93) we find that

$$X_{2n+2}' = o(|\ w\ |^{p-1}). \tag{4.97}$$

Differentiating (4.92), and doing an analysis similar to the one above we obtain

$$g_w(w,t) = o(|\ w\ |^{p-1}).\Diamond$$

**Proposition 4.2.**
*If $\hat{H}_1$ satisfies (H5),(H1), (H2), (H7) and (H8) then there are constants $\bar{r} > 0, \bar{q} > 2, \bar{c} > 0$ so that $\bar{q}+2 > p > \bar{q} > 2$ and*

$$(\hat{H}_{3w}(w,t),w) \geq \bar{q}\hat{H}_3(w,t) \geq \bar{c}\ |\ w\ |^p$$

$$\forall w \in \mathbf{R}^{2n}, |\ w\ | \leq \bar{r}, \quad \forall t \in \mathbf{R}.$$

**Proof.** Let us consider the following functions

$$a(w,t) = (A_1 w, G_1(w,t)) + \frac{1}{2}(A_1 G_1(w,t), G_1(w,t)), \tag{4.98}$$

$$b(w,t) = \frac{1}{2}(A_2 h(G(w,t),t), h(G(w,t),t)), \tag{4.99}$$

$$c(w,t) = \hat{H}_1(G(w,t), h(G(w,t),t),t), \tag{4.100}$$

Then

$$\hat{H}_3(w,t) = a(w,t) + b(w,t) + c(w,t) + g(w,t).$$

We analyze each term separately. Differentiating (4.98) and taking the inner product with $w$ yields

$$(a'(w,t),w) = (A_1 w, G_1(w,t)) + (A_1 w, G_1'(w,t)w) + (A_1 G_1(w,t), G_1'(w,t)w). \quad (4.101)$$

From Lemma 4.6 we have

$$(a'(w,t),w) = O(|\, w\,|^{\beta(p)+2}) + O(|\, w\,|^{2\beta(p)+2}). \quad (4.102)$$

By the definition of $\beta(p)$, and noting that $p > 2$

$$\beta(p) + 2 = \text{Min}\{2p-2, p+1\} \geq p + \delta$$

where $\delta = p - 2$. Also $2\beta(p) + 2 \geq p + \delta$, so that

$$(a'(w,t),w) = O(|\, w\,|^{p+\delta}). \quad (4.103)$$

Differentiating (4.99) and taking the inner product with $w$ we obtain

$$(b'(w,t),w) = (A_2 h(G(w,t),t), G'^{\tau} h'(G(w,t),t)w).$$

Then from Lemma 4.6

$$(b'(w,t),w) = O(|\, w\,|^{p+\delta}). \quad (4.104)$$

Finally let's analyze the function c. Differentiating (4.100), and taking the inner product with $w$

$$\begin{aligned}(c'(w,t),w) = &(\hat{H}_{1\xi_1}(G(w,t), h(G(w,t),t),t), G'(w,t)w) \\ &+ (\hat{H}_{1\xi_2}(G(w,t), h(G(w,t),t),t), h'(G(w,t),t)G'(w,t)w)\end{aligned} \quad (4.105)$$

By Taylor's theorem and Lemma 4.6 and using that $\hat{H}_{1zz}(z,t) = o(1)$ we have

$$\hat{H}_1(G(w,t), h(G(w,t),t),t) = \hat{H}_{1\xi_1}(w,0,t) + o(|\, w\,|^{p-1}). \quad (4.106)$$

In a similar fashion we obtain

$$(\hat{H}_{1\xi_2}(G(w,t), h(G(w,t),t),t), h'(G(w,t),t)G'(w,t)w) = o(|\, w\,|^{p}). \quad (4.107)$$

From (4.106), (4.107) and (4.104)

$$(c'(w,t),w) = (\hat{H}_{1\xi_1}(w,0,t),w) + o(|\, w\,|^{p}). \quad (4.108)$$

From (4.77), (4.103), (4.104) and (4.108) we obtain

$$(\hat{H}_{3w}(w,t),w) = (\hat{H}_{1\xi_1}(w,0,t),w) + o(|\ w\ |^p). \tag{4.109}$$

By analyzing a, b, c and g in the same way we also obtain

$$\hat{H}(w,0,t) = \hat{H}_3(w,t) + o(|\ w\ |^p). \tag{4.110}$$

From (H7), and (4.109) we have

$$(\hat{H}_{3w}(w,t),w) \geq q\hat{H}_1(w,0,t) + o(|\ w\ |^p), \tag{4.111}$$

thus by (4.110), (4.111) and (H7) there is $\epsilon > 0$ so that for $r_1 > 0$ small enough

$$(\hat{H}_{3w}(w,t),w) \geq (q-\epsilon)\hat{H}_3(w,t), \tag{4.112}$$

$$\forall w \in \mathbf{R}^{2n}, |\ w\ |\leq r_1, \quad \forall t \in \mathbf{R}.$$

The constant $\epsilon > 0$ can be chosen so that $\bar{q} = q - \epsilon$ satisfies $\bar{q} + 2 > p > \bar{q} > 2$. On the other hand from (H7) and (4.110)

$$\hat{H}_3(w,t) = \hat{H}_1(w,0,t) + o(|\ w\ |^p) \geq c\ |\ w\ |^p + o(|\ w\ |^p)$$

so that there is a constant $r_2 > 0$ so that for a constant $\bar{c}$ and $|\ w\ |\leq r_2$

$$(q-\epsilon)\hat{H}_3(w,t) \geq \bar{c}\ |\ w\ |^p$$

by setting $\bar{r} = \mathrm{Min}\{r_1, r_2\}$ we complete the proof.$\diamondsuit$

**Remark 4.9** As we said before, all the results we have proved in this section can be obtained if we assume that $\hat{H}$ satisfies an adequate hypothesis. We preferred to impose the conditions on $\hat{H}_1$ instead,for simplicity.

## §4.5 Proof of theorem II.

This is a direct application of Theorem I noting that by Remark 4.4 $H_3$ satisfies (H1). We only want to mention that in the case the system has $-1$ as a Floquet *exponent*, the reduced system (4.33), or system (4.71) obtained from (4.33) by applying Proposition 4.2:

$$\dot{w} = JA_1 w + J\hat{H}_{3w}(w,t)$$

is $R$-periodic. A change of variables we perform now allows us to obtain (4.72) to which theorem I is applicable.

**Lemma 4.7.**

*There is a symplectic matrix function $R(t)$ so that $w(t)$ is an $Rk$-periodic solution of*

$$\dot{w} = JA_1 w + J\hat{H}_{3w}(w,t)$$

*if and only if $z(t) = R(t)w(t)$ is a $k$-periodic solution of*

$$\dot{z} = JB(t)z + J\hat{K}_z(z,t)$$

*with the matrix $B(t)$ defined by $B(t) = -J\dot{R}(t)R^{-1}(t) + R^{-1\tau}(t)A_1 R^{-1}(t)$ symmetric and the higher order term given by $\hat{K}(z,t) = \hat{H}_3(R(t)^{-1}z,t)$.*

**Proof.** The proof corresponds to the nonlinear reduced version of Lemma 3.2. The matrix $R$ we are using here corresponds to the block $R_1$ we defined in Section 4.1. We can define $R(t)$ in the same way as we did in Lemma 3.2. The proof follows directly.◊

## APPENDIX A. LOGARITHM OF A MATRIX.

This appendix is devoted to the definition of the logarithm of a real matrix. We collect here some elementary properties of the logarithm, as they are used in the text. We will define the logarithm of a matrix using the Cauchy formula for the logarithm in the complex plane. We will follow the presentation of Yakubovich and Starzhinskii [16], and we refer to this book for the proofs we omit.

