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Analysis of Three- and Four-Sided Uniform Waveguides With Unusual Cross-Section/Boundary Conditions

by
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Research Department



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NAVAL WEAPONS CENTER
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FOREWORD

The research described in this report was performed at the Naval Weapons Center during fiscal years 1985 through 1989. The work was supported by 6.1 Independent Research Funds.

This study provides theoretical analysis for those working with uniform waveguides of complicated cross-sectional form. It is intended to be an interim working document; the research it describes is not complete.

This report has been reviewed for technical accuracy by D. J. White.

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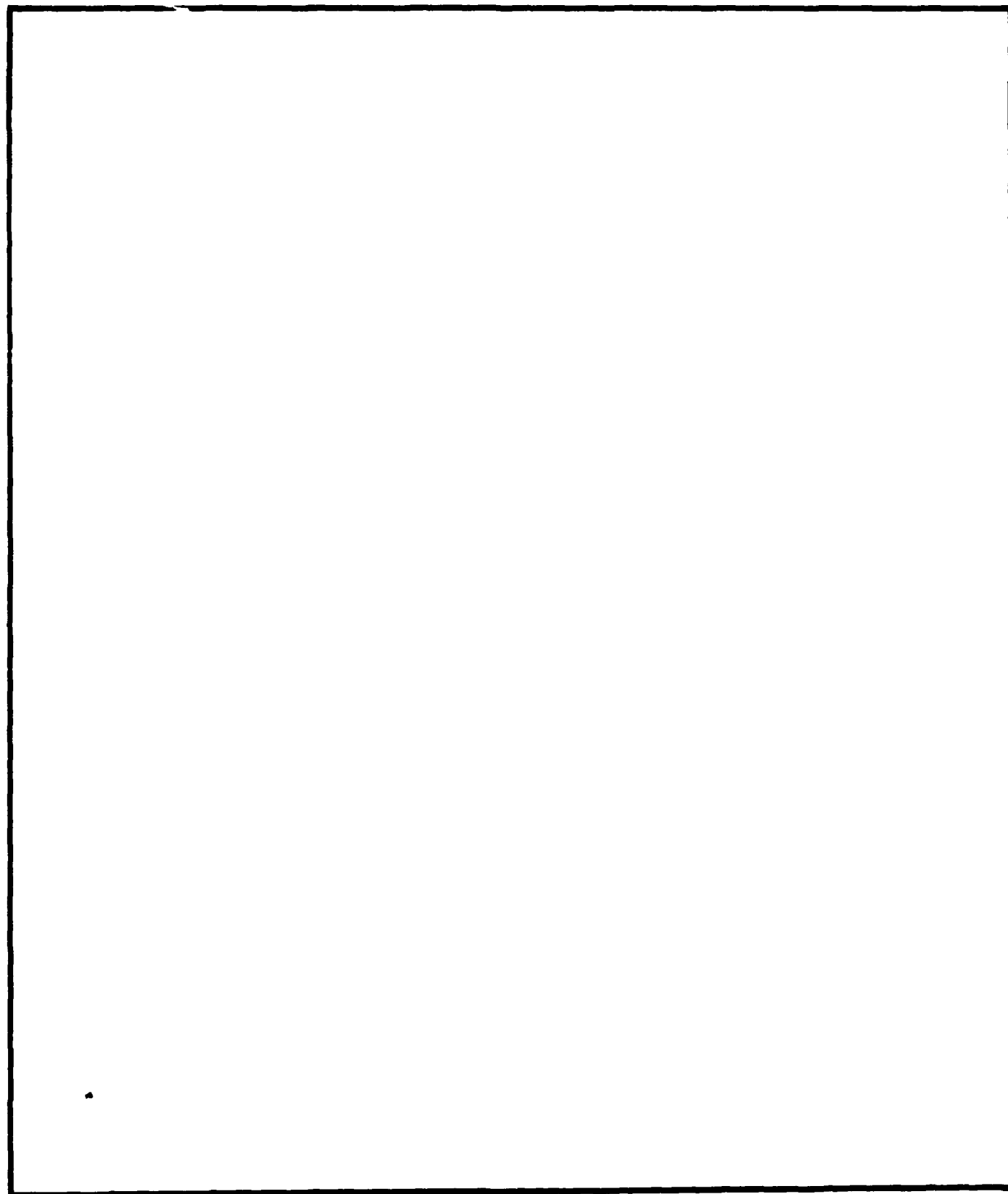
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I. INTRODUCTION

Closed-form expressions for the transverse electric (TE) and transverse magnetic (TM) modes of uniform waveguides with unusual cross-sectional geometry are useful in several areas of electromagnetic theory. Both microstrip-antenna analysis using cavity models (Reference 1) and high-power microwave applications (Reference 2) require solution methods capable of calculating accurate resonant frequencies and electromagnetic field components. Typically, while analysis involving unusual geometries is handled via approximate numerical techniques (References 3 through 6), closed-form solutions are especially desirable for their physical and computational simplicity as well as for their ability to provide checks on the accuracy of numerical solutions.

In the past, closed-form TE and TM mode expressions for four perfectly conducting uniform waveguides of triangular cross section were determined using the superposition of plane waves technique (References 7 through 9). All of these solutions have the general form of finite sums of rectangular harmonics. Although each rectangular harmonic term alone satisfies the Helmholtz equation, only the entire solution with particular relationships among the eigenvalues satisfies the boundary conditions (either Dirichlet or Neumann) as well.

