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TECHNICAL MEMORANDUM

# TWO ENHANCEMENTS OF THE LOGARITHMIC LEAST-SQUARES METHOD FOR ANALYZING SUBJECTIVE COMPARISONS



MICHAEL L. McCURDY

25 MARCH 1989

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This memorandum presents two extensions of the logarithmic least-squares method for analyzing subjective comparisons. The first is a test of consistency that is based on analysis-of-variance techniques. Unlike Saaty's eigenvalue-based consistency ratio, the consistency metric developed here is independent of any scale used to elicit comparisons. It is easily applied to sets of comparisons that are incomplete or contain multiple comparisons. This metric provides valuable insights into the trade-off between data set com-

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pleteness and our ability to detect inconsistency. The second extension is to use robust regression for calculating the ratio scale that best represents most of the data. These extensions combine to make the logarithmic least-squares method an attractive alternative to Saaty's eigenvalue method.

# ABSTRACT

This memorandum presents two extensions of the logarithmic least-squares method for analyzing subjective comparisons. The first is a test of consistency that is based on analysis-of-variance techniques. Unlike Saaty's eigenvalue-based consistency ratio, the consistency metric developed here is independent of any scale used to elicit comparisons. It is easily applied to sets of comparisons that are incomplete or contain multiple comparisons. This metric provides valuable insights into the trade-off between data set completeness and our ability to detect inconsistency. The second extension is to use robust regression for calculating the ratio scale that best represents most of the data. These extensions combine to make the logarithmic least-squares method an attractive alternative to Saaty's eigenvalue method.

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TWO EXTENSIONS OF THE LOGARITHMIC LEAST-SQUARES METHOD FOR ANALYZING SUBJECTIVE COMPARISONS

## INTRODUCTION

1. This technical memorandum presents two extensions of the logarithmic least-squares (LLS) method for analyzing sets of subjective comparisons. The first is a test of consistency based on analysis-of-variance techniques. Unlike Saaty's eigenvalue-based consistency ratio, the consistency metric developed here does not depend on a scale for eliciting comparisons. This metric is easily applied to comparisons sets that do not contain comparisons between all possible pairs of entities or that contain multiple comparisons between some pairs. Furthermore, this consistency metric provides valuable insights into the trade-off between the number of comparisons in a set and our ability to detect inconsistency.

2. The second extension is a robust regression approach to finding a ratio scale that best represents most of the comparisons. Because the LLS method is a least-squares technique, it is sensitive to outliers, i.e., comparisons that contain relatively large errors. One or more outliers in a comparisons set can seriously degrade the validity of the ratio scale derived from it. The robust regression method selectively de-emphasizes outliers, providing a ratio scale that best represents most of the comparisons.

## CONTENTS

3. This memorandum defines terminology, then describes and compares the logarithmic least-squares and eigenvalue methods for estimating ratio scales and assessing consistency. It discusses the two extensions of the LLS method described above and presents examples of their application. The memorandum concludes by summarizing the extensions.

## DEFINITIONS

4. Paralleling Crawford's definitions [1], we define a collection of n

entities $\{E_i\}$ and the existence of an unknown ratio scale $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$, where $x_i$ is the weight of the $i^{th}$ entity and $x_i/x_j$ is the relative value of $E_i$ to $E_j$. We call the scale $\mathbf{x}$ the underlying ratio scale for the relative values of the entities and refer to its entries as weights. We also assume a set of comparisons

$$A = \{a_{ijk}\} \qquad i,j = 1,\ldots,n \qquad k = 1,\ldots,n_{ij}$$

where $a_{ijk}$ is the $k^{th}$ comparison between entities i and j. A comparison is an estimate of the relative value of $E_i$ and $E_j$, i.e., of $x_i/x_j$. To assure symmetry, we require that $a_{jik} = 1/a_{ijk}$. We also require that $a_{ijk}$ be positive for all i, j, and k. We use the set of comparisons to estimate the underlying ratio scale. It is common practice to compare a single set of entities on the basis of several attributes to obtain a ratio scale for each attribute. The relative importance of the attributes to the overall goal is then determined, and the ratio scale for each of the attributes is combined with the ratio scale for the overall goal to provide an estimate of the values of the entities relative to the overall goal. This is the basis for Saaty's analytic hierarchy process [2]. However, this memorandum deals only with estimating ratio scales from sets of comparisons.

5. Two methods that are frequently used to estimate $\mathbf{x}$ are Saaty's eigenvalue method [2] and Crawford and Williams' logarithmic least-squares method [3]. These methods are discussed below. Both methods are approaches to filtering data in the comparisons set to eliminate errors in estimating the ratios.

6. In addition to estimating the values of the weights, we are concerned with the consistency of the comparisons set. If A is a set of comparisons, A is perfectly consistent if $a_{ijk} = a_{ilk}a_{ljk}$ for all i, j, k, and l and all comparisons of entities i and j are the same, i.e., $a_{ijk}$ is independent of k for all i and j. We consider randomness as the antithesis of consistency, and accordingly test for consistency by hypothesizing that the comparisons set is random in the hope of rejecting this hypothesis. If we cannot reject this hypothesis, the weights derived from the comparisons set are probably not meaningful and should not be used.

7. If $n_{ij} < 1$ for some i and j, there is at least one pair of entities for which there are no comparisons in the set, and we say that the comparisons set is incomplete. Conversely, if $n_{ij} \geq 1$ for all i and j, the comparisons set is complete. Similarly, if $n_{ij} > 1$ for some i and j, there is a pair of entities for which the set contains multiple comparisons. Thus a comparisons set may simultaneously be either complete or incomplete and contain multiple comparisons. We also define a minimally complete comparisons set as one for which $n_{ij} = 1$ for all i and j. For a minimally complete comparisons set, we drop the subscript k and arrange the set as a matrix, referring to the matrix as a (complete) subjective judgment matrix (SJM). Because such a matrix has only positive entries and $a_{ji} = 1/a_{ij}$, an SJM is a positive reciprocal matrix. It will also be useful to refer to an incomplete comparisons set that contains no multiple comparisons as an incomplete SJM. An SJM must contain at least n-1 comparisons to allow us to compute a ratio scale. Figure 1 shows the relationships between these differing classifications of comparisons sets.

8. Because a complete SJM is a positive reciprocal matrix, it contains $N = n(n-1)/2$ independent comparisons. The remaining comparisons are dependent, assuring that the matrix is positive reciprocal. My convention in this memorandum is that the entries above the diagonal of an SJM are the independent comparisons. In addition, we define the fraction complete of an incomplete SJM as

$$\sum_{i=1}^{n} \sum_{j>i} \delta_{ij} \, / \, N$$

where $\delta_{ij}$ equals 1 if $n_{ij} \geq 1$ and 0 otherwise.

## EIGENVALUE (EV) METHOD

9. For complete SJMs, Saaty [2] argues that the best estimate of the ratio scale $\mathbf{x}$ is provided by the principal right eigenvector of the SJM. This vector is obtained by solving the matrix equation

$$\mathbf{Au} = \lambda_{max}\mathbf{u} \tag{1}$$

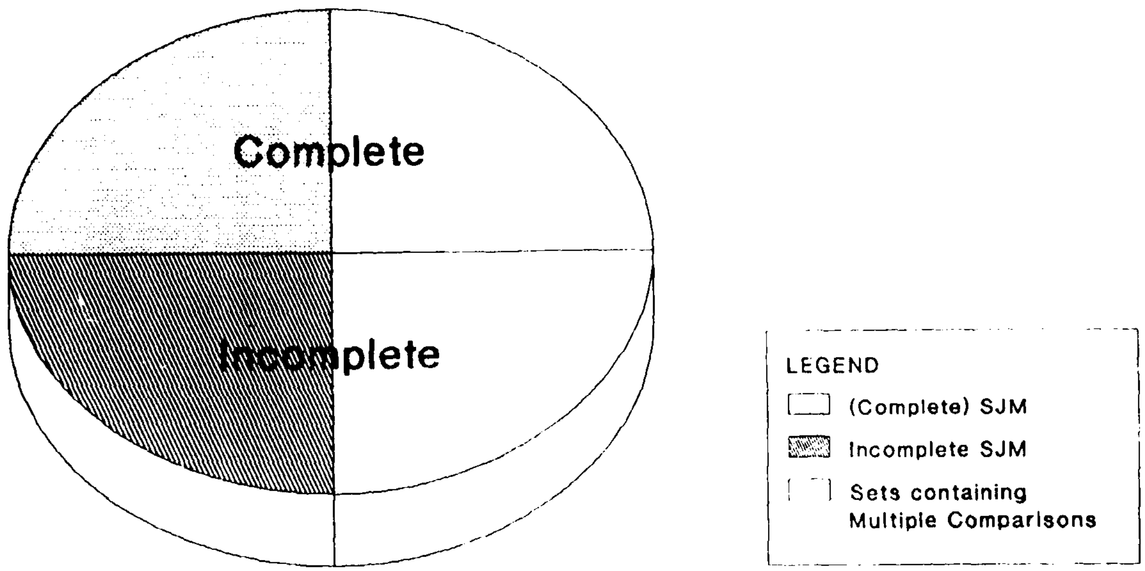


Figure 1. Summary of Terminology for Comparisons Sets

where **A** is an SJM, **u** is the principal right eigenvector corresponding to the maximum eigenvalue, and $\lambda_{max}$ is the maximum eigenvalue.

10. Saaty also shows that the maximum eigenvalue is a measure of the consistency of **A**. If **A** is perfectly consistent, $\lambda_{max}$ equals n, the order of the SJM. As **A** becomes less consistent, the value of $\lambda_{max}$ increases. Saaty uses this fact to construct a consistency index for **A**, which is defined as

$$CI(A) = (\lambda_{max} - n) / (n - 1).$$

11. By itself, the consistency index does not provide a basis for testing whether **A** is random. Saaty proposes using a consistency ratio as the basis for such a test. The consistency ratio is

$$CR(A) = CI(A) / RI(n) \tag{2}$$

where CR(**A**) is the consistency ratio for **A** and RI(n) is a random index for nxn SJMs, i.e., the average consistency index for a sample of randomly-generated nxn SJMs. The consistency ratio increases with the inconsistency of the comparisons set. Saaty recommends we conclude that **A** is adequately consistent if its consistency ratio is less than 0.10.

12. The consistency ratio test of randomness is scale-dependent. Random indexes have been generated for a variety of scales, which represent the distribution of values in the randomly-generated matrices. To use the test statistic given in equation 2, a random index must be available for the same scale used to generate **A**. The most popular set of random indexes are those for Saaty's nine-value scale, which limits the comparisons in **A** to values between 1/9 and 9. The appropriateness of this scale and whether a scale should be used in eliciting comparisons has been a subject of debate [1].

## LOGARITHMIC LEAST-SQUARES METHOD

13. As an alternative to the eigenvalue method, Crawford and Williams [3] advocate using the LLS method to estimate the underlying ratio scale. This

method assumes that the comparisons are subject to a random multiplicative error $\varepsilon_{ijk}$, i.e.,

$$a_{ijk} = (x_i/x_j)\varepsilon_{ijk}.$$

In keeping with the requirement that the comparisons be reciprocally symmetric, we require that the error distribution also be reciprocally symmetric, i.e., $f(\varepsilon_{jik}) = f(1/\varepsilon_{ijk})$, where $f(x)$ is the probability density function of the errors. The LLS method minimizes the sum of the squares of the logarithms of the errors, i.e., it minimizes

$$SSE = S(e_{ijk}^{\;2}) \tag{3}$$

where

$$S(z_{ijk}) = \sum_{i=1}^{n} \sum_{j>i} \sum_{k=1}^{n_{ij}} z_{ijk},$$

$e_{ijk}$ = estimated logarithmic error
= $\ln a_{ijk} - \ln w_i + \ln w_j$,
and $w_i$ = LLS estimate of $x_i$.

The estimated logarithmic error is also frequently written as

$$e_{ijk} = y_{ijk} - b_i + b_j \tag{4}$$

where $y_{ijk} = \ln a_{ijk}$
and $b_i = \ln w_i$.

Equation 4 can also be written as

$$e_{ijk} = y_{ijk} - \hat{y}_{ijk} \tag{5}$$

where $\hat{y}_{ijk} = \sum_{l=1}^{n} b_l X_{ijkl}$

and $X_{ijkl}$ is the $l^{th}$ independent variable used to estimate $y_{ijk}$. For any k, $X_{ijkl}$ equals 1 if $l = i \neq j$, -1 if $l = j \neq i$, and zero otherwise. Equation 5 is the linear model used in the LLS method.

14. For a complete SJM, calculating the weight of the $i^{th}$ entity by the LLS method is equivalent to taking the geometric mean of the comparisons in the $i^{th}$ row of the matrix. (For this reason, the LLS method is frequently referred to as the geometric mean (GM) method.) Mathematically,

$$w_i = c \left( \prod_{j=1}^{n} a_{ij} \right)^{1/n} \tag{6}$$

where c is an arbitrary constant.

15. In 1987, Crawford [1] developed a test of consistency for the LLS method that is based on a transformation of the random indexes used to calculate Saaty's consistency ratio. Hence, this test is also scale-dependent. Until Crawford developed his test, Saaty's was the only one available. In some early applications of the LLS method, notably the State-of-the-Art Contingency Analysis (SOTACA) model, this has led ˜o using an eigenvalue-based test of consistency to assess the validity of a ratio scale generated using the LLS method. However, the EV and LLS weights are not the same except when the SJM is perfectly consistent. One of the extensions of the LLS method presented in this memorandum is an analysis-of-variance test of randomness that complements the LLS approach.

COMPARISON OF FIGENVALUE AND LOGARITHMIC LEAST-SQUARES METHODS

16. Table 1 presents the advantages and limitations of the EV and LLS methods. The EV method uses all the information in an SJM to obtain weights,

TABLE 1. COMPARISON OF EIGENVALUE AND LOGARITHMIC LEAST-SQUARES METHODS

| EV Method | LLS Method |
| --- | --- |
| **Advantages** | |
| Ratio scale depends on all comparisons | Works with comparisons sets that are incomplete or contain multiple comparisons |
| **Limitations** | |
| Does not work with comparisons sets that are incomplete or contain multiple comparisons | Consistency metric is scale-dependent |
| Consistency metric is scale-dependent | Ratio scale depends only on direct comparisons |
| Consistency ratio test is not uniform for SJMs of different orders | Sensitive to outliers |

i.e.. it uses the value of $a_{ik}/a_{jk}$ to provide indirect evidence on the relative value of entities i and j in addition to the direct evidence provided by $a_{ij}$. In contrast, the LLS method uses only the direct comparisons between an entity i and other entities to determine its weight. This characteristic is inherent in the regression methodology, which minimizes equation 3 by setting the partial derivative of SSE with respect to each of the weights equal to zero. This approach eliminates consideration of any comparisons that do not involve the entity whose weight is being calculated.

17. As equation 1 shows, the EV method is inherently a matrix method. This raises the question of how to proceed if comparisons have not been made between all pairs of entities. In contrast, the LLS method is not a matrix method, although it is convenient to use matrix-like notation to remind ourselves of which entities are being compared. As Crawford and Williams [3] point out, the LLS method is adaptable to comparisons sets that are incomplete or contain several comparisons between a pair of entities.

18. An additional limitation of the EV method is that it does not provide a uniform criterion for testing the hypothesis of randomness as the order of the matrix changes. Saaty's consistency test is based on the value of the consistency ratio itself rather than its probability distribution. In [2], Saaty states that for random matrices of orders 4 through 9, the maximum eigenvalues, and hence the random indexes, approximately follow a truncated normal distribution. He also gives values for the means and variances of the maximum eigenvalues. These are shown in Table 2, which also gives the probability, based on the normal distribution, that the consistency ratio of a random matrix is less than 0.10. This probability is given by

$$\alpha_S = (P(u) - P(n)) / (1 - P(n))$$

where $u = n + 0.1\ CI(n)\ (n - 1)$

and $P(x)$ is the cumulative normal probability distibution with the mean and variance taken from Table 2. We will refer to $\alpha_S$ as Saaty's risk. Saaty's risk is the probability of falsely rejecting the hypothesis that the matrix is random, i.e., the probability we will conclude that A is non-random when in fact it is. Table 2 shows that as the order of the matrix increases, the consistency ratio test imposes increasingly stringent criteria for rejecting a random matrix. This may be undesirable because the higher number of comparisons in larger SJMs make consistency harder to attain, suggesting that a more lenient test rather than a more stringent one would be appropriate as SJM order increases.