Let $X$ be an $n \times n$ matrix with coefficients in $\mathbf{C}$. Let $\lambda_1, \lambda_2, ..., \lambda_k$ be the different eigenvalues of $X$ (without counting multiplicity). We assume that $X$ is nonsingular, so that all eigenvalues are different from zero. Let $l$ be a ray in the complex plane going from the origin to infinity, not containing any of the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$, and not containing 1. In $\mathbf{C} \setminus l$ we can define a single valued branch of the logarithm in the usual way

$$\log z = \log \mid z \mid + i \arg z \tag{A.1}$$

where $\arg z$ is the branch of the argument function satisfying $\arg 1 = 0$.

Let $\Gamma_j$ be circles, centered at the points $\lambda_j, j = 1, 2, ...k$, which are mutually disjoint and do not intersect $l$. Let $G_j$ denote the open disk with boundary $\Gamma_j$. If $G = \bigcup_{j=1}^{k} G_j$ then $\log z$ is well defined and analytic in an open set containing $\bar{G}$.

Using the Cauchy formula we now define a logarithm function for the matrix $X$ by

$$\log X = \frac{1}{2\pi i} \sum_{j=1}^{k} \int_{\Gamma_j} (zI_n - X)^{-1} \log z \, dz. \tag{A.2}$$

**Lemma A.1.**
*(i) If $K = \log X$ then $X = \exp K$.*
*(ii) If $Q$ is a nonsingular matrix then*

$$\log(Q^{-1}XQ) = Q^{-1}(\log X)Q.$$

**Proof.** For (i) see [16], page 55. (ii) is direct from the definition.◇

**Remark A.1.** A more general notion of logarithm can be defined if in each $G_j$ a different branch of the complex logarithm is considered. If $m_j \in \mathbf{Z}$, then define

$$\log(z)_{m_j} = \log z + 2\pi i m_j \tag{A.3}$$

then we define

$$\log(X)_{\vec{m}} = \frac{1}{2\pi i} \sum_{j=1}^{k} \int_{\Gamma_j} (zI_n - X)^{-1} \log(z)_{m_j} \, dz, \tag{A.4}$$

where $\vec{m} = (m_1, m_2, ..., m_k)$. This logarithm of $X$ also satisfies Lemma A.1. For our purposes it will be enough to take $m_j = 0, j = 1, 2, ..., k$. See [16].

**Remark A.2.** If the matrix X is real, one would like $\log X$ to be real. This is not always possible, however the following lemma tell us when it is so.

**Lemma A.2.**

*If $X$ is a real matrix with no eigenvalues on the negative real axis then $\log X$ is a real matrix.*

**Proof.** See [16], page 56.◇

When the matrix $X$ is a symplectic real matrix, i.e.

$$X^\tau J X = J$$

then the logarithm of X satisfies a special property:

**Lemma A.3.**

*If $X$ is a symplectic matrix then*

$$(\log X)^* J = -J \log X. \tag{A.5}$$

**Proof.** See [16], page 211.◇

**Remark A.3.** In (A.5) we have used $*$ for conjugate transpose instead of $\tau$ because in general, as we said before, the matrix $\log X$ is not real.

The block property for the logarithm of a matrix is obvious, but we make it explicit. Let the matrix $X$ have the following block form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

where $X_i$ is an $n_i \times n_i$ matrix, $i = 1, 2$. Let us further assume that the eigenvalues of $X_1$ are different from the eigenvalues of $X_2$, then we obtain directly from the definition that

$$\log X = \begin{pmatrix} \log X_1 & 0 \\ 0 & \log X_2 \end{pmatrix}. \tag{A.6}$$

Here we understand the logarithm of the blocks defined as in (A.2) but restrict the sum to the eigenvalues in the corresponding block.

**Lemma A.4.**

*If $X$ is as before and both $\log(-X)$ and $\log(X)$ are well defined then*

$$\log(-X) = i\pi I_n + \log X$$

**Proof.** For every $z \in \mathbf{C}$ such that $z$ and $-z$ do not belong to $l$ we have

$$\log(-z) = i\pi + \log z.$$

Changing the variable $z$ to $-z$ in the integral (A.2), the result follows.$\diamondsuit$

Finally we have a lemma concerning the relation between the eigenvalues of a matrix and its logarithm.

**Lemma A.5.**

*Let $A$ be a real matrix. If $\exp A$ has one eigenvalue equal to 1 then $\exists m \in \mathbf{Z}$ so that $A - 2mV$ has zero as an eigenvalue. Here the matrix $V$ is defined by*

$$V = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix},$$

*and the matrix $K$ is defined by*

$$K = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \dots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

**Proof.** Let $\mathcal{J}$ be the Jordan canonical form of $A$, i.e. for some nonsingular matrix $Q$

$$Q\mathcal{J}Q^{-1} = A.$$

Since $\exp A$ has 1 as an eigenvalue $\exists v \in \mathbf{R}^n, v \neq 0$ such that

$$\exp Av = v$$

and then

$$\exp(pA)v = v, \qquad \forall p \in \mathbf{Z}.$$

Using the Jordan canonical form we obtain, by taking $w = Q^{-1}v$

$$\exp(p\mathcal{J})w = w \qquad \forall p \in \mathbf{Z},$$

and by blocks

$$\exp(p\mathcal{J}_j)w_j = w_j \qquad \forall p \in \mathbf{Z}, j = 1, 2, ..., k.$$

Since $v \neq 0$, for at least one index the vector $w_j$ is not zero. Considering this index, but droping it from the formulae we have

$$\exp(p\mathcal{J})w = w \qquad \forall p \in \mathbf{Z}, \tag{A.6}$$

with $\mathcal{J} = \lambda I + U$ and $U$ is a upper diagonal matrix. Actually the matrix $U$ has the form

$$U = \begin{pmatrix} U_1 & 0 & \dots & 0 \\ 0 & U_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U_k \end{pmatrix}$$

where

$$U_j = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

If $\lambda = \alpha + i\beta$, then (A.6) can be written as

$$e^{\alpha p} \exp(ip\beta I + pU)w = w$$

Since this has to be true $\forall p \in \mathbb{Z}$, and $w \neq 0$, we conclude that $\alpha = 0$. Consequently $\lambda$ is purely imaginary.

Using the block structure of $U$ we have that for some $U_j$, the corresponding vector $w_j \neq 0$ and

$$e^{i\beta p} \exp(pU_j)w_j = w_j, \qquad \forall p \in \mathbb{Z} \tag{A.7}$$

For simplicity, let us drop the index $j$ in (A.7). We can calculate $\exp(pU)$ explicitly

$$\exp(pU) = \begin{pmatrix} 1 & p & \frac{p^2}{2!} & \dots & \frac{p^n}{n!} \\ 0 & 1 & p & \dots & \frac{p^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Thus

$$\exp(pU)w = (w_1 + pw_2 + ... + \frac{p^n}{n!}w_n, ..., w_n).$$

All components of $\exp(pU)w$ have to be bounded, in particular the first one, and this is possible only if $w_2 = w_3 = ... = w_n = 0$, consequently $w = (w_1, 0, ..., 0)$, and then (A.7) implies

$$e^{ip\beta} = 1 \qquad \forall p \in \mathbb{Z},$$

in particular $e^{i\beta} = 1$, thus $\beta = 2\pi m$, for some $m \in \mathbb{Z}$.

We conclude that $A$ has an eigenvalue $\lambda = 2\pi i m$, for $m \in \mathbb{Z}$.

If $m = 0$, then certainly $A - 2\pi mV$ has zero as an eigenvalue. If $m \neq 0$, since $A$ is real, $\bar{\lambda} = -2\pi im$ is also an eigenvalue of A. Then the conclusion follows since the real representation of the pair $(\lambda, \bar{\lambda})$ is the matrix

$$\begin{pmatrix} 0 & -2\pi m \\ 2\pi m & 0 \end{pmatrix},$$

then $A - 2\pi mV$ has zero as an eigenvalue. The multiplicity of 0 is in this case at least 2.◇

## APPENDIX B. BRIEF PERTURBATION THEORY FOR SYMMETRIC MATRICES.

In this appendix we collect some basic facts about the perturbation theory for eigenvalue problems for symmetric matrices and we give a proof of Propositions 2.1, 2.2 and 2.3.