Beginning with an initial solution formed using a finite sum of rectangular harmonics and utilizing the symmetry properties (Reference 10) of a particular waveguide geometry as well as the Riemann-Schwarz reflection principle (References 11 and 12), we have found some closed-form solutions for the TE and TM modes of certain three- and four-sided uniform waveguides with unusual cross-sectional geometry. In the majority of cases, we have assumed perfectly conducting walls; however, in some instances, combinations of perfect electric ($\hat{n} \times \vec{E} = 0$, $\hat{n} \cdot \vec{B} = 0$) and perfect magnetic walls ($\hat{n} \times \vec{H} = 0$, $\hat{n} \cdot \vec{D} = 0$) have been assumed. Although infinite sets of TE and TM modes based on finite sums of rectangular harmonics have been determined for each geometry, complete sets of modes (Reference 13) (in the sense of the rectangular waveguide) are not found. We believe that this incompleteness of the mode sets is a consequence of solving geometries with nonorthogonal boundaries that possess multiple reflections and periodic extensions that cannot cover all of space. We believe that the "missing" modes cannot have the form of finite sums of rectangular harmonics and speculate that they may be determined from either infinite sums of

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rectangular harmonics or purely nonseparable solutions of the Helmholtz equation (Reference 14).

Section II presents TE and TM mode expressions for the following waveguide geometries and boundary conditions:

- A. Isosceles right triangular waveguide with two electric and one magnetic wall at $x = a$
- B. Isosceles right triangular waveguide with two electric and one magnetic wall at $y = 0$
- C. Isosceles right triangular waveguide with two electric and one magnetic wall at $y = x$
- D. 45- and 135-degree parallelogram waveguide with perfectly conducting walls
- E. 90- and 90- and 135- and 45-degree waveguide with perfectly conducting walls
- F. 45- and 135-degree trapezoidal waveguide with perfectly conducting walls
- G. 60- and 120-degree parallelogram waveguide with perfectly conducting walls
- H. 60- and 120-degree trapezoidal waveguide with perfectly conducting walls.

In Section III, cutoff formulas and lowest order (i.e., the lowest found by the above method) mode plots are presented. Section IV contains our conclusions.

II. CLOSED-FORM TRANSVERSE ELECTRIC AND MAGNETIC SOLUTIONS FOR UNUSUAL WAVEGUIDE GEOMETRIES

In this report, all waveguides are assumed to be uniform, i.e., with $e^{-\gamma z}$ $e^{i\omega t}$ dependence and free space inside and outside the guide. In all cases,

$$k_1^2 + k_2^2 = k_0^2 + \gamma^2 \quad (1)$$

where

$$\gamma = \alpha + i\beta \quad (2)$$

$$k_o^2 = \omega^2 \epsilon_o \mu_o = \left(\frac{2\pi}{\lambda_o} \right)^2 \quad (3)$$

and k_1 and k_2 are the transverse wave numbers.

A. ISOSCELES RIGHT TRIANGULAR WAVEGUIDE WITH TWO ELECTRIC AND ONE MAGNETIC WALL AT $x = a$

Using the general form of solution for the perfectly conducting waveguide of isosceles right triangular cross section (see Figure 1) as given in Reference 9, we find by inspection that

$$E_z = \sin k_1 x \sin k_2 y - \sin k_2 x \sin k_1 y \quad (4)$$

for the TM modes and

$$H_z = \cos k_1 x \cos k_2 y + \cos k_2 x \cos k_1 y \quad (5)$$

for the TE modes with the transverse wave numbers

$$k_1 = \frac{m\pi}{2a}, \quad k_2 = \frac{n\pi}{2a} \quad (6)$$

will satisfy the appropriate boundary conditions. There are restrictions on the integer indices, m and n , such that for TM modes

$$m \neq n \quad (7)$$

and for all modes *both m and n must be odd.*

Equations 4 and 6 must satisfy the boundary conditions

$$E_z = 0 \text{ on } y = 0, \quad y = x \quad (8a)$$

and

$$\frac{\partial E_z}{\partial x} = 0 \text{ on } x = a \text{ (or } E_z = \text{extremum on } x = a) \quad (8b)$$

(An extremum can be either a maximum or a minimum.) By inspection of Equation 4, we see that Equation 8a is satisfied and that

$$\frac{\partial E_z}{\partial x} = k_1 \cos k_1 x \sin k_2 y - k_2 \cos k_2 x \sin k_1 y \quad (9)$$

can be 0 on $x = a$ only when Equation 6 is used with both m and n odd integers. The TE modes are usually of more practical importance than the TM modes in waveguiding applications. Equation 5 must satisfy the boundary conditions

$$\frac{\partial H_z}{\partial y} = 0 \text{ on } y = 0 \quad (10a)$$

$$\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} = 0 \text{ on } y = x \quad (10b)$$

$$H_z = 0 \text{ on } x = a \quad (10c)$$

Equation 10a is satisfied on $y = 0$ automatically. H_z is 0 on $x = a$ provided m and n in Equation 6 are both odd, just as for the TM modes. To show that Equation 10b is true, we simply write

$$\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x} = (-k_1 \sin k_1 x \cos k_2 x - k_2 \sin k_2 x \cos k_1 x) \\ + (k_2 \cos k_1 x \sin k_2 x + k_1 \cos k_2 x \sin k_1 x) = 0 \quad (11)$$

The lowest order mode found from Equations 5 and 6 is the TE_{11} mode ($m = n$ is allowed for TE modes) with a distribution (see Figure 12)

$$H_z (TE_{11}) = 2 \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a} \quad (12)$$

The TE_{13} and TM_{13} are the second lowest order modes found in this way.

