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ABSTRACT

This work is concerned with the study of existence and multiplicity of periodic solutions of Hamiltonian systems of ordinary differential equations

$$\dot{z} = J(H_z(z,t) + f(t))$$

when the Hamiltonian $H(z,t) = H(p,q,t)$ is periodic in the variable q and superlinear in the variable p.

By imposing a growth condition on the derivative of H, we obtain the existence of at least $n+1$ periodic solutions, where n is the dimension of the system.

The existence of periodic solutions is obtained by using a Saddle Point Theorem recently proved by Lui. We consider a functional over $E \times M$, where E is a Hilbert space and M is a compact manifold, satisfying a saddle point condition on E, uniformly on M. We present a proof of this Saddle Point Theorem using standard minimax techniques based on the cup length of M.

AMS (MOS) Subject Classification: 34C25, 35J60, 58E05, 58F05, 58F22.

Key words and phrases: Ljusternik-Schnirelmann category, cup length, critical points, Hamiltonian system, periodic Hamiltonian systems, periodic solutions, multiple solutions.

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PERIODIC SOLUTIONS OF SPATIALLY PERIODIC HAMILTONIAN SYSTEMS

0 Introduction

In this paper we study the existence of periodic solutions for Hamiltonian systems of ordinary differential equations

$$\dot{z} = J(H_z(z,t) + f(t)) \tag{0.1}$$

where $H : \mathbf{R}^{2n} \times \mathbf{R} \longrightarrow \mathbf{R}$ and $f : \mathbf{R} \longrightarrow \mathbf{R}^{2n}$. Here $z = (p,q) \in \mathbf{R}^{2n}$, $\dot{}$ denotes derivative with respect to t, H_z is the partial derivative of H with respect to z and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is the standard symplectic form in $\mathbf{R}^{2n}$. We consider the following basic hypotheses on H and f

(H0) H is of class C^1,

(H1) $H(p,q,t) = H(p,q,t+2\pi) \quad \forall (p,q) \in \mathbf{R}^{2n},\ \forall t \in \mathbf{R}.$

(f0) f is continuous, and

(f1) $f(t) = f(t+2\pi) \quad \forall t \in \mathbf{R}.$

In the study of equation (0.1) the assumption that H is superquadratic has been considered by many authors. This condition is usually expressed in the following form

(S) There are constants $\mu > 2,\ r > 0$ such that

$$0 < \mu H(z,t) \le z \cdot H_z(z,t) \quad \forall z \in \mathbf{R}^{2n},\ |z| \ge r,\ \forall t \in \mathbf{R}.$$

Here, and in the rest of the paper, $\cdot$ denotes the usual inner product in $\mathbf{R}^{2n}$ and $|\cdot|$ its associated norm.

There has been a considerable amount of work in the study of periodic solutions of (0.1) and variations under condition (S). See, for example [16], [18], [2], [1] and [12]. We also mention the recent book by Mawhin and Willem [14] where the reader can find an extensive bibliography.

In this work we assume that the Hamiltonian H satisfies the following periodicity condition on the q-variables

(H2) $H(p,q,t) = H(p,q+m,t) \quad \forall (p,q) \in \mathbf{R}^{2n},\ \forall m \in \mathbf{Z}^n,\ \forall t \in \mathbf{R}.$

We call such an H a spatially periodic Hamiltonian. Under hypothesis (H2), condition (S) cannot longer be true. Instead, we assume a version of (S) for the variables p

(H3) There are constants $\mu > 1$, $r > 0$ such that

$$0 < \mu H(p,q,t) \leq p \cdot H_p(p,q,t) \quad \forall \mid p \mid \geq r,\ \forall q \in \mathbf{R}^n,\ \forall t \in \mathbf{R}.$$

We note that we only require $\mu > 1$, i.e. the Hamiltonian has to be superlinear in the variable p.

To begin we will consider the case of the function f being identically zero.

Theorem 0.1 *Suppose $f = 0$ and H satisfies* (H0) - (H3) *and*

(H4) *There are constants $a, b > 0$ and $s \leq \mu$ such that*

$$\mid H_q(p,q,t) \mid \leq a \mid p \mid^s + b \quad \forall (p,q) \in \mathbf{R}^{2n},\ \forall t \in \mathbf{R}.$$

Then the system

$$\dot{z} = JH_z(z,t) \tag{0.2}$$

possesses at least $n+1$ 2π-periodic solutions.

Strengthening hypothesis (H4) we can also treat the case of a forced Hamiltonian. Set $f(t) = (f_p(t), f_q(t))$ and consider

(f2) $\int_0^{2\pi} f_q(t)\,dt = 0.$

We will prove the following theorem.

Theorem 0.2 *Suppose f satisfies* (f0) - (f2) *and H satisfies* (H0) - (H3) *and*

(H4') *There are constants $a, b > 0$ and $s < \mu$ such that*

$$\mid H_z(p,q,t) \mid \leq a \mid p \mid^s + b \quad \forall (p,q) \in \mathbf{R}^{2n},\ \forall t \in \mathbf{R}.$$

Then the system (0.1) possesses at least $n+1$ 2π-periodic solutions.

Remark 0.1 By replacing 2π by any $T > 0$ we obtain the corresponding results for T-periodic solutions for (0.1) and (0.2). We can also generalize (H2) by assuming that H has a different period for each component of q.

Remark 0.2 Theorem 0.2 generalizes Theorem 1.5 of Rabinowitz in [20].

Equation (0.1), under spatially periodic assumptions has been studied by several authors. However in all cases the growth of the Hamiltonian is assumed to be at most quadratic. When

$$H(p,q,t) = L(q,t)p \cdot p + V(q,t) \tag{0.3}$$

with $L(q,t)$ an $n \times n$ symmetric matrix, H satisfying (H0), (H1) and (H2), and $f = (0, f_q)$ satisfying (f0), (f1) and (f2), Rabinowitz in [20] showed the existence of at least $n+1$ 2π-periodic solutions for (0.1). See also results of Fonda and Mawhin [9], Liu [13], where V is assumed periodic only in some of the variables. In [13] some resonant problems are also considered.

If $H(p,q,t)$ satisfies (H0), (H1) and (H2) and also

$$H(p+m,q,t) = H(p,q,t) \quad \forall (p,q) \in \mathbf{R}^{2n},\ \forall m \in \mathbf{Z}^M,\ \forall t \in \mathbf{R} \tag{0.4}$$

and H is of class C^2, $f \equiv 0$ the existence of at least $2n+1$ 2π-periodic solutions for (0.1), was proved by Conley and Zehnder [5]. Another proof was given by Chang [3]. Assuming that H is only C^1, Liu [13] and Szulkin [23] obtained the same conclusion.

When

$$H(p,q,t) = \frac{1}{2} Ap \cdot p + G(p,q,t) \tag{0.5}$$

with A an $n \times n$ symmetric matrix, satisfies (H0), (H1) and (H2), and G_{zz} is bounded, Chang [4] showed the existence of at least $n+1$ 2π-periodic solutions of (0.1). See also Fonda and Mawhin [9]. In [4], some intermediate situations, assuming that H is periodic only in some of the variables q and some resonant problems are considered. In [13] and [23] similar results where obtained assuming that G is only C^1. We also mention the work of Josellis [11] for related results.

The proof of Theorems 0.1 and 0.2 is based on a generalization of the Saddle Point Theorem of Rabinowitz [17]. We consider a functional $I : E \times M \longrightarrow \mathbf{R}$ of class C^1, where E is a Hilbert space and M is a compact manifold. Assuming that I satisfies a saddle point condition on E, uniformly on M, we prove that I possesses at least $c\ell(M)+1$ critical points, where $c\ell(M)$ denotes the cup length of the manifold M. This generalization of the Saddle Point Theorem was proved recently by Liu [13] using a notion of pseudo-category and a Galerkin approximation. Our version of the theorem considers slightly different hypotheses and the proof proceeds in a more standard way. The key ingredient in the proof is an intersection result we prove in Appendix A.

This paper is organized in 3 sections and 2 appendices. In Section 1 we prove some Saddle Point type theorems that we will use in the applications. In Section 2 we begin the proof of Theorems 0.1 and 0.2 by introducing the framework in which we study the problem. We introduce several splittings of the Sobolev space of interest and also we prove some estimates for various norms in the subspaces introduced. In Section 3 we prove Theorems 0.1 and 0.2 by verifying the hypothesis of the Saddle Point type theorems proved in Section 1. Appendix A is devoted to the proof of an intersection lemma. Appendix B has some remarks about the deformation Lemma.