Table 2. Effect of Matrix Order on
Probability of Falsely Rejecting a Random Matrix

| Matrix | Maximum Eigenvalue | | Probability of |
|---|---|---|---|
| Order | Mean | Variance | False Rejection |
| 4 | 6.650 | 3.370 | 0.025 |
| 5 | 9.418 | 4.424 | 0.011 |
| 6 | 12.313 | 4.413 | 0.002 |
| 7 | 15.000 | 4.123 | < 0.001 |
| 8 | 17.952 | 3.627 | << 0.001 |
| 9 | 20.565 | 3.327 | << 0.001 |

EXTENSIONS OF THE LOGARITHMIC LEAST-SQUARES METHOD

19. This memorandum presents two extensions of the LLS method: a scale-independent test for the randomness of comparisons, and a robust regression method for finding weights that best represent most of comparisons.

Scale-Independent Test of Randomness

20. As shown above, the LLS method for determining weights is based on log-linear regression of the comparisons set. This raises the possibility of using analysis-of-variance techniques to formulate a test statistic for the randomness of a comparisons set. As Appendix A shows, formulating such a test statistic is possible. However, it is not the test statistic used for more general regression models. The general linear regression model is

$$y_i = \sum_{k=0}^{p-1} \beta_l X_{il} + \varepsilon_i \tag{7}$$

where $y_i$ = the variable being fitted,
$\beta_l$ = the $l^{th}$ regression coefficient,
$X_{il}$ = the $l^{th}$ independent variable for dependent variable $y_i$,
$X_{i0}$ = 1 for all i,
and $\varepsilon_i$ = error term.

For this model, the total sum of squares (SSTO), defined as

$$SSTO = \sum_{i=1}^{n} (y_i - \bar{y})^2,$$

can be partitioned into error and regression sums of squares. These are

$$SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

and

$$SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2,$$

respectively. Their mean squares are

$$MSE = SSE \,/\, (n-p)$$

and

$$MSR = SSR \,/\, (p-1),$$

where p is the number of regression coefficients estimated. From these values, we define a test statistic F*:

$$F^* = MSR / MSE.$$

The expected value of MSE is $\sigma^2$, the variance of the error terms. The expected value of MSR is $\sigma^2$ plus a function of the estimated weights. If the dependent variables $y_i$ are random variations around a constant mean, $\beta_1, \ldots, \beta_{p-1}$ are all zero, and MSR equals MSE. Hence we expect that if the $y_i$'s are random, F* will be near 1. In addition, if the errors $\varepsilon_i$ are normally distributed, F* will be F-distributed with (p-1, n-p) degrees of freedom. In this case, the probability that a set of dependent variables is random is

$$s = \int_0^{F^*} f(x;\, p-1,\, n-p)\, dx \tag{8}$$

where f(x; p-1, n-p) is the probability density function of the F distribution. The probability $\alpha$ of falsely rejecting the hypothesis of randomness is

$$\alpha = 1 - s. \tag{9}$$

21. Appendix A shows that the general regression model of equation 7 is inappropriate for estimating weights from sets of comparisons. Unlike the general regression model, the LLS model of equation 5 has no constant term. This term's presence assures that the total sum of squares can be separated into error and regression sums of squares. Its absence in equation 5 precludes using the F* statistic to test whether an SJM is random. However, Appendix A shows that, for the LLS model, we can formulate an alternative test statistic that is similar to F*. This statistic, denoted $F^*_U$, is given by

$$F^*_U = MSRU / MSE$$

where MSRU is the uncorrected regression mean square given by

$$MSRU = SSRU / n$$

and

$$SSRU = S(\hat{y}_{ij}^{\,2}). \tag{10}$$

If the error terms are normally distributed, $F^*_U$ will be F-distributed with (n, N-n) degrees of freedom. The significance level and risk for the hypothesis of randomness can then be determined as in equations 8 and 9.

22. Both SSE and SSRU have intuitively appealing interpretations. SSE is the distance, in logarithms-of-errors terms, between the SJM being analyzed and the nearest consistent matrix, whose entries are $w_i/w_j$. Similarly, SSRU is the distance, in the same terms, between the nearest consistent matrix and a unit matrix, i.e., one having all its entries equal to 1. If the comparisons in the SJM are random, we expect the nearest consistent matrix to be nearly a unit matrix, indicating that the weights of all the entities are equal. Although there is nothing inherently wrong with equal weights, generating comparisons that provide them is easy and accurate. In contrast, we expect a randomly-generated SJM to have a high value of MSE and hence a low value of $F^*_U$.

23. The consistency test based on $F^*_U$ is distribution-dependent, i.e., it assumes that the distribution of the errors $\varepsilon_{ij}$ is log-normal or equivalently that the distribution of the logarithms of the errors $e_{ij}$ is normal. This assumption is easy to test. Neter and Wasserman [4] state that the F test is fairly robust in tolerating departures of the errors from normality, and that it is most sensitive to the skewness of the error distribution. Accordingly, it is best to test the assumption that errors are normally distributed using the skewness of the observed error distribution. Appendix B presents statistics for testing the normality of the error distribution based on its skewness. These statistics are based on Monte Carlo simulation results and extend the skewness statistics in the Biometrika tables [5] to sample sizes as small as 3. The statistics in Appendix B are for SJMs of orders 3 through 22.

Statistics for incomplete comparison sets can be interpolated from the tables in Appendix B.

24. Even if the assumption of normally-distributed errors can not be sustained, we can use the test statistic $F^*_U$ in a first-order test of randomness. We could, for example, reject the hypothesis of randomness if $F^*_U > 10$. Such a test is in the spirit of Saaty's consistency ratio test and has the additional advantage of being scale-independent. I have frequently observed values of $F^*_U$ much higher than 10. However, $F^*_U$ depends on the estimated weights. Thus a comparisons set that contains high values for comparisons will have a high $F^*_U$ value provided that the comparisons are fairly consistent. Hence a test based on the F distribution in conjunction with the skewness test of normality is preferable to using the $F^*_U$ test statistic directly.

## Incomplete Comparisons Sets and Multiple Comparisons

25. In [3], Crawford and Williams state that the LLS method is easily extended to solve more general problems. One such problem is to minimize

$$SSE = S[(y_{ijk} - \hat{y}_{ijk})^2]$$

where $y_{ijk} = \ln a_{ijk}$

and $\hat{y}_{ijk} = \sum_{l=1}^{n} b_l X_{ijkl}$.

This is the same model given in equation 3. Except in the special case where $n_{ij}$ equals one for all i and j, minimizing equation 3 over the set of weights does not yield a closed-form solution. Using the method of steepest descents [6] easily overcomes this problem. This method iteratively calculates the weights using

$$b_l^{(m)} = b_l^{(m-1)} + zf_l \qquad l=1,\dots,n$$

where $b_l^{(m)}$ is the estimate of $\beta_l$ at the $m^{th}$ iteration,

$$z = S[(y_{ijk} - b_i + b_j)(f_i - f_j)] \Big/ \sum_{i=1}^{n} \sum_{j>i} (f_i - f_j)^2,$$

and $$f_l = 2 \sum_{i=1}^{n} \sum_{k=1}^{n_{ij}} (y_{ilk} - b_i + b_l).$$

It is convenient to initialize these calculations by setting $b_l^{(0)} = 1/n$ for all l. We calculate the weights iteratively until SSE converges within prescribed limits. Thus, in contrast to the EV method, it is easy to modify the LLS method to accomodate comparisons sets that are incomplete or contain multiple comparisons.

26. This leads to an additional application for the LLS method. If the comparisons set is incomplete, the LLS method can be used to estimate the weights, which in turn can be used to estimate the missing comparisons of an incomplete comparisons set using $w_i/w_j$, allowing use of the EV method. (Of course, weights estimated using other approaches can be used similarly.) However, the consistency of the completed SJM will reflect in part the estimated values of the missing comparisons. If the approach above is used, the estimated comparisons that complete the SJM are automatically consistent, and the consistency index of the completed SJM may indicate a high degree of consistency in the matrix as a whole. This tends to happen especially when the SJM is sparse, i.e., when it contains relatively few comparisons that were not estimated from weights. Furthermore, because weights calculated using the EV and LLS methods converge as an SJM approaches perfect consistency, applying the EV method to a sparse SJM that has been completed using LLS weights would yield a set of weights little different from those derived using the LLS method. Although I have not studied this matter in detail, I estimate that the EV method will indicate a falsely high consistency if another method is used to estimate comparisons for an SJM that is less than 90% complete.

27. Because the LLS method is not matrix-based, there is no need to estimate missing comparisons. Furthermore, the approach used in Appendix A to develop a scale-independent test of consistency remains valid with only minor adjustments. These include taking the sums in equations 3 and 10 only over the comparisons that have been made and adjusting the number of degrees of freedom in calculating MSE, MSRU, and the significance or risk level. Thus the test statistic $F^*_U$ becomes

$$F^*_U = MSRU' / MSE$$

where

$$MSRU' = SSRU' / n,$$

$$SSRU' = S(\hat{y}_{ijk}^2),$$

$$MSE = SSE / \nu_2,$$

and $$\nu_2 = \sum_{i=1}^{n} \sum_{j>i} n_{ij} - n.$$

Under the hypothesis of randomness, $F^*_U$ will be F-distributed with $(n, \nu_2)$ degrees of freedom, and the significance and risk levels for the hypothesis that an SJM is random are given by equations 8 and 9.

28. This result provides valuable insight into how the LLS method's ability to detect inconsistency varies with the number of comparisons. Figure 2 shows the relationship between the risk of falsely concluding that a random matrix is non-random (i.e., consistent) and the number of comparisons. The figure shows that, for a 12x12 SJM, risk is largely insensitive to both the value of $F^*_U$ and the number of degrees of freedom associated with SSE once the latter value exceeds approximately 10. Because the number of degrees of freedom for SSE is the number of SJM entries in excess of that required to form a ratio scale, Figure 2 shows that little further ability to detect inconsistency is

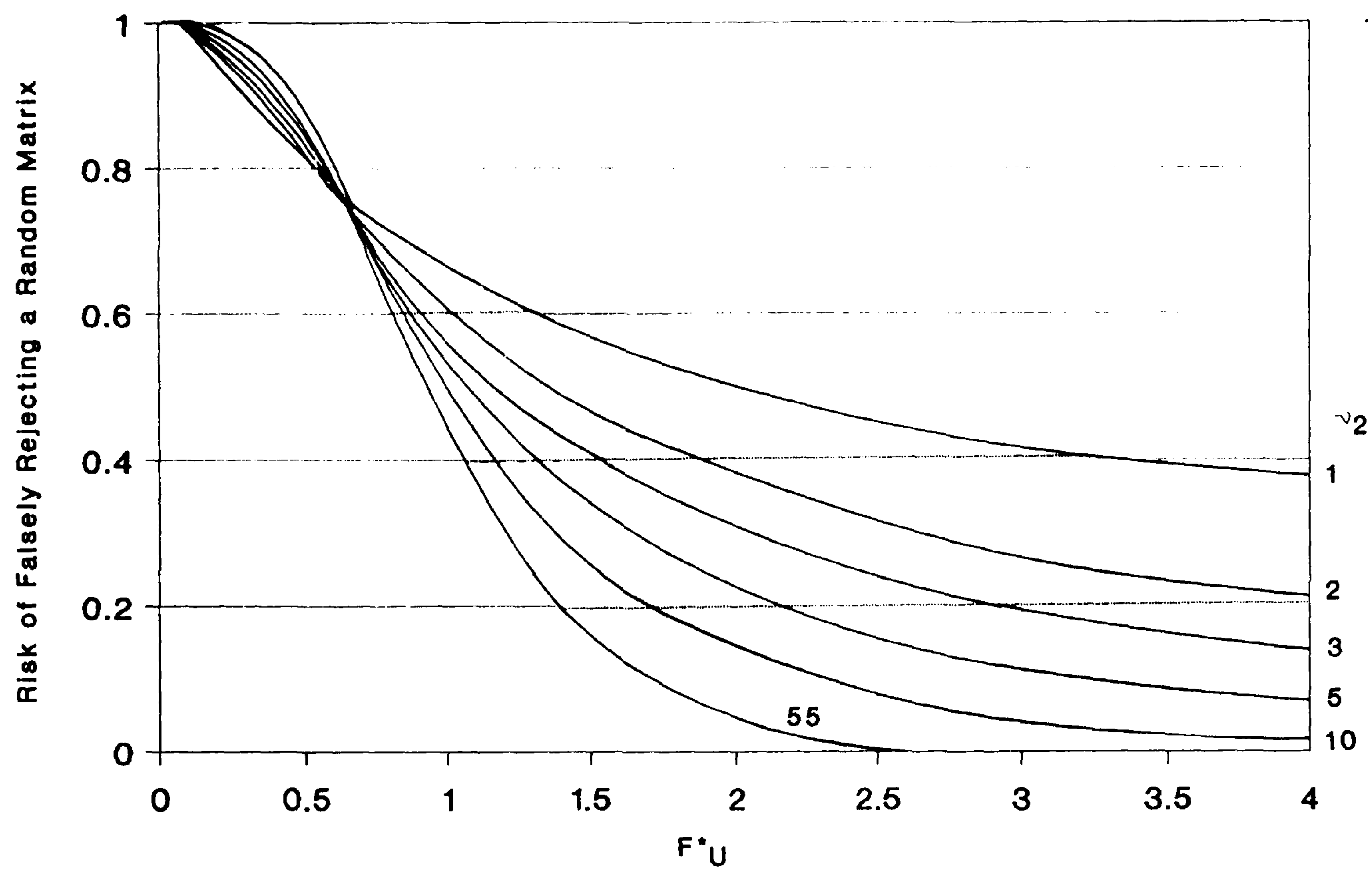


Figure 2. Effect of Comparison Set Incompleteness on Ability to Detect Inconsistency (12x12 matrix)

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obtained if there are more than $n + \nu_2 = 10 + 11 = 21$ independent comparisons. The number of independent comparisons in a complete 12x12 SJM is 66. Thus a 12x12 SJM that is one-third complete can be adequately tested for consistency using the LLS method. (For smaller SJMs, higher fractions of completeness would probably be required to test consistency adequately; graphs similar to Figure 2 can be generated for these SJMs.) We can not test such a matrix using the consistency ratio approach unless we can estimate its missing comparisons. If the approach suggested in paragraph 26 is used, the EV method will provide an artificially low consistency ratio and a set of weights little different from those derived using the LLS method.

## Best Fit to Most of the Comparisons

29. As with all least-squares regression methods, the LLS method is sensitive to outliers. These are comparisons whose errors are large relative to those of other comparisons in the set. Because the least-squares regression algorithm chooses weights that minimize the sum of squares of errors, an outlier causes the algorithm to choose weights that do not best represent most of the comparisons. An alternative method, known as robust regression (RR), selectively de-emphasizes outliers.

30. Least-squares regression's sensitivity to outliers arises from its use of the square of an error as the penalty for that error. An alternative is to use a penalty that is less sensitive to the magnitude of the error. One such penalty function is

$$g(e_{ijk}) = 1 - \exp(-e_{ijk}^{2})$$

As Figure 3 shows, this function resembles an inverted normal density function centered at $e_{ijk} = 0$. This function has an upper limit of 1, whereas the square of the error is unbounded.