B. ISOSCELES RIGHT TRIANGULAR WAVEGUIDE WITH TWO ELECTRIC AND ONE MAGNETIC WALL AT $y = 0$

Consider the isosceles right triangular waveguide in Figure 2. This solution is difficult to achieve by symmetry and inspection. (We could, of course, determine it by rotation and translation—it is the same situation as in Equations 4, 5, and 6.) If we use superposition of plane waves and substitute boundary conditions as in Reference 9, we find that (for TM modes)

$$E_z = \cos k_1 x \cos k_2 y - \cos k_2 x \cos k_1 y \quad (13a)$$

with

$$k_1 = \frac{m\pi}{2a} , \quad k_2 = \frac{n\pi}{2a} \quad (13b)$$

will match the boundary conditions, i.e.,

$$E_z = 0 \text{ on } x = a , \quad y = x \quad (14a)$$

$$\frac{\partial E_z}{\partial y} = 0 \text{ on } y = 0 \quad (14b)$$

k_1 and k_2 are given by Equation 13b, with both m and n odd integers, $m \neq n$. The lowest order TM mode is the TM_{13} . The TE modes must satisfy

$$\frac{\partial H_z}{\partial x} = 0 \text{ on } x = a \quad (15a)$$

$$\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} = 0 \text{ on } x = y \quad (15b)$$

$$H_z = 0 \text{ on } y = 0 \quad (15c)$$

Again using symmetry and an initial form based on a finite sum of rectangular harmonic terms, we find that the TE modes are given by

$$H_z = \sin k_1 x \sin k_2 y + \sin k_2 x \sin k_1 y \quad (16)$$

with k_1 and k_2 as in Equation 13b. The boundary conditions in Equation 15 are again satisfied provided both m and n are odd integers. For the TE modes, $m = n$ is allowed. The TE_{11} mode is the lowest order mode with the form of Equation 16 and has a distribution (see Figure 13)

$$H_z (TE_{11}) = 2 \sin \frac{\pi x}{2a} \sin \frac{\pi y}{2a} \quad (17)$$

Naturally, it is the same distribution—only rotated—as for Section II, Part A. The TE_{13} and TM_{13} are the second lowest order modes.

C. ISOSCELES RIGHT TRIANGULAR WAVEGUIDE WITH TWO ELECTRIC AND ONE MAGNETIC WALL AT $y = x$

For TM modes, we attempt a solution of the form (see Figure 3)

$$E_z = \sin k_1 x \sin k_2 y + \sin k_2 x \sin k_1 y \quad (18)$$

with

$$k_1 = \frac{m\pi}{a} , \quad k_2 = \frac{n\pi}{a} \quad (19)$$

For this situation, we must satisfy

$$E_z = 0 \text{ on } y = 0 \text{ and } x = a \quad (20a)$$

$$\frac{\partial E_z}{\partial x} - \frac{\partial E_z}{\partial y} = 0 \text{ on } y = x \quad (20b)$$

By inspection, we see that the boundary conditions in Equation 20 are satisfied and place no restrictions on the values of m and n (except that they must be integers and $m \neq 0, n \neq 0$). Thus, the TM_{11} is the lowest order TM mode.

The TE modes must satisfy

$$\frac{\partial H_z}{\partial y} = 0 \text{ on } y = 0 \quad (21a)$$

$$\frac{\partial H_z}{\partial x} = 0 \text{ on } x = a \quad (21b)$$

$$H_z = 0 \text{ on } y = x \quad (21c)$$

Using

$$H_z = \cos k_1 x \cos k_2 y - \cos k_2 x \cos k_1 y \quad (22)$$

with

$$k_1 = \frac{m\pi}{a} , \quad k_2 = \frac{n\pi}{a} \quad (23)$$

the boundary conditions in Equation 21 are satisfied by inspection. The only restriction is $m \neq n$ in Equation 22. The lowest order mode of this structure is a TE_{10} mode (see Figure 14) given by

$$H_z (TE_{10}) = \cos \frac{\pi x}{a} - \cos \frac{\pi y}{a} \quad (24)$$

The second lowest order mode is the TM_{11} , i.e.,

$$E_z(TM_{11}) = 2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \quad (25)$$

The next lowest modes are the TE_{12} and TM_{12} .

D. 45- AND 135-DEGREE PARALLELOGRAM WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

Consider the parallelogram geometry in Figure 4. Since the 45- and 135-degree parallelogram is composed of two isosceles right triangles oriented back to back, we use the general form of solution in Reference 9 and find, by inspection, that for TM modes possible solutions are given by

$$E_z = \sin k_1 x \sin k_2 y - \sin k_2 x \sin k_1 y \quad (26)$$

with

$$k_1 = \frac{m\pi}{a}, \quad k_2 = \frac{n\pi}{a} \quad (27)$$

The boundary conditions for this parallelogram are

$$E_z = 0 \text{ on } y=0, y=a, y=x, \text{ and } y=x-a \quad (28)$$

Obviously, Equation 26 is zero for $y=0$, $y=a$, and $y=x$. The fourth boundary condition, $y=x-a$, when substituted into Equation 26 gives

$$\begin{aligned} E_z = \sin k_1 x (\sin k_2 x \cos k_2 a - \cos k_2 x \sin k_2 a) \\ - \sin k_2 x (\sin k_1 x \cos k_1 a - \cos k_1 x \sin k_1 a) \end{aligned} \quad (29)$$

Equation 29 can be zero only if

$$\sin k_1 a = 0 \text{ and } \sin k_2 a = 0 \quad (30a)$$

as well as

$$\cos k_1 a = \cos k_2 a \quad (30b)$$

Equation 30a is satisfied by any integers m and n , but Equation 30b requires that (using Equation 27)

$$\cos m\pi = \cos n\pi \quad (31)$$

This implies that both m and n must be even or they both must be odd, with $m \neq 0$, $n \neq 0$, and $m \neq n$. For E_z , as given by Equations 26 and 27, no m even, n odd (or vice versa) modes can satisfy all four boundary conditions simultaneously. This means that the TM_{13} is the lowest order TM mode found by the above procedure. We see from Equation 26 that E_z is 0 not only on the walls but also along the line $x = a$ (see Figure 4). This means that we have found a set of modes that is "odd" about the line $x = a$; thus, there should exist a corresponding set of modes "even" about $x = a$, i.e., E_z should be a maximum there. This observation originally motivated the isosceles right triangular solutions with a magnetic wall at $x = a$. Unfortunately, any attempt to use Equations 4 and 6 as the starting point for a set of TM solutions that are maximum on $x = a$ but still zero on the walls is doomed to failure. From Equations 4 and 6, we see that there is no way for $E_z = 0$ on $y = 0$ and on $y = a$ simultaneously if E_z is to be maximum on $x = a$.