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1 Saddle Point Type Theorems

In this section we prove a saddle point type theorem for functionals defined in $E \times M$, where E is a Hilbert space and M is a compact manifold. When M reduces to a point our theorem becomes the Saddle Point Theorem of Rabinowitz [17] and [2].

We assume that E has a splitting $E = X \oplus Y$ with $X \neq \{0\}$. The subspaces X and Y are not necessarily orthogonal and both of them can be infinite dimensional. Let $\|\cdot\|$ denotes the norm in E and let

$$Q = \{x \in X \,/\, \| x \| \leq R\}$$

where $R > 0$. We note that $\partial Q = \{x \in X \,/\, \| x \| = R\}$ and Y link in the sense of Benci and Rabinowitz [2] and [19].

We consider a functional $I : E \times M \longrightarrow \mathbf{R}$ of class C^1 having the following structure

$$I(z,\theta) = \frac{1}{2} < Lz, z > + b(z,\theta) \tag{1.1}$$

where $< \cdot, \cdot >$ denotes the inner product in E, and

(I1) $L : E \longrightarrow E$ is a linear, bounded, selfadjoint operator,

(I2) $b : E \times M \longrightarrow \mathbf{R}$ is a functional of class C^1 such that $\nabla_z b : E \times M \longrightarrow E$ is compact in the sense that $\nabla_z b(B \times M)$ is precompact whenever $B \subset E$ is bounded. Here $\nabla_z b$ represents the partial gradient of b with respect to $z \in E$, and

(I3) The subspace X is invariant for the linear operator L.

We will also need some compactness for the functional I as expressed via the Palais-Smale condition:

(P.S) Every sequence $\{(z_k, \theta_k)\}_{k \in \mathbf{N}} \subset E \times M$ such that

$$| I(z_k, \theta_k) | \leq c \quad \forall k \in \mathbf{N} \quad \text{and} \quad dI(z_k, \theta_k) \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

possesses a convergent subsequence.

Here we denote by dI the differential of I.

Our saddle point type theorem is a multiplicity result expressed in terms of a topological invariant of the manifold M, namely the cup length. Let Z be a topological space. Let us consider $\check{H}^*(Z)$, the Alexander-Spanier (A.S.) cohomology of Z with coefficients in $\mathbf{R}$, and let us denote by $\smile$ the cup product in $\check{H}^*(Z)$. See Spanier [22] for definition and properties of A.S. cohomology theory.

Definition 1.1 *We say that the cup length of Z is n, denoted by $c\ell(Z) = n$, if there are n elements $u_i \in \check{H}^{n_i}(Z)$, $n_i > 0$, such that $u_1 \smile \ldots \smile u_n \neq 0$ and n is maximal with this property. If such an n does not exist we set $c\ell(Z) = \infty$.*

Now we can state our theorem.

Theorem 1.1 *Let $I : E \times M \longrightarrow \mathbf{R}$ be a C^1 function satisfying the P.S. condition. Suppose that I satisfies* (I1), (I2), (I3) *and*
(I4) *There are constants $\alpha < \beta$ such that*
(i) $I(x,\theta) \leq \alpha \quad \forall x \in \partial Q,\ \forall \theta \in M$
(ii) $I(y,\theta) \geq \beta \quad \forall y \in Y,\ \forall \theta \in M$
(iii) *There is a constant γ such that*
$I(x,\theta) \leq \gamma \quad \forall x \in Q,\ \forall \theta \in M.$

Then I possesses at least $c\ell(M) + 1$ critical points with critical values greater than or equal to β.

The proof of this theorem goes in the standard way. We start by defining certain classes of functions and sets.

A function h belongs to the class Γ if it satisfies the following conditions:

(Γ1) $h : [0,1] \times E \times M \longrightarrow E \times M$ is continuous, and $h(t,\cdot,\cdot)$ is a homeomorphism for every $t \in [0,1]$,

(Γ2) $h(0,z,\theta) = (z,\theta) \quad \forall (z,\theta) \in E \times M.$

(Γ3) $P_E h(t,x,\theta) = x \quad \forall (x,\theta) \in \partial Q \times M,\ \forall t \in [0,1],$

(Γ4) $P_E h(t,z,\theta) = \exp(\nu(t,z,\theta)L)z + K(t,z,\theta)$ where $K : [0,1] \times E \times M \longrightarrow E$ is compact and $\nu : [0,1] \times E \times M \longrightarrow \mathbf{R}$ is continuous and it maps bounded sets into bounded sets.

Here P_E denotes the projection from $E \times M$ onto E. We note that the class Γ is not empty because $h(t,z,\theta) \equiv (z,\theta)$ belongs to Γ. Also the composition of two functions in Γ belongs to Γ as can be easily seen.

Now, for every $k = 1,2,\ldots,m$, with $m = c\ell(M) + 1$, we define a class of sets $\mathcal{A}_k$. A set A belongs to the class $\mathcal{A}_k$ if it satisfies:

(a1) $A = h(1, Q \times M \setminus K)$ where $h \in \Gamma$, $K \subset Q \times M$ and

$$\mathrm{cat}_{Q\times M}(K) \leq m - k$$

Here cat denotes the Ljusternik Schnirelmann (L.S.) category. We refer the reader to [21] for its definition and its basic properties that we will use later. The classes of sets $\mathcal{A}_k$ that we have defined are clearly not empty since $\mathcal{A}_m$ contains $Q \times M$. Also they are ordered by inclusion

$$\mathcal{A}_1 \supset \mathcal{A}_2 \ldots \supset \mathcal{A}_m.$$

The following intersection lemma is a key ingredient in our proof. We delay its proof to Appendix A.

Lemma 1.1 (Intersection Lemma) *If* $h \in \Gamma$ *and we define*

$$\mathcal{S}_h = \{(x,\theta) \in Q \times M \ / \ h(1,x,\theta) \in Y \times M\}$$

then $\mathrm{cat}_{Q\times M}(\mathcal{S}_h) \geq c\ell(M) + 1$.

In what follows we will use the following notation. For $c \in \mathbf{R}$ we set
$I^c = \{(x,\theta) \in E \times M \ / \ I(z,\theta) \leq c\}, \quad K = \{(z,\theta) \in E \times M \ / \ dI(z,\theta) = 0\}$
and $\quad K_c = \{(z,\theta) \in E \times M \ / \ I(z,\theta) = c, \ dI(z,\theta) = 0\}$

Proof of Theorem 1.1. We define the following values

$$c_k = \inf_{A\in\mathcal{A}_k} \sup_{(z,\theta)\in A} I(z,\theta) \quad k = 1,\ldots,m.$$

By (I4) (iii) the values c_k are bounded from above. On the other hand, if $A \in \mathcal{A}_k$ then $A = h(1, Q \times M \setminus K)$ with $\mathrm{cat}_{Q\times M}(K) \leq m - k$, and since by Lemma 1.1 $\mathrm{cat}_{Q\times M}(\mathcal{S}_h) \geq m$, we have $\mathrm{cat}_{Q\times M}(\mathcal{S}_h \setminus K) \geq k \geq 1$ and then $A \cap (Y \times M) \neq \emptyset$. Thus, by (I4) (ii), the numbers c_k are bounded from below by β. Then, taking in account the ordering in the sets $\mathcal{A}_k$, we have that

$$\beta \leq c_1 \leq \ldots \leq c_m \leq \gamma.$$

Let us show now that each value c_k is a critical value of the functional I. We write $c = c_k$. Let us assume that $K_c = \emptyset$ and define $\bar{\epsilon} = \frac{1}{2}(\beta - \alpha)$. From the Deformation Lemma (Appendix B), we get $0 < \epsilon < \bar{\epsilon}$ and η. Choose $A \in \mathcal{A}_k$ such that

$$\sup_{z\in A} I(z) \leq c + \epsilon.$$

We claim that $\eta(1, A) \in \mathcal{A}_k$. In fact, noting that from (I4) (i) we have that for every $x \in \partial Q$

$$I(x,\theta) \leq \alpha < c - \bar{\epsilon},$$

by the Deformation Lemma, (d0), (d2) and (d6), we conclude that $\eta \in \Gamma$. If $h \in \Gamma$ is such that $A = h(1, Q \times M \setminus K)$ then $\eta \circ h \in \Gamma$ and then $\eta(1, A) \in \mathcal{A}_k$.