31. By analogy with equation 3, we can define a robust error function f' that is analogous to the error sum of squares. This function is

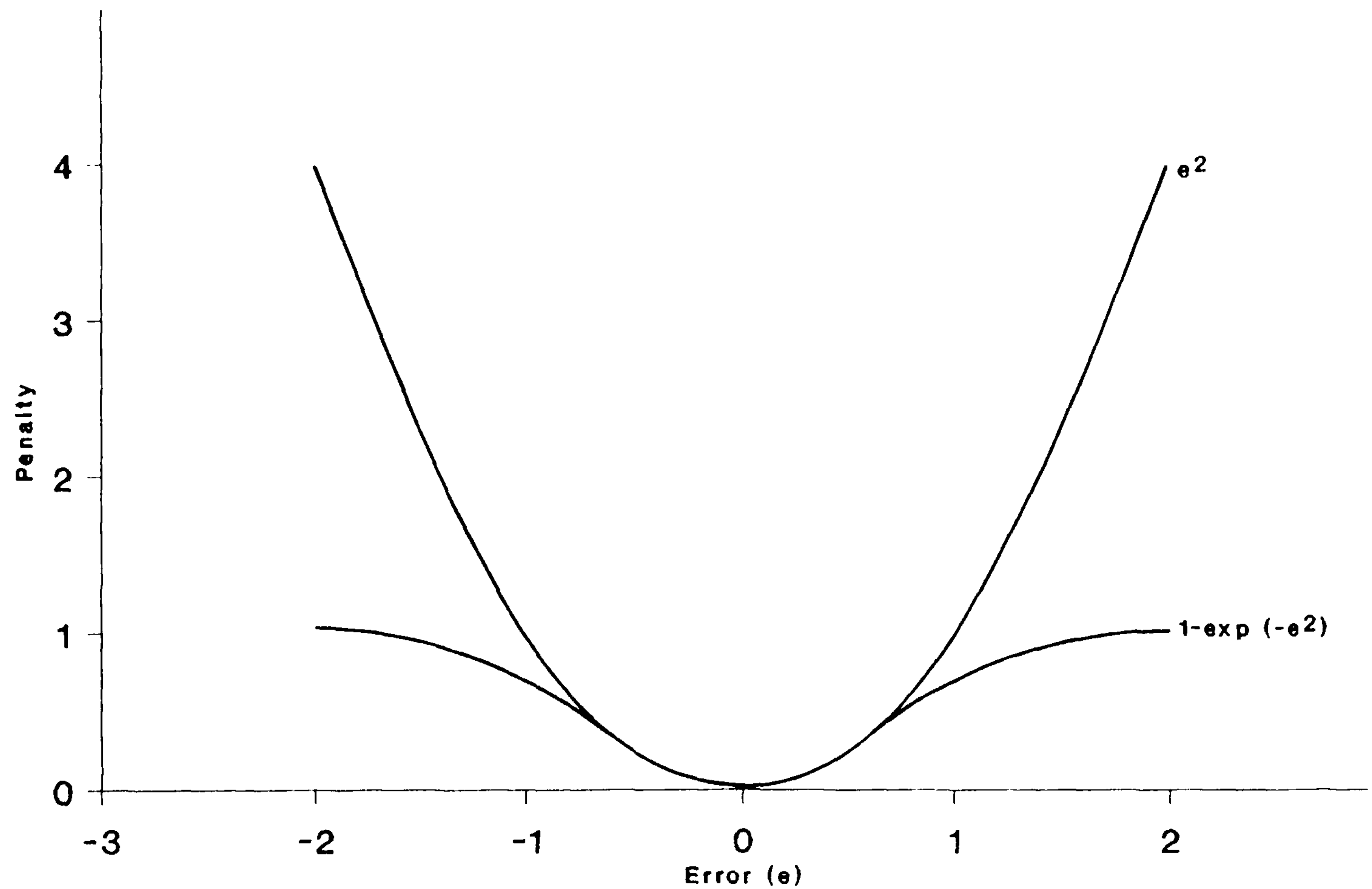


Figure 3. Comparison of Least-Squares and Robust Regression Penalty Functions

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$$f' = S(g(e_{ijk})). \tag{11}$$

To minimize the robust error function for a given weight, we equate the partial derivative of $f'$ with respect to the weight to zero and solve. This yields

$$\sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} (w_{ljk}\, e_{ljk}) = 0 \qquad l=1,\ldots,n \tag{12}$$

where $w_{ljk} = \exp(-e_{ljk}^2)$. The factor $w_{ijk}$ is frequently referred to as the Welsh weight associated with $y_{ijk}$.

32. To solve equations 12, we expand $e_{ljk}$ as $(y_{ljk} - b_l + b_j)$ to get

$$b_l = \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} w_{ljk}\,(y_{ljk} + b_j) \Big/ \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} w_{ljk} \qquad l=1,\ldots,n.$$

Unlike equations 6, each of these equations depends on weights other than $b_l$. We solve this set of equations using an iterative technique. For the $m^{th}$ iteration we calculate

$$b_l^{(m)} = \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} w_{ljk}^{(m-1)}\,(y_{ljk} + b_j^{(m-1)}) \Big/ \sum_{j=1}^{n} \sum_{k=1}^{n_{ij}} w_{ljk}^{(m-1)} \qquad l=1,\ldots,n \tag{13}$$

where the superscript (m) indicates that the factor was calculated during the $m^{th}$ iteration. Iteration continues until $f'^{(m)} - f'^{(m-1)}$ converges within prescribed limits.

33. As with other iterative techniques, we need starting values for the weights. We could set the weights equal, e.g., $b_l^{(0)} = 1/n$ for all l, as we did in the LLS method for incomplete comparisons sets. However, the number of iterations required for f′ to converge depends on how well the initial values of $b_l$ estimate the weights. The approach taken here is to use the LLS method to provide the initial estimates.

34. The robust regression method provides weights that best represent most of the comparisons. These weights are based on all the comparisons in a set rather than only on direct comparisons, as in the LLS method. This overcomes one of Saaty and Vargas' objections to the LLS method [7]. Furthermore, because the robust regression method assigns low Welsh weights to outliers, these can be identified by examining the Welsh weights. However, because of the nature of $g(e_{ijk})$, we cannot partition an analogue of the uncorrected total sum of squares into error and regression terms, so we lose the ability to perform an F-test of randomness. If the RR method does not significantly modify the LLS weights, an F-test based on the LLS weights should remain valid. This will occur if the comparisons set contains no outliers. Such a situation exists if $f'^{(0)}$, the value of equation 11 calculated using LLS weights, is nearly the same as the final value of $f'^{(m)}$ calculated using the weights given by equations 13.

## Summary of Extensions

35. Table 3 compares the EV method with the enhanced LLS and RR methods. The EV method's advantages and limitations are nearly the same as those shown in Table 1. The only change is that the eigenvalue method can be adapted to SJMs that are up to perhaps 10% incomplete, given another method for estimating the weights. The LLS method is one such method. However, if the SJM is sparsely populated with comparisons, and missing comparisons are estimated using $w_i/w_j$, the SJM will appear to be almost perfectly consistent and the weights calculated by the EV method will be only slightly different than those calculated by the non-EV method used to estimate the missing comparisons.

TABLE 3. COMPARISON OF EIGENVALUE, ENHANCED LOGARITHMIC LEAST-SQUARES, AND ROBUST REGRESSION METHODS

| EV Method | Enhanced LLS Method | RR Method |
|---|---|---|
| | Advantages | |
| Ratio scale depends on all comparisons<br>Works with incomplete comparisons sets, within limits | Works with comparisons sets that are incomplete or contain multiple comparisons<br>Consistency metric is scale-independent | Works with comparisons sets that are incomplete or contain multiple comparisons<br>Ratio scale depends on all comparisons<br>Insensitive to outliers |
| | Limitations | |
| Consistency metric is scale-dependent<br>Consistency ratio test is not uniform for SJMs of different orders<br>Does not work with comparisons sets that contain multiple comparisons | Consistency test is distribution-dependent<br>Sensitive to outliers | No consistency metric |

36. The enhancements discussed above eliminate several of the LLS method's limitations without significantly degrading its advantages. There is now a scale-independent consistency metric for both complete and incomplete comparisons sets. Furthermore, the relationship between risk and the number of comparisons provides insight into the number of comparisons needed to detect inconsistency. This consistency metric can be used to perform tests of consistency that are uniform for comparisons sets of different sizes.

37. In addition, the robust regression technique overcomes the LLS method's sensitivity to outliers by selectively de-emphasizing the comparisons that are most inconsistent, thereby providing a set of weights that best represent most of the comparisons.

## EXAMPLES

### Example 1

38. Table 4 shows an SJM that contains known inconsistencies. During data collection, the comparisons involving entities F, G, H, and I were recorded incorrectly, i.e., the reciprocals of the correct comparisons were recorded. (I have found that recording errors of this kind are fairly common.) Table 5 presents selected statistics for this matrix, including the EV, LLS, and robust regression weights for the entities. Because the matrix contains only a single comparison greater than 9, using the random index for Saaty's nine-value scale is reasonable. The consistency ratio is 0.72, much higher than required to accept the hypothesis of randomness using Saaty's test. However, Saaty's risk of falsely rejecting a random matrix with this consistency ratio is 0.038. This shows that a random SJM of the same order as the SJM in Table 4 is unlikely to have a consistency ratio less than 0.72. Similarly, the $F^*_U$ value for the SJM is 2.65 and the associated risk of false rejection, based on the F test, is 0.024. Table 5 also shows a significant difference between the LLS and robust regression weights, especially for the entities involved in incorrectly recorded comparisons. Comparing $f'^{(0)}$ and $f'^{(m)}$ confirms this difference, indicating that the robust regression method substantially revised the LLS values used to initialize it. Table 6, the

TABLE 4. SUBJECTIVE JUDGMENT MATRIX FOR EXAMPLE 1

| | Entity | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Entity | A | B | C | D | E | F | G | H | I |
| A | 1.000 | 0.333 | 0.250 | 0.200 | 5.000 | 0.500 | 0.111 | 3.000 | 5.000 |
| B | 3.000 | 1.000 | 0.333 | 0.333 | 3.000 | 1.000 | 0.333 | 3.000 | 7.000 |
| C | 4.000 | 3.000 | 1.000 | 0.333 | 7.000 | 5.000 | 0.200 | 3.000 | 9.000 |
| D | 5.000 | 3.000 | 3.000 | 1.000 | 8.000 | 7.000 | 0.333 | 9.000 | 10.000 |
| E | 0.200 | 0.333 | 0.143 | 0.125 | 1.000 | 0.250 | 0.200 | 0.500 | 3.000 |
| F | 2.000 | 1.000 | 0.200 | 0.143 | 4.000 | 1.000 | 0.333 | 0.143 | 0.143 |
| G | 9.000 | 3.000 | 5.000 | 3.000 | 5.000 | 3.000 | 1.000 | 0.125 | 0.125 |
| H | 0.333 | 0.333 | 0.333 | 0.111 | 2.000 | 7.000 | 8.000 | 1.000 | 4.000 |
| I | 0.200 | 0.143 | 0.111 | 0.100 | 0.333 | 7.000 | 8.000 | 0.250 | 1.000 |

TABLE 5. SELECTED STATISTICS FOR EXAMPLE 1

| | Original Matrix | | | Corrected Matrix | | |
|---|---|---|---|---|---|---|
| | EV | LLS | RR | EV | LLS | RR |
| Consistency Test Statistic* | 0.72 | 2.65 | -- | 0.11 | 22.64 | -- |
| Risk** | 0.038 | 0.024 | -- | 0 | 0 | -- |
| Skewness Probability | -- | 0.089 | -- | -- | 0.468 | -- |
| Robust Error (f′) | -- | 32.28 | 20.08 | -- | 15.36 | 15.07 |
| Entity | Ratio Scales | | | | | |
| A | 0.073 | 0.064 | 0.046 | 0.052 | 0.053 | 0.049 |
| B | 0.098 | 0.107 | 0.085 | 0.080 | 0.089 | 0.090 |
| C | 0.144 | 0.180 | 0.129 | 0.152 | 0.150 | 0.139 |
| D | 0.231 | 0.300 | 0.236 | 0.245 | 0.251 | 0.251 |
| E | 0.031 | 0.031 | 0.023 | 0.025 | 0.026 | 0.026 |
| F | 0.032 | 0.042 | 0.078 | 0.083 | 0.083 | 0.090 |
| G | 0.162 | 0.143 | 0.358 | 0.318 | 0.301 | 0.308 |
| H | 0.129 | 0.093 | 0.033 | 0.030 | 0.032 | 0.034 |
| I | 0.100 | 0.042 | 0.011 | 0.014 | 0.014 | 0.013 |

* Consistency ratio for EV, $F^*_U$ for LLS

** Probability of falsely rejecting hypothesis of randomness. Saaty's risk for EV method, based on $F^*_U$ for LLS

TABLE 6. WELSH WEIGHTS FOR EXAMPLE 1

| | Entity | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Entity | A | B | C | D | E | F | G | H | I |
| A | 1.000 | 0.783 | 0.881 | 1.000 | 0.431 | 0.973 | 0.978 | 0.552 | 0.967 |
| B | 0.783 | 1.000 | 0.635 | 0.995 | 0.961 | 0.994 | 0.889 | 0.973 | 0.993 |
| C | 0.881 | 0.635 | 1.000 | 0.781 | 0.951 | 0.293 | 0.704 | 0.935 | 0.935 |
| D | 1.000 | 0.995 | 0.781 | 1.000 | 0.942 | 0.491 | 0.628 | 0.945 | 0.565 |
| E | 0.431 | 0.961 | 0.951 | 0.942 | 1.000 | 0.973 | 0.279 | 0.897 | 0.876 |
| F | 0.973 | 0.994 | 0.293 | 0.491 | 0.973 | 1.000 | 0.838 | 0.000 | 0.000 |
| G | 0.978 | 0.889 | 0.704 | 0.628 | 0.279 | 0.838 | 1.000 | 0.000 | 0.000 |
| H | 0.552 | 0.973 | 0.935 | 0.945 | 0.897 | 0.000 | 0.000 | 1.000 | 0.921 |
| I | 0.967 | 0.993 | 0.935 | 0.565 | 0.876 | 0.000 | 0.000 | 0.921 | 1.000 |

table of Welsh weights for this SJM, shows that very low Welsh weights were assigned to comparisons between entities F, G, H, and I, clearly showing that comparisons involving these entities are outliers.

39. Once this inconsistency was detected, the respondent who provided the comparisons was consulted. He corrected the SJM by changing all outlier comparisons to their reciprocals. Table 5 also presents statistics for the corrected SJM. The corrections bring the SJM to the threshold of passing Saaty's consistency ratio test. However, Saaty's risk for the corrected matrix is nil. There is a dramatic increase in the value of $F^*_U$, and the risk of false rejection based on the F test is nil. The LLS and robust regression weights for the corrected matrix are only slightly different, as are the values of $f'^{(0)}$ and $f'^{(m)}$, suggesting that there are no additional outliers.

40. Also note in Table 5 that the RR ratio scale for the original SJM is much closer to the EV and LLS scales for the corrected matrix than these scales for the original one. Although correcting the outlier comparisons has a significant effect on the EV and LLS ratio scales, this effect is only slightly larger than that of the RR method de-emphasizing the outliers in the original. In addition, the RR ratio scales for the original and corrected matrices are nearly the same. This suggests that, when a comparisons set is inconsistent and the respondent cannot be contacted in a timely manner to review his comparisons, the RR ratio scale should be used.

Example 2

41. Table 7 shows an incomplete SJM Smith [8] used to illustrate an approach to completing an incomplete SJM and resolving inconsistencies. In Table 7, zeroes indicate missing comparisons. The matrix is sparse, i.e., it contains only one-third of the comparisons in a complete SJM. Furthermore, Smith's matrix is 12x12, the order for which Figure 2 was generated. As shown there, we obtain little additional ability to detect inconsistency by completing more than one-third of a 12x12 matrix.

TABLE 7. SUBJECTIVE COMPARISON MATRIX FOR EXAMPLE 2

| | Entity | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Entity | A | B | C | D | E | F | G | H | I | J | K | L |
| A | 1.000 | 1.300 | 1.300 | 1.400 | 3.900 | 1.800 | 1.800 | 1.100 | 1.900 | 1.800 | 1.500 | 10.000 |
| B | 0.769 | 1.000 | 0.000 | 9.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.600 | 0.000 | 0.000 | 0.000 |
| C | 0.769 | 0.000 | 1.000 | 0.000 | 1.100 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.600 |
| D | 0.714 | 0.111 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 2.700 | 0.000 | 0.000 |
| E | 0.256 | 0.000 | 0.909 | 0.000 | 1.000 | 0.000 | 0.700 | 0.000 | 0.000 | 0.000 | 0.000 | 0.500 |
| F | 0.556 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.900 | 0.000 |
| G | 0.556 | 0.000 | 0.000 | 0.000 | 1.429 | 0.000 | 1.000 | 0.000 | 0.000 | 1.700 | 0.000 | 0.000 |
| H | 0.909 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.200 |
| I | 0.526 | 1.667 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| J | 0.556 | 0.000 | 0.000 | 0.370 | 0.000 | 0.000 | 0.588 | 0.000 | 0.000 | 1.000 | 0.000 | 1.200 |
| K | 0.667 | 0.000 | 0.000 | 0.000 | 0.000 | 1.111 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| L | 0.100 | 0.000 | 1.667 | 0.000 | 2.000 | 0.000 | 0.000 | 5.000 | 0.000 | 0.833 | 0.000 | 1.000 |

42. Table 8 presents selected statistics for Smith's matrix. As the table shows, this SJM is inconsistent, and rejecting the hypothesis of randomness entails a high risk. Because the matrix was designed to exemplify an approach to resolving inconsistency, this result is not surprising. The LLS and robust regression methods produce ratio scales that accurately represent the incomplete SJM. In this case, the consistency ratio of the SJM completed using LLS weight estimates is 0.06, sufficiently low to pass Saaty's consistency test. However, this consistency ratio is artificially low because the estimated comparisons that comprise two-thirds of the matrix are automatically consistent. Hence Saaty's test does not provide an adequate basis for testing consistency in this case.