For the TE modes, we choose

$$H_z = \cos k_1 x \cos k_2 y + \cos k_2 x \cos k_1 y \quad (32)$$

$$k_1 = \frac{m\pi}{a}, \quad k_2 = \frac{n\pi}{a} \quad (33)$$

subject to $\frac{\partial H_z}{\partial n} = 0$ on the walls. Just as for the TM modes, we find that either both m and n must be even or both m and n must be odd and $m = n$ is allowed. Thus, the TE_{11} mode is the lowest order TE mode found using Equations 32 and 33. Its distribution is (see Figure 15)

$$H_z(TE_{11}) = 2 \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} \quad (34)$$

The next lowest mode is the TE_{20} , followed by the TE_{22} , and then the TM_{13} and TE_{13} modes.

In comparing the perfectly conducting isosceles right triangular waveguide with the 45- and 135-degree parallelogram waveguide, it is evident that only about half of the modes that satisfy the triangular boundary conditions also satisfy the parallelogram boundary conditions. Any even-odd correspondence (and vice versa) between the mode indices (m and n) has been eliminated by the $y = x - a$ boundary condition. Since the isosceles right triangular modes form a complete set (Reference 13), we conclude that some of the modes for the 45- and 135-degree parallelogram are missing, including the true lowest order mode.

Using finite difference analysis (Reference 15) of the 45- and 135-degree parallelogram waveguide, we can indeed show that the above TE_{11}

distribution is actually only the third lowest order mode of the structure. There are two modes with lower cutoff wave numbers, and we conclude that these two modes cannot be found in closed form using a finite sum of rectangular harmonics.

E. 90- AND 90- AND 135- AND 45-DEGREE WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

Consider the four-sided waveguide cross section shown in Figure 5. It is a combination of a square and an isosceles right triangle. Using the isosceles right triangular solution for TM modes given previously, we assume that

$$E_z = \sin k_1 x \sin k_2 y - \sin k_2 x \sin k_1 y \quad (35)$$

with

$$k_1 = \frac{m\pi}{a}, \quad k_2 = \frac{n\pi}{a} \quad (36)$$

as usual for this type of boundary. The boundary conditions in Equation 37 for the TM modes of this geometry are

$$E_z = 0 \text{ on } y=0, \quad x=0, \quad x=a, \quad y=x+a \quad (37)$$

By inspection, we see that the first three conditions in Equation 37 are met. For the fourth condition, E_z becomes

$$\begin{aligned} E_z &= \sin k_1 x \sin k_2 (x+a) - \sin k_2 x \sin k_1 (x+a) \\ &= \sin k_1 x (\sin k_2 x \cos k_2 a + \cos k_2 x \sin k_2 a) \\ &\quad - \sin k_2 x (\sin k_1 x \cos k_1 a + \cos k_1 x \sin k_1 a) \end{aligned} \quad (38)$$

Thus, for E_z to be zero on $y = x + a$, we must have

$$\sin k_1 a = 0 \text{ and } \sin k_2 a = 0 \quad (39a)$$

simultaneously, along with

$$\cos k_1 a = \cos k_2 a \quad (39b)$$

To satisfy the conditions of Equation 39a, m and n must be integer. To satisfy the condition of Equation 39b, we must have both m and n odd or both m and n even, $m \neq 0$, $n \neq 0$, $m \neq n$. This is the same solution and integer restrictions found for the 45- and 135-degree parallelogram. However, in this instance, we have another TM solution.

If

$$E_z = \sin k_1 x \sin k_2 y + \sin k_2 x \sin k_1 y \quad (40)$$

with

$$k_1 = \frac{m\pi}{a} , \quad k_2 = \frac{n\pi}{a} \quad (41)$$

we see that the first three boundary conditions are again satisfied term by term and the $y = x + a$ condition forces

$$\sin k_1 a = 0 , \quad \sin k_2 a = 0 \quad (42a)$$

but now

$$\cos k_1 a = -\cos k_2 a \quad (42b)$$

Again, m and n must be integer, but now if m is odd, n must be even and vice versa. Thus, the solution in Equation 40 allows a TM_{12} mode, which satisfies all boundary conditions simultaneously and is the lowest order TM mode found using the initial forms given by Equations 35 and 40.