Since $A \subset I^{c+\epsilon}$ by (d5) of the Deformation Lemma, $\eta(1, A) \subset I_{c-\epsilon}$ i.e.

$$\sup_{z\in\eta(1,A)} I(z) \leq c - \epsilon$$

contradicting the definition of c. If $c_1 < c_2 < \ldots < c_m$ then the proof is complete. Let us assume now that for $j \geq 1$ we have

$$c \equiv c_k = c_{k+1} = \ldots = c_{k+j}$$

and that $\mathrm{cat}_{Q\times M}(K_c) \leq j$. Let $\bar{\epsilon} = \frac{1}{2}(\beta - \alpha)$ and let U be a neighborhood of K_c such that

$$\mathrm{cat}_{E\times M}(K_c) = \mathrm{cat}_{E\times M}(U).$$

Let $\epsilon > 0$ and η obtained from the Deformation Lemma. By definition of c there is $A \in \mathcal{A}_{k+j}$ such that

$$\sup_{(z,\theta)\in A} I(z,\theta) \leq c + \epsilon$$

By definition $A = h(1, Q \times M \setminus K)$ with $h \in \Gamma$ and $\mathrm{cat}_{Q\times M}(K) \leq m - (k+j)$. Since $h(1,\cdot,\cdot)$ is a homeomorphism, by the invariance and monotonicity properties of L.S. category we have

$$\mathrm{cat}_{Q\times M}(h^{-1}(1,U) \cap (Q \times M)) \leq j.$$

Let $\tilde{K} = h^{-1}(1,U) \cap (Q \times M)$, then by the subadditive property of L.S. category

$$\mathrm{cat}_{Q\times M}(K \cup \tilde{K}) \leq m - (j+k) + j = m - k,$$

which implies

$$\tilde{A} = h^{-1}(1, Q \times M \setminus (K \cup \tilde{K})) \in \mathcal{A}_k \quad \text{and} \quad \tilde{A} \subset I_{c+\epsilon} \setminus U.$$

Then, arguing as before we have $\eta(1,\tilde{A}) \in \mathcal{A}_k$, and by the Deformation Lemma

$$\sup_{(z,\theta)\in\eta(1,\tilde{A})} I(z,\theta) \leq c - \epsilon$$

contradicting the definition of c.□

Remark 1.1 Theorem 1.1, with slightly different hypothesis, was proved by Liu [13]. He considers a notion of pseudo-category in order to obtain the intersection property needed to show that the critical values are bounded. The infinite dimensional nature of the problem is treated using a Galerkin approximation.

Remark 1.2 A result in the spirit of our theorem was obtained by Szulkin [23] by using a notion of relative category designed to treat the problem. See Theorem 3.8 in [23].

In our applications we will need a variation of Theorem 1.1 we consider next. Let $G : E \longrightarrow E$ be a homeomorphism satisfying

(g1) $G(x) = x \quad \forall x \in X,$

(g2) If $h \in \Gamma$ such that $P_E h(t,z,\theta) = \exp(\nu(t,z,\theta)L)z + K(t,z,\theta)$ then

$$G^{-1}(P_E h(t,z,\theta)) = \exp(\nu(t,z,\theta)L)x + \bar{K}(t,x,\theta)$$

$\forall (x,\theta) \in X \times M \; \forall t \in [0,1]$, where $\bar{K} : [0,1] \times X \times M \longrightarrow E$ is compact.

Theorem 1.2 *Let $I : E \times M \longrightarrow \mathbf{R}$ be a C^1 functional satisfying the P.S. condition. Suppose that I also satisfies* (I1), (I2), (I3) *and there is a homeomorphism G in E satisfying* (g1) *and* (g2) *for which*

(I4') *There are constants $\alpha < \beta$ such that*

(i) $I(x,\theta) \leq \alpha \quad \forall x \in \partial Q, \forall \theta \in M$

(ii) $I(G(y),\theta) \geq \beta \quad \forall y \in Y, \forall \theta \in M$

(iii) *There is a constant γ such that* $I(x,\theta) \leq \gamma \quad \forall x \in Q, \forall \theta \in M.$

Then I possesses at least $c\ell(M)+1$ critical points with critical values greater than or equal to β.

Proof. The proof follows the same lines as that of Theorem 1.1. Only the Intersection Lemma changes. We need to obtain an estimate on the category of the set

$$\mathcal{S}_h = \{(x,\theta) \in Q \times M \ / \ h(1,x,\theta) \in G(Y) \times M\},$$

for $h \in \Gamma$. But $h(1,x,\theta) \in G(Y) \times M$ is equivalent to

$$G^{-1} P_E h(1,x,\theta) \in Y.$$

Then by (g2) we see that

$$\mathcal{S}_h = \{(x,\theta) \in Q \times M \ / \ \exp(\nu(1,x,\theta)L)x + \bar{K}(1,x,\theta) \in Y\}.$$

Now we are in the same situation we were in Theorem 1.1.□

2 Application to spatially periodic Hamiltonian systems. Preliminaries

Our goal is to apply the Saddle Point type theorems proved in Section 1 to the study of periodic solutions of spatially periodic Hamiltonian system. In this section we set up the basic framework in which we treat the problem.

We consider the Hilbert space $\mathcal{E} = W^{1/2,2}(S^1,\mathbf{R}^{2n})$ consisting of functions $z \in L^2(S^1,\mathbf{R}^{2n})$ with

$$\| z \|_{\mathcal{E}}^2 = \sum_{j\in\mathbf{Z}} (1+ \mid j \mid) \mid a_j \mid^2 < \infty \tag{2.1}$$

where $z(t) = \sum_{j\in\mathbf{Z}} a_j e^{ijt}, \quad a_j = \bar{a}_j \in \mathbf{C}^{2n}$ is the Fourier series expansion of z. For $z = (p,q), \xi = (\varphi,\psi) \in \mathcal{E}$ and smooth we define

$$B(z,\xi) = \int_0^{2\pi} p \cdot \dot{\psi} + \varphi \cdot \dot{q}\, dt \quad \text{and} \quad A(z) = \frac{1}{2} B(z,z). \tag{2.2}$$

Both A and B can be extended continuously to the whole space $\mathcal{E}$, and the bilinear form B induces a linear, bounded selfadjoint, operator $L : \mathcal{E} \to \mathcal{E}$ defined by

$$B(z,\xi) = < Lz, \xi > \quad \forall z, \xi \in \mathcal{E}. \tag{2.3}$$

In (2.3) $< \cdot, \cdot >$ denotes the inner product in $\mathcal{E}$. We refer the reader to [19] for more details. Let us consider now a Hamiltonian H of class C^1 that satisfies the growth condition

(G) $\quad H(z,t) \leq a \mid z \mid^s + b \quad \forall z \in \mathbf{R}^{2n},\ \forall t \in \mathbf{R}$

for some constants $a, b > 0$ and $s > 1$. Then we can define the functional

$$\mathcal{H}(z) = \int_0^{2\pi} H(z(t), t)\, dt \tag{2.4}$$

on $\mathcal{E}$. The functional $\mathcal{H}$ is well defined. Moreover $\mathcal{H}$ is of class C^1 and its derivative $\mathcal{H}'$ is compact. See [19]. When f satisfies (f0) and (f1) we can define on $\mathcal{E}$ the functional

$$I(z) = A(z) - \mathcal{H}(z) - \int_0^{2\pi} f(t) \cdot z(t)\, dt \tag{2.5}$$

The following proposition gives the relation between the critical points of I and the solution of (0.1).See [19].

Proposition 2.1 *$z \in \mathcal{E}$ is a critical point of I if and only if z is 2π-periodic, continuously differentiable and it satisfies*

$$\dot{z} = J(H_z(z,t) + f). \tag{2.6}$$

We devote the rest of the section to defining some splittings of $\mathcal{E}$ and to obtaining some estimates involving the norms of various subspaces of $\mathcal{E}$.

Let us consider first the usual decomposition of $\mathcal{E}$. Let $\{e_1, \dots e_{2n}\}$ be the canonical basis of $\mathbf{R}^{2n}$ and define the following subspaces

$$\begin{aligned} E^+ &= \overline{\text{span}}^{\mathcal{E}}\{\sin(jt)e_k - \cos(jt)e_{k+n}, \\ &\qquad \cos(jt)e_k + \sin(jt)e_{k+n} \;/\; j \in \mathbf{N},\ 1 \leq k \leq n\}, \\ E^- &= \overline{\text{span}}^{\mathcal{E}}\{\sin(jt)e_k + \cos(jt)e_{k+n}, \\ &\qquad \cos(jt)e_k - \sin(jt)e_{k+n} \;/\; j \in \mathbf{N},\ 1 \leq k \leq n\}, \quad \text{and} \\ E^0 &= \text{span}\{e_1, \dots, e_{2n}\}. \end{aligned}$$

Then we have the decomposition: $\mathcal{E} = E^+ \oplus E^- \oplus E^0$. We observe that A is positive over E^+, negative over E^- and it vanishes over E^0. Moreover an equivalent and more convenient norm on $\mathcal{E}$ is defined by

$$\| z \|^2 = A(z^+) - A(z^-) + \mid z^0 \mid^2 \tag{2.7}$$

where $z = z^+ + z^- + z^0$, $z^+ \in E^+$, $z^- \in E^-$ and $z^0 \in E^0$, and $\mid \cdot \mid$ denotes the usual norm in $\mathbf{R}^{2n}$. We will consider also the space $L^\sigma \equiv L^\sigma(S^1, \mathbf{R}^{2n})$ for $\sigma > 1$. The following inequality relates the norm $\| \cdot \|$ to the usual norm $\| \cdot \|_\sigma$ in L^σ

$$\| z \| \geq \alpha_\sigma \| z \|_\sigma \quad \forall z \in \mathcal{E} \tag{2.8}$$

for certain constant $\alpha_\sigma > 0$. See [10] for a proof of (2.8).

