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**STRONG MODERATE DEVIATION THEOREMS FOR m -DEPENDENT
RANDOM VARIABLES**

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Strong Moderate Deviation Theorems for m-Dependent Random Variables†

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Abstract

Consider a stationary sequence $\{X_1, X_2, \dots\}$ of m-dependent random variables. Let $S_n = \sum_{i=1}^n X_i$ be the partial sum. Under some moment conditions, in this paper we obtain asymptotic expression for the probability of moderate deviations, $P(S_n > x_n)$, where $x_n = O(\sqrt{\log(n)})$. This result extends some well known results obtained for independent and identically distributed sequences of random variables.

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1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1. Let $S_n = \sum_{i=1}^n X_i$ be the n^{th} partial sum. The theory of moderate deviations introduced by Rubin and Sethuraman (1965a) is concerned with obtaining asymptotic expression for

$$(1.1) \quad P(S_n > x_n)$$

where $x_n = O(\sqrt{\log(n)})$, under some moment conditions which are less restrictive than the assumption of finiteness of the moment generating function of X_1 . In a subsequent paper Rubin and Sethuraman (1965b) showed that the asymptotic expression for (1.1) is useful to compare test statistics via Bayes risk efficiency. We shall call a result which gives the asymptotic expression for (1.1) a weak moderate deviation result. On the other hand a strong moderate deviation theorem gives an asymptotic expression to the probability of the event $\{S_n > x\}$ which is valid uniformly in the interval $-A \leq x \leq c\sqrt{\log(n)}$. In this paper we obtain strong moderate deviation theorem for the partial sums of a stationary sequence $\{X_n, n \geq 1\}$ of m -dependent random variables. Note that a sequence $\{X_n, n \geq 1\}$ is said to be m -dependent if $(X_1, \dots, X_r)$ and $(X_s, X_{s+1}, \dots)$ are independent whenever $s - r > m$. The sequence is said to be stationary if $(X_{i+1}, \dots, X_{i+k})$ has the same distribution as $(X_{j+1}, \dots, X_{j+k})$ for all $k \geq 1$ and $i \neq j$.

2. Main Results. In this section we establish the main theorem of this paper. Theorem 2.1 below obtains a strong moderate deviation theorem for partial sums of m -dependent sequence of random variables.

Theorem 2.1. Consider a stationary sequence $\{X_n, n \geq 1\}$ of m -dependent random variables. Let $E(X_1) = 0$ and $\sigma^2 = \text{Var}(X_1) + 2 \sum_{j=1}^m \text{Cov}(X_1, X_{1+j})$ be finite. Let $S_n = \sum_{i=1}^n X_i$. If $E|X_1|^p < \infty$ for some $p > c^2 + 2$, where $c > 0$, then

$$(2.1) \quad P\left(\frac{S_n}{\sqrt{n}\sigma} > x\right) = [1 - \Phi(x)] \left[1 + o\left(\frac{1}{\log(n)}\right)\right]$$

uniformly in the region $-A \leq x \leq c\sqrt{\log(n)}$, where $A \geq 0$ is a constant and Φ denotes the distribution function of standard normal.

Vandemaele and Veraverbeke (1982) obtained strong moderate deviation theorems for L-statistics which are functions of independent and identically distributed sequences of random variables. A special case of their Theorem 1 yields the following Lemma 2.2. We will need Lemma 2.2 in the proof of Theorem 2.1.

Lemma 2.2. Let $\{X_n, n \geq 1\}$ be sequence of i.i.d. random variables with mean zero and variance σ^2 . Let $S_n = \sum_{i=1}^n X_i$ be the n^{th} partial sum. If $E|X_1|^p < \infty$ for some $p > c^2 + 2$, ($c > 0$) then (2.1) holds uniformly in the region $-A \leq x \leq c\sqrt{\log(n)}$, where $A \geq 0$ is a constant.

In the case $m = 0$, Theorem 2.1 yields Lemma 2.2 and in this sense our main result generalizes the result of Vandemaele and Veraverbeke (1982) to m -dependent random variables.

Proof of Theorem 2.1. We shall use the blocking technique used by Hoeffding and Robbins (1948) in proving central limit theorem for m -dependent random variables. Let $0 < \alpha < \min\{1, 1/p\}$ be fixed and $k = \max\{2m, [n^\alpha]\}$. We can write $n = k\nu + r$, where

$0 \leq r \leq k$. For n sufficiently large we can partition the n^{th} partial sum as follows:

$$\begin{aligned}
 S_n &= [X_1 + \dots + X_{k-m}] + [X_{k-m+1} + \dots + X_k] + [X_{k+1} + \dots + X_{2k-m}] \\
 &\quad + [X_{2k-m+1} + \dots + X_{2k}] + \dots + [X_{\nu k-m+1} + \dots + X_{\nu k}] \\
 &\quad + [X_{\nu k+1} + \dots + X_n]. \\
 (2.2) \quad &= U_{n1} + R_{n1} + U_{n2} + R_{n2} + \dots + U_{n\nu} + R_{n\nu} + T_{nr} \\
 &= [U_{n1} + \dots + U_{n\nu}] + [R_{n1} + R_{n2} + \dots + R_{n\nu}] + T_{nr} \\
 &= U_n + R_n + T_{nr}. \quad (\text{say}).
 \end{aligned}$$