For the TE modes, we have a similar situation. One solution is given by

$$H_z = \cos k_1 x \cos k_2 y + \cos k_2 x \cos k_1 y \quad (43)$$

with the eigenvalues as in Equation 41. We must satisfy

$$\frac{\partial H_z}{\partial n} = 0 \quad \text{on } x=0 , \quad y=0 , \quad x=a , \quad y=x+a \quad (44)$$

As for the TM modes, the first three conditions in Equation 44 are seen to be satisfied by Equation 43 by inspection. On $y = x + a$, we must satisfy

$$\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x+a} = 0 \quad (45)$$

Upon taking the derivatives and substituting in the fourth boundary condition, we have

$$\begin{aligned} \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x+a} &= [-k_1 \sin k_1 x \cos k_2 (x+a) \\ &\quad - k_2 \sin k_2 x \cos k_1 (x+a)] - [-k_2 \cos k_1 x \sin k_2 (x+a) \\ &\quad - k_1 \cos k_2 x \sin k_1 (x+a)] \end{aligned} \quad (46)$$

$$\begin{aligned} \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x+a} &= -k_1 \sin k_1 x (\cos k_2 x \cos k_2 a - \sin k_2 x \sin k_2 a) \\ &\quad - k_2 \sin k_2 x (\cos k_1 x \cos k_1 a - \sin k_1 x \sin k_1 a) \\ &\quad + k_2 \cos k_1 x (\sin k_2 x \cos k_2 a + \cos k_2 x \sin k_2 a) \\ &\quad + k_1 \cos k_2 x (\sin k_1 x \cos k_1 a + \cos k_1 x \sin k_1 a) \end{aligned} \quad (47)$$

For this to be zero, we see that

$$\sin k_1 a = 0 \text{ and } \sin k_2 a = 0 \quad (48a)$$

and

$$\cos k_1 a = \cos k_2 a \quad (48b)$$

Thus, both m and n are even or both m and n are odd, with $m = n$ allowed.

The other set of TE solutions is given by

$$H_z = \cos k_1 x \cos k_2 y - \cos k_2 x \cos k_1 y \quad (49)$$

with

$$k_1 = \frac{m\pi}{a}, \quad k_2 = \frac{n\pi}{a} \quad (50)$$

As for the TM modes, again we see that (for the $y = x + a$ boundary condition)

$$\sin k_1 a = 0 \text{ and } \sin k_2 a = 0 \quad (51a)$$

but now

$$\cos k_1 a = -\cos k_2 a \quad (51b)$$

Thus, m must be even, n odd, and vice versa. This second TE solution allows a TE_{10} mode as the lowest order mode derived from finite sums of rectangular harmonics found for this guiding structure. The TE_{10} mode has the form (see Figure 16)

$$H_z(TE_{10}) = \cos \frac{\pi x}{a} - \cos \frac{\pi y}{a} \quad (52)$$

The next lowest modes are the TE_{11} (see Figure 17) from Equation 43 and the TM_{12} from Equation 40. Just as for the 45- and 135-degree parallelogram waveguide, we do not have a complete set of modes and the true lowest order mode is missing. A finite difference analysis shows that there is one mode below the TE_{10} of Equation 52 that has not been determined in closed form (Reference 16).

F. 45- AND 135-DEGREE TRAPEZOIDAL WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

Consider the trapezoidal geometry oriented as in Figure 6. For the TM modes, we must satisfy

$$E_z = 0 \text{ on } x = 0, a ; y = \pm \left(x + \frac{a}{2} \right) \quad (53)$$

As previously, we attempt a solution of the form

$$E_z = \sin k_1 x \sin k_2 y - \sin k_2 x \sin k_1 y \quad (54)$$

with

$$k_1 = \frac{2m\pi}{a}, \quad k_2 = \frac{2n\pi}{a} \quad (55)$$

By inspection, E_z in Equation 54 is zero on $x = 0$ and a . Substituting the ($y = x + a/2$) boundary condition into Equation 54,

$$\begin{aligned} E_z = \sin k_1 x \left(\sin k_2 x \cos \frac{k_2 a}{2} + \cos k_2 x \sin \frac{k_2 a}{2} \right) \\ - \sin k_2 x \left(\sin k_1 x \cos \frac{k_1 a}{2} + \cos k_1 x \sin \frac{k_1 a}{2} \right) \end{aligned} \quad (56)$$

Equation 56 can be zero when

$$\sin \frac{k_1 a}{2} = 0 \text{ and } \sin \frac{k_2 a}{2} = 0 \quad (57a)$$

simultaneously with

$$\cos \frac{k_1 a}{2} = \cos \frac{k_2 a}{2} \quad (57b)$$

Equation 57 becomes (using Equation 55)

$$\sin m\pi = 0 \text{ and } \sin n\pi = 0 \quad (58a)$$

with

$$\cos m\pi = \cos n\pi \quad (58b)$$

Thus, m and n integer satisfy Equation 58a, and Equation 58b implies that either both m and n are even or both m and n are odd. [The $y = -(x + a/2)$ condition gives the same result as above.] By considering the form of Equation 56, we see that another solution is

$$E_z = \sin k_1 x \sin k_2 y + \sin k_2 x \sin k_1 y \quad (59)$$

using Equation 55.

Using Equation 59 and substituting in $y = x + a/2$, we now have the conditions

$$\sin m\pi = 0 \text{ and } \sin n\pi = 0 \quad (60a)$$

and

$$\cos m\pi = -\cos n\pi \quad (60b)$$

for E_z in Equation 59 to be zero on $y = \pm(x + a/2)$. Now if m is an even integer, n must be odd and vice versa.

Equations 54 and 59 are solutions that are zero on $y = 0$. Thus, these solutions are odd with respect to the center of the trapezoid, and we would like also to find solutions that are even with respect to $y = 0$. We attempt a solution of the form

$$E_z = \sin k_1 x \cos k_2 y \pm \sin k_2 x \cos k_1 y \quad (61)$$

with

$$k_1 = \frac{(2m-1)\pi}{a}, \quad k_2 = \frac{(2n-1)\pi}{a} \quad (62)$$

Again, $E_z = 0$ on $x = 0, a$ by inspection.