There is another natural splitting of $\mathcal{E}$ by the components of the functions. Let us define the following subspaces of $\mathcal{E}$

$$\begin{aligned}
E_p &= \overline{\operatorname{span}}^{\mathcal{E}}\{\sin(jt)e_k, \cos(jt)e_k \;/\; j \in \mathbf{N}\cup\{0\},\ 1 \le k \le n\}, \\
E_q &= \overline{\operatorname{span}}^{\mathcal{E}}\{\sin(jt)e_{k+n}, \cos(jt)e_{k+n} \;/\; j \in \mathbf{N},\ 1 \le k \le n\}, \\
E_p^0 &= \operatorname{span}\{e_1, \ldots, e_n\} \quad \text{and} \\
E_q^0 &= \operatorname{span}\{e_{n+1}, \ldots, e_{2n}\}.
\end{aligned}$$

Remark 2.1 The space E_p contains the constant functions with zero q coordinate, but E_q does not contain constant functions other than zero.

Clearly the following decomposition of $\mathcal{E}$ holds $\mathcal{E} = E_p \oplus E_q \oplus E_q^0$. In our applications we will need another decomposition of $\mathcal{E}$, namely

$$\mathcal{E} = (E^- \oplus E_p^0) \oplus (E_q \oplus E_q^0). \tag{2.9}$$

By analyzing the Fourier series of elements in $\mathcal{E}$ it is easy to see that (2.9) holds. We define $X = E^- \oplus E_p^0$, $Y = E_q$ and $E = X \oplus Y$. Then we have a decomposition of $\mathcal{E}$ as $\mathcal{E} = X \oplus Y \oplus E_q^0$. Since the definition of the subspaces of $\mathcal{E}$ was made in terms of Fourier series, we can define corresponding subspaces of L^s. Thus we have the subspaces F^+, F^-, F^0, F_p, F_q, F_p^0, and F_q^0. The following two inequalities are obtained directly, by noting that the projection operators are bounded. If $z = z^- + z_p^0$, with $z^- \in F^-$ and $z_p^0 \in F_p^0$, then

$$\| z_p^0 \|_s \le \beta_0 \| z \|_s \tag{2.10}$$

and

$$\| z^- \|_s \le \beta_- \| z \|_s \tag{2.11}$$

for some constants β_0, $\beta_- > 0$.

Let us now consider $z \in F_q$. Then z can be decomposed as $z = z^+ + z^-$, with $z^+ \in F^+$ and $z^- \in F^-$ and we have the following lemma.

Lemma 2.1 *There are constants γ_1, $\gamma_2 > 0$ such that*

$$\gamma_1 \| z^\pm \|_s \le \| z \|_s \le \gamma_2 \| z^\pm \|_s, \quad \forall z \in F_q \tag{2.12}$$

and if $z \in E_q$ then

$$\| z^+ \| = \| z^- \| . \tag{2.13}$$

Proof. Given a function $u \in L^s(S^1, \mathbf{R}^n)$ with Fourier series $u = \sum_{j=0}^\infty a_j \cos(jt) + b_j \sin(jt)$ where a_j, $b_j \in \mathbf{R}^n$, we define the conjugate of u by

$$\bar{u} = \sum_{j=0}^\infty -b_j \cos(jt) + a_j \sin(jt). \tag{2.14}$$

If $z \in F_q$ then $z = (0, q)$ with $q \in L^s(S^1, \mathbf{R}^n)$, and we define

$$\tilde{z} = (\bar{q}, 0) \tag{2.15}$$

With this notation we have

$$\| \tilde{z} \|_s \leq k_s \| z \|_s \tag{2.16}$$

for some constant $k_s > 0$. This result is known as Theorem of M. Riesz and its proof can be found in Edwards [6]. Given $z \in F_q$, we can decompose it as

$$z = \frac{1}{2}(z - \tilde{z}) + \frac{1}{2}(z + \tilde{z}), \tag{2.17}$$

and by analyzing the Fourier series we see that $z^+ = \frac{1}{2}(z - \tilde{z}) \in F^+$ and $z^- = \frac{1}{2}(z + \tilde{z}) \in F^-$. Using the triangle inequality and (2.16) we obtain

$$\| z^+ \|_s \leq \frac{1}{2}(1 + k_s) \| z \|_s . \tag{2.18}$$

Since, $z(t) \in \{0\} \times \mathbf{R}^n$ and $\tilde{z}(t) \in \mathbf{R}^n \times \{0\}$ we easily see that

$$\| z \|_s^s \leq \| z - \bar{z} \|_s^s . \tag{2.19}$$

Then from (2.17) and (2.19) we have

$$\| z \|_s \leq 2 \| z^+ \|_s, \tag{2.20}$$

Inequalities (2.18) and (2.20) prove (2.12) for z^+. A similar argument gives the result for z^-. Finally, recalling that $z = (0, q)$ and $\tilde{z} = (\bar{q}, 0)$, and assuming z is smooth

$$A(\frac{1}{2}(z - \tilde{z})) = \frac{1}{2} \int_0^{2\pi} -\bar{q}.q \, dt \quad \text{and} \quad A(\frac{1}{2}(z + \tilde{z})) = \frac{1}{2} \int_0^{2\pi} \bar{q}.q \, dt \tag{2.21}$$

Then, from the definition of $\| \cdot \|$ in (2.7) and (2.21), it is clear that (2.13) is true for z smooth, and then it is true for every $z \in E_q$.□

3 Proof of Theorems 0.1 and 0.2

Here we prove Theorems 0.1 and 0.2, using Theorems 1.1 and 1.2. We will denote by a_i, b_i, c_i various constants appearing in the estimates.

Let us assume that the Hamiltonian H satisfies hypothesis (H0)-(H3) and (H4'). From hypothesis (H3) and (H4'), after integrating we obtain the following inequalities

$$| H(p, q, t) | \leq a_1 | p |^{s+1} + a_2 \quad \text{and} \tag{3.1}$$

$$H(p, q, t) \geq a_3 | p |^{\mu} - a_4. \tag{3.2}$$

By (3.1) the Hamiltonian H satisfies the growth condition (G), and assuming (f0) and (f1) we can define the functional I as in (2.5).

We can define a $\mathbb{Z}^n$-action on $\mathcal{E}$ by the following formula

$$mz = (p, q + m), \quad m \in \mathbb{Z}^n, \; z = (p, q) \in \mathcal{E}. \tag{3.3}$$

If we consider $\mathcal{E} = E \oplus E_q^0$, we easily see that $\mathcal{E}/\mathbb{Z}^n \cong E \times T^n$, where T^n denotes the n-dimensional torus. Because of (H2), (f2) and the definition of A, we see that $I(mz) = I(z) \quad \forall m \in \mathbb{Z}^n, \forall z \in \mathcal{E}$ so that I is $\mathbb{Z}^n$-invariant and then we can define I on $E \times T^n$. If $(w, \theta) \in E \times T^n$ we see that

$$I(w, \theta) = \frac{1}{2} < Lw, w > + b(w, \theta) \tag{3.4}$$

where $b(w, \theta)$ is given by

$$b(w, \theta) = -\int_0^{2\pi} H(z, t) + f(t) \cdot z(t) \, dt. \tag{3.5}$$

Here, and also in the future, we identify (w, θ) with $z(t) = w(t) + q_0$ where q_0 is a representative of the class θ.

In E, we consider the splitting $E = X \oplus Y$, where $X = E^- \oplus E_p^0$ and $Y = E_q$. Then one can see that X is an invariant subspace for L. This observation and the discussion given above prove the following lemma.

Lemma 3.1 *The functional $I : E \times T^n \longrightarrow \mathbb{R}$ satisfies hypothesis* (I0)-(I2). *If the space E is decomposed as above, then I also satisfies* (I3).