## SUMMARY

43. This technical memorandum has presented two extensions of the LLS method for analyzing subjective comparisons. The first of these is a scale-independent test for the consistency of a comparisons set. This test, based on analysis-of-variance techniques, uses a test statistic that is F-distributed if the error terms are log-normal-distributed and the comparisons are random. It is easy to apply this test to comparisons sets that are incomplete or contain multiple comparisons between pairs of entities. The risk of falsely rejecting the hypothesis of randomness based on this test statistic also provides valuable insights into the trade-off between comparisons set completeness and our ability to detect inconsistency.

44. The second extension was using robust regression to find a ratio scale that best represents most of the comparisons in a set. Because the LLS method is a least-squares method, its results are sensitive to outliers. The robust regression method selectively de-emphasizes outliers. This method also allows comparisons that do not directly involve an entity to affect its weight, overcoming one of Saaty and Vargas' objections to the LLS method.

45. These enhancements to the LLS method eliminate many of the limitations of the basic LLS method without introducing substantial new limitations. Along with the previously-known advantages of the LLS method, these extensions

TABLE 8. SELECTED STATISTICS FOR EXAMPLE 2

| | EV | LLS | RR |
|---|---|---|---|
| Consistency Test Statistic* | 0.06 | 0.82 | -- |
| Risk** | 0 | 0.635 | -- |
| Skewness Probability | -- | 0.431 | -- |
| Robust Error (f′) | -- | 11.48 | 8.20 |
| Entity | Ratio Scale | | |
| A | 0.163 | 0.140 | 0.130 |
| B | 0.173 | 0.169 | 0.074 |
| C | 0.055 | 0.057 | 0.058 |
| D | 0.062 | 0.062 | 0.122 |
| E | 0.038 | 0.042 | 0.044 |
| F | 0.071 | 0.078 | 0.073 |
| G | 0.067 | 0.073 | 0.077 |
| H | 0.043 | 0.040 | 0.113 |
| I | 0.133 | 0.139 | 0.091 |
| J | 0.047 | 0.047 | 0.060 |
| K | 0.081 | 0.088 | 0.084 |
| L | 0.066 | 0.066 | 0.075 |

* Consistency ratio for EV, $F^*_U$ for LLS

** Probability of falsely rejecting hypothesis of randomness. Saaty's risk for EV method, based on $F^*_U$ for LLS contribute to making the logarithmic least-squares method an attractive alternative to Saaty's eigenvalue method for analyzing subjective comparisons.

## APPENDIX A

### Statistical Test for Random Comparisons

A.1 This appendix presents a statistical test for whether the entries in a subjective comparisons set are random. The test is based on the logarithmic least-squares (LLS) method for estimating the relative values or weights of the entities represented in the comparisons set. As Crawford and Williams [3] showed, the logarithmic least-squares method is equivalent to log-linear regression, allowing us to use analysis-of-variance techniques to formulate a test of randomness for comparisons sets. However, because of a special characteristic of the log-linear model used to fit the comparisons set, the usual analysis-of-variance approach to testing randomness is inappropriate. This appendix develops an alternative approach, also based on the analysis of variance, that tests randomness and provides intuitively appealing metrics for the consistency of a comparisons set. It develops this approach for a (complete) subjective judgment matrix (SJM), but it is easy to generalize the result to comparisons sets that are incomplete or contain multiple comparisons.

A.2 An SJM is a positive reciprocal matrix whose entries are subjective estimates of the relative importance of entities i and j. A positive reciprocal matrix is one whose entries are all positive and satisfy $a_{ji} = 1/a_{ij}$ for all i and j. The entries in an SJM are usually viewed as estimates of $x_i/x_j$, where $x_i$ is the weight of entity i. In addition to estimating the values of the weights, we are concerned with the consistency of the SJM. If $A = \{a_{ij}\}$ is an SJM, A is perfectly consistent if $a_{ij} = a_{ik}a_{kj}$ for all i, j, and k. We want to test the hypothesis that the $a_{ij}$ are random. If they are, then A is not consistent. This is a hypothesis we would like to reject.

A.3 Crawford and Williams [3] showed that the logarithmic least-squares (LLS) method of estimating the ratio scale underlying an SJM minimizes

$$m(A,C) = S((\ln a_{ij} - \ln w_i + \ln w_j)^2) \qquad (A.1)$$

where

$$S(z_{ij}) = \sum_{i=1}^{n} \sum_{j>i} z_{ij}$$

and m(A,C) is the sum of squares of logarithmic errors between A and the nearest consistent matrix C. The metric m(A,C) will be referred to below as the m-distance between matrices A and C, and the consistent matrix having the smallest m-distance to A is the m-closest consistent matrix. C is a consistent matrix whose entries $c_{ij}$ equal $w_i/w_j$, and $w_i$ is the weight associated with entity i. This method gives

$$w_i = c * \left( \prod_{j=1}^{n} a_{ij} \right)^{1/n}, \qquad i=1,\ldots,n \qquad \text{(A.2)}$$

where c is an arbitrary scaling constant. The weight of entity i is the geometric mean of the entries in the $i^{th}$ row of A. (For this reason, the LLS method is frequently referred to as the geometric mean (GM) method.) Crawford and Williams [3] show that using equation A.2 to calculate weights is equivalent to doing a log-linear regression on the entries of A. This regression has the form

$$y_{ij} = \sum_{k=1}^{n} \beta_k X_{ijk} + \varepsilon_{ij} \qquad \text{(A.3)}$$

where $y_{ij}$ = ln $a_{ij}$,
$\beta_k$ = ln $w_k$,
$X_{ijk}$ = the k'th independent variable for the (i,j) entry of A,
and $\varepsilon_{ij}$ = error term.

$X_{ijk}$ equals 1 if $k = i \neq j$, -1 if $k = j \neq i$, and zero otherwise. Equation A.1 is the sum of squared errors for the log-linear regression, i.e., $m(A,C) = S(e_{ij}^2)$, where $e_{ij}$ is an estimate of $\varepsilon_{ij}$.

A.4 One noteworthy characteristic of equation A.3 is that it does not contain a constant term, i.e., there is no k for which $X_{ijk} = 1$ for all i and j. In a general regression model, there is usually a constant term that is fitted in the same manner as the other regression coefficients. This constant term represents the estimated value of y when all of the $X_{ij}$ values equal zero, i.e., the y-intercept. In such a "fitted-intercept" model, we're assured that the error terms sum to zero. Suppose the model is

$$y_i = \sum_{k=0}^{p-1} \beta_k X_{ik} + \varepsilon_i \tag{A.4}$$

where $X_{i0} = 1$ for all i. Then

$$SSE = \sum_{i=1}^{n} e_i^{\,2} \tag{A.5}$$

and

$$\partial SSE/\partial b_0 = 2 \sum_{i=1}^{n} \varepsilon_i [\partial \varepsilon_i/\partial b_0]$$

where $b_0$ is the least-squares estimator of $\beta_0$. The residual errors $e_i$ equal $y_i - \hat{y}_i$, where

$$\hat{y}_i = \sum_{k=0}^{p-1} b_k X_{ik},$$

so $\partial e_i/\partial b_0 = -X_{i0} = -1$

and

$$\partial SSE/\partial b_0 = -2 \sum_{i=1}^{n} e_i .$$

But $\partial SSE/\partial b_0 = 0$ is a necessary condition for minimizing equation A.5. It follows that

$$\sum_{i=1}^{n} e_i = 0$$

is a necessary condition for minimizing SSE in the fitted-intercept model given by equation A.4. In the zero-intercept modelof equation A.3, there is no constant term and the sum of errors is unconstrained. As will be shown below, this has a significant impact on designing a scale-free test of randomness.

A.5 In regression theory, an F test is frequently used to assess whether there is any meaningful relationship between the dependent and independent variables. If there is no relationship between the variables, a regression of the form

$$y_i = \bar{y} + e_i,$$

where $\bar{y}$ is the average of the $y_i$'s, suffices. This model is the same as that of equation A.4, but with $b_k = 0$, $k=1,\ldots,p-1$. The F test addresses the hypothesis that the $y_i$'s are random variations about a constant mean. An F test examines this hypothesis using the criterion that $b_k = 0$, $k=1,\ldots,p-1$. This test uses an analysis-of-variance approach to subdivide the variance of the $y_i$'s into separate variances that represent the variance of the dependent variables around the regression line and the variance of the regression line around the mean value. Mathematically, for the model given by equation A.4,

$$SSTO = SSE + SSR \tag{A.6}$$

where SSTO is the total sum of squares (i.e., the variance of the $y_i$'s), SSE is error sum of squares, and SSR is the regression sum of squares. SSTO, SSE, and SSR are given by

$$SSTO = \sum_{i=1}^{n} (y_i - \bar{y})^2,$$

$$SSE = \sum_{i=1}^{n} e_i^2,$$

and $$SSR = \sum_{i=1}^{n} \sum_{k=0}^{p-1} (b_k X_{ik} - \bar{y})^2,$$

where n is the number of independent y values.

A.6 For the fitted-intercept model, SSE and SSR are used to calculate a test statistic F* given by

$$F^* = MSR \;/\; MSE \tag{A.7}$$

where MSR and MSE are the regression and error mean squares, respectively. They are defined as

$$MSR = SSR \;/\; (p-1)$$

and

$$MSE = SSE \;/\; (n-p).$$

If, as is usually assumed, the error terms in equation A.4 are normally distributed with mean 0 and variance $\sigma^2$, MSR and MSE are $\chi^2$-distributed with p-1 and n-p degrees of freedom, respectively. The expected value of MSE is

$\sigma^2$. The ratio of two independent $\chi^2$-distributed random variables is F-distributed. Furthermore, if $b_k = 0$, $k=1,\ldots,p-1$, the expected value of MSR is $\sigma^2$, and F*'s expected value is 1. Hence the significance level for the hypothesis that the y values are random, i.e., the probability of concluding correctly that the $y_i$'s are random, using the criterion that $b_k = 0$, $k=1,\ldots,p-1$, is

$$s = \int_0^{F^*} f(x;\ p-1,\ n-p)\ dx$$

where s is the significance level and $f(x;\ p-1,\ n-p)$ is the density function of the F distribution with (p-1, n-p) degrees of freedom. Because the value of s increases as F* increases, the hypothesis of randomness is increasingly supported by increasing values of F*. Conversely, the probability of falsely rejecting the hypothesis that the $y_i$'s are random is

$$\alpha = 1 - s$$

where $\alpha$ is frequently referred to as the risk associated with the hypothesis of randomness. This risk decreases as F* increases.

A.7 In their derivation of equation A.6, Neter and Wasserman [4] use the fact that, in the fitted-intercept model, the error terms sum to zero. In the zero-intercept model of equation A.3, the error terms do not generally sum to zero. Defining SSE as in equation A.5, we have

$$\partial SSE/\partial b_l = -2 \sum_{i=1}^{n} (e_i \sum_{k=1}^{p} X_{ik}\ \partial b_k/\partial b_l). \tag{A.8}$$

Because $\partial b_k/\partial b_l = 1$ for $k = l$, $l=1,\ldots p$, and zero otherwise, equation A.8 reduces to

$$\partial SSE/\partial b_l = -2 \sum_{i=1}^{n} e_i X_{il} \qquad l = 1,\ldots,p. \tag{A.9}$$

A necessary condition for minimizing SSE is that the p equations given by equation A.9 all equal zero. However, for the zero-intercept model, there is no value of l for which $X_{il}$ equals 1 for all i and j. Accordingly, the error terms need not sum to zero as a necessary condition for minimizing SSE in the zero-intercept model.

A.8 In the zero-intercept case, it can be shown that, using the notation of equation A.6,

$$SSTO = SSE - 2\bar{y}S(e_{ij}) + SSR \tag{A.10}$$

Hence, unlike the fitted-intercept case, the total sum of squares cannot be separated into error and regression sums of squares. Accordingly, F*, the F-test statistic given by equation A.7, does not completely represent the total sum of squares. Although it is possible to formulate a test statistic similar to F* based on equation A.10, it will not be F-distributed. However, if the error terms have mean values of zero, $S(e_{ij})$ will usually be small and the test statistic will be approximately F-distributed.

A.9 Alternatively, we can use a test based on partitioning the uncorrected total sum of squares (SSTOU). SSTOU is given by

$$SSTOU = S({y_{ij}}^2) = m(A,U)$$

where U is the unit matrix, i.e., a matrix all of whose entries equal 1. It is easy to show that

$$SSTOU = SSE + SSRU \tag{A.11}$$

where SSRU is the uncorrected regression sum of squares given by

$$SSRU = S(\hat{y}_{ij}^{\ 2})$$

and $$\hat{y}_{ij} = \sum_{k=1}^{n} b_k X_{ijk}.$$

Furthermore, both SSE and SSRU have intuitively appealing interpretations. SSE is the m-distance between A and the m-closest consistent matrix C, whose entries $c_{ij} = w_i/w_j$; SSRU is the m-distance between C and U. Thus equation A.11 can be rewritten as

$$m(A,U) = m(A,C) + m(C,U).$$

If the entries of A are random, we expect its m-closest consistent matrix to be nearly a unit matrix, and SSRU = m(C,U) will be small.

A.10 Under the assumption that errors are normally distributed, SSE and SSRU are $\chi^2$-distributed with N-n and n degrees of freedom, respectively. Here $N = n(n-1)/2$, the number of independent entries in the SJM. The mean squares associated with SSE and SSRU are

$$MSE = SSE\ /\ (N-n)$$

and

$$MSRU = SSRU\ /\ n.$$

The expected values of the mean squares are

$$E(MSE) = \sigma^2$$

and

$$E(MSRU) = \sigma^2 + (n-1) \sum_{k=1}^{n} \beta_k^{\ 2}\ /\ n - \sum_{k \neq l} \beta_k\ \beta_l$$

where $\beta_i$ is the expected value of $b_i$. Under the hypothesis of randomness, $\beta_k$ equals zero for all k and E(MSRU) equals $\sigma^2$. This allows us to formulate an uncorrected F test statistic

$$F^*_U = MSRU / MSE$$

that is F-distributed with (n, N-n) degrees of freedom. The significance level associated with the hypothesis of randomness is

$$s = \int_0^{F^*_U} f(x;\ n,\ N-n)\ dx \qquad (A.12)$$

and the risk of falsely rejecting the hypothesis is

$$\alpha = 1 - s. \qquad (A.13)$$

A.11 These results are readily extended to comparisons sets that are incomplete or contain multiple comparisons between entities. All that is required is to take sums over all comparisons. Thus the summation operator defined in paragraph A.3 becomes

$$S(z_{ijl}) = \sum_{i=1}^{n} \sum_{j>i} \sum_{l=1}^{n_{ij}} z_{ijl}$$

where $n_{ij}$ is the number of comparisons between entities i and j. The number of degrees of freedom associated with SSE is N-n, where N is redefined as

$$N = \sum_{i=1}^{n} \sum_{j>i} n_{ij}.$$

For the special case of a complete SJM, N equals n(n-1)/2. The significance and risk levels are given by equations A.12 and A.13.

APPENDIX B

Tables for Tests of Error Normality and Independence

B.1 This appendix presents tabulated data that can be used to test the hypothesis that errors in complete subjective judgment matrices (SJMs) are independent and identically distributed. The tables can also be used to test this hypothesis for comparisons sets that are incomplete or contain multiple comparsions between entities.