Using $y = x + a/2$ in Equation 61, we have

$$E_z = \sin k_1 x \left(\cos k_2 x \cos \frac{k_2 a}{2} - \sin k_2 x \sin \frac{k_2 a}{2} \right) \\ \pm \sin k_2 x \left(\cos k_1 x \cos \frac{k_1 a}{2} - \sin k_1 x \sin \frac{k_1 a}{2} \right) \quad (63)$$

Equation 63 can be zero if

$$\cos \frac{k_2 a}{2} = 0 \quad \text{and} \quad \cos \frac{k_1 a}{2} = 0 \quad (64a)$$

simultaneously with

$$\begin{cases} \sin \frac{k_2 a}{2} = -\sin \frac{k_1 a}{2} & \text{for the plus sign in Equation 61} \\ \sin \frac{k_2 a}{2} = \sin \frac{k_1 a}{2} & \text{for the minus sign in Equation 61} \end{cases} \quad (64b)$$

Thus, we see from Equation 64 that for the minus sign in Equation 61, we must have both m and n even or both m and n odd. For the plus sign, we must have m even, n odd, and vice versa. To summarize the TM modes, we have

$$E_z^{(1)} = \sin k_1 x \sin k_2 y - \sin k_2 x \sin k_1 y ; \quad (65a)$$

both m, n even (odd)

$$E_z^{(2)} = \sin k_1 x \sin k_2 y + \sin k_2 x \sin k_1 y ; \quad (65b)$$

m even, n odd (or vice versa) with

$$k_1 = \frac{2m\pi}{a}, \quad k_2 = \frac{2n\pi}{a} \quad (65c)$$

$m \neq 0, n \neq 0$ for $E_z^{(1)}$ and $E_z^{(2)}$; $m \neq n$ for $E_z^{(1)}$ only.

$$E_z^{(3)} = \sin k_1 x \cos k_2 y - \sin k_2 x \cos k_1 y ; \quad (66a)$$

both m, n even (or odd)

$$E_z^{(4)} = \sin k_1 x \cos k_2 y + \sin k_2 x \cos k_1 y ; \quad (66b)$$

m even, n odd (or vice versa) with

$$k_1 = \frac{(2m-1)\pi}{a} , \quad k_2 = \frac{(2n-1)\pi}{a} \quad (66c)$$

and $m \neq n$ for $E_z^{(3)}$ only. The TE modes must satisfy

$$\frac{\partial H_z}{\partial x} = 0 \text{ on } x = 0, a \quad (67a)$$

$$\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} = 0 \text{ on } y = \pm \left(x + \frac{a}{2} \right) \quad (67b)$$

We use the TE solution for the isosceles right triangular waveguide as usual and

$$H_z = \cos k_1 x \cos k_2 y + \cos k_2 x \cos k_1 y \quad (68)$$

with

$$k_1 = \frac{2m\pi}{a} , \quad k_2 = \frac{2n\pi}{a} \quad (69)$$

Equation 67a is satisfied immediately. Substituting $y = x + a/2$ into Equation 67b,

$$\begin{aligned} \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x+\frac{a}{2}} &= \left[-k_1 \sin k_1 x \cos k_2 \left(x + \frac{a}{2} \right) \right. \\ &\quad \left. - k_2 \sin k_2 x \cos k_1 \left(x + \frac{a}{2} \right) \right] - \left[-k_2 \cos k_1 x \sin k_2 \left(x + \frac{a}{2} \right) \right. \\ &\quad \left. - k_1 \cos k_2 x \sin k_1 \left(x + \frac{a}{2} \right) \right] \end{aligned} \quad (70)$$

$$\begin{aligned}
 \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \bigg|_{y=x+\frac{a}{2}} &= -k_1 \sin k_1 x \left(\cos k_2 x \cos \frac{k_2 a}{2} - \sin k_2 x \sin \frac{k_2 a}{2} \right) \\
 &\quad - k_2 \sin k_2 x \left(\cos k_1 x \cos \frac{k_1 a}{2} - \sin k_1 x \sin \frac{k_1 a}{2} \right) \\
 &\quad + k_2 \cos k_1 x \left(\sin k_2 x \cos \frac{k_2 a}{2} + \cos k_2 x \sin \frac{k_2 a}{2} \right) \\
 &\quad + k_1 \cos k_2 x \left(\sin k_1 x \cos \frac{k_1 a}{2} + \cos k_1 x \sin \frac{k_1 a}{2} \right)
 \end{aligned} \tag{71}$$

Equation 71 is zero provided

$$\sin \frac{k_1 a}{2} = 0 \quad \text{and} \quad \sin \frac{k_2 a}{2} = 0 \tag{72a}$$

along with

$$\cos \frac{k_1 a}{2} = \cos \frac{k_2 a}{2} \tag{72b}$$

Thus, both m and n must be even integers or both m and n are odd integers, just as for $E_z^{(1)}$. [We obtain the same result when $y = -(x + a/2)$.] Also from Equation 71, we see that

$$H_z = \cos k_1 x \cos k_2 y - \cos k_2 x \cos k_1 y \tag{73}$$

along with Equation 68, is a solution provided m is an even integer, while n is odd and vice versa. Just as for the TM modes, we have two more solutions, i.e.,

$$H_z = \cos k_1 x \sin k_2 y \pm \cos k_2 x \sin k_1 y \tag{74}$$

with

$$k_1 = \frac{(2m-1)\pi}{a}, \quad k_2 = \frac{(2n-1)\pi}{a} \tag{75}$$

which now go to zero along $y = 0$. Equation 74 obeys Equation 67a by inspection. Equation 67b becomes (using Equation 74)

$$\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x+\frac{a}{2}} = \left[-k_1 \sin k_1 x \sin k_2 \left(x + \frac{a}{2} \right) \right. \\ \left. + k_2 \sin k_2 x \sin k_1 \left(x + \frac{a}{2} \right) \right] - \left[k_2 \cos k_1 x \cos k_2 \left(x + \frac{a}{2} \right) \right. \\ \left. \pm k_1 \cos k_2 x \cos k_1 \left(x + \frac{a}{2} \right) \right] \quad (76)$$

$$\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial y} \right) \Big|_{y=x+\frac{a}{2}} = -k_1 \sin k_1 x \left(\sin k_2 x \cos \frac{k_2 a}{2} + \cos k_2 x \sin \frac{k_2 a}{2} \right) \\ + k_2 \sin k_2 x \left(\sin k_1 x \cos \frac{k_1 a}{2} + \cos k_1 x \sin \frac{k_1 a}{2} \right) \\ - k_2 \cos k_1 x \left(\cos k_2 x \cos \frac{k_2 a}{2} - \sin k_2 x \sin \frac{k_2 a}{2} \right) \\ + k_1 \cos k_2 x \left(\cos k_1 x \cos \frac{k_1 a}{2} - \sin k_1 x \sin \frac{k_1 a}{2} \right) \quad (77)$$