We show next that the functional I satisfies the P.S. condition.

Lemma 3.2 *$I : E \times T^n \longrightarrow \mathbb{R}$ satisfies the P.S. condition.*

Proof. Let us consider a sequence $(w_k, \theta_k) \in E \times T^n$ such that

$$I(w_k, \theta_k) \leq c \quad \text{and} \quad dI(w_k, \theta_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.6}$$

Certainly $\{\theta_k\}_{k \in \mathbb{N}}$ has a convergent subsequence. Let us analyze $\{w_k\}_{k \in \mathbb{N}}$ and show that it is bounded in E.

Let $w = w_k$, and decompose it as $w = w_p + w_q$ with $w_p \in E_p$ and $w_q \in E_q$. By (3.6) and for large k we have

$$\begin{aligned} c + \| w_p \| \;\geq\; & I(w, z) - I'(w, z) w_p = A(w) - A'(w) w_p - \\ & - \int_0^{2\pi} H(w + q_0, t) - H_p(w + q_0, t) \cdot w_p \, dt - \int_0^{2\pi} f \cdot w_q \, dt \end{aligned} \tag{3.7}$$

But $A'(w) w_p = B(w, w_p) = A(w)$, so from (3.7) and using (H3) in the standard way

$$c + \| w_p \| \geq (1 - \frac{1}{\mu}) \mu \int_0^{2\pi} H(w + q_0, t) \, dt - \int_0^{2\pi} f \cdot w_q \, dt. \tag{3.8}$$

Using the Schwarz inequality and (3.2) in (3.8), and then using (2.8) we obtain

$$\| w_p \|_\mu^\mu \leq a_5 \| w \| + a_9. \tag{3.9}$$

On the other hand, for large k we have from (3.6)

$$| A'(w)\xi - \int_0^{2\pi} H_z(w + q_0, t) \cdot \xi \, dt - \int_0^{2\pi} f \cdot \xi \, dt | \leq \| \xi \| \tag{3.10}$$

for every $\xi \in E$. Then, taking $\xi = w^+$ in (3.10), using Schwarz inequality, (2.8) and the definition of $\| \cdot \|$ we obtain

$$2 \| w^+ \|^2 \leq | \int_0^{2\pi} H_z(w + q_0, t) \cdot w^+ \, dt | + a_6 \| w^+ \| . \tag{3.11}$$

By hypothesis (H4'), Hölder inequality and (2.8) with $\sigma = \mu/\mu - s$ we have

$$| \int_0^{2\pi} H_z(w + q_0, t) \cdot w^+ \, dt | \leq a_7(1 + \| w_p \|_\mu^s) \| w^+ \| . \tag{3.12}$$

Thus, from (3.11) and (3.12) we obtain

$$\| w^+ \| \leq a_8(1 + \| w_p \|_\mu^s). \tag{3.13}$$

By taking $\xi = w^-$ in (3.10) and proceeding analogously we obtain a similar inequality for w^-, that together with (3.9) and (3.13) gives

$$\| w^+ \| + \| w^- \| \leq a_9(1 + \| w \|^{s/\mu}). \tag{3.14}$$

From (3.9) and considering (2.10) we have

$$| p^0 | \leq a_{10}(1 + \| w \|^{1/\mu}). \tag{3.15}$$

Inequalities (3.14) and (3.15) give

$$\| w \| \leq a_{11}(1 + \| w \|^{s/\mu} + \| w \|^{1/\mu}) \tag{3.16}$$

from which, noting that $\mu > 1$ and $\mu > s$, we conclude that the sequence $\{w_k\}_{k \in \mathbf{N}}$ is bounded in E. In particular $\{p_k^0\}_{k \in \mathbf{N}}$ is bounded. Now a standard argument shows that $\{w_k\}_{k \in \mathbf{N}}$ has a convergent subsequence. See [19].□

Next we will give a proof of Theorem 0.1 by showing that the hypothesis (I4) is satisfied.

Proof of Theorem 0.1. Since we are only assuming (H4), the functional (2.4) may not be defined on all of $\mathcal{E}$. Using a procedure employed by Rabinowitz, we will define a modified Hamiltonian that satisfies (H4') and then show that the solutions obtained for this modified problem are indeed solutions of the original problem.

Let $K > 0$ and $\chi \in C^\infty(\mathbf{R}^+, \mathbf{R}^+)$ such that $\chi(y) = 1$ if $0 \leq y \leq K$, $\chi(y) = 0$ if $y \geq K + 1$ and $\chi'(y) < 0$ for $K < y < K + 1$. We define

$$H_K(p, q, t) = \chi(| p |)H(p, q, t) + (1 - \chi(| p |))M \, | p |^\mu \tag{3.17}$$

where $M = M(K)$ is a number satisfying

$$M \geq \max_{K \leq |p| \leq K+1} \frac{H(p,q,t)}{|\, p \,|^{\mu}}. \tag{3.18}$$

Certainly H_K satisfies (H0), (H1) and (H2) and it is not hard to check that it also satisfies (H3). Note that the inequality (3.2) still holds for H_K with a_3 and a_4 independent of K if K is large. We also see that for some constants the following inequalities hold

$$H_K(p,q,t) \leq b_1 \mid p \mid^{\mu} + b_2 \tag{3.19}$$

and

$$\mid H_{Kp}(p,q,t) \mid \leq b_3 \mid p \mid^{\mu-1} + b_4. \tag{3.20}$$

From (3.20) and (H4) we see that (H4') is satisfied. Therefore the functional

$$I_K(z,\theta) = A(z) - \int_0^{2\pi} H_K(z + q_0, t)\, dt \tag{3.21}$$

is well defined in $\mathcal{E}$, it is of class C^1, and by Lemmas 3.1 and 3.2, it satisfies (I1)-(I3) and the P.S. condition. Here we are considering the splitting $E = X \oplus Y$ where $X = E^- \oplus E_p^0$ and $Y = E_q$.

Let us now show that the hypothesis (I4) of Theorem 1.1 is also satisfied. If $z \in E_q = Y$, then we have

$$I_K(z) = A(z) - \int_0^{2\pi} H_K(z,t)\, dt = -\int_0^{2\pi} H_K(z,t)\, dt. \tag{3.22}$$

But, since $z \in E_q$, by (H1) and (H2) there is a constant such that $H_K(z,t) \leq b_5$ so that, by taking $\beta = -2\pi b_5$ we have from (3.22) that

$$I_K(z) \geq \beta \quad \forall z \in E_q. \tag{3.23}$$

If we consider $x = z^- + p^0 \in X = E^- \oplus E_p^0$, then we have

$$I_K(x) = - \| z^- \|^2 - \int_0^{2\pi} H_K(z^- + p^0, t)\, dt. \tag{3.24}$$

Using (3.2) and the projection from F_p into F_p^0 we obtain from (3.24)

$$\begin{aligned} I_K(x) &\leq - \| z^- \|^2 - \int_0^{2\pi} a_3 \mid P_p(z^- + p^0) \mid^{\mu} - a_4\, dt \\ &\leq - \| z^- \|^2 - b_6 \mid p^0 \mid^{\mu} + b_7. \end{aligned} \tag{3.25}$$

Note that b_6 and b_7 are independent of K. From (3.25) follows the existence of a constant c such that

$$I_K(x) \leq c \quad \forall z^- + p^0 \in E^- \oplus E_p^0, \tag{3.26}$$

and there exists $R > 0$ so that for $\| x \| = R$

$$I_K(x) \leq \beta - 1 \equiv \alpha. \tag{3.27}$$

Inequality (3.23) still holds if we change z by $z + q_0$, where $q_0 \in E_q^0$. The same is true for inequalities (3.26) and (3.27). Then we see that hypothesis (I4) of Theorem 1.1 is satisfied. It is well known the cup length of the n-dimensional torus T^n is n, see for example [21]. Consequently, from Theorem 1.1, I_K possesses at least $n+1$ critical points.

We show next that the critical points of I_K obtained by the minimax method of Theorem 1.1 are bounded independent of K. Let $(w_K, \theta_K) \in E \times T^n$ be a critical point obtained from Theorem 1.1. Then

$$I_K(w_K, \theta_K) \leq \sup_{\|z^- + p^0\| \leq R} I_K(z^- + p^0, \theta) \leq c \tag{3.28}$$

where, as we noted earlier c is independent of K. Letting q_K^0 be a representative of θ_K and decomposing $w_K = p_K + q_K$ we have from (H3) and (3.2) that

$$c \geq (1 - \frac{1}{\mu})\mu(a_3 \| p \|_\mu^\mu - 2\pi a_4), \tag{3.29}$$

which implies there is a constant b_8 independent of K such that

$$\| p_K \|_\mu^\mu \leq b_8. \tag{3.30}$$

Inequality (3.30) shows that

$$| p_K^0 | \leq b_9 \tag{3.31}$$

for some constant b_9 independent of K.