B.2 Tables B.1 through B.20 present values of six statistics for SJMs of order 3 through 22. The statistics are:

XBAR - the mean error;

VAR - the variance $\sigma^2$ of the errors;

|GAMMA1| - the absolute value of error skewness, i.e.,

$$|GAMMA1| = | \mu_3 / \sigma_3 |$$

where $\mu_3 = \sum_{i=1}^{N} (X_i - XBAR)^3) / (N-1)$,

$$N = n (n-1) / 2,$$

and n is the order of the SJM;

GAMMA2 - the kurtosis of errors, i.e.,

$$GAMMA2 = ( \mu_4 / \sigma^4 ) - 3$$

where $\mu_4 = (\sum_{i=1}^{N} (X_i - XBAR)^4) / (N-1)$;

CHISQC - a $\chi^2$ statistic that compares the number of positive and negative error terms;

and PC2C - the probability that CHISQC is $\chi^2$-distributed with one degree of freedom.

Each table gives the mean and variance of each statistic as well as selected percentage points.

Table B.1: Statistics for 3x3 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00815 | 1.08184 | 0.37452 | -2.00000 | 1.00000 | 0.55641 |
| Var. | 0.34821 | 1.25151 | 0.03058 | 0.00000 | 1.33454 | 0.04332 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -1.80137 | 0.00081 | 0.00010 | -2.00127 | 0.33333 | 0.43630 |
| 1.00 | -1.36577 | 0.01233 | 0.00766 | -2.00017 | 0.33333 | 0.43630 |
| 5.00 | -0.97805 | 0.05438 | 0.05036 | -2.00000 | 0.33333 | 0.43630 |
| 10.00 | -0.76288 | 0.11348 | 0.09974 | -2.00000 | 0.33333 | 0.43630 |
| 15.00 | -0.61534 | 0.19208 | 0.14244 | -2.00000 | 0.33333 | 0.43630 |
| 20.00 | -0.48616 | 0.25596 | 0.19106 | -2.00000 | 0.33333 | 0.43630 |
| 25.00 | -0.38749 | 0.30064 | 0.23804 | -2.00000 | 0.33333 | 0.43630 |
| 30.00 | -0.29612 | 0.36002 | 0.27542 | -2.00000 | 0.33333 | 0.43630 |
| 35.00 | -0.21624 | 0.44432 | 0.31905 | -2.00000 | 0.33333 | 0.43630 |
| 40.00 | -0.13248 | 0.52956 | 0.35090 | -2.00000 | 0.33333 | 0.43630 |
| 45.00 | -0.06626 | 0.61857 | 0.38170 | -2.00000 | 0.33333 | 0.43630 |
| 50.00 | -0.00089 | 0.70627 | 0.41762 | -2.00000 | 0.33333 | 0.43630 |
| 55.00 | 0.09669 | 0.82845 | 0.44924 | -2.00000 | 0.33333 | 0.43630 |
| 60.00 | 0.17998 | 0.93104 | 0.47111 | -2.00000 | 0.33333 | 0.43630 |
| 65.00 | 0.26256 | 1.05952 | 0.49540 | -2.00000 | 0.33333 | 0.43630 |
| 70.00 | 0.33256 | 1.26693 | 0.51749 | -2.00000 | 0.33333 | 0.43630 |
| 75.00 | 0.41378 | 1.48020 | 0.53452 | -2.00000 | 0.33333 | 0.43630 |
| 80.00 | 0.51155 | 1.72608 | 0.55133 | -2.00000 | 3.00000 | 0.91674 |
| 85.00 | 0.62798 | 2.03814 | 0.56248 | -2.00000 | 3.00000 | 0.91674 |
| 90.00 | 0.74359 | 2.59913 | 0.56946 | -2.00000 | 3.00000 | 0.91674 |
| 95.00 | 0.96504 | 3.34821 | 0.57553 | -1.99999 | 3.00000 | 0.91674 |
| 99.00 | 1.29685 | 5.08265 | 0.57717 | -1.99979 | 3.00000 | 0.91674 |
| 100.00 | 1.97259 | 7.95851 | 0.57735 | -1.99755 | 3.00000 | 0.91674 |

Table B.2: Statistics for 4x4 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00775 | 1.02735 | 0.45069 | -1.21155 | 0.98364 | 0.45329 |
| Var. | 0.17240 | 0.42520 | 0.11442 | 0.24271 | 1.73584 | 0.12547 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -1.50905 | 0.05281 | 0.00091 | -2.12499 | 0.00000 | 0.00000 |
| 1.00 | -1.03679 | 0.11829 | 0.00552 | -2.03941 | 0.00000 | 0.00000 |
| 5.00 | -0.68521 | 0.22560 | 0.03464 | -1.85245 | 0.00000 | 0.00000 |
| 10.00 | -0.53083 | 0.32342 | 0.06739 | -1.75170 | 0.00000 | 0.00000 |
| 15.00 | -0.43037 | 0.41961 | 0.10592 | -1.69707 | 0.00000 | 0.00000 |
| 20.00 | -0.33919 | 0.49794 | 0.14171 | -1.65787 | 0.00000 | 0.00000 |
| 25.00 | -0.27161 | 0.56509 | 0.17705 | -1.59373 | 0.00000 | 0.00000 |
| 30.00 | -0.19908 | 0.62402 | 0.22012 | -1.54824 | 0.00000 | 0.00000 |
| 35.00 | -0.14042 | 0.67940 | 0.26429 | -1.48396 | 0.66667 | 0.58579 |
| 40.00 | -0.08046 | 0.75092 | 0.29992 | -1.42215 | 0.66667 | 0.58579 |
| 45.00 | -0.03157 | 0.81667 | 0.33557 | -1.35539 | 0.66667 | 0.58579 |
| 50.00 | 0.01517 | 0.89961 | 0.38237 | -1.28935 | 0.66667 | 0.58579 |
| 55.00 | 0.08020 | 0.96531 | 0.41940 | -1.23223 | 0.66667 | 0.58579 |
| 60.00 | 0.12683 | 1.05688 | 0.47153 | -1.16751 | 0.66667 | 0.58579 |
| 65.00 | 0.17451 | 1.14153 | 0.53010 | -1.08139 | 0.66667 | 0.58579 |
| 70.00 | 0.22871 | 1.24833 | 0.58578 | -0.98463 | 0.66667 | 0.58579 |
| 75.00 | 0.27797 | 1.33452 | 0.64333 | -0.89837 | 0.66667 | 0.58579 |
| 80.00 | 0.35003 | 1.48134 | 0.72162 | -0.79006 | 2.66667 | 0.89751 |
| 85.00 | 0.44184 | 1.64925 | 0.81187 | -0.67169 | 2.66667 | 0.89751 |
| 90.00 | 0.52263 | 1.89103 | 0.95534 | -0.52925 | 2.66667 | 0.89751 |
| 95.00 | 0.68186 | 2.28010 | 1.13038 | -0.25083 | 2.66667 | 0.89751 |
| 99.00 | 0.94866 | 2.97527 | 1.40082 | 0.13967 | 6.00000 | 0.98571 |
| 100.00 | 1.45058 | 5.07716 | 1.61022 | 0.46403 | 6.00000 | 0.98571 |

Table B.3: Statistics for 5x5 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00331 | 0.97609 | 0.42357 | -0.77916 | 0.99964 | 0.47790 |
| Var. | 0.09662 | 0.20302 | 0.11757 | 0.45613 | 1.85987 | 0.10668 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -1.02221 | 0.12505 | 0.00111 | -1.81560 | 0.00000 | 0.00000 |
| 1.00 | -0.75197 | 0.23622 | 0.01011 | -1.74502 | 0.00000 | 0.00000 |
| 5.00 | -0.51788 | 0.38264 | 0.03455 | -1.56581 | 0.00000 | 0.00000 |
| 10.00 | -0.40477 | 0.47720 | 0.06001 | -1.46896 | 0.00000 | 0.00000 |
| 15.00 | -0.32945 | 0.53710 | 0.08936 | -1.41197 | 0.00000 | 0.00000 |
| 20.00 | -0.25809 | 0.59528 | 0.11615 | -1.33927 | 0.00000 | 0.00000 |
| 25.00 | -0.20925 | 0.64496 | 0.14953 | -1.27283 | 0.00000 | 0.00000 |
| 30.00 | -0.16082 | 0.69551 | 0.17970 | -1.20239 | 0.40000 | 0.47291 |
| 35.00 | -0.11585 | 0.74166 | 0.22062 | -1.14119 | 0.40000 | 0.47291 |
| 40.00 | -0.07877 | 0.79236 | 0.25702 | -1.07502 | 0.40000 | 0.47291 |
| 45.00 | -0.04716 | 0.85213 | 0.30441 | -1.00986 | 0.40000 | 0.47291 |
| 50.00 | -0.00013 | 0.90199 | 0.33467 | -0.94066 | 0.40000 | 0.47291 |
| 55.00 | 0.03628 | 0.95633 | 0.38614 | -0.86284 | 0.40000 | 0.47291 |
| 60.00 | 0.06701 | 1.01378 | 0.43509 | -0.76880 | 0.40000 | 0.47291 |
| 65.00 | 0.10205 | 1.08596 | 0.48581 | -0.67513 | 0.40000 | 0.47291 |
| 70.00 | 0.15241 | 1.15441 | 0.54758 | -0.56115 | 1.60000 | 0.79410 |
| 75.00 | 0.19833 | 1.22527 | 0.60934 | -0.44167 | 1.60000 | 0.79410 |
| 80.00 | 0.26398 | 1.33124 | 0.68627 | -0.29822 | 1.60000 | 0.79410 |
| 85.00 | 0.32332 | 1.44223 | 0.79643 | -0.11645 | 1.60000 | 0.79410 |
| 90.00 | 0.39659 | 1.55806 | 0.90904 | 0.16985 | 3.60000 | 0.94221 |
| 95.00 | 0.51619 | 1.82081 | 1.07517 | 0.54825 | 3.60000 | 0.94221 |
| 99.00 | 0.69592 | 2.25392 | 1.46316 | 1.36536 | 6.40000 | 0.98862 |
| 100.00 | 0.91614 | 3.22417 | 1.84403 | 2.39093 | 10.00000 | 0.99840 |

Table B.4: Statistics for 6x6 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.01263 | 0.99995 | 0.41590 | -0.49314 | 0.94327 | 0.50159 |
| Var. | 0.06343 | 0.14579 | 0.10820 | 0.60896 | 1.60581 | 0.07561 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.80077 | 0.17810 | 0.00009 | -1.67330 | 0.06667 | 0.20375 |
| 1.00 | -0.59972 | 0.33211 | 0.00395 | -1.58005 | 0.06667 | 0.20375 |
| 5.00 | -0.39328 | 0.46555 | 0.03713 | -1.38502 | 0.06667 | 0.20375 |
| 10.00 | -0.31143 | 0.54101 | 0.07169 | -1.27677 | 0.06667 | 0.20375 |
| 15.00 | -0.24637 | 0.61323 | 0.09901 | -1.18561 | 0.06667 | 0.20375 |
| 20.00 | -0.20196 | 0.66488 | 0.12727 | -1.13122 | 0.06667 | 0.20375 |
| 25.00 | -0.15682 | 0.72344 | 0.15999 | -1.04351 | 0.06667 | 0.20375 |
| 30.00 | -0.12363 | 0.76712 | 0.19933 | -0.98299 | 0.06667 | 0.20375 |
| 35.00 | -0.09277 | 0.80950 | 0.23019 | -0.91018 | 0.06667 | 0.20375 |
| 40.00 | -0.05996 | 0.86113 | 0.26133 | -0.83825 | 0.06667 | 0.20375 |
| 45.00 | -0.02497 | 0.90366 | 0.29669 | -0.74237 | 0.60000 | 0.56142 |
| 50.00 | 0.01349 | 0.95213 | 0.34466 | -0.66334 | 0.60000 | 0.56142 |
| 55.00 | 0.04299 | 1.00447 | 0.37927 | -0.57844 | 0.60000 | 0.56142 |
| 60.00 | 0.07803 | 1.04919 | 0.41911 | -0.48622 | 0.60000 | 0.56142 |
| 65.00 | 0.10933 | 1.10115 | 0.48232 | -0.37550 | 0.60000 | 0.56142 |
| 70.00 | 0.14049 | 1.16702 | 0.52592 | -0.25475 | 0.60000 | 0.56142 |
| 75.00 | 0.17750 | 1.21933 | 0.59073 | -0.16493 | 1.66667 | 0.80330 |
| 80.00 | 0.22375 | 1.30232 | 0.66756 | -0.01498 | 1.66667 | 0.80330 |
| 85.00 | 0.27933 | 1.38968 | 0.75484 | 0.21719 | 1.66667 | 0.80330 |
| 90.00 | 0.35362 | 1.51700 | 0.86755 | 0.54332 | 3.26667 | 0.92931 |
| 95.00 | 0.43224 | 1.72053 | 1.04184 | 1.02088 | 3.26667 | 0.92931 |
| 99.00 | 0.57477 | 2.05082 | 1.46593 | 2.18778 | 5.40000 | 0.97984 |
| 100.00 | 0.81547 | 2.59706 | 1.99716 | 3.93414 | 8.06667 | 0.99548 |

Table B.5: Statistics for 7x7 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00685 | 0.99978 | 0.35359 | -0.40782 | 1.04398 | 0.50935 |
| Var. | 0.04849 | 0.10713 | 0.07851 | 0.50571 | 2.22114 | 0.08004 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.67332 | 0.29053 | 0.00008 | -1.71364 | 0.04762 | 0.17274 |
| 1.00 | -0.53068 | 0.35914 | 0.00354 | -1.41901 | 0.04762 | 0.17274 |
| 5.00 | -0.37841 | 0.53304 | 0.02898 | -1.23956 | 0.04762 | 0.17274 |
| 10.00 | -0.29187 | 0.60891 | 0.05234 | -1.12762 | 0.04762 | 0.17274 |
| 15.00 | -0.24113 | 0.67137 | 0.08520 | -1.02577 | 0.04762 | 0.17274 |
| 20.00 | -0.19271 | 0.72862 | 0.11624 | -0.94829 | 0.04762 | 0.17274 |
| 25.00 | -0.15344 | 0.76641 | 0.14051 | -0.88160 | 0.04762 | 0.17274 |
| 30.00 | -0.11447 | 0.80023 | 0.17341 | -0.82695 | 0.04762 | 0.17274 |
| 35.00 | -0.08695 | 0.83666 | 0.19888 | -0.75960 | 0.42857 | 0.48730 |
| 40.00 | -0.06451 | 0.87623 | 0.22866 | -0.69334 | 0.42857 | 0.48730 |
| 45.00 | -0.03467 | 0.91916 | 0.26278 | -0.61617 | 0.42857 | 0.48730 |
| 50.00 | -0.00662 | 0.96781 | 0.29473 | -0.55072 | 0.42857 | 0.48730 |
| 55.00 | 0.01703 | 1.00652 | 0.32640 | -0.47841 | 0.42857 | 0.48730 |
| 60.00 | 0.05070 | 1.05781 | 0.36451 | -0.40761 | 0.42857 | 0.48730 |
| 65.00 | 0.07675 | 1.10792 | 0.40246 | -0.32018 | 1.19048 | 0.72476 |
| 70.00 | 0.10614 | 1.15263 | 0.44116 | -0.21973 | 1.19048 | 0.72476 |
| 75.00 | 0.13831 | 1.20523 | 0.49512 | -0.11362 | 1.19048 | 0.72476 |
| 80.00 | 0.17992 | 1.26851 | 0.54772 | 0.00127 | 1.19048 | 0.72476 |
| 85.00 | 0.22121 | 1.32878 | 0.62165 | 0.21134 | 2.33333 | 0.87336 |
| 90.00 | 0.27473 | 1.43204 | 0.72504 | 0.48597 | 2.33333 | 0.87336 |
| 95.00 | 0.35174 | 1.58285 | 0.89673 | 0.88641 | 3.85714 | 0.95045 |
| 99.00 | 0.50401 | 1.92196 | 1.27804 | 2.05589 | 5.76190 | 0.98363 |
| 100.00 | 0.66945 | 2.99929 | 1.97804 | 5.63063 | 10.71429 | 0.99897 |

Table B.6: Statistics for 8x8 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | $\lvert \text{GAMMA1} \rvert$ | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.02209 | 0.99919 | 0.31221 | -0.32576 | 1.06286 | 0.50434 |
| Var. | 0.03935 | 0.07266 | 0.06260 | 0.44503 | 2.22502 | 0.09222 |