The subject of perturbation of eigenvalue problems is well known, and we refer the reader to the book by Kato [8] for more information. Also the book by Rellich [14] is worth mentioning.

After enunciating the basic theorem, we describe the calculations necessary to obtain the power series for the eigenvalues of the perturbed problem. We then specialize the results to perturbation linear in $\epsilon$ and then apply these results to our situation.

Suppose we have a power series

$$A(\epsilon) = A_0 + \epsilon A_1 + \epsilon^2 A_2 + ... \qquad (B.1)$$

which converges for small $\epsilon$, where the $A_i$ are $n \times n$ symmetric matrices. We consider the eigenvalue problem

$$A(\epsilon)v = \lambda v. \qquad (B.2)$$

**Theorem B.1.**

*Suppose $\lambda$ is an eigenvalue for the matrix $A(0) = A_0$ of multiplicity $k \geq 1$, and suppose that the interval $(\lambda - \alpha, \lambda + \beta)$ contains no eigenvalues of $A_0$ other than $\lambda$, where $\alpha, \beta$ are positive real numbers.*

*Then there exist convergent power series*

$$\lambda_i(\epsilon) = \lambda + \lambda_i^{(1)}\epsilon + \lambda_i^{(2)}\epsilon^2 + ..., \quad i = 1, 2, ..., k \qquad (B.3)$$

*and*

$$v_i(\epsilon) = v_i^{(0)} + v_i^{(1)}\epsilon + v_i^{(2)}\epsilon^2 + ..., \quad i = 1, 2, ..., k \qquad (B.4)$$

*where $\lambda_i^{(j)} \in \mathbf{R}$, $v_i^{(j)} \in \mathbf{R}^n$, such that*

$$A(\epsilon)v_i(\epsilon) = \lambda_i(\epsilon)v_i(\epsilon) \quad i = 1, 2, ..., k \qquad (B.5)$$

$$\text{and} \quad (v_i(\epsilon), v_j(\epsilon)) = \delta_{ij} \quad i,j = 1,2,...,k. \tag{B.6}$$

*For $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$, there is $\delta > 0$ such that in the interval $(\lambda - \alpha', \lambda + \beta')$, $\lambda_i(\epsilon)$ are the only eigenvalues of $A(\epsilon)$ for $| \epsilon |< \delta$.*

For a direct proof of this theorem see [14]. See also [8].

Our aim now is to give a formula to compute the coefficients $\lambda_i^{(j)}$ of the power series (B.3). Given a symmetric matrix $A$, we can consider its spectral decomposition

$$A = \sum_{i=1}^{m} \lambda_i P_i \tag{B.7}$$

where $\{\lambda_1, \lambda_2, ..., \lambda_m\} = \sigma(A)$ is the spectrum of $A$ and $P_i$ is the matrix of orthogonal projection onto the eigenspace associated to $\lambda_i$.

The resolvent of the matrix $A$ is defined for every $\xi \in \mathbb{C} \setminus \sigma(A)$ by

$$R(\xi) = (A - \xi I)^{-1} \tag{B.8}$$

We can obtain a power series expansion about an eigenvalue, say $\lambda_1$, using (B.7): for large $\xi \in \mathbb{C}$

$$\begin{aligned} (A-\xi I)^{-1} &= -\frac{1}{\xi}\sum_{j=0}^{\infty}(\frac{1}{\xi}A)^j \\ &= -\frac{1}{\xi}\sum_{j=1}^{\infty}(\frac{1}{\xi})^j(\sum_{i=1}^{m}\lambda_i^j P_i) \\ &= \sum_{i=1}^{m} -\frac{1}{\xi}\sum_{j=1}^{\infty}(\frac{\lambda_i}{\xi})^j P_i \\ &= \sum_{i=1}^{m}\frac{1}{\lambda_i-\xi}P_i, \end{aligned} \tag{B.9}$$

where we used the facts that the series is absolutely convergent and that the projections are orthogonal. We observe that (B.9) makes sense for every $\xi \notin \sigma(A)$.

Reordering terms, for $\xi$ close to $\lambda_1$, we have

$$\begin{aligned} \sum_{i=2}^{m}\frac{1}{\lambda_i-\xi}P_i &= \sum_{i=2}^{m}\frac{1}{\lambda_i-\lambda_1}\sum_{j=0}^{\infty}(\xi-\lambda_1)^j\frac{1}{(\lambda_i-\lambda_1)^j}P_i \\ &= \sum_{j=0}^{\infty}(\xi-\lambda_1)^j[\sum_{i=2}^{m}\frac{1}{\lambda_i-\lambda_1}P_i]^{j+1}. \end{aligned} \tag{B.10}$$

Then we can write, from (B.9) and (B.10)

$$R(\xi) = \frac{1}{\lambda_1-\xi}P_1 + \sum_{j=0}^{\infty}(\xi-\lambda_1)^j S^{j+1} \tag{B.11}$$

where the matrix $S$ is called the reduced resolvent and it is given by

$$S = \sum_{i=2}^{m} \frac{1}{\lambda_i - \lambda_1} P_i.$$

From (B.11) we obtain that

$$-\frac{1}{2\pi i} \int_{\Gamma} R(\xi)\, d\xi = P_1, \tag{B.12}$$

where $\Gamma$ is a small circle around $\lambda_1$. Considering the matrix valued function, $A(\epsilon)$, we can define the resolvent

$$R(\xi, \epsilon) = (A(\epsilon) - \xi I)^{-1}. \tag{B.13}$$

If $\lambda$ is an eigenvalue of $A(0)$ of multiplicity $k$, then by the perturbation theorem, for small $\epsilon$ in an open interval about $\lambda$ the only eigenvalues of $A(\epsilon)$ are $\lambda_1(\epsilon), ..., \lambda_k(\epsilon)$. If we denote by $P(\epsilon)$ the projection onto the eigenspace associated to $\lambda_1(\epsilon), ..., \lambda_k(\epsilon)$ then using essentially the argument given above, we can prove that

$$-\frac{1}{2\pi i} \int_{\Gamma} R(\xi, \epsilon)\, d\xi = P(\epsilon) \tag{B.14}$$

where $\Gamma$ is a small circle containing $\lambda_1(\epsilon), ..., \lambda_k(\epsilon)$. Noting that

$$(A(\epsilon) - \lambda I) R(\xi, \epsilon) = I + (\xi - \lambda) R(\xi, \epsilon),$$

from (B.14) we obtain that

$$(A(\epsilon) - \lambda I) P(\epsilon) = -\frac{1}{2\pi i} \int_{\Gamma} (\xi - \lambda) R(\xi, \epsilon)\, d\xi. \tag{B.15}$$

On the other hand, if $B(\epsilon) = \sum_{i=1}^{\infty} A_i \epsilon^i$,

$$\begin{aligned} R(\xi, \epsilon) &= (A_0 + B(\epsilon) - \xi I)^{-1} \\ &= (A_0 - \xi I)^{-1} (I + B(\epsilon)(A_0 - \xi I)^{-1})^{-1} \\ &= R(\xi, 0)(I + B(\epsilon) R(\xi, 0))^{-1}. \end{aligned}$$

Then, a power series expansion leads to

$$R(\xi, \epsilon) = R(\xi) \sum_{p=0}^{\infty} (-B(\epsilon) R(\xi))^p. \tag{B.16}$$

Finally, from (B.15) and (B.16)

$$(A(\epsilon) - \lambda I)P(\epsilon) = \sum_{p=0}^{\infty}(-1)^{p+1}\frac{1}{2\pi i}\int_{\Gamma}(\xi - \lambda)R(\xi)(B(\lambda)R(\xi))^p \, d\xi. \qquad (B.17)$$

Knowing the power series $B(\epsilon)$ allows to obtain a power series

$$(A(\epsilon) - \lambda)P(\epsilon) = \sum_{p=1}^{\infty} \epsilon^p \tilde{A}_p \qquad (B.18)$$

where the first integral, for $p = 0$, vanishes by (B.15). Dividing (B.18) by $\epsilon$ we obtain

$$\tilde{A}^{(1)}(\epsilon) = \frac{1}{\epsilon}(A(\epsilon) - \lambda)P(\epsilon) = \sum_{p=0}^{\infty} \epsilon^p \tilde{A}_{p+1}. \qquad (B.19)$$

The matrix series $\tilde{A}^{(1)}(\epsilon)$ is symmetric and for every eigenvalue $\lambda^{(1)}$ of $\tilde{A}_1$ we can use the perturbation theorem to obtain a power series of eigenvalues of $\tilde{A}^{(1)}(\epsilon)$. Thus the coefficients $\lambda_i^{(1)}$, $1 \leq i \leq k$, in (B.3) correspond to the eigenvalues of $\tilde{A}_1$.