Equation 77 can be zero provided

$$\cos \frac{k_1 a}{2} = 0 \text{ and } \cos \frac{k_2 a}{2} = 0 \quad (78a)$$

along with

$$\begin{cases} \sin \frac{k_1 a}{2} = \sin \frac{k_2 a}{2} & \text{for the plus sign in Equation 74} \\ \sin \frac{k_1 a}{2} = -\sin \frac{k_2 a}{2} & \text{for the minus sign in Equation 74} \end{cases} \quad (78b)$$

Thus, the TE solutions follow restrictions on integers m and n similar to the TM modes. That is, for the plus sign in Equation 74, both m and n must be even or both m and n must be odd; for the minus sign, m must be even and n odd or vice versa.

To summarize the TE modes, we have

$$H_z^{(1)} = \cos k_1 x \cos k_2 y + \cos k_2 x \cos k_1 y \quad (79a)$$

both m, n even (or odd)

$$H_z^{(2)} = \cos k_1 x \cos k_2 y - \cos k_2 x \cos k_1 y \quad (79b)$$

m even, n odd (or vice versa) with

$$k_1 = \frac{2m\pi}{a}, \quad k_2 = \frac{2n\pi}{a} \quad (79c)$$

and

$$H_z^{(3)} = \cos k_1 x \sin k_2 y + \cos k_2 x \sin k_1 y \quad (80a)$$

both m, n even (or odd)

$$H_z^{(4)} = \cos k_1 x \sin k_2 y - \cos k_2 x \sin k_1 y \quad (80b)$$

m even, n odd (or vice versa) with

$$k_1 = \frac{(2m-1)\pi}{a}, \quad k_2 = \frac{(2n-1)\pi}{a} \quad (80c)$$

The lowest order mode of this trapezoidal geometry (note that the top length and the width of the trapezoid are both equal) determined from Equations 79 and 80 is the TE_{11} mode given by $H_z^{(3)}$. Its distribution is (see Figure 18)

$$H_z^{(3)}(TE_{11}) = 2 \cos \frac{\pi x}{a} \sin \frac{\pi y}{a} \quad (81)$$

The exact same mode can be found from Equation 80b with $m = 0, n = 1$. The next lowest mode is the TE_{10} mode given by Equation 79b, i.e.,

$$H_z^{(2)}(TE_{10}) = \cos \frac{2\pi x}{a} - \cos \frac{2\pi y}{a} \quad (82)$$

G. 60- AND 120-DEGREE PARALLELOGRAM WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

Using the solutions found previously for the equilateral triangular waveguide (Reference 9), we can achieve some solutions for a 60- and 120-degree parallelogram oriented as in Figure 7. We assume that (for TM modes)

$$E_z^{(1)} = \sin 2k_1x \sin 2k_2y + \sin k_3x \sin k_4y + \sin k_5x \sin k_6y \quad (83)$$

with

$$k_1 = \frac{2m\pi}{a\sqrt{3}}, \quad k_2 = \frac{2n\pi}{3a}, \quad k_3 = \frac{(m+n)2\pi}{a\sqrt{3}}, \quad k_4 = \frac{(n-3m)2\pi}{3a},$$

$$k_5 = \frac{(m-n)2\pi}{a\sqrt{3}}, \quad k_6 = \frac{(n+3m)2\pi}{3a} \quad (84)$$

We know that this solution for $E_z^{(1)}$ is zero on $x = a\sqrt{3}/2$ and $x = y\sqrt{3}$. Also, we see immediately that it is zero for $x = 0$. So we need to check only the fourth boundary condition of the 60- and 120-degree parallelogram geometry, i.e., $y = x/\sqrt{3} - a/2$ or $x = y\sqrt{3} + \sqrt{3}a/2$. Substituting this into Equation 83, we have

$$E_z^{(1)} = \sin 2k_1 \sqrt{3} \left(y + \frac{a}{2} \right) \sin 2k_2y + \sin k_3 \sqrt{3} \left(y + \frac{a}{2} \right) \sin k_4y$$

$$+ \sin k_5 \sqrt{3} \left(y + \frac{a}{2} \right) \sin k_6y \quad (85)$$

Using the k_i ($i = 1$ to 6) given above, the arguments of the sine terms in Equation 85 become

$$2k_1 \sqrt{3} \left(y + \frac{a}{2} \right) = \frac{4m\pi y}{a} + 2m\pi \quad (86a)$$

$$2k_2y = \frac{4n\pi y}{3a} \quad (86b)$$

$$k_3 \sqrt{3} \left(y + \frac{a}{2} \right) = \frac{2\pi(m+n)y}{a} + \pi(m+n) \quad (86c)$$

$$k_4 y = \frac{(n - 3m)2\pi y}{3a} \quad (86d)$$

$$k_5 \sqrt{3} \left(y + \frac{a}{2} \right) = \frac{2\pi(m - n)y}{a} + \pi(m - n) \quad (86e)$$

$$k_6 y = \frac{(n + 3m)2\pi y}{3a} \quad (86f)$$

Now let

$$A = \frac{2m\pi y}{a}, \quad B = \frac{2n\pi y}{3a} \quad (87)$$

Using Equations 86 and 87 and substituting into $E_z^{(1)}$ in Equation 85, we have

$$\begin{aligned} E_z^{(1)} = & \sin(2A + 2m\pi) \sin 2B + \sin[A + 3B + \pi(m + n)] \sin(B - A) \\ & + \sin[A - 3B + \pi(m - n)] \sin(B + A) \end{aligned} \quad (88)$$