Percentage Points of Statistics

| % | XBAR | VAR | $\lvert \text{GAMMA1} \rvert$ | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.66801 | 0.37311 | 0.00045 | -1.54153 | 0.00000 | 0.00000 |
| 1.00 | -0.49556 | 0.48682 | 0.00289 | -1.29857 | 0.00000 | 0.00000 |
| 5.00 | -0.34508 | 0.59135 | 0.02066 | -1.11343 | 0.00000 | 0.00000 |
| 10.00 | -0.27497 | 0.67435 | 0.04049 | -1.01090 | 0.00000 | 0.00000 |
| 15.00 | -0.22425 | 0.72997 | 0.06606 | -0.91962 | 0.14286 | 0.29455 |
| 20.00 | -0.18603 | 0.77044 | 0.09118 | -0.83390 | 0.14286 | 0.29455 |
| 25.00 | -0.15556 | 0.80423 | 0.11678 | -0.79278 | 0.14286 | 0.29455 |
| 30.00 | -0.12424 | 0.83962 | 0.14562 | -0.73583 | 0.14286 | 0.29455 |
| 35.00 | -0.09716 | 0.86489 | 0.17733 | -0.66327 | 0.14286 | 0.29455 |
| 40.00 | -0.07501 | 0.90692 | 0.19934 | -0.60069 | 0.14286 | 0.29455 |
| 45.00 | -0.05254 | 0.94025 | 0.22824 | -0.55020 | 0.57143 | 0.55031 |
| 50.00 | -0.02538 | 0.96787 | 0.25723 | -0.48349 | 0.57143 | 0.55031 |
| 55.00 | 0.00127 | 1.00376 | 0.29122 | -0.41031 | 0.57143 | 0.55031 |
| 60.00 | 0.02596 | 1.03904 | 0.31806 | -0.31743 | 0.57143 | 0.55031 |
| 65.00 | 0.05247 | 1.08563 | 0.35097 | -0.22917 | 1.28571 | 0.74315 |
| 70.00 | 0.08324 | 1.12089 | 0.39208 | -0.12779 | 1.28571 | 0.74315 |
| 75.00 | 0.11103 | 1.16879 | 0.44365 | -0.01892 | 1.28571 | 0.74315 |
| 80.00 | 0.14531 | 1.22073 | 0.50085 | 0.12418 | 1.28571 | 0.74315 |
| 85.00 | 0.18241 | 1.28877 | 0.57342 | 0.28232 | 2.28571 | 0.86943 |
| 90.00 | 0.23320 | 1.36823 | 0.65184 | 0.54521 | 2.28571 | 0.86943 |
| 95.00 | 0.31002 | 1.48722 | 0.80659 | 0.90316 | 3.57143 | 0.94121 |
| 99.00 | 0.41740 | 1.69588 | 1.06319 | 1.96780 | 7.00000 | 0.99182 |
| 100.00 | 0.66026 | 1.91652 | 1.48894 | 2.98311 | 17.28572 | 1.00000 |

Table B.7: Statistics for 9x9 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.01449 | 1.01835 | 0.29373 | -0.22450 | 0.97656 | 0.48951 |
| Var. | 0.02725 | 0.05813 | 0.05616 | 0.42751 | 1.77247 | 0.09142 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.53123 | 0.43106 | 0.00250 | -1.49912 | 0.00000 | 0.00000 |
| 1.00 | -0.35780 | 0.55253 | 0.00539 | -1.22275 | 0.00000 | 0.00000 |
| 5.00 | -0.24785 | 0.66171 | 0.02279 | -1.01732 | 0.00000 | 0.00000 |
| 10.00 | -0.20140 | 0.72375 | 0.03931 | -0.88921 | 0.00000 | 0.00000 |
| 15.00 | -0.16118 | 0.76853 | 0.06268 | -0.80626 | 0.11111 | 0.26112 |
| 20.00 | -0.13230 | 0.80759 | 0.08200 | -0.74280 | 0.11111 | 0.26112 |
| 25.00 | -0.09876 | 0.84645 | 0.10157 | -0.68282 | 0.11111 | 0.26112 |
| 30.00 | -0.07034 | 0.87943 | 0.12781 | -0.60883 | 0.11111 | 0.26112 |
| 35.00 | -0.04329 | 0.91529 | 0.15626 | -0.53436 | 0.11111 | 0.26112 |
| 40.00 | -0.02005 | 0.94106 | 0.17948 | -0.48167 | 0.44444 | 0.49501 |
| 45.00 | -0.00274 | 0.96573 | 0.21440 | -0.41542 | 0.44444 | 0.49501 |
| 50.00 | 0.01365 | 0.99587 | 0.23989 | -0.34401 | 0.44444 | 0.49501 |
| 55.00 | 0.03329 | 1.02532 | 0.27495 | -0.25978 | 0.44444 | 0.49501 |
| 60.00 | 0.05924 | 1.05682 | 0.30775 | -0.18932 | 0.44444 | 0.49501 |
| 65.00 | 0.07931 | 1.09183 | 0.34275 | -0.13489 | 1.00000 | 0.68269 |
| 70.00 | 0.09891 | 1.12798 | 0.38146 | -0.04237 | 1.00000 | 0.68269 |
| 75.00 | 0.12230 | 1.17419 | 0.42105 | 0.06178 | 1.00000 | 0.68269 |
| 80.00 | 0.15316 | 1.21673 | 0.47502 | 0.20951 | 1.77778 | 0.81758 |
| 85.00 | 0.18539 | 1.26821 | 0.53255 | 0.38286 | 1.77778 | 0.81758 |
| 90.00 | 0.22856 | 1.33448 | 0.61060 | 0.57001 | 2.77778 | 0.90440 |
| 95.00 | 0.28052 | 1.43599 | 0.76976 | 0.94686 | 4.00000 | 0.95448 |
| 99.00 | 0.41290 | 1.63916 | 1.03606 | 2.10575 | 5.44444 | 0.98035 |
| 100.00 | 0.62502 | 2.09891 | 1.34140 | 3.87976 | 9.00000 | 0.99732 |

Table B.8: Statistics for 10x10 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00262 | 1.00007 | 0.26071 | -0.17996 | 0.95022 | 0.49854 |
| Var. | 0.02100 | 0.04811 | 0.04242 | 0.38934 | 1.60737 | 0.08058 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.43397 | 0.48739 | 0.00012 | -1.26288 | 0.02222 | 0.11850 |
| 1.00 | -0.33856 | 0.55785 | 0.00417 | -1.15601 | 0.02222 | 0.11850 |
| 5.00 | -0.23587 | 0.68262 | 0.02242 | -0.96351 | 0.02222 | 0.11850 |
| 10.00 | -0.17603 | 0.73338 | 0.04044 | -0.85357 | 0.02222 | 0.11850 |
| 15.00 | -0.14537 | 0.77469 | 0.06335 | -0.76318 | 0.02222 | 0.11850 |
| 20.00 | -0.12290 | 0.81386 | 0.08486 | -0.67927 | 0.02222 | 0.11850 |
| 25.00 | -0.09683 | 0.85235 | 0.09971 | -0.61801 | 0.20000 | 0.34528 |
| 30.00 | -0.07910 | 0.87587 | 0.12149 | -0.55201 | 0.20000 | 0.34528 |
| 35.00 | -0.06054 | 0.90122 | 0.14349 | -0.49073 | 0.20000 | 0.34528 |
| 40.00 | -0.03980 | 0.92963 | 0.15987 | -0.43649 | 0.20000 | 0.34528 |
| 45.00 | -0.01910 | 0.95341 | 0.18403 | -0.37625 | 0.20000 | 0.34528 |
| 50.00 | -0.00197 | 0.97973 | 0.21120 | -0.30165 | 0.55556 | 0.54394 |
| 55.00 | 0.01947 | 1.00672 | 0.23793 | -0.22638 | 0.55556 | 0.54394 |
| 60.00 | 0.03844 | 1.03021 | 0.26355 | -0.15477 | 0.55556 | 0.54394 |
| 65.00 | 0.06128 | 1.06098 | 0.29910 | -0.07801 | 1.08889 | 0.70328 |
| 70.00 | 0.07895 | 1.08918 | 0.33435 | 0.01126 | 1.08889 | 0.70328 |
| 75.00 | 0.10213 | 1.13725 | 0.36746 | 0.11909 | 1.08889 | 0.70328 |
| 80.00 | 0.12860 | 1.18201 | 0.41511 | 0.24380 | 1.80000 | 0.82029 |
| 85.00 | 0.15380 | 1.22437 | 0.47418 | 0.41395 | 1.80000 | 0.82029 |
| 90.00 | 0.18695 | 1.30206 | 0.56094 | 0.64412 | 2.68889 | 0.89893 |
| 95.00 | 0.24350 | 1.38299 | 0.68496 | 1.05637 | 3.75556 | 0.94736 |
| 99.00 | 0.34831 | 1.58461 | 0.86535 | 1.78730 | 6.42222 | 0.98872 |
| 100.00 | 0.43478 | 1.92315 | 1.17668 | 2.71917 | 9.80000 | 0.99823 |

Table B.9: Statistics for 11x11 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00051 | 0.99066 | 0.24650 | -0.14316 | 1.03081 | 0.50232 |
| Var. | 0.01891 | 0.03521 | 0.03950 | 0.34739 | 2.18882 | 0.08496 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.51548 | 0.45546 | 0.00011 | -1.18467 | 0.01818 | 0.10726 |
| 1.00 | -0.31148 | 0.60322 | 0.00381 | -1.04224 | 0.01818 | 0.10726 |
| 5.00 | -0.22131 | 0.71343 | 0.01598 | -0.85125 | 0.01818 | 0.10726 |
| 10.00 | -0.17210 | 0.76451 | 0.03563 | -0.74136 | 0.01818 | 0.10726 |
| 15.00 | -0.14810 | 0.79693 | 0.05786 | -0.66022 | 0.01818 | 0.10726 |
| 20.00 | -0.12215 | 0.82530 | 0.07945 | -0.59152 | 0.01818 | 0.10726 |
| 25.00 | -0.09846 | 0.85212 | 0.10065 | -0.53103 | 0.16364 | 0.31417 |
| 30.00 | -0.07260 | 0.88324 | 0.11765 | -0.47671 | 0.16364 | 0.31417 |
| 35.00 | -0.05552 | 0.91041 | 0.13455 | -0.42756 | 0.16364 | 0.31417 |
| 40.00 | -0.03606 | 0.93018 | 0.15819 | -0.36971 | 0.16364 | 0.31417 |
| 45.00 | -0.01960 | 0.95275 | 0.17511 | -0.31768 | 0.45455 | 0.49981 |
| 50.00 | -0.00339 | 0.97607 | 0.20051 | -0.27210 | 0.45455 | 0.49981 |
| 55.00 | 0.01445 | 1.00038 | 0.22449 | -0.20735 | 0.45455 | 0.49981 |
| 60.00 | 0.03177 | 1.02542 | 0.25589 | -0.14704 | 0.89091 | 0.65478 |
| 65.00 | 0.04779 | 1.05562 | 0.28618 | -0.07422 | 0.89091 | 0.65478 |
| 70.00 | 0.06635 | 1.08538 | 0.31650 | 0.02158 | 1.47273 | 0.77507 |
| 75.00 | 0.09450 | 1.11543 | 0.34498 | 0.11200 | 1.47273 | 0.77507 |
| 80.00 | 0.12079 | 1.14351 | 0.37950 | 0.24114 | 1.47273 | 0.77507 |
| 85.00 | 0.14799 | 1.18101 | 0.43569 | 0.36459 | 2.20000 | 0.86199 |
| 90.00 | 0.17937 | 1.23455 | 0.50733 | 0.62537 | 2.20000 | 0.86199 |
| 95.00 | 0.22935 | 1.31975 | 0.61529 | 0.94102 | 4.09091 | 0.95686 |
| 99.00 | 0.30881 | 1.46826 | 0.88918 | 1.88365 | 6.56364 | 0.98958 |
| 100.00 | 0.45266 | 1.84344 | 1.35711 | 4.09483 | 15.29091 | 1.00000 |

Table B.10: Statistics for 12x12 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00317 | 0.99844 | 0.22751 | -0.14736 | 0.93565 | 0.48747 |
| Var. | 0.01498 | 0.02939 | 0.03075 | 0.25186 | 1.56145 | 0.08772 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.40626 | 0.48667 | 0.00007 | -1.23879 | 0.00000 | 0.00000 |
| 1.00 | -0.28197 | 0.63963 | 0.00225 | -1.03516 | 0.00000 | 0.00000 |
| 5.00 | -0.19525 | 0.73649 | 0.01976 | -0.78969 | 0.00000 | 0.00000 |
| 10.00 | -0.15639 | 0.78705 | 0.03619 | -0.67503 | 0.00000 | 0.00000 |
| 15.00 | -0.12901 | 0.82425 | 0.05772 | -0.61168 | 0.06061 | 0.19446 |
| 20.00 | -0.10645 | 0.85001 | 0.07459 | -0.54909 | 0.06061 | 0.19446 |
| 25.00 | -0.08183 | 0.87798 | 0.08856 | -0.49824 | 0.06061 | 0.19446 |
| 30.00 | -0.06067 | 0.90359 | 0.10866 | -0.44408 | 0.06061 | 0.19446 |
| 35.00 | -0.04601 | 0.92192 | 0.12839 | -0.38689 | 0.24242 | 0.37754 |
| 40.00 | -0.02698 | 0.94608 | 0.14722 | -0.33457 | 0.24242 | 0.37754 |
| 45.00 | -0.01305 | 0.96607 | 0.16980 | -0.27925 | 0.24242 | 0.37754 |
| 50.00 | 0.00426 | 0.98645 | 0.18871 | -0.22100 | 0.54545 | 0.53982 |
| 55.00 | 0.02164 | 1.00884 | 0.21072 | -0.17254 | 0.54545 | 0.53982 |
| 60.00 | 0.03465 | 1.02919 | 0.23718 | -0.11231 | 0.54545 | 0.53982 |
| 65.00 | 0.04837 | 1.05591 | 0.26092 | -0.06199 | 0.96970 | 0.67524 |
| 70.00 | 0.06840 | 1.08015 | 0.28974 | 0.02035 | 0.96970 | 0.67524 |
| 75.00 | 0.08770 | 1.10690 | 0.32401 | 0.09299 | 1.51515 | 0.78163 |
| 80.00 | 0.10576 | 1.13416 | 0.36783 | 0.19168 | 1.51515 | 0.78163 |
| 85.00 | 0.12424 | 1.16804 | 0.41086 | 0.31436 | 2.18182 | 0.86035 |
| 90.00 | 0.16033 | 1.22026 | 0.47491 | 0.50835 | 2.18182 | 0.86035 |
| 95.00 | 0.21158 | 1.31458 | 0.55566 | 0.78019 | 3.87879 | 0.95109 |
| 99.00 | 0.29036 | 1.43828 | 0.74381 | 1.45655 | 6.06061 | 0.98619 |
| 100.00 | 0.34516 | 1.65202 | 1.12910 | 2.49610 | 8.72727 | 0.99688 |

Table B.11: Statistics for 13x13 Subjective Judgment Matrices

## Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00289 | 1.00742 | 0.20037 | -0.11891 | 1.01855 | 0.50906 |
| Var. | 0.01359 | 0.02814 | 0.02408 | 0.22939 | 1.82731 | 0.08472 |

## Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.45176 | 0.57806 | 0.00089 | -1.07879 | 0.00000 | 0.00000 |
| 1.00 | -0.27757 | 0.65076 | 0.00477 | -0.91152 | 0.00000 | 0.00000 |
| 5.00 | -0.19005 | 0.74951 | 0.01741 | -0.74117 | 0.00000 | 0.00000 |
| 10.00 | -0.15093 | 0.79656 | 0.03183 | -0.64397 | 0.05128 | 0.17915 |
| 15.00 | -0.12572 | 0.83284 | 0.04610 | -0.57226 | 0.05128 | 0.17915 |
| 20.00 | -0.10341 | 0.86822 | 0.06395 | -0.51109 | 0.05128 | 0.17915 |
| 25.00 | -0.07915 | 0.89583 | 0.08160 | -0.46301 | 0.05128 | 0.17915 |
| 30.00 | -0.06337 | 0.91651 | 0.09445 | -0.40471 | 0.20513 | 0.34939 |
| 35.00 | -0.04842 | 0.93694 | 0.11177 | -0.35677 | 0.20513 | 0.34939 |
| 40.00 | -0.03446 | 0.95946 | 0.13141 | -0.31242 | 0.20513 | 0.34939 |
| 45.00 | -0.01880 | 0.97768 | 0.14856 | -0.24783 | 0.46154 | 0.50309 |
| 50.00 | -0.00316 | 0.99721 | 0.16699 | -0.20334 | 0.46154 | 0.50309 |
| 55.00 | 0.01052 | 1.01685 | 0.18977 | -0.14181 | 0.46154 | 0.50309 |
| 60.00 | 0.02518 | 1.03787 | 0.20939 | -0.08601 | 0.82051 | 0.63498 |
| 65.00 | 0.03965 | 1.06120 | 0.22979 | -0.01864 | 0.82051 | 0.63498 |
| 70.00 | 0.05656 | 1.08623 | 0.25475 | 0.05118 | 1.28205 | 0.74248 |
| 75.00 | 0.07491 | 1.10955 | 0.28372 | 0.12370 | 1.28205 | 0.74248 |
| 80.00 | 0.09576 | 1.14168 | 0.31748 | 0.21308 | 1.84615 | 0.82578 |
| 85.00 | 0.11633 | 1.17610 | 0.35289 | 0.34022 | 1.84615 | 0.82578 |
| 90.00 | 0.15044 | 1.22677 | 0.41635 | 0.51854 | 2.51282 | 0.88707 |
| 95.00 | 0.18829 | 1.30374 | 0.48979 | 0.80724 | 3.28205 | 0.92997 |
| 99.00 | 0.28062 | 1.45031 | 0.69819 | 1.45445 | 6.20513 | 0.98729 |
| 100.00 | 0.36349 | 1.65806 | 0.92051 | 2.31334 | 10.05128 | 0.99844 |

Table B.12: Statistics for 14x14 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00031 | 0.99516 | 0.19272 | -0.08523 | 1.00072 | 0.51039 |
| Var. | 0.01023 | 0.02157 | 0.02311 | 0.22773 | 1.99211 | 0.07739 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.35076 | 0.60193 | 0.00003 | -1.05385 | 0.01099 | 0.08349 |
| 1.00 | -0.22857 | 0.68912 | 0.00229 | -0.87129 | 0.01099 | 0.08349 |
| 5.00 | -0.16285 | 0.76230 | 0.01510 | -0.71432 | 0.01099 | 0.08349 |
| 10.00 | -0.12604 | 0.81425 | 0.02792 | -0.61038 | 0.01099 | 0.08349 |
| 15.00 | -0.10498 | 0.84496 | 0.04603 | -0.53496 | 0.01099 | 0.08349 |
| 20.00 | -0.08358 | 0.86639 | 0.05997 | -0.48629 | 0.09890 | 0.24685 |
| 25.00 | -0.06825 | 0.89039 | 0.07548 | -0.42127 | 0.09890 | 0.24685 |
| 30.00 | -0.05514 | 0.91382 | 0.09293 | -0.36969 | 0.27473 | 0.39982 |
| 35.00 | -0.04209 | 0.93261 | 0.10944 | -0.32557 | 0.27473 | 0.39982 |
| 40.00 | -0.03116 | 0.95055 | 0.12512 | -0.28071 | 0.27473 | 0.39982 |
| 45.00 | -0.01568 | 0.96782 | 0.14141 | -0.22614 | 0.27473 | 0.39982 |
| 50.00 | -0.00232 | 0.99036 | 0.15719 | -0.17459 | 0.53846 | 0.53693 |
| 55.00 | 0.01055 | 1.00706 | 0.17564 | -0.10879 | 0.53846 | 0.53693 |
| 60.00 | 0.02484 | 1.02729 | 0.19483 | -0.05332 | 0.89011 | 0.65456 |
| 65.00 | 0.03730 | 1.04497 | 0.22151 | 0.01733 | 0.89011 | 0.65456 |
| 70.00 | 0.05250 | 1.06213 | 0.24742 | 0.08995 | 0.89011 | 0.65456 |
| 75.00 | 0.06842 | 1.08871 | 0.27657 | 0.17671 | 1.32967 | 0.75113 |
| 80.00 | 0.08527 | 1.11075 | 0.30448 | 0.27013 | 1.32967 | 0.75113 |
| 85.00 | 0.10561 | 1.14197 | 0.34690 | 0.37705 | 1.85714 | 0.82705 |
| 90.00 | 0.12708 | 1.18061 | 0.39562 | 0.57470 | 2.47253 | 0.88414 |
| 95.00 | 0.16815 | 1.24696 | 0.48065 | 0.82258 | 3.96703 | 0.95358 |
| 99.00 | 0.23796 | 1.37935 | 0.67250 | 1.31528 | 6.86813 | 0.99120 |
| 100.00 | 0.31674 | 1.66742 | 1.03333 | 1.98715 | 13.46154 | 1.00000 |

Table B.13: Statistics for 15x15 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00505 | 0.99898 | 0.17836 | -0.09901 | 0.99626 | 0.50433 |
| Var. | 0.00952 | 0.01852 | 0.01825 | 0.18072 | 1.90045 | 0.08177 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.34322 | 0.59349 | 0.00031 | -0.96264 | 0.00952 | 0.07774 |
| 1.00 | -0.23798 | 0.70732 | 0.00176 | -0.83563 | 0.00952 | 0.07774 |
| 5.00 | -0.16632 | 0.78187 | 0.01192 | -0.65822 | 0.00952 | 0.07774 |
| 10.00 | -0.12957 | 0.82348 | 0.02790 | -0.55566 | 0.00952 | 0.07774 |
| 15.00 | -0.10309 | 0.85008 | 0.04167 | -0.48942 | 0.00952 | 0.07774 |
| 20.00 | -0.08475 | 0.87971 | 0.05541 | -0.44536 | 0.08571 | 0.23030 |
| 25.00 | -0.06952 | 0.90345 | 0.07284 | -0.39342 | 0.08571 | 0.23030 |
| 30.00 | -0.05583 | 0.92299 | 0.08492 | -0.34364 | 0.08571 | 0.23030 |
| 35.00 | -0.03950 | 0.94318 | 0.10697 | -0.29604 | 0.23810 | 0.37441 |
| 40.00 | -0.02736 | 0.96182 | 0.12441 | -0.24931 | 0.23810 | 0.37441 |
| 45.00 | -0.01782 | 0.98107 | 0.13925 | -0.20931 | 0.46667 | 0.50547 |
| 50.00 | -0.00856 | 0.99639 | 0.15352 | -0.16881 | 0.46667 | 0.50547 |
| 55.00 | 0.00533 | 1.01569 | 0.16746 | -0.12534 | 0.46667 | 0.50547 |
| 60.00 | 0.01799 | 1.03366 | 0.18288 | -0.07379 | 0.77143 | 0.62023 |
| 65.00 | 0.03149 | 1.04870 | 0.20425 | -0.01126 | 0.77143 | 0.62023 |
| 70.00 | 0.04556 | 1.06998 | 0.22796 | 0.05157 | 1.15238 | 0.71695 |
| 75.00 | 0.05867 | 1.09146 | 0.25112 | 0.11730 | 1.15238 | 0.71695 |
| 80.00 | 0.07748 | 1.11635 | 0.28039 | 0.18541 | 1.60952 | 0.79544 |
| 85.00 | 0.09769 | 1.13859 | 0.31997 | 0.28203 | 2.14286 | 0.85679 |
| 90.00 | 0.12073 | 1.17763 | 0.36422 | 0.42582 | 2.75238 | 0.90287 |
| 95.00 | 0.15496 | 1.21867 | 0.44414 | 0.71132 | 4.20000 | 0.95955 |
| 99.00 | 0.22253 | 1.32212 | 0.58923 | 1.28618 | 5.95238 | 0.98532 |
| 100.00 | 0.36057 | 1.49480 | 0.83878 | 2.30351 | 11.66667 | 0.99938 |

Table B.14: Statistics for 16x16 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00583 | 1.00341 | 0.16866 | -0.06000 | 1.00767 | 0.51146 |
| Var. | 0.00796 | 0.01732 | 0.01766 | 0.17154 | 1.79328 | 0.08206 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.26300 | 0.67478 | 0.00001 | -0.86597 | 0.00000 | 0.00000 |
| 1.00 | -0.20974 | 0.73698 | 0.00231 | -0.79116 | 0.00000 | 0.00000 |
| 5.00 | -0.14451 | 0.79708 | 0.01362 | -0.61372 | 0.00000 | 0.00000 |
| 10.00 | -0.11249 | 0.83376 | 0.02664 | -0.53458 | 0.03333 | 0.14487 |
| 15.00 | -0.08577 | 0.86932 | 0.03738 | -0.47459 | 0.03333 | 0.14487 |
| 20.00 | -0.06926 | 0.89058 | 0.04815 | -0.40758 | 0.13333 | 0.28500 |
| 25.00 | -0.05515 | 0.91187 | 0.06274 | -0.35417 | 0.13333 | 0.28500 |
| 30.00 | -0.03893 | 0.93082 | 0.07768 | -0.30641 | 0.13333 | 0.28500 |
| 35.00 | -0.02582 | 0.94556 | 0.09239 | -0.25421 | 0.30000 | 0.41612 |
| 40.00 | -0.01390 | 0.96438 | 0.10944 | -0.21684 | 0.30000 | 0.41612 |
| 45.00 | -0.00520 | 0.97880 | 0.12428 | -0.16532 | 0.53333 | 0.53479 |
| 50.00 | 0.00557 | 0.99551 | 0.14173 | -0.11349 | 0.53333 | 0.53479 |
| 55.00 | 0.01554 | 1.01314 | 0.15644 | -0.06586 | 0.53333 | 0.53479 |
| 60.00 | 0.02615 | 1.03212 | 0.17244 | -0.01288 | 0.83333 | 0.63870 |
| 65.00 | 0.04032 | 1.05063 | 0.19234 | 0.03655 | 0.83333 | 0.63870 |
| 70.00 | 0.05166 | 1.06566 | 0.21618 | 0.09285 | 1.20000 | 0.72667 |
| 75.00 | 0.06620 | 1.09146 | 0.24053 | 0.15898 | 1.20000 | 0.72667 |
| 80.00 | 0.07844 | 1.11650 | 0.27022 | 0.25764 | 1.63333 | 0.79876 |
| 85.00 | 0.09739 | 1.13907 | 0.30396 | 0.34723 | 2.13333 | 0.85589 |
| 90.00 | 0.12409 | 1.16847 | 0.35211 | 0.47592 | 2.70000 | 0.89963 |
| 95.00 | 0.15635 | 1.22838 | 0.44211 | 0.71106 | 4.03333 | 0.95537 |
| 99.00 | 0.19254 | 1.34582 | 0.54155 | 1.19377 | 6.53333 | 0.98940 |
| 100.00 | 0.29737 | 1.46845 | 0.76109 | 1.62261 | 10.80000 | 0.99902 |

Table B.15: Statistics for 17x17 Subjective Judgment Matrices

## Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00256 | 1.00037 | 0.15783 | -0.07865 | 0.99650 | 0.49998 |
| Var. | 0.00760 | 0.01557 | 0.01447 | 0.13073 | 2.09215 | 0.08371 |

## Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.29859 | 0.61413 | 0.00001 | -0.94676 | 0.00000 | 0.00000 |
| 1.00 | -0.21350 | 0.74408 | 0.00295 | -0.74785 | 0.00000 | 0.00000 |
| 5.00 | -0.14132 | 0.79674 | 0.01368 | -0.57328 | 0.00000 | 0.00000 |
| 10.00 | -0.11517 | 0.84231 | 0.02411 | -0.49074 | 0.02941 | 0.13617 |
| 15.00 | -0.09192 | 0.86862 | 0.03715 | -0.43501 | 0.02941 | 0.13617 |
| 20.00 | -0.07324 | 0.89794 | 0.05055 | -0.36967 | 0.02941 | 0.13617 |
| 25.00 | -0.05954 | 0.91764 | 0.06312 | -0.32743 | 0.11765 | 0.26840 |
| 30.00 | -0.04888 | 0.93505 | 0.07653 | -0.28258 | 0.11765 | 0.26840 |
| 35.00 | -0.03730 | 0.95022 | 0.08804 | -0.24561 | 0.26471 | 0.39309 |
| 40.00 | -0.02741 | 0.96383 | 0.10574 | -0.20445 | 0.26471 | 0.39309 |
| 45.00 | -0.01602 | 0.97692 | 0.12380 | -0.15966 | 0.47059 | 0.50728 |
| 50.00 | -0.00331 | 0.99388 | 0.13666 | -0.12009 | 0.47059 | 0.50728 |
| 55.00 | 0.00832 | 1.00997 | 0.14891 | -0.08359 | 0.47059 | 0.50728 |
| 60.00 | 0.01935 | 1.02167 | 0.16520 | -0.04090 | 0.73529 | 0.60883 |
| 65.00 | 0.02974 | 1.03784 | 0.18127 | 0.00162 | 0.73529 | 0.60883 |
| 70.00 | 0.04111 | 1.05759 | 0.20172 | 0.05722 | 1.05882 | 0.69651 |
| 75.00 | 0.05413 | 1.07908 | 0.22479 | 0.11378 | 1.44118 | 0.77004 |
| 80.00 | 0.06885 | 1.10203 | 0.25276 | 0.19037 | 1.88235 | 0.82994 |
| 85.00 | 0.08916 | 1.13243 | 0.28010 | 0.27082 | 1.88235 | 0.82994 |
| 90.00 | 0.10681 | 1.15957 | 0.31593 | 0.39650 | 2.38235 | 0.87728 |
| 95.00 | 0.13616 | 1.21412 | 0.38469 | 0.57191 | 3.55882 | 0.94076 |
| 99.00 | 0.20998 | 1.31564 | 0.49904 | 1.02926 | 7.52941 | 0.99395 |
| 100.00 | 0.29930 | 1.44939 | 0.81570 | 1.99064 | 14.23529 | 1.00000 |

Table B.16: Statistics for 18x18 Subjective Judgment Matrices

## Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00119 | 0.99752 | 0.15011 | -0.05410 | 0.[illegible]6302 | 0.49530 |
| Var. | 0.00619 | 0.01273 | 0.01441 | 0.13448 | 1.73580 | 0.08330 |

## Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.24342 | 0.64337 | 0.00003 | -0.87953 | 0.00654 | 0.06443 |
| 1.00 | -0.18399 | 0.75072 | 0.00173 | -0.69734 | 0.00654 | 0.06443 |
| 5.00 | -0.13070 | 0.81457 | 0.01081 | -0.57235 | 0.00654 | 0.06443 |
| 10.00 | -0.10238 | 0.85104 | 0.02228 | -0.48790 | 0.00654 | 0.06443 |
| 15.00 | -0.08219 | 0.88080 | 0.03350 | -0.40802 | 0.05882 | 0.19164 |
| 20.00 | -0.06394 | 0.89903 | 0.04353 | -0.35682 | 0.05882 | 0.19164 |
| 25.00 | -0.05095 | 0.91891 | 0.05776 | -0.31693 | 0.05882 | 0.19164 |
| 30.00 | -0.03953 | 0.93641 | 0.06877 | -0.27139 | 0.16340 | 0.31395 |
| 35.00 | -0.02876 | 0.94880 | 0.08181 | -0.22359 | 0.16340 | 0.31395 |
| 40.00 | -0.01984 | 0.96639 | 0.09492 | -0.18152 | 0.32026 | 0.42855 |
| 45.00 | -0.00759 | 0.98290 | 0.10902 | -0.14667 | 0.32026 | 0.42855 |
| 50.00 | 0.00274 | 0.99540 | 0.12123 | -0.10436 | 0.52941 | 0.53315 |
| 55.00 | 0.01213 | 1.00842 | 0.13685 | -0.06505 | 0.52941 | 0.53315 |
| 60.00 | 0.02348 | 1.02387 | 0.15036 | -0.02419 | 0.79085 | 0.62616 |
| 65.00 | 0.03379 | 1.03866 | 0.16932 | 0.03692 | 0.79085 | 0.62616 |
| 70.00 | 0.04459 | 1.05609 | 0.18966 | 0.08985 | 1.10458 | 0.70673 |
| 75.00 | 0.05607 | 1.07408 | 0.21752 | 0.14895 | 1.47059 | 0.77473 |
| 80.00 | 0.06812 | 1.09432 | 0.24109 | 0.22955 | 1.47059 | 0.77473 |
| 85.00 | 0.08225 | 1.11375 | 0.27762 | 0.30823 | 1.88889 | 0.83068 |
| 90.00 | 0.10022 | 1.14764 | 0.31699 | 0.42514 | 2.35948 | 0.87547 |
| 95.00 | 0.12205 | 1.18431 | 0.37345 | 0.60672 | 3.45752 | 0.93706 |
| 99.00 | 0.17851 | 1.25472 | 0.51895 | 1.11051 | 6.28105 | 0.98782 |
| 100.00 | 0.28168 | 1.44906 | 0.66495 | 1.48588 | 8.94771 | 0.99724 |