Since (w_K, θ_K) is a critical point of I_K, from Proposition 2.1

$$\dot{p}_K = -\frac{\partial H_K}{\partial q}(w_K + q_K^0, t) \tag{3.32}$$

so that by hypothesis (H4)

$$\| \dot{p}_K \|_1 \leq a \| p_K \|_\mu^\mu + 2\pi b. \tag{3.33}$$

If $\tilde{p}_K = p_K - p_K^0$, then

$$\tilde{p}_K(t) - \tilde{p}_K(s) = \int_s^t \dot{p}_K(r)\, dr, \tag{3.34}$$

integrating (3.34) with respect to s and taking absolute values we obtain

$$2\pi \mid \tilde{p}_K(t) \mid \leq \int_0^{2\pi} \int_s^t \mid \dot{p}_K(r) \mid \, dr\, dt \leq 2\pi \| \dot{p}_K \|_1 . \tag{3.35}$$

Consequently, from (3.30), (3.33) and (3.35) we find a constant b_{10} such that

$$| \tilde{p}_K(t) | \leq b_{10} \quad \forall t \in \mathbf{R}$$

and this inequality together with (3.31) implies that $p_K(t)$ is bounded by a constant independent of K. Thus, if K is large enough the $n+1$, or more, critical points of I_K are really solutions of the original equation (0.2) completing the proof of the theorem.□

Next we prove Theorem 0.2. Here the introduction of the forcing term f, in particular its q-component, produces difficulties due to the degeneracy of H in the q direction. For this reason we cannot apply Theorem 1.1. However with some modifications, introducing a homeomorphism G we can use Theorem 1.2. We start by stating a simple lemma whose proof is obtained by standard calculus.

Lemma 3.3 *Let $\gamma > 2$ and consider the real valued function $g : \mathbf{R}_0^+ \times [0,1) \longrightarrow \mathbf{R}$ defined by*

$$g(\lambda,\alpha) = (1-\alpha^2)\lambda^2 - c_1(1-\alpha)^\gamma\lambda^\gamma - c_2(1+\alpha)\lambda - c_3.$$

Then there exists a continuous function $\bar{\alpha} : \mathbf{R}_0^+ \longrightarrow [0,1)$ and a constant C depending on c_1, c_2 and c_3 such that

$$g(\lambda, \bar{\alpha}(\lambda)) \geq C \quad \forall \lambda \in \mathbf{R}_0^+.$$

Proof of Theorem 0.2. Since H satisfies (H0)-(H3) and (H4'), from Lemma 3.1 and 3.2, the functional

$$I(z) = A(z) - \int_0^{2\pi} H(z,t) + f\cdot z\, dt$$

defined in $E \times T^n$ satisfies the hypothesis (I1)-(I3) of Theorem 1.2. Now we prove it also satisfies (I4') for a certain G. Since H satisfies (H4'), after integrating we obtain constants c_4, c_5

$$H(p,q,t) \leq c_4 \mid p \mid^{s+1} + c_5 \tag{3.36}$$

and then, perhaps with a different constant c_6

$$H(p,q,t) \leq c_4 \mid p \mid^{\gamma} + c_6 \tag{3.37}$$

with $\gamma > 2$. We consider as before the splitting $E = X \oplus Y$. We consider $z \in E$ decomposed as

$$z = z^- + p^0 + q, \quad z^- + p^0 \in X,\ q \in Y.$$

Since E also admits the splitting $E = E^+ \oplus E^- \oplus E^0$, every $q \in Y$ can be decomposed as

$$q = q^+ + q^-, \quad q^+ \in E^+,\ q^- \in E^-.$$

We define $G : E \longrightarrow E$ as $G(z) = z^- + p^0 + q^+ - \bar{\alpha}(\| q \|_\gamma)q^-$ where $\bar{\alpha}$ is the function defined in Lemma 3.3. It is easy to check that G is a homeomorphism by giving an explicit formula for G^{-1}.

From the definition of G, it is clear that for every $x = x^- + p^0 \in X$, $G(x) = x$. Thus (g1) is true. In order to check (g2), let us consider P_X and P_Y, the projections onto X and Y induced by the splitting $E = X \oplus Y$, and P_{E^-} the projection onto

E^- induced by the splitting $E = E^+ \oplus E^- \oplus E^0$. Let $h \in \Gamma$, where Γ is the class of functions defined in Section 1. Then

$$P_E h(t, z, \theta) = \exp(\nu(t, z, \theta)L)z + K(t, z, \theta)$$

where K is compact. If $x \in X$, $x = x^- + p^0$, $\theta \in T^n$, then

$$P_E h(t, z, \theta) = \exp(\nu(t, x, \theta)L)x + P_X K(t, x, \theta) + P_Y K(t, x, \theta) \tag{3.38}$$

so for $F = G^{-1}$ we have

$$\begin{aligned} F(P_E h(t, x, \theta)) &= \exp(\nu(t, x, \theta)L)x + K(t, x, \theta) - \\ &\quad (\bar{\alpha}(\| P_X K(t, x, \theta) \|_\gamma) - 1)P_- P_X K(t, x, \theta). \end{aligned} \tag{3.39}$$

Since $\bar{\alpha}$ is a bounded function and $P_- \circ P_X$ is continuous

$$\bar{K}(t, x, \theta) = K(t, x, \theta) - (\bar{\alpha}(\| P_X K(t, x, \theta) \|_\gamma) - 1)P_- P_X K(t, x, \theta)$$

is compact, so that (g2) is satisfied.

Now we study hypothesis (I4'). First consider I over $G(Y)$. Let $q \in Y$, and $z = G(q)$. By the definition of G, $z = q^+ + \alpha q^-$, where $q = q^+ + q^-$, $q^+ \in E^+$, $q^- \in E^-$ and $\alpha = \bar{\alpha}(\| q \|_\gamma)$. By Lemma 2.1, and (2.8) we have

$$A(G(q)) = \| q^+ \|^2 - \alpha^2 \| q^- \|^2 \geq c_7(1 - \alpha^2) \| q \|_\gamma^2, \tag{3.40}$$

For $q \in Y$, let $\tilde{q}$ be the conjugate of q as defined in (2.15). Then from (2.17)

$$\begin{aligned} q^+ + \alpha q^- &= \frac{1}{2}(q - \tilde{q}) + \frac{1}{2}\alpha(q + \tilde{q}) \\ &= \frac{1}{2}(1 + \alpha)q + \frac{1}{2}(\alpha - 1)\tilde{q}. \end{aligned} \tag{3.41}$$

We note that $\tilde{q} \in E_p$. Hence if P_p denotes the projection onto E_p induced by the splitting $E = E_p \oplus E_q$, and using (2.16)

$$\| P_p(q^+ + \alpha q^-) \|_\gamma = \frac{1}{2}(1 - \alpha) \| \tilde{q} \|_\gamma \leq c_8(1 - \alpha) \| q \|_\gamma . \tag{3.42}$$

Thus by (3.37) and (3.42)

$$\int_0^{2\pi} H(G(q), t)\, dt \leq \int_0^{2\pi} c_4 \mid P_p(q^+ + \alpha q^-) \mid^\gamma + c_6\, dt \leq c_9(1 - \alpha)^\gamma \| q \|_\gamma^\gamma + 2\pi c_6. \tag{3.43}$$

Finally, by the Hölder inequality, Lemma 2.1 and the definition of G we have

$$\int_0^{2\pi} f \cdot G(q)\, dt \leq c_{10}(1 + \alpha) \| q \|_\gamma . \tag{3.44}$$

Then, (3.40), (3.43) and (3.44) imply

$$\begin{aligned} I(G(q)) \ \geq \ & c_8(1-\alpha^2) \parallel q \parallel_\gamma^2 - c_9(1-\alpha)^\gamma \parallel q \parallel_\gamma^\gamma - \\ & c_{10}(1+\alpha) \parallel q \parallel_\gamma - 2\pi c_6. \end{aligned} \tag{3.45}$$

Recalling that $\alpha = \bar{\alpha}(\parallel q \parallel_\gamma)$, and taking $\lambda = \parallel q \parallel_\gamma$, Lemma 3.3 yields the existence of a constant β so that

$$I(G(q)) \geq \beta \quad \forall q \in Y. \tag{3.46}$$

We consider now I over $X = E^- \oplus E_p^0$. Let $x \in E^- \oplus E_p^0$ with $x = x^- + p^0$, $x^- \in E^-$, $p^0 \in E_p^0$. Then

$$I(x) = A(x^-) - \int_0^{2\pi} H(x^- + p^0, t)\, dt - \int_0^{2\pi} f \cdot (x^- + p^0)\, dt. \tag{3.47}$$