Note that U_n is the sum of ν i.i.d. random variables and R_n is also the sum of ν i.i.d. random variables. Let $\delta_n = 1/(\log(n))^2$. It is easy to verify that the following important identity holds:

$$\begin{aligned}
 (2.3) \quad &P\left(\frac{U_n}{\sqrt{n}\sigma} > x + 2\delta_n\right) - P\left(\frac{R_n}{\sqrt{n}\sigma} < -\delta_n\right) - P\left(\frac{T_{nr}}{\sqrt{n}\sigma} < -\delta_n\right) \\
 &\leq P\left(\frac{S_n}{\sqrt{n}\sigma} > x\right) \\
 &\leq P\left(\frac{U_n}{\sqrt{n}\sigma} > x - 2\delta_n\right) + P\left(\frac{R_n}{\sqrt{n}\sigma} > \delta_n\right) + P\left(\frac{T_{nr}}{\sqrt{n}\sigma} > \delta_n\right).
 \end{aligned}$$

Since $[1 - \Phi(x)]^{-1} = O((c\sqrt{\log(n)})n^{c^2/2})$ uniformly in $-A \leq x \leq c\sqrt{\log(n)}$, ($A > 0, c > 0$), the proof of the theorem will be complete once we establish the following Lemma 2.3.

Lemma 2.3. *Let U_n, R_n and T_{nr} be as defined above. Then under the hypothesis of Theorem 2.1 we have the following:*

$$\begin{aligned}
 (A) \quad &P\left(\frac{U_n}{\sqrt{n}\sigma} > x \pm 2\delta_n\right) = [1 - \Phi(x)] \left[1 + o\left(\frac{1}{\log(n)}\right)\right] \\
 &\text{uniformly in } -A \leq x \leq c\sqrt{\log(n)}.
 \end{aligned}$$

$$(B) \ P \left(\left| \frac{R_n}{\sqrt{n}\sigma} \right| > \delta_n \right) = o \left((\log(n))^{-3/2} n^{-c^2/2} \right).$$

$$(C) \ P \left(\left| \frac{T_{nr}}{\sqrt{n}\sigma} \right| > \delta_n \right) = o \left((\log(n))^{-3/2} n^{-c^2/2} \right).$$

Proof of (A). Note that $U'_n = \frac{U_n}{k}$ is the sum of ν i.i.d. random variables with mean zero and variance given by

$$(2.4) \quad M^2 = \frac{1}{k^2} [(k-m) \text{Var}(X_1) + 2 \sum_{j=1}^m (k-m-j) \text{Cov}(X_1, X_{1+j})].$$

Now,

$$(2.5) \quad \begin{aligned} P \left(\frac{U_n}{\sqrt{n}\sigma} > x \right) &= P \left(\frac{\sqrt{\nu}kM}{\sqrt{n}\sigma} \frac{U'_n}{M\sqrt{\nu}} > x \right) \\ &= P \left(\frac{U'_n}{M\sqrt{\nu}} > xc_n \right) \end{aligned}$$

where

$$(2.6) \quad \begin{aligned} c_n &= \frac{\sqrt{n}\sigma}{M\sqrt{\nu}k} = \left[\frac{\sigma^2}{kM^2} \right]^{1/2} \left[\frac{n}{k\nu} \right]^{1/2} \\ &= \left[\frac{\sigma^2}{kM^2} \right]^{1/2} \left[1 + \frac{r}{k\nu} \right]^{1/2}. \end{aligned}$$

Note that

$$(2.7) \quad \begin{aligned} kM^2 &= \frac{1}{k} [(k-m) \text{Var}(X_1) + 2 \sum_{j=1}^m (k-m-j) \text{Cov}(X_1, X_{1+j})] \\ &= \sigma^2 - \frac{1}{k} [m \text{Var}(X_1) + 2 \sum_{j=1}^m (m+j) \text{Cov}(X_1, X_{1+j})] \\ &= \sigma^2 + O(n^{-\alpha}) = \sigma^2 [1 + O(n^{-\alpha})] \end{aligned}$$

since $k = O(n^\alpha)$. Thus we get

$$(2.8) \quad \begin{aligned} c_n &= [1 + O(n^{-\alpha})]^{1/2} \left(1 + \frac{r}{k\nu} \right)^{1/2} \\ &= [1 + O(n^{-\beta})] \quad \text{for some } \beta > 0. \end{aligned}$$