This process called Reduction Process can be continued to obtain the higher order coefficients of (B.3). The matrix $\tilde{A}^{(1)}(\epsilon)$ given in (B.19) is a perturbation of $\tilde{A}_1 = \tilde{A}^{(1)}(0)$; *thus for each of the eigenvalues* $\lambda_1^{(1)}$ *of* $\tilde{A}_1$ *we can apply the* Reduction Process to obtain the coefficients $\lambda_i^{(2)}$. By applying it several times we can obtain any coefficient in (B.3).

So far we have described the general perturbation problem. We now specialize to linear perturbations in $\epsilon$. In particular we are interested in obtaining a formula for $\tilde{A}_p$ in the case all $\tilde{A}_i$ are zero for $i < p$. This will give a formula for the first nonzero coefficient in (B.3).

If $A(\epsilon) = A_0 + \epsilon A_1$ then $B(\epsilon) = \epsilon A_1$ and formula (B.17) greatly simplifies:

$$\tilde{A}_p = (-1)^{p+1}\frac{1}{2\pi i}\int_{\Gamma} \xi R(\xi)(A_1 R(\xi))^p \, d\xi, \qquad (B.20)$$

where we assumed, for simplicity that $\lambda = 0$. The resolvent (B.11) is given by

$$R(\xi) = -\frac{1}{\xi}P + \sum_{j=0}^{\infty}(\xi)^j S^{j+1} \qquad (B.21)$$

If we rewrite $R(\xi)$ as

$$R(\xi) = \sum_{j=-1}^{\infty} \xi^j R^{(j)} \qquad (B.22)$$

with the obvious identification with (B.21) to define the matrices $R^{(j)}$, then we see that given $p$

$$\tilde{A}_p = \sum_{\nu \in \Omega_p} (-1)^{p+1} R^{(\nu_0)} A_1 R^{(\nu_1)} A_1 ... A_1 R^{(\nu_p)}, \qquad (B.23)$$

where $\Omega_p = \{\nu = (\nu_0, \nu_1, ..., \nu_p) / \quad \nu_i \geq -1, \quad \sum_{i=0}^{p} \nu_i = -2\}$. Thus for example,

$$\tilde{A}_1 = PA_1P, \quad \text{and}$$

$$\tilde{A}_2 = -PA_1PA_1S - PA_1SA_1P - SA_1PA_1P.$$

Formula (B.23) gives the matrix $\tilde{A}_p$ in terms of $P$, $S$ and $A_1$, for any positive integer $p$. We are interested only in the first nonzero matrix $\tilde{A}_p$, and in this situation the formula is even simpler.

**Lemma B.1**

*If $\tilde{A}_q = 0$ for all $1 \leq q \leq p$ then*

$$\tilde{A}_{p+1} = (-1)^{p+2} PA_1(SA_1)^p P \qquad (B.24)$$

**Proof.** Our proof is based on the following simple combinatorial fact whose proof we present later:

*for $q \in \mathbb{N}$, if $\nu \in \Omega_q$ and $\nu \neq (-1, 0, ..., 0, -1)$ then there is $j$ such that $0 \leq j < q$* for which $\sum_{i=0}^{j} \nu_i = -2$ or 0.

We prove the lemma using an induction argument on $p$. If $p = 1$, then $\tilde{A}_1 = 0$ implies $\tilde{A}_2 = -PA_1SA_1P$ as we can easily see from the formulae given above. Now assume that the result of the lemma is true for every $r$ such that $1 \leq r \leq p-1$, and let us assume that $\tilde{A}_p = 0$ for all $1 \leq q \leq p$. Let $\nu \in \Omega_{p+1}$ and let us assume that $\nu \neq (-1, 0, ..., 0, -1)$. Then from our claim, there is $0 \leq j < p$ such that

$$\begin{aligned} Q(\nu) &= R^{(\nu_0)} A_1 R^{(\nu_1)} ... A_1 R^{(\nu_{p+1})} \\ &= (R^{(\nu_0)} A_1 R^{(\nu_1)} ... A_1 R^{(\nu_j)}) A_1 (R^{(\nu_{j+1})} ... A_1 R^{(\nu_{p+1})}) \end{aligned}$$

where i) $\sum_{i=0}^{j} \nu_i = -2$ or ii) $\sum_{i=j+1}^{p+1} \nu_i = -2$. Case i) We have two possibilities. a) $(\nu_0, ..., \nu_j) = (-1, 0, ..., 0, -1)$ or b) $(\nu_0, ..., \nu_j) = (-1, 0, ..., 0, -1)$. If a) by induction hypothesis $R^{(\nu_0)} A_1 ... R^{(\nu_j)} = \tilde{A}_j$. Consequently, from our assumption, (B.24) shows $\tilde{A}_j = 0$ and hence $Q(\nu) = 0$. If b) then for $\nu^1 = (\nu_0, ..., \nu_j) \in \Omega_j$ and $Q(\nu^1) = R^{(\nu_0)} A_1 R^{(\nu_1)} ... A_1 R^{(\nu_j)}$ we can iterate the argument. We note that $j < p+1$. Case ii) We proceed as in Case ii). After a finite number of steps we will find $Q(\nu) = 0$. Formula (B.23) finally finishes the proof.

Now we prove the claim we made at the beginning. We have three cases: $\nu_0 = -1$, $\nu_0 = 0$ or $\nu_0 > 0$. Case 1. If $\nu_0 = -1$ then either $\nu_1 = ... = \nu_{(q-1)} = 0$ which is not allowed, or there is $k \leq q-1$ such that $\nu_1 = ... = \nu_{k-1} = 0$ and $\nu_k \neq 0$. If $\nu_k = -1$ we can take $j = k$. If $\nu_k = 1$ then again we can take $j = k$. The remaining situation corresponds to $\sum_{i=0}^{k} \nu_i > 0$. Now, since $\sum_{i=0}^{q} \nu_i = -2$ and $\nu_i \geq -1$, there will be $j < q$ such that $\sum_{i=0}^{j} \nu_i = 0$. Case 2. If $\nu_0 = 0$ then $j = 0$. Case 3. If $\nu_0 > 0$ then $\sum_{i=0}^{k} \nu_i > 0$ for $k = 0$, and we proceed as we did in Case 1. $\diamond$

In what follows we apply the perturbation theory as presented above to prove Propositions 2.1, 2.2.

**Proof of Proposition 2.1.** First we identify the terms according to the notation used in the Appendix.

Let

$$T = \sigma W J \quad \text{and} \quad T^{(1)} = VJ. \tag{B.25}$$

Then (2.3) can be written as

$$(T + \epsilon T^{(1)})w + \lambda w = 0. \tag{B.26}$$

We apply the reduction process to (B.26).