Table B.17: Statistics for 19x19 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | 0.00131 | 1.00218 | 0.14682 | -0.04358 | 1.00357 | 0.50582 |
| Var. | 0.00586 | 0.01185 | 0.01288 | 0.12249 | 1.78927 | 0.08366 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.24941 | 0.72855 | 0.00008 | -0.81430 | 0.00585 | 0.06096 |
| 1.00 | -0.16690 | 0.77533 | 0.00165 | -0.66645 | 0.00585 | 0.06096 |
| 5.00 | -0.12470 | 0.82701 | 0.01233 | -0.53454 | 0.00585 | 0.06096 |
| 10.00 | -0.09730 | 0.86152 | 0.02389 | -0.45290 | 0.00585 | 0.06096 |
| 15.00 | -0.07941 | 0.88981 | 0.03503 | -0.38347 | 0.05263 | 0.18145 |
| 20.00 | -0.06613 | 0.90993 | 0.04588 | -0.33176 | 0.05263 | 0.18145 |
| 25.00 | -0.05364 | 0.92526 | 0.05482 | -0.29317 | 0.14620 | 0.29781 |
| 30.00 | -0.04201 | 0.94168 | 0.06893 | -0.25376 | 0.14620 | 0.29781 |
| 35.00 | -0.03034 | 0.95600 | 0.08098 | -0.20123 | 0.14620 | 0.29781 |
| 40.00 | -0.01697 | 0.97091 | 0.09404 | -0.16449 | 0.28655 | 0.40756 |
| 45.00 | -0.00489 | 0.98491 | 0.10755 | -0.12827 | 0.28655 | 0.40756 |
| 50.00 | 0.00347 | 0.99695 | 0.12133 | -0.08610 | 0.47368 | 0.50870 |
| 55.00 | 0.01273 | 1.00920 | 0.13792 | -0.04254 | 0.70760 | 0.59976 |
| 60.00 | 0.02217 | 1.02502 | 0.15157 | -0.00261 | 0.70760 | 0.59976 |
| 65.00 | 0.03196 | 1.04027 | 0.16889 | 0.04254 | 0.98830 | 0.67984 |
| 70.00 | 0.04192 | 1.05732 | 0.18660 | 0.09737 | 0.98830 | 0.67984 |
| 75.00 | 0.05422 | 1.07276 | 0.21204 | 0.15995 | 1.31579 | 0.74864 |
| 80.00 | 0.06517 | 1.09282 | 0.23071 | 0.21138 | 1.69006 | 0.80641 |
| 85.00 | 0.07757 | 1.11318 | 0.26844 | 0.28734 | 2.11111 | 0.85379 |
| 90.00 | 0.09479 | 1.14339 | 0.31156 | 0.41861 | 3.09357 | 0.92141 |
| 95.00 | 0.12552 | 1.19870 | 0.36531 | 0.61609 | 3.65497 | 0.94409 |
| 99.00 | 0.18480 | 1.25209 | 0.49018 | 1.01420 | 5.61988 | 0.98221 |
| 100.00 | 0.26682 | 1.45466 | 0.57527 | 1.57119 | 11.84210 | 0.99944 |

Table B.18: Statistics for 20x20 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00062 | 1.00201 | 0.13699 | -0.03912 | 0.94329 | 0.49941 |
| Var. | 0.00514 | 0.01061 | 0.01112 | 0.11477 | 1.62271 | 0.08101 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.20625 | 0.66775 | 0.00009 | -0.84644 | 0.00000 | 0.00000 |
| 1.00 | -0.16349 | 0.75965 | 0.00252 | -0.64227 | 0.00000 | 0.00000 |
| 5.00 | -0.11345 | 0.83845 | 0.01273 | -0.48679 | 0.00000 | 0.00000 |
| 10.00 | -0.09148 | 0.87312 | 0.02216 | -0.42335 | 0.02105 | 0.11536 |
| 15.00 | -0.07469 | 0.89385 | 0.03435 | -0.36603 | 0.02105 | 0.11536 |
| 20.00 | -0.06150 | 0.91058 | 0.04447 | -0.31870 | 0.08421 | 0.22833 |
| 25.00 | -0.05197 | 0.93173 | 0.05502 | -0.28177 | 0.08421 | 0.22833 |
| 30.00 | -0.04122 | 0.94665 | 0.06556 | -0.23833 | 0.18947 | 0.33664 |
| 35.00 | -0.03068 | 0.95995 | 0.07643 | -0.20847 | 0.18947 | 0.33664 |
| 40.00 | -0.02247 | 0.97115 | 0.08945 | -0.17093 | 0.33684 | 0.43834 |
| 45.00 | -0.01410 | 0.98571 | 0.10469 | -0.12803 | 0.33684 | 0.43834 |
| 50.00 | -0.00362 | 1.00198 | 0.11492 | -0.07984 | 0.52632 | 0.53184 |
| 55.00 | 0.00753 | 1.01492 | 0.12759 | -0.04382 | 0.52632 | 0.53184 |
| 60.00 | 0.01814 | 1.02765 | 0.14248 | 0.00647 | 0.75789 | 0.61602 |
| 65.00 | 0.02780 | 1.03974 | 0.15503 | 0.04738 | 0.75789 | 0.61602 |
| 70.00 | 0.03902 | 1.05427 | 0.17836 | 0.09982 | 1.03158 | 0.69021 |
| 75.00 | 0.04909 | 1.07021 | 0.19412 | 0.14831 | 1.34737 | 0.75425 |
| 80.00 | 0.05930 | 1.08649 | 0.21651 | 0.20763 | 1.70526 | 0.80840 |
| 85.00 | 0.07058 | 1.10668 | 0.24283 | 0.26996 | 2.10526 | 0.85323 |
| 90.00 | 0.09323 | 1.13387 | 0.28126 | 0.37784 | 2.54737 | 0.88951 |
| 95.00 | 0.11678 | 1.17167 | 0.33344 | 0.59012 | 3.55789 | 0.94073 |
| 99.00 | 0.16961 | 1.24566 | 0.47241 | 1.07502 | 6.08421 | 0.98638 |
| 100.00 | 0.27720 | 1.35356 | 0.67606 | 1.37094 | 11.13684 | 0.99914 |

Table B.19: Statistics for 21x21 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00083 | 0.99987 | 0.12595 | -0.05188 | 0.98462 | 0.50338 |
| Var. | 0.00468 | 0.00982 | 0.00914 | 0.08817 | 1.82501 | 0.08182 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.23580 | 0.66768 | 0.00037 | -0.69795 | 0.00000 | 0.00000 |
| 1.00 | -0.16453 | 0.78771 | 0.00193 | -0.59643 | 0.00000 | 0.00000 |
| 5.00 | -0.11434 | 0.83865 | 0.00920 | -0.47746 | 0.00000 | 0.00000 |
| 10.00 | -0.08949 | 0.87272 | 0.01968 | -0.40411 | 0.01905 | 0.10977 |
| 15.00 | -0.07214 | 0.89725 | 0.03092 | -0.34781 | 0.07619 | 0.21747 |
| 20.00 | -0.05975 | 0.91504 | 0.04101 | -0.30439 | 0.07619 | 0.21747 |
| 25.00 | -0.04742 | 0.93374 | 0.05237 | -0.26478 | 0.07619 | 0.21747 |
| 30.00 | -0.03613 | 0.94968 | 0.06267 | -0.22815 | 0.17143 | 0.32115 |
| 35.00 | -0.02569 | 0.96174 | 0.07389 | -0.19062 | 0.17143 | 0.32115 |
| 40.00 | -0.01707 | 0.97443 | 0.08600 | -0.15825 | 0.30476 | 0.41909 |
| 45.00 | -0.00812 | 0.98671 | 0.09513 | -0.12436 | 0.30476 | 0.41909 |
| 50.00 | -0.00018 | 0.99819 | 0.10588 | -0.08571 | 0.47619 | 0.50984 |
| 55.00 | 0.00851 | 1.00818 | 0.11854 | -0.04733 | 0.68571 | 0.59238 |
| 60.00 | 0.01789 | 1.02120 | 0.12932 | -0.00988 | 0.68571 | 0.59238 |
| 65.00 | 0.02717 | 1.03577 | 0.14253 | 0.03381 | 0.93333 | 0.66601 |
| 70.00 | 0.03531 | 1.04764 | 0.15980 | 0.07097 | 1.21905 | 0.73045 |
| 75.00 | 0.04606 | 1.06206 | 0.17700 | 0.12437 | 1.21905 | 0.73045 |
| 80.00 | 0.05655 | 1.08194 | 0.19987 | 0.18685 | 1.54286 | 0.78581 |
| 85.00 | 0.06863 | 1.10267 | 0.22235 | 0.25149 | 1.90476 | 0.83246 |
| 90.00 | 0.08248 | 1.13039 | 0.26944 | 0.33172 | 2.74286 | 0.90229 |
| 95.00 | 0.10986 | 1.17004 | 0.30720 | 0.47384 | 3.73333 | 0.94665 |
| 99.00 | 0.16025 | 1.24209 | 0.40792 | 0.73353 | 6.17143 | 0.98704 |
| 100.00 | 0.22484 | 1.34792 | 0.62186 | 1.44753 | 11.90476 | 0.99946 |

Table B.20: Statistics for 22x22 Subjective Judgment Matrices

Means and Variances of Statistics

| | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| Mean | -0.00073 | 0.99937 | 0.12144 | -0.03658 | 0.96357 | 0.49885 |
| Var. | 0.00416 | 0.00866 | 0.00887 | 0.09109 | 1.66709 | 0.08270 |

Percentage Points of Statistics

| % | XBAR | VAR | \|GAMMA1\| | GAMMA2 | CHISQC | PC2C |
|---|---|---|---|---|---|---|
| 0.09 | -0.17272 | 0.74084 | 0.00022 | -0.72286 | 0.00433 | 0.05246 |
| 1.00 | -0.15452 | 0.79226 | 0.00169 | -0.57188 | 0.00433 | 0.05246 |
| 5.00 | -0.10214 | 0.85113 | 0.01039 | -0.44804 | 0.00433 | 0.05246 |
| 10.00 | -0.08020 | 0.88636 | 0.02016 | -0.38200 | 0.00433 | 0.05246 |
| 15.00 | -0.06620 | 0.90350 | 0.02796 | -0.33352 | 0.03896 | 0.15647 |
| 20.00 | -0.05649 | 0.91820 | 0.03919 | -0.29574 | 0.03896 | 0.15647 |
| 25.00 | -0.04713 | 0.93336 | 0.04791 | -0.25078 | 0.10823 | 0.25783 |
| 30.00 | -0.03826 | 0.94669 | 0.06028 | -0.21528 | 0.10823 | 0.25783 |
| 35.00 | -0.02640 | 0.96225 | 0.07054 | -0.18057 | 0.21212 | 0.35489 |
| 40.00 | -0.01951 | 0.97190 | 0.08004 | -0.14391 | 0.21212 | 0.35489 |
| 45.00 | -0.01088 | 0.98338 | 0.08915 | -0.10994 | 0.35065 | 0.44625 |
| 50.00 | -0.00260 | 0.99393 | 0.10025 | -0.07087 | 0.52381 | 0.53078 |
| 55.00 | 0.00772 | 1.00767 | 0.11210 | -0.03497 | 0.52381 | 0.53078 |
| 60.00 | 0.01434 | 1.02009 | 0.12539 | -0.00171 | 0.73160 | 0.60764 |
| 65.00 | 0.02318 | 1.03330 | 0.13997 | 0.04391 | 0.97403 | 0.67632 |
| 70.00 | 0.03413 | 1.04460 | 0.15554 | 0.08195 | 0.97403 | 0.67632 |
| 75.00 | 0.04356 | 1.06101 | 0.17064 | 0.13288 | 1.25108 | 0.73665 |
| 80.00 | 0.05393 | 1.07621 | 0.18966 | 0.18281 | 1.56277 | 0.78874 |
| 85.00 | 0.06770 | 1.09784 | 0.21393 | 0.25614 | 1.90909 | 0.83295 |
| 90.00 | 0.08228 | 1.12374 | 0.25471 | 0.36081 | 2.70563 | 0.89999 |
| 95.00 | 0.10595 | 1.15619 | 0.30077 | 0.50454 | 3.64069 | 0.94361 |
| 99.00 | 0.14983 | 1.21267 | 0.39351 | 0.88435 | 5.92641 | 0.98510 |
| 100.00 | 0.22273 | 1.32238 | 0.64989 | 1.19565 | 11.25974 | 0.99919 |

B.3 The statistics in Tables B.1 through B.20 were generated using Monte Carlo simulation. Each table is based on 1100 randomly-generated matrices. Errors in estimating entries in SJMs were assumed to be ratio errors that are log-normally distributed, i.e.,

$$a_{ij} = (w_i/w_j)\varepsilon_{ij}$$

where $a_{ij}$ = (i,j) entry in subjective judgment matrix A,
$w_i$ = weight for entity i,
and $\varepsilon_{ij}$ = error in estimating $w_i/w_j$.

The logarithmic least-squares approach to estimating the weights minimizes the sum of squares of the logarithms of errors. Accordingly, the entries in the randomly-generated matrices had a normal distribution with zero mean and unit variance.

Statistics Addressing Normality of Errors

B.4 The first four statistics in each table, XBAR, VAR, |GAMMA1|, and GAMMA2, can be used to test whether the error terms are normally distributed. Because we want to assess consistency of SJMs by using an F test for the equality of variances and the F test is most sensitive to the skewness of the error distribution [4], a test of skewness is most appropriate for assessing whether the errors are normally distributed. We are interested here in whether the error distribution is skewed, not whether it's positively or negatively skewed. Accordingly, I have tabulated the absolute value of skewness for randomly-generated errors, allowing us to use it in a two-tailed test of normality.

B.5 Note that the data for |GAMMA1| and GAMMA2 in Tables B.1 through B.20 extend Geary and Pearson's tables [5] for the skewness and kurtosis of normally-distributed data to smaller sample sizes. The sample size for each table is equal to n(n-1)/2, where n is the order of the matrix.

## Statistics Addressing Independence of Errors

B.6 The statistics CHISQC and PC2C address the independence of the error terms. My initial approach to testing independence of errors used a $\chi^2$ statistic based on a 2x2 contingency table. This approach led to a test that was little different from one based on the signs of individual error terms and had little intuitive appeal. A simple $\chi^2$ test that compares the numbers of positive and negative error terms with their expected values was substituted for the contingency table approach. The CHISQC statistics in Tables B.1 through B.20 are based on the latter approach.

B.7 PC2C is the $\chi^2$ probability for CHISQC. If CHISQC is $\chi^2$-distributed with one degree of freedom, PC2C is uniformly distributed between zero and one, and has a mean of 0.5 and a variance of 1/12. Furthermore, the values of PC2C will be approximately equal to the percentages in the left column of each table if CHISQC is $\chi^2$-distributed. The tables show that these relationships do not hold for matrices smaller than about 8x8. This occurs because the numbers of positive and negative error terms are integers, so only discrete values of CHISQC occur. For example, it is easily shown that for 3x3 SJMs, only two values of CHISQC, 1/3 and 3, can occur. Their probabilities are 0.75 and 0.25, respectively, under the assumption that positive and negative errors are equally likely.

B.8 Because the values of CHISQC are discrete, especially for small SJMs, a $\chi^2$ test for independence based on the numbers of positive and negative error terms will yield only gross indications of whether the error terms are independent. A better test would use the values of CHISQC given in the tables. However, the $\chi^2$ test is easier to implement, so it was selected as the test of independence of errors.

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