Using Schwarz inequality, (2.8), (2.10) and (3.2) in (3.47) we obtain

$$I(x) \leq - \parallel x^- \parallel^2 - c_{11} \mid p_0 \mid^\mu + c_{12} \parallel x^- \parallel + c_{13} \mid p^0 \mid + 2\pi b_2. \tag{3.48}$$

From (3.48) follows the existence of a constant c so that

$$I(x) \leq c \quad \forall x \in X \tag{3.49}$$

and for some $R > 0$

$$I(x) \leq \beta - 1 \equiv \alpha \quad \forall x \in X, \ \parallel x \parallel = R. \tag{3.50}$$

Introducing $\theta \in T^n$ in the argument of I in (3.46), (3.49) and (3.50) does not alter the inequalities then hypothesis (I4') of Theorem 1.2 is satisfied, and so the existence of at least $n+1$ critical points of I follows. This with Proposition 2.1 completes the proof.□

4 Appendix A.

This Appendix is devoted to the proof of the Intersection Lemma. This lemma can be proved by studying a more general problem having to do with the continuation of solutions of certain equations.

Let us consider a Hilbert space X and let $Q = \{x \in X \ / \ \parallel x \parallel \leq R\}$ where $R > 0$ and $\parallel \cdot \parallel$ denotes the norm in X. Let M be a compact manifold and let us assume we have a homotopy

$$F : [0,1] \times Q \times M \longrightarrow X$$

satisfying the following conditions:

(E1) $F(t,x,\theta) = x + K(t,x,\theta)$ where $K : [0,1] \times Q \times M \longrightarrow X$ is compact,

(E2) $F(0,x,\theta) = x \quad \forall x \in Q, \ \forall \theta \in M$ and

(E3) $\| F(t,x,\theta) \| \geq \alpha > 0 \quad \forall x \in \partial Q,\ \forall t \in [0,1],\ \forall \theta \in M.$

We consider the projection $g : Q \times M \longrightarrow M$, and we abuse notation by denoting also by g several of its restrictions. We define the set

$$\mathcal{S} = \{(x,\theta) \in Q \times M \ / \ F(1,x,\theta) = 0\}$$

then we have the following proposition

Proposition 4.1 *If $g : \mathcal{S} \longrightarrow M$ is the projection then*

$$g^* : \check{H}^*(M) \longrightarrow \check{H}^*(\mathcal{S})$$

is a monomorphism.

Here $\check{H}^*$ denotes the A.S. cohomology.

Remark 4.1 The set

$$\mathcal{S}_0 = \{(x,\theta) \in Q \times M \ / \ F(0,x,\theta) = 0\}$$

is exactly $\{0\} \times M$. Thus $g : \mathcal{S}_0 \longrightarrow M$ induces an isomorphism in cohomology. The hypothesis on the homotopy H allow us to continue $\mathcal{S}_0$ to $\mathcal{S}$ without losing the fact that g induces a monomorphism.

Proof. We start noting that the set $\mathcal{S}$ is a compact subset of $Q \times M$. Hence by the continuity property of A.S. cohomology $\check{H}^*(\mathcal{S}) = \lim_{\rightarrow} \check{H}^*(V)$, where the direct limit is taken over all neighborhoods of $\mathcal{S}$ in $Q \times M$. Since M is a compact manifold, its cohomology ring is finitely generated. Consequently in order to prove the proposition, it is enough to show that $g^* : \check{H}^*(M) \longrightarrow \check{H}^*(V)$ is a monomorphism, where $V \supset \mathcal{S}$ is an open neighborhood of $\mathcal{S}$ in $Q \times M$.

Now we reduce the problem to a finite dimensional one. Since the map K is compact, given $\epsilon > 0$ there exists a finite dimensional subspace $\tilde{X}$ in X and a continuous function K_ϵ such that $K_\epsilon : [0,1] \times Q \times M \longrightarrow \tilde{X}$,

$$\| K_\epsilon(t,x,\theta) - K(t,x,\theta) \| \leq \epsilon \quad \forall (x,\theta) \in Q \times M,\ \forall t \in [0,1], \tag{4.1}$$

and

$$K_\epsilon(0,x,\theta) = 0 \quad \forall (x,\theta) \in Q \times M. \tag{4.2}$$

See [21]. Define the set

$$\mathcal{S}_\epsilon = \{(x,\theta) \in Q \times M \ / \ x + K_\epsilon(1,x,\theta) = 0\}.$$

Then one can show that for ϵ small enough, $\mathcal{S}_\epsilon \subset V$. By making ϵ smaller if necessary, we can assume $\epsilon < \frac{\alpha}{2}$. Define the function $f : [0,1] \times \tilde{Q} \times M \longrightarrow \tilde{X}$ by restricting $I + K_\epsilon$ to $[0,1] \times \tilde{Q} \times M$, where $\tilde{Q} = Q \cap \tilde{X}$. If we define the set

$$\tilde{\mathcal{S}} = \{(x,\theta) \in \tilde{Q} \times M \ / \ f(1,x,\theta) = 0\}$$

then we have $\tilde{\mathcal{S}} \subset \mathcal{S}_\epsilon \subset \mathcal{S}$. Defining $\tilde{g} : \tilde{\mathcal{S}} \longrightarrow M$ as the projection, and denoting by i the inclusion map we have the following commutative diagram

$$\begin{array}{ccc} \check{H}^*(\tilde{\mathcal{S}}) & \xleftarrow{\tilde{g}^*} & \check{H}^*(M) \\ {\scriptstyle i^*}\nwarrow & & \swarrow{\scriptstyle g^*} \\ & \check{H}^*(V) & \end{array}$$

from which it is clear that g^* is a monomorphism if $\tilde{g}^*$ is a monomorphism. By using the continuity property of A.S. cohomology again, we see that it is enough to prove that $\tilde{g}^* : \check{H}^*(M) \longrightarrow \check{H}^*(\tilde{V})$ is a monomorphism, where $\tilde{V}$ is a neighborhood of $\tilde{\mathcal{S}}$ in $\tilde{Q} \times M$. for notational convenience we suppress the tilde from now on. Let us define

$$\bar{Z} = \{(x, \theta) \in Q \times M \ / \ f(t, x, \theta) \neq 0 \ \forall t \in [0, 1]\}$$

and consider the following homotopy of pairs

$$h : [0, 1] \times (Q \times M, \bar{Z}) \longrightarrow (Q \times M, (Q \setminus \{0\}) \times M)$$

defined by $h(t, x, \theta) = (f(t, x, \theta), g(x, \theta))$. By (E3), (4.1) and the fact that $\epsilon < \frac{\alpha}{2}$ we have the following commutative diagram

$$\begin{array}{ccc} H^*(Q \times M, \bar{Z}) & \xleftarrow{h_0^*} & H^*(Q \times M, (Q \setminus \{0\} \times M) \\ {\scriptstyle j^*}\nwarrow & & \swarrow{\scriptstyle i^*} \\ & H^*(Q \times M, \partial Q \times M) & \end{array}$$

where i and j denote inclusion maps. Since i^* is an isomorphism we conclude that h_0^* is a monomorphism. Here we use singular cohomology since we are working with open subsets of an euclidean space and with manifolds.

Then, from the homotopy axiom for cohomology we obtain that

$$h_1 = (f_1, g) : (Q \times M, \bar{Z}) \longrightarrow (Q \times M, (Q \setminus \{0\}) \times M)$$

induces a monomorphism in cohomology. Let

$$Z' = \{(x, \theta) \in Q \times M \ / \ f_1(x, \theta) \neq 0 \ \}.$$

Since $\bar{Z} \subset Z'$ we obtain that

$$\tilde{h}_1 : (Q \times M, Z') \longrightarrow (Q \times M, (Q \setminus \{0\}) \times M)$$

induces a monomorphism. Since $\bar{V}^c$ is open and $\bar{V}^c \subset \mathrm{int}(Z') = Z'$, defining $Z = Z' \cap V$ and using the excision axiom for cohomology, we conclude that

$$h_1 : (V, Z) \longrightarrow (Q \times M, (Q \setminus \{0\}) \times M)$$

induces a monomorphism in cohomology. Let e be a generator of $H^n(Q, Q \setminus \{0\})$ and $u \in H^i(M)$ where $n = \dim(M)$, $i \geq 0$. Then

$$h_1^*(e \times u) = \Delta^*(f_1^*(e) \times g^*(u)) = f_1^*(e) \smile g^*(u) \tag{4.3}$$

by the basic relation between cross product and cup product in cohomology. Since h_1^* is a monomorphism, from (4.3) it follows that g^* is a monomorphism, and so the proof is complete.□

Now we can prove the Intersection Lemma. We will need to introduce the cup length of a subspace of a topological space.