We first compute the matrix

$$\tilde{T}^{(1)} = PT^{(1)}P \tag{B.27}$$

where P is the projection matrix onto the eigenspace associated to the eigenvalue 0 of $T$. Displaying the actual form of $T$ we see that the eigenspace associated to zero is span $\{e_1, e_{2l}\}$ so that $P$ takes the simple form

$$P = \text{diag}(1, 0, ..., 0, 1). \tag{B.28}$$

From the definitions given in (1.9), (1.19) and (B.25) we have

$$T^{(1)} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \tag{B.29}$$

and then the coefficients

$$T^{(1)}_{11} = T^{(1)}_{2l\,2l} = 0. \tag{B.30}$$

From (B.28) and (B.30) we see then that

$$\tilde{T}^{(1)} = PT^{(1)}P = 0. \tag{B.31}$$

This indicates, according to the reduction process, that $\lambda^1_j = 0, \quad j = 1, 2$, in (2.5).

The next step in the reduction process requires the introduction of the reduced resolvent of $T$

$$S = \sum_{\lambda_j \neq 0} \frac{1}{\lambda_j} P_j \tag{B.32}$$

where $\lambda_j$ is an eigenvalue of $T$ and $P_j$ is the projection matrix onto the eigenspace associated with the eigenvalue $\lambda_j$.

In order to compute $S$ we first obtain the nonzero eigenvalues of $T$. After a calculation one obtains that

$$\det(T - \lambda I) = \begin{cases} \lambda^2(\lambda-\sigma)^2(\lambda^2-1)^{l-2} & \text{if } l \text{ even} \\ \lambda^2(\lambda^2-1)^{l-1} & \text{if } l \text{ odd.} \end{cases} \tag{B.33}$$

The eigenvalues of $T$ different from zero are $\sigma$ and $-\sigma$. Since $\sigma$ takes the values 1 or $-1$, from (B.32) and the spectral decomposition of $T$ we see that

$$S = T. \tag{B.34}$$

For a given $p \in \mathbb{N}$ we compute $(ST^{(1)})^{p-1}$, from (B.25) and (B.34)

$$(ST^{(1)})^{p-1} = \sigma^{p-1}(WJVJ)^{p-1}. \tag{B.35}$$

From the definition of $V$ and $W$ given in (1.19) we have

$$WJVJ = \begin{pmatrix} d & 0 \\ 0 & d^{\tau} \end{pmatrix}, \tag{B.36}$$

so that, from (B.35) and (B.36)

$$(ST^{(1)})^{p-1} = \sigma^{p-1}\begin{pmatrix} d^{p-1} & 0 \\ 0 & (d^{\tau})^{p-1} \end{pmatrix}. \tag{B.37}$$

Then, from (B.29) and (B.37) we obtain

$$T^{(1)}(ST^{(1)})^{p-1} = \sigma^{p-1}\begin{pmatrix} Kd^{p-1} & 0 \\ 0 & K(d^{\tau})^{p-1} \end{pmatrix}. \tag{B.38}$$

If $p < l$ then, using Lemma B.1 and (B.38)

$$\tilde{T}^{(p)} = (-1)^{p+1} PT^{(1)}(ST^{(1)})^{p-1}P = 0, \tag{B.39}$$

and if $p = l$ then, using Lemma B.1 and (B.38) again

$$\begin{aligned} \tilde{T}^{(l)} &= (-1)^{(l+1)} PT^{(1)}(ST^{(1)})^{l-1}P \\ &= (-1)^{l+1}\sigma^{l-1}\operatorname{diag}(1,0,\dots,0,1). \end{aligned} \tag{B.40}$$

Thus, the first nonzero coefficients in (2.5) are the nonzero eigenvalues of $-\tilde{T}^{(l)}$ so that

$$\lambda_j^{(l)} = (-1)^l \sigma^{l-1}, \qquad j = 1, 2. \tag{B.41}$$

This finishes the proof.◇

**Proof of Proposition 2.2.** In this case

$$T = \begin{pmatrix} 0 & -d \\ -d^{\tau} & 0 \end{pmatrix}, \quad \text{and} \quad T^{(1)} = -iJ. \qquad (B.42)$$

$T$ has $\lambda = 0$ as an eigenvalue of double multiplicity, with eigenvectors $e_1$ and $e_{2l}$. The projection matrix is given by $P$ =diag$(1,0,...,0,1)$, and as before

$$\tilde{T}^{(1)} = PT^{(1)}P = 0.$$

We compute the reduced resolvent $S$. A calculation gives

$$\det(T - \lambda I) = \lambda^2(\lambda^2 - 1)^{l-1}, \qquad (B.43)$$

thus the eigenvalues of T are $\lambda = 0, \lambda = 1$ and $\lambda = -1$, both 1 and $-1$ of multiplicity $l - 1$.

From (B.32), and the spectral decomposition of T we see that

$$S = T. \qquad (B.44)$$

From (B.42) and (B.44) we get

$$ST^{(1)} = i \begin{pmatrix} -d & o \\ 0 & d^{\tau} \end{pmatrix}. \qquad (B.45)$$

Then, for $p \in \mathbb{N}$

$$(ST^{(1)})^{p-1} = i^{p-1} \begin{pmatrix} (-1)^{p-1}d^{p-1} & 0 \\ 0 & (d^{\tau})^{p-1} \end{pmatrix} \qquad (B.46)$$

and from (B.42)

$$T^{(1)}(ST^{(1)})^{p-1} = i^p \begin{pmatrix} 0 & (d^{\tau})^{p-1} \\ (-1)^p d^{p-1} & 0 \end{pmatrix}. \qquad (B.47)$$

Thus if $p < l$,

$$\tilde{T}^{(p)} = (-1)^{p+1} PT^{(1)}(ST^{(1)})^{p-1}P = 0, \qquad (B.48)$$

and if $p = l$

$$\tilde{T}^{(l)} = (-1)^{p+1}(i)^l K \text{diag}((-1)^l, 0, ..., 0, 1). \qquad (B.49)$$

Now, recalling that in this case $l$ is odd,

$$\tilde{T}^{(l)} = \pm iK\text{diag}(-1, 0, ..., 0, 1),$$

whose nonzero eigenvalues are 1 and $-1$, giving

$$\lambda_1^{(l)} = 1 \qquad \text{and} \qquad \lambda_2^{(l)} = -1.$$

from which (2.10) follows. ◇

## APPENDIX C. THE CENTER MANIFOLD THEOREM.

In this appendix we provide the modifications necessary to be made to the standard Center Manifold Theorem in order to suit our needs.

We consider the following system of ordinary differential equations

$$\begin{aligned} \dot{\theta} &= 1 \\ \dot{x} &= Ax + \tilde{X}(\theta, x, y, z) \\ \dot{y} &= By + \tilde{Y}(\theta, x, y, z) \\ \dot{z} &= Cz + \tilde{Z}(\theta, x, y, z) \end{aligned} \tag{C.1}$$

here $x$, $y$ and $z$ are vectors in $\mathbf{R}^{n(\alpha)}$, for $n(x)$, $n(y)$ and $n(z)$ their corresponding dimensions.

For system (C.1) we look for a locally invariant manifold parametrized by the variable $y$, such a manifold is known as the center manifold. For more details we refer the reader to [5] and [9].

We make the following assumptions:

(h1) $A, B, C$ are constant matrices in real canonical form. $A$ has eigenvalues with negative real part, $B$ has eigenvalues with zero real part and $C$ has eigenvalues with positive real part,

(h2) $\tilde{X}, \tilde{Y}, \tilde{Z}$ are vector valued functions of class $C^2$ in a neighborhood

$$N_\delta\{(\theta, x, y, z) / \mid x \mid^2 + \mid y \mid^2 + \mid z \mid^2 \leq \delta^2\},$$

(h3) $\tilde{X}, \tilde{Y}, \tilde{Z}$ and their derivatives with respect to $\theta$, $x$, $y$ and $z$ are all zero for arbitrary values of $\theta$ when $(x, y, z) = 0$.