Therefore we have shown that

$$P\left(\frac{U_n}{\sqrt{n}\sigma} > x\right) = P\left(\frac{U'_n}{M\sqrt{\nu}} > c_n x\right)$$

where $c_n = [1 + O(n^{-\beta})]$. Hence,

$$\begin{aligned} P\left(\frac{U_n}{\sqrt{n}\sigma} > x \pm \frac{2}{(\log(n))^2}\right) &= P\left(\frac{U'_n}{M\sqrt{\nu}} > c_n \left[x \pm \frac{2}{(\log(n))^2}\right]\right) \\ (2.10) \qquad \qquad \qquad &= P\left(\frac{U'_n}{M\sqrt{\nu}} > x_n\right) \end{aligned}$$

where

$$\begin{aligned} x_n &= c_n \left[x \pm \frac{2}{(\log(n))^2}\right] \\ (2.11) \qquad \qquad \qquad &= [1 + O(n^{-\beta})] \left[x \pm \frac{2}{(\log(n))^2}\right]. \end{aligned}$$

Let δ be such that $\sqrt{p-2} > c + \delta$. Then applying Lemma 2.2 with c replaced by $c + \delta$ we get

$$(2.12) \qquad P\left(\frac{U'_n}{M\sqrt{\nu}} > x_n\right) = [1 - \Phi(x_n)] \left[1 + o\left(\frac{1}{\log(n)}\right)\right]$$

uniformly in the interval $-A \leq x_n \leq (c + \delta)\sqrt{\log(n)}$. Also using Lemma A1 of Vandermale and Veraverbeke (1982) and (2.11) we get

$$(2.13) \qquad [1 - \Phi(x_n)] = [1 - \Phi(x)] \left[1 + o\left(\frac{1}{\log(n)}\right)\right].$$

Combining (2.10), (2.12) and (2.13) we have

$$P\left(\frac{U_n}{\sqrt{n}\sigma} > x \pm \frac{2}{(\log(n))^2}\right) = [1 - \Phi(x)] \left[1 + o\left(\frac{1}{\log(n)}\right)\right]$$

uniformly in $-A \leq x \leq c\sqrt{\log(n)}$. This completes the proof of (A).

Proof of (B). Note that R_n is the sum of ν i.i.d. random variables with mean zero and variance

$$(2.14) \quad \sigma_1^2 = m \text{Var}(X_1) + 2 \sum_{j=1}^{m-1} (m-j) \text{Cov}(X_1, X_{1+j}).$$

Fix constants α and c_1 such that $\alpha < 1 - \frac{c^2}{p-2}$, $c_1^2 > \frac{c^2}{(1-\alpha)}$ and $p > 2 + c_1^2$. Applying Theorem 1 of Rubin and Sethuraman (1965a) we get for sufficiently large n ,

$$(2.15) \quad \begin{aligned} P \left(\left| \frac{R_n}{\sqrt{n}\sigma} \right| > \delta_n \right) &= P \left(\left| \frac{R_n}{\sqrt{\nu}} \right| > \frac{\sigma\sqrt{n}}{\sqrt{\nu}(\log(n))^2} \right) \\ &\leq P \left(\left| \frac{R_n}{\sqrt{\nu}} \right| > \sigma_1 c_1 \sqrt{\frac{\log(\nu)}{\nu}} \right) \\ &\sim \frac{\nu^{-c_1^2/2}}{c_1 \sqrt{2\pi \log(\nu)}} \end{aligned}$$

since $p > c_1^2 + 2$. Using the fact $\nu = O(n^{1-\alpha})$ and $(1-\alpha)c_1^2 > c^2$ and (2.15), we get that

$$P \left(\left| \frac{R_n}{\sqrt{n}\sigma} \right| > \delta_n \right) = o \left((\log(n))^{-3/2} n^{-c^2/2} \right).$$

This proves (B).

Proof of (C). Applying Chebyshev's inequality we get

$$(2.16) \quad \begin{aligned} P \left(\left| \frac{T_{nr}}{\sqrt{n}\sigma} \right| > \delta_n \right) &\leq \frac{E|T_{nr}|^p}{(\sigma\sqrt{n}\delta_n)^p} \\ &\leq \text{const.} \frac{k^p}{(\sigma\sqrt{n}\delta_n)^p}. \end{aligned}$$

Note that $k = O(n^\alpha)$ and hence

$$\begin{aligned}
(2.17) \quad n^{c^{2/2}} (\log(n))^{3/2} \frac{k^p}{(\sqrt{n}\delta_n)^p} &= \frac{n^{c^{2/2}} n^{p\alpha}}{n^{p/2}} (\log(n))^{2p+3/2} \\
&= n^{-\frac{1}{2}(p-2p\alpha-c^2)} (\log(n))^{2p+3/2} \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

since $0 < \alpha < \frac{1}{p}$ and $p > c^2 + 2$. The proof of (C) now follows combining (2.16) and (2.17).

Remark 2.4. Theorem 2.1 suggests that it is possible to obtain strong moderate deviation theorems for L-statistics and U-statistics which are functions of m-dependent sequences of random variables in the same spirit as Vandemaële and Veraverbeke(1982) and Ghosh(1974).

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