Definition 4.1 *Given a topological space Y and $A \subset Y$, and letting $j_A : A \to Y$ be the inclusion, we say that $c\ell_Y(A) = n$ if there exist $u_i \in \check{H}^{n_i}(Y)$, $n_i > 0$, $i = 1, \ldots, n$ such that*

$$j_A^*(u_1 \smile \ldots \smile u_n) \neq 0$$

and n is maximal with this property. If such an n does not exist then we set $c\ell_Y(A) = \infty$.

The following proposition relates the cup length with the L.S. category. The proof of this proposition can be obtained by modifying appropriately the arguments given in [21] to prove the case $A = Y$. For a proof in details see [8].

Proposition 4.2 *If Y is a topological space and $A \subset Y$, then*

$$\mathrm{cat}_Y(A) \geq c\ell_Y(A) + 1$$

Proof of the Intersection Lemma. We are given $h \in \Gamma$ and we have

$$\mathcal{S}_h = \{(x, \theta) \in Q \times M \ / \ h(1, x, \theta) \in Y \times M\}.$$

From Γ1-Γ4 and (I3) we can define

$$F(t, x, \theta) = x + \exp(-\nu(t, x, \theta)L)P_X K(t, x, \theta)$$

and we see that

$$\mathcal{S}_h = \{(x, \theta) \in Q \times M \ / \ F(1, x, \theta) = 0\}.$$

We have that F satisfying (E1)-(E3). Then in view of Proposition 4.2, it is enough to show that

$$c\ell_{Q \times M}(\mathcal{S}) \geq c\ell(M).$$

Let us consider the following commutative diagram

$$\begin{tikzcd} \check{H}^*(Q \times M) & & \check{H}^*(M) \arrow[ll, "\pi^*"'] \arrow[dl, "g^*"] \\ & \check{H}^*(\mathcal{S}) \arrow[from=ul, "j^*"] & \end{tikzcd}$$

where j is the inclusion and π and g are the projections into M. Since g^* is a monomorphism and π^* is an isomorphism, if $u_i \in \check{H}^{n_i}(M)$, $i = 1, \ldots, k$, $n_i > 0$ and $u_1 \smile \ldots \smile u_k \neq 0$, then $j^*(\pi^* v_1 \smile \ldots \smile \pi^* v_k) = g(u_1 \smile \ldots \smile u_k) \neq 0$. Thus the proof is complete.□

5 Appendix B

In this Appendix we state the Deformation Lemma as needed in the proof of the Saddle Point type theorem given in Section 2.1.

Lemma 5.1 (Deformation Lemma) *Let E be a Hilbert space and M be a compact manifold. Let $I : E \times M \longrightarrow \mathbf{R}$ be a functional of class C^1 satisfying* (I1), (I2) *and the P.S. condition.*

For every $c \in \mathbf{R}$, $\bar{\epsilon} > 0$ and U a neighborhood of K_c, there exists $\epsilon > 0$, $\epsilon > \bar{\epsilon}$ and a homotopy $\eta : [0,1] \times M \to M$ such that

(d0). $\eta(0, z, \theta) = (z, \theta)$ $\forall (z, \theta) \in E \times M$.

(d1). $\eta(t, \cdot)$ *is a homeomorphism* $\forall t \in [0,1]$.

(d2). $\eta(t, z, \theta) = (z, \theta)$ *if* $| I(z,\theta) - c | \geq \bar{\epsilon}$ $\quad \forall t \in [0,1]$.

(d3). $I(\eta(t, z, \theta)) \leq I(z, \theta)$ $\quad \forall (z, \theta) \in E \times M$, $\forall t \in [0,1]$.

(d4). $\eta(1, I^{c+\epsilon} \setminus U) \subset I^{c-\epsilon}$.

(d5). *If* $K_c = \emptyset$ *then* $\eta(1, I^{c+\epsilon}) \subset I^{c-\epsilon}$.

(d6). $P_E \eta(t, z, \theta) = \exp(\nu(t, z, \theta) L) z + K(t, z, \theta)$ *where* $0 \leq \nu(t, z, \theta) \leq 1$ *and* $K : [0,1] \times E \times M \longrightarrow E$ *is compact.*

Proof. We don't give a proof of this lemma since it follows with minor modifications from similar results. First (d0)-(d5) is obtained as a consequence of the general result for Riemannian manifolds given, for example, in [15]. The extra structure of the deformation map are obtained by the assumptions (I1) and (I2). The deformation η is constructed as a solution of a initial value problem

$$\frac{d\eta}{dt} = -\omega(\eta) V(\eta), \quad \eta(0, z, \theta) = (z, \theta), \tag{5.1}$$

where V is a pseudo-gradient of I and $0 \leq \omega \leq 1$. When (5.1) is projected onto E we obtain a situation similar to that of Proposition A.18 in [19]. Following the proof of that proposition with minor modifications we obtain (d6). See Lemma 3.4 in [23] were some of the details are carried out. □.

References

[1] A. Bahri and H. Berestycki, *Forced vibrations of superquadratic Hamiltonian systems*, Acta Math. 152, 143-197(1984).

[2] V. Benci and P. Rabinowitz, *Critical points theorems for indefinite functionals*, Invent. Math. 52, 241-273 (1979).

[3] K. C. Chang, *Applications of homology theory to some problems in differential equations.* In Nonlinear Analysis and Applications (F.E. Browder, ed), Proc. Symp. Pure Math. 45, AMS, Providence. 253-262(1986).

[4] K. C. Chang, *On the periodic nonlinearity and the multiplicity of solutions,* Nonlinear Analysis, 13 (1989), 527-538.

[5] C. Conley and E. Zehnder, *The Birkhoff-Lewis fixed point theorem and a conjecture of V. Arnold,* Invent. Math. 73 (1983), 33-49.

[6] R. E. Edwards, *Fourier Series, a Modern Introduction* Vol II, Holt, Rinehart and Winston, Inc. N.Y., 1967.

[7] E. Fadell, *Cohomological methods in non-free G-spaces with applications to general Borsuk-Ulam theorems and critical point theorems for invariant functionals,* Nonlinear Functional Analysis and its Applications, ed by S.P. Singh. Nato ASI Series C, Vol.173, 1986, 1-45.

[8] E. Fadell, *Lectures in cohomological index theories of G-spaces with applications to critical point theory,* Raccolta Di, Universita della Calabria, 1987, Cosenza, Italy.

[9] A. Fonda, J. Mawhin, *Multiple periodic solutions of conservative systems with periodic nonlinearity,* Preprint.

[10] A. Friedman, *Partial Differential Equations*, Holt, Rinehart, and Winston, Inc., New York, 1969.

[11] F. Josellis, *Forced oscillations on $T^n \times \mathbf{R}^n$ having a prescribed rotation vector*, Preprint.

[12] Y. Long, *Periodic solutions of perturbed superquadratic Hamiltonian systems*, CMS Technical Summary Report #88-23, Madison, 1987.

[13] J.Q. Liu, *A generalized saddle point theorem*, Preprint.

[14] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences 74, Springer Verlag, New York, 1989.

[15] R. Palais, *The Ljusternik-Schnirelmann theory on Banach manifolds*, Topology 5, 115-132(1966).

[16] P. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math., 31, 157-184(1978).

[17] P. Rabinowitz, *Some minimax theorems and applications to nonlinear partial differential equations*, Nonlinear Analysis: A collection of papers in honor of Erich Röthe, Academic Press, New York, 1978, 161-177.

[18] P. Rabinowitz, *On subharmonic solutions of Hamiltonian Systems*, Comm. Pure Appl. Math., 33, 609-633 (1980).

[19] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, C.B.M.S. Regional Conference Series in Mathematics, 65, A.M.S., Providence, 1986.

[20] P. Rabinowitz, *On a class of functionals invariant under a $\mathbb{Z}^n$-action,* Trans. Amer. Math. Soc. Vol 310, No 1, Nov. 1988. 303-311.

[21] J. Schwartz, *Nonlinear Functional Analysis,* Gordon and Breach Sc. Pub. Inc. New York. 1969.

[22] E.H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[23] A. Szulkin, *A relative category and applications to critical point theory for strongly indefinite functionals*, Preprint.