(h4) For matrices $R_x$, $R_y$ and $R_z$ of the form

$$R_\alpha = \begin{pmatrix} -I_{1,\alpha} & 0 \\ 0 & I_{2,\alpha} \end{pmatrix}$$

with $I_{1,\alpha}$ and $I_{2,\alpha}$ identity matrices of given dimensions, and $\alpha = x, y, z$, we have the following identities

$$\begin{aligned} \tilde{X}(\theta + 1, x, y, z) &= R_x \tilde{X}(\theta, R_x x, R_y y, R_z z) \\ \tilde{Y}(\theta + 1, x, y, z) &= R_y \tilde{Y}(\theta, R_x x, R_y y, R_z z) \\ \tilde{Z}(\theta + 1, x, y, z) &= R_z \tilde{Z}(\theta, R_x x, R_y y, R_z z) \end{aligned}$$

and $R_x A = A R_x$, $R_y B = B R_y$ and $R_z C = C R_z$,

(h5) For some constants $c > 0$, $p > 0$ and $\delta_1 > 0$

$$\frac{\partial \tilde{X}}{\partial y}(\theta,0,y,0) \le c \mid y \mid^p, \quad \frac{\partial \tilde{X}}{\partial \theta}(\theta,0,y,0) \le c \mid y \mid^{p+1},$$
$$\frac{\partial \tilde{Z}}{\partial y}(\theta,0,y,0) \le c \mid y \mid^p, \quad \frac{\partial \tilde{Z}}{\partial \theta}(\theta,0,y,0) \le c \mid y \mid^{p+1},$$

for every y so that $\mid y \mid \le \delta_1$.

**Remark C.1** We note that (h4) implies that the functions $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ are 2-periodic in $\theta$. Then we can apply the Center Manifold Theorem to system (C.1). We want to use (h4) and (h5) to obtain extra properties for the center manifold.

**Theorem C.1**

*For system (C.1) there is a center manifold given by*

$$M = \{(\theta,x,y,z)/\, \theta \,\text{arbitrary},\ \mid y \mid < \delta_2,\ x = u(\theta,y),\ z = v(\theta,y)\}$$

*where the functions $u$ and $v$ are defined and of class $C^2$ in a neighborhood*

$$N_{\delta_2} = \{(\theta,y)/\ \mid y \mid < \delta_2\}$$

*and they satisfy*

$$\begin{aligned} u(\theta,R_y y) &= R_x u(\theta,y) \\ v(\theta,R_y y) &= R_z v(\theta,y) \end{aligned} \qquad (C.2)$$

*And for some constants $K$ and $\delta_3$*

$$\begin{aligned} &\mid \frac{\partial u}{\partial y}(\theta,y) \mid \le K \mid y \mid^p, \quad \mid \frac{\partial u}{\partial \theta}(\theta,y) \mid \le K \mid y \mid^{p+1}, \\ &\mid \frac{\partial v}{\partial y}(\theta,y) \mid \le K \mid y \mid^p, \quad \mid \frac{\partial v}{\partial \theta}(\theta,y) \mid \le K \mid y \mid^{p+1}, \end{aligned} \qquad (C.3)$$

*for all $\mid y \mid \le \delta_3$.*

**Proof.** This theorem is proved by using the contraction principle in an adequate Banach space. See [9]. We start describing a change of variables designed to deal with the local nature of the problem. Let $\phi(r)$ be a smooth real valued function so that $\phi(r) \equiv 1$ if $0 \le r \le 1/2$, and $\phi(r) \equiv 0$ if $1 \le r < \infty$. For large $R$ and for $\lambda$ in $\mathbb{R}$ we define

$$\begin{aligned} X(\theta,x,y,z,\lambda) &= \phi(\mid x \mid^2 + \mid y \mid^2 + \mid z \mid^2 + R\lambda^2)\lambda^{-1}\tilde{X}(\theta,\lambda x,\lambda y,\lambda z) \\ Y(\theta,x,y,z,\lambda) &= \phi(\mid x \mid^2 + \mid y \mid^2 + \mid z \mid^2 + R\lambda^2)\lambda^{-1}\tilde{Y}(\theta,\lambda x,\lambda y,\lambda z)\,. \\ Z(\theta,x,y,z,\lambda) &= \phi(\mid x \mid^2 + \mid y \mid^2 + \mid z \mid^2 + R\lambda^2)\lambda^{-1}\tilde{Z}(\theta,\lambda x,\lambda y,\lambda z) \end{aligned}$$

With this change of variables system (C.1) is transformed into

$$\begin{aligned}\dot{\theta} &= 1\\ \dot{x} &= Ax + X(\theta, x, y, z, \lambda)\\ \dot{y} &= By + Y(\theta, x, y, z, \lambda)\\ \dot{z} &= Cz + Z(\theta, x, y, z, \lambda)\end{aligned} \tag{C.4}$$

and the functions $X$, $Y$ and $Z$ satisfy the following properties:

$(h2)_\lambda$ $X$, $Y$ and $Z$ are defined and continuous, and for each $\lambda$ fixed they are of class $C^2$ in $(\theta, x, y, z)$,

$(h3)_\lambda$ $X, Y, Z$ and their derivatives with respect to $\theta$, $x$, $y$ and $z$ are all zero for arbitrary values of $\theta$ when $(x, y, z) = 0$. And $D^\rho_{(\theta,x,y,z)}(X, Y, Z) \to 0$ uniformly in $(\theta, x, y, z)$ as $\lambda \to 0$ for $0 \leq |\rho| \leq 2$,

$(h4)_\lambda$

$$\begin{aligned}X(\theta + 1, x, y, z) &= R_x X(\theta, R_x x, R_y y, R_z z),\\ Y(\theta + 1, x, y, z) &= R_y Y(\theta, R_x x, R_y y, R_z z),\\ Z(\theta + 1, x, y, z) &= R_z Z(\theta, R_x x, R_y y, R_z z),\end{aligned}$$

$(h5)_\lambda$ For some constants $c > 0$, $p > 0$ and $\delta_1 > 0$ and for $\lambda \leq 1$

$$\begin{aligned}\frac{\partial X}{\partial y}(\theta, 0, y, 0, \lambda) \leq c \mid y \mid^p, &\quad \frac{\partial X}{\partial \theta}(\theta, 0, y, 0, \lambda) \leq c \mid y \mid^{p+1},\\ \frac{\partial Z}{\partial y}(\theta, 0, y, 0, \lambda) \leq c \mid y \mid^p, &\quad \frac{\partial Z}{\partial \theta}(\theta, 0, y, 0, \lambda) \leq c \mid y \mid^{p+1},\end{aligned}$$

for every $y$ so that $\mid y \mid \leq \delta_1$.

We define the following space of functions

$$\mathcal{H}^1 = \{(u(\theta, y), v(\theta, y))/u \text{ and } v \text{ satisfy(f1)} - \text{(f4)}\}$$

(f1) $u$ and $v$ are of class $C^1$ for all $\theta$ and $y$,

(f2) $u$ and $v$ have period 2 in $\theta$,

(f3) $u(\theta, 0)$, $v(\theta, 0)$, $\frac{\partial u}{\partial y}(\theta, 0)$ $\frac{\partial v}{\partial y}(\theta, 0)$ are all zero,

(f4)

$$\| (u, v) \| = \max_{0 \leq |\rho| \leq 1} \sup_{(\theta, y)} \mid D^\rho_{(\theta,y)}(u, v) \mid < \infty.$$

$\mathcal{H}^1$ provided with the norm given in (f4) is a Banach space. For $(u, v) \in \mathcal{H}^1$ and for $\theta_0, y_0$ we define $\psi(t, \theta_0, y_0, \lambda)$ and $\eta(t, \theta_0, y_0, \lambda)$ as the unique solution of the differential equation

$$\begin{aligned}\dot{\theta} &= 1\\ \dot{y} &= By + Y(\theta, u(\theta, y), y, v(\theta, y), \lambda)\end{aligned} \tag{C.5}$$

and for which $\psi(0,\theta_0,y_0,\lambda) = \theta_0$ and $\eta(0,\theta_0,y_0,\lambda) = y_0$. Obviously the function $\psi$ is given simply by $\psi(t) = \theta_0 + t$. We define now our operator $T$ in $\mathcal{H}^1$. The $u$-component of $T$ is given by the formula

$$T_u(u,v)(\theta_0,y_0) = \int_{-\infty}^{0} e^{-A\sigma} X(\psi(\sigma), u(\psi(\sigma),\eta(\sigma)),\eta(\sigma), v(\psi(\sigma),\eta(\sigma)),\lambda)\, d\sigma \qquad (C.6)$$

and similarly the $v$-component is defined by

$$T_v(u,v)(\theta_0,y_0) = \int_{\infty}^{0} e^{-C\sigma} Z(\psi(\sigma), u(\psi(\sigma),\eta(\sigma)),\eta(\sigma), v(\psi(\sigma),\eta(\sigma)),\lambda)\, d\sigma \qquad (C.7)$$

In [9] it is shown that for $\lambda$ small enough the operator $T$ is a contraction in the unit ball $\tilde{\mathcal{H}}^1$ of $\mathcal{H}^1$. We will consider the following closed subset of $\tilde{\mathcal{H}}^1$:

$$\tilde{\mathcal{H}}^1(K) = \{(u,v) \in \tilde{\mathcal{H}}^1 / \quad (u,v) \text{ satisfies (C.2) and (C.3)}\}$$

In what follows we show that for a certain constant $K$ the operator $T$ is a contraction in $\tilde{\mathcal{H}}^1(K)$. We first show that if $(u,v)$ satisfies (C.2) then $(T_u(u,v), T_v(u,v))$ also satisfies (C.2). We claim that

$$\eta(t,\theta_0+1,y_0,\lambda) = R_y \eta(t,\theta_0,R_y y_0,\lambda). \qquad (C.8)$$

Lets assume for the moment that (C.8) is true. Since $(u,v)$ satisfies (C.2), by $(\mathrm{h4})_\lambda$ and (C.8) we have

$$\begin{aligned}
&T_v(u,v)(\theta_0+1,y_0) = \\
&= \int_{-\infty}^{0} e^{-A\sigma} X(\theta_0+1+\sigma, u(\theta_0+1+\sigma, \eta(\sigma,\theta_0+1,y_0,\lambda)), \\
&\qquad \eta(\sigma,\theta_0+1,y_0,\lambda)), v(\theta_0+1+\sigma,\eta(\sigma,\theta_0+1,y_0,\lambda)),\lambda)\, d\sigma = \\
&= \int_{-\infty}^{0} e^{-A\sigma} R_x X(\theta_0+\sigma, R_x u(\theta_0+1+\sigma, \eta(\sigma,\theta_0+1,y_0,\lambda)), \\
&\qquad R_y\eta(\sigma,\theta_0+1,y_0,\lambda)), R_z v(\theta_0+1+\sigma,\eta(\sigma,\theta_0+1,y_0,\lambda)),\lambda)\, d\sigma \\
&= R_x \int_{-\infty}^{0} e^{-A\sigma} X(\theta_0+\sigma, u(\theta_0+\sigma, R_y\eta(\sigma,\theta_0+1,y_0,\lambda)), \\
&\qquad \eta(\sigma,\theta_0,R_y y_0,\lambda)), v(\theta_0+\sigma, R_y\eta(\sigma,\theta_0+1,y_0,\lambda)),\lambda)\, d\sigma \\
&= R_x \int_{-\infty}^{0} e^{-A\sigma} X(\theta_0+\sigma, u(\theta_0+\sigma, \eta(\sigma,\theta_0,R_y y_0,\lambda)), \\
&\qquad \eta(\sigma,\theta_0,R_y y_0,\lambda)), v(\theta_0+\sigma,\eta(\sigma,\theta_0,R_y y_0,\lambda)),\lambda)\, d\sigma \\
&= R_x T(u,v)(\theta_0,R_y y_0)
\end{aligned}$$

In other words we have proved

$$T_u(u,v)(\theta_0+1,y_0)=R_xT_u(u,v)(\theta_0,R_yy_0) \tag{C.9}$$

In a similar fashion we can show

$$T_v(u,v)(\theta_0+1,y_0)=R_zT_v(u,v)(\theta_0,R_yy_0) \tag{C.10}$$

In order to complete the proof we check that (C.8) is correct. Let us consider the function

$$\eta^*(t)=R_y\eta(t,\theta_0,R_yy_0). \tag{C.11}$$

Then $\eta^*(0)=y_0$, and since $(u,v)$ satisfies (C.2) and $Y$ satifies $(\mathrm{h4})_\lambda$ we have

$$\begin{aligned}
&\frac{d\eta^*}{dt}=R_y\frac{d\eta}{dt}(t,\theta_0,R_yy_0)=\\
=&R_yB\eta(t,\theta_0,R_yy_0)+R_yY(\theta_0+t,u(\theta_0,\eta),\eta,v(\theta_0+t,\eta),\lambda)\\
=&B\eta^*+Y(\theta_0+1+t,R_xu(\theta_0+t,\eta),R_y\eta,R_zv(\theta_0+t,\eta),\lambda)\\
=&B\eta^*+Y(\theta_0+1+t,u(\theta_0+1+t,\eta^*),\eta^*,v(\theta_0+1+t,\eta^*),\lambda)
\end{aligned}$$

Then, by uniqueness of the solution of (C.5) we have

$$\eta^*(t)=\eta(t,\theta_0+1,y_0,\lambda),$$

i.e. (C.8) is satisfied.

We now show that a constant $K$ exists so that given $(u,v)\in\mathcal{H}^1$ that satisfies (C.3) the functions $(T_u(u,v),T_v(u,v))$ also satisfies (C.3). We will use the following result on the behaviour of $\eta(t,\theta_0,y_0,\lambda)$ that can be prove using the ideas given in [9], Lemma 3: for $\lambda$ small enough and $(u,v)\in\tilde{\mathcal{H}}^1$ we have

$$|\ \eta(t,\theta_0,y_0,\lambda)\ |\le e^{2\gamma|t|}\ |\ y_0\ | \tag{C.12}$$

$$|\ \frac{\partial\eta}{\partial y}(t,\theta_0,y_0,\lambda)\ |\le e^{2\gamma|t|}, \tag{C.13}$$

$$|\ \frac{\partial\eta}{\partial\theta}(t,\theta_0,y_0,\lambda)\ |\le(1+\ |\ t\ |)e^{2\gamma|t|}\ |\ y_0\ |\ . \tag{C.14}$$

The constant $\gamma$ has been chosen so that $-a+2\gamma(p+2)<0$ where $a$ and $-a$ are lower and upper bound for the real part of the eigenvalues of the matrices $A$ and $C$ respectively.

We have to find $K$ so that if $(u,v)$ satisfies (C.3) then $T(u,v)$ also satisfies (C.3). Let $g(\theta_0,y_0)=T_u(u,v)(\theta_0,y_0)$, then we have

$$\frac{\partial g}{\partial y}(\theta_0,y_0)=\int_{-\infty}^{0}e^{-A\sigma}\{\frac{\partial X}{\partial x}\frac{\partial u}{\partial y}\frac{\partial\eta}{\partial y}+\frac{\partial X}{\partial y}\frac{\partial\eta}{\partial y}+\frac{\partial X}{\partial z}\frac{\partial u}{\partial y}\frac{\partial\eta}{\partial y}\}\,d\sigma. \tag{C.15}$$

From (C.3) and (C.12) we have

$$\left| \frac{\partial u}{\partial y}(\theta_0+\sigma,\eta(\sigma,\theta_0,y_0,\lambda)) \right| \leq K \mid \eta(\sigma,\theta_0,y_0,\lambda) \mid^p \leq Ke^{2p\gamma|\sigma|} \mid y_0 \mid^p, \qquad (C.16)$$

and by integrating (C.3) we obtain

$$\mid u(\theta_0,y_0) \mid \leq \frac{K}{p+1} \mid y_0 \mid^{p+1} \qquad (C.17)$$

and

$$\mid v(\theta_0,y_0) \mid \leq \frac{K}{p+1} \mid y_0 \mid^{p+1} . \qquad (C.18)$$

Let us estimate the middle term in (C.15). Noting that $X$ is of class $C^2$, by the Mean Value Theorem and using $(h3)_\lambda$, we see that for $\lambda$ small enough

$$\begin{aligned} &\left| \frac{\partial X}{\partial y}(\theta_0+\sigma,u,\eta,v,\lambda) - \frac{\partial X}{\partial y}(\theta_0+\sigma,0,\eta,0,\lambda) \right| \leq \\ &\leq \frac{p+1}{2}\{\mid u(\theta_0+\sigma,\eta(\sigma,\theta_0,y_0,\lambda)) \mid + \mid v(\theta_0+\sigma,\eta(\sigma,\theta_0,y_0,\lambda)) \mid\} \\ &\leq K \mid \eta(\sigma,\theta_0,y_0,\lambda) \mid^{p+1} \\ &\leq Ke^{2\gamma(p+1)|\sigma|} \mid y_0 \mid^{p+1}, \end{aligned} \qquad (C.19)$$

here we used (C.12), (C.17) and (C.18). Then, by $(h5)_\lambda$, (C.13) and (C.19) we obtain

$$\left| \int_{-\infty}^0 e^{-A\sigma} \frac{\partial X}{\partial y}\frac{\partial \eta}{\partial y}\, d\sigma \right| \leq \{ \int_{-\infty}^0 e^{(a-2\gamma)\sigma}(c+Ke^{-2\gamma(p+1)\sigma} \mid y_0 \mid)\, dt\} \mid y_0 \mid^p \qquad (C.20)$$

Choose $K$ so that

$$\int_{-\infty}^0 e^{(a-2\gamma)\sigma} c\, d\sigma \leq \frac{K}{6} \qquad (C.21)$$

and choose $\delta_4$ so that,

$$\int_{-\infty}^0 \delta_4 e^{(a-2\gamma(p+2))\sigma}\, d\sigma \leq \frac{1}{6}. \qquad (C.22)$$

Then, from (C.20), (C.21) and (C.22) we conclude that for all $\mid y_0 \mid \leq \delta_4$

$$\left| \int_{-\infty}^0 e^{-A\sigma} \frac{\partial X}{\partial y}\frac{\partial \eta}{\partial y}\, d\sigma \right| \leq \frac{K}{3} \mid y_0 \mid^p . \qquad (C.23)$$

Now we estimate the first term in the right hand side of (C.15). By (C.13) and (C.16)

$$\left| \int_{-\infty}^0 e^{-A\sigma} \frac{\partial X}{\partial x}\frac{\partial u}{\partial y}\frac{\partial \eta}{\partial y}\, d\sigma \right| \leq (\int_{-\infty}^0 e^{(a-2\gamma(p+1))\sigma} K \left| \frac{\partial X}{\partial x} \right| d\sigma) \mid y_0 \mid^p \qquad (C.24)$$

From $(h3)_\lambda$ and by restricting $\lambda$ if necessary, we have

$$\int_{-\infty}^{0} e^{(a-2\gamma(p+1))\sigma} \mid \frac{\partial X}{\partial x} \mid d\sigma \leq \frac{1}{3} \tag{C.25}$$

and then from (C.24) and (C.25)

$$\mid \int_{-\infty}^{0} e^{-A\sigma} \frac{\partial X}{\partial x} \frac{\partial u}{\partial y} \frac{\partial \eta}{\partial y} d\sigma \mid \leq \frac{K}{3} \mid y_0 \mid^p \tag{C.26}$$

In an analogous fashion we can prove that

$$\mid \int_{-\infty}^{0} e^{-A\sigma} \frac{\partial X}{\partial z} \frac{\partial v}{\partial y} \frac{\partial \eta}{\partial y} d\sigma \mid \leq \frac{K}{3} \mid y_0 \mid^p . \tag{C.27}$$

Thus, from (C.18), (C.23), (C.26) and (C.27) we see that there is $\delta_5 > 0$ and $K$ so that for all $\mid y_0 \mid \leq \delta_5$

$$\mid \frac{\partial g}{\partial y}(\theta_0, y_0) \mid \leq K \mid y_0 \mid^p . \tag{C.28}$$

A similar argument can be given to show that if $h(\theta_0, y_0) = T_v(u,v)(\theta_0, y_0)$ then

$$\mid \frac{\partial h}{\partial y}(\theta_0, y_0) \mid \leq K \mid y_0 \mid^p . \tag{C.29}$$

In order to show that the derivative with respect to $\theta$ of $g$ and $h$ satisfies condition (C.3) we proceed in a similar fashion. We may need to reduce further $\lambda$ and increase $K$. We omit the details. ◇

## BIBLIOGRAPHY.

[1] Abraham, R. and Marsden, J. *Foundations of Classical Mechanics.* Benjamin, New York, 1967.

[2] Benci,V. and Fortunato, D. *A 'Birkhoff-Lewis' type result for a class of Hamiltonian systems.* Manuscripta Math., 59, 441-456 (1987).

[3] Birkhoff, G.D. and Lewis, D.C. *On the periodic motion near a given periodic motion of a dynamical system.* Ann. Mat. Pure Appl. 12, 117-133 (1933).

[4] Bryuno, A.D. *The normal form of a Hamiltonian system.* Russian Math. Surveys 43:1 25-66(1988)

[5] Carr, J. *Applications of Centre Manifold Theory.* Applied Mathematical Sciences 35. Springer Verlag, New York. 1981.

[6] Coddington, E. and Levinson, N. *Theory of Ordinary Differential Equations.* R. Krieger Pub. Co. Florida, 1985.

[7] Harris, T.C. *Periodic solutions of arbitrary long periods in Hamiltonian systems.* J. Diff. Eq. 4, 131-141(1968)

[8] Kato, T. *A Short Introduction to Perturbation Theory for Linear Operators.* Springer Verlag, New York. 1982.

[9] Kelley, A. *The Stable, Center-stable, Center, Center-unstable and Unstable Manifold.* Appendix C of R. Abraham and J. Robbin *Transversal Mappings and Flows.* W. A. Benjamin, New York. 1967.

[10] Laub, A. and Meyer, K. *canonical forms for symplectic and Hamiltonian matrices.* Celestial Mechanics 9, 213-238 (1974).

[11] Moser, J. *Proof of a generalized form of a fixed point theorem due to G.D. Birkhoff.* Springer Verlag, Lecture Notes in Math. 597, 464-494. 1977.

[12] Rabinowitz, P. *On subharmonic solutions of Hamiltonian systems.*Comm. Pure and App. Math. Vol XXXIII, 609-633 (1980).

[13] Rabinowitz, P. *Minimax Methods in Critical Point Theory with Applications to Differential Equations.* C.B.M.S. Regional Conference Series in Mathematics, 65, A.M.S., Providence, 1986.

[14] Rellich, F. *Perturbation Theory of Eigenvalue Problems.* Gordon and Breach, New York, 1969.

[15] Williamson, J. *On the Algebraic problem concerning the normal forms of linear dynamical systems.* Amer. J. Math. 58 ,129-140 (1936). MR 2-206.

[16] Yakubovich, V. A. and Starzhinskii, V.M. *Linear Differential Equations with Periodic Coefficients.* Vol I, John Willey & Sons, 1975.