Using the double-angle formulas, i.e.,

$$\sin(A \pm 3B) = \sin A \cos 3B \pm \cos A \sin 3B = U_1 \pm U_2 \quad (89a)$$

$$\sin(B \pm A) = \sin B \cos A \pm \cos B \sin A = V_1 \pm V_2 \quad (89b)$$

and

$$\sin[A \pm 3B + \pi(m \pm n)] = \sin(A \pm 3B) \cos \pi(m \pm n)$$

(provided m and n are integer)

$$= \begin{cases} + \sin(A \pm 3B) & \text{for } \begin{Bmatrix} m+n \\ m-n \end{Bmatrix} \text{ even} \\ - \sin(A \pm 3B) & \text{for } \begin{Bmatrix} m+n \\ m-n \end{Bmatrix} \text{ odd} \end{cases} \quad (90)$$

we see that

$$E_z^{(1)} = \sin 2A \sin 2B \pm (U_1 + U_2)(V_1 - V_2) \pm (U_1 - U_2)(V_1 + V_2) \quad (91)$$

$$= \sin 2A \sin 2B \pm 2(U_1 V_1 - U_2 V_2) \quad (92)$$

$$= \sin 2A \sin 2B \pm 2[\sin A \cos A (\sin B \cos 3B - \cos B \sin 3B)] \quad (93)$$

$$= \sin 2A \sin 2B \pm \sin 2A \sin (-2B) \quad (94)$$

At this point, we see that the plus sign in Equation 90 must be used to force $E_z^{(1)}$ to be 0. Therefore, we must have both $(m + n)$ and $(m - n)$ even to match all the boundary conditions. For the other solution, i.e., a solution that is even about $y = 0$ (Reference 9), we have

$$E_z^{(2)} = \sin 2k_1 x \cos 2k_2 y - \sin k_3 x \cos k_4 y - \sin k_5 x \cos k_6 y \quad (95)$$

along with Equation 84. In a similar manner as for $E_z^{(1)}$, we can show that both $(m + n)$ and $(m - n)$ must be even in order for our solution to obey the fourth boundary condition. As in the equilateral triangular waveguide, $E_z^{(1)}$ is zero on $y = 0$, the "odd" solution, whereas, $E_z^{(2)}$ is an extremum on $y = 0$, the "even" solution. For the odd types of TM modes, we must have $m \neq 0$, $n \neq 0$, $m \neq n$, and $n \neq 3m$. For the even TM modes, we must have $m \neq 0$ and $m \neq n$. The lowest order TM mode is given by $E_z^{(2)}$, and it is a TM_{20} mode.

The TE modes are given also by the corresponding equilateral triangular TE solutions, i.e.,

$$H_z^{(1)} = \cos 2k_1 x \cos 2k_2 y + \cos k_3 x \cos k_4 y + \cos k_5 x \cos k_6 y \quad (96a)$$

$$H_z^{(2)} = \cos 2k_1 x \sin 2k_2 y - \cos k_3 x \sin k_4 y - \cos k_5 x \sin k_6 y \quad (96b)$$

with the k_i 's given by Equation 84. Again, we find that both $(m + n)$ and $(m - n)$ must be even to satisfy the condition on the fourth boundary. Furthermore, for the $H_z^{(2)}$ solutions, $n \neq 0$ and $n \neq 3m$. The TE_{11} mode is the lowest order mode determined in the above way for this structure. Its distribution is (see Figure 19)

$$H_z^{(1)}(TE_{11}) = 2 \cos \frac{4\pi x}{a\sqrt{3}} \cos \frac{4\pi y}{3a} + \cos \frac{8\pi y}{3a} \quad (97a)$$

which is even about $y = 0$ and (see Figure 20)

$$H_z^{(2)}(TE_{11}) = 2 \cos \frac{4\pi x}{a\sqrt{3}} \sin \frac{4\pi y}{3a} - \sin \frac{8\pi y}{3a} \quad (97b)$$

which is 0 on $y = 0$.

H. 60- AND 120-DEGREE TRAPEZOIDAL WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

The 60- and 120-degree trapezoidal waveguide, oriented as in Figure 8, must satisfy the following boundary conditions for TM mode solutions. These are

$$E_z = 0 \text{ on } x = 0, \quad x = \frac{a\sqrt{3}}{2}, \quad y = \pm \left(\frac{x}{\sqrt{3}} + \frac{a}{2} \right) \quad (98)$$

Again, we attempt to use the equilateral triangular waveguide solutions, i.e.,

$$E_z \sin 2k_1 x \sin 2k_2 y + \sin k_3 x \sin k_4 y + \sin k_5 x \sin k_6 y \quad (99)$$

with the k_i as in Equation 84. We know that E_z is zero on $x = 0$ and $x = a\sqrt{3}/2$ by inspection. To check the $y = x/\sqrt{3} + a/2$ condition, we use Equations 99 and 84 to give

$$\begin{aligned} E_z = & \sin 2k_1 \sqrt{3} \left(y - \frac{a}{2} \right) \sin 2k_2 y + \sin k_3 \sqrt{3} \left(y - \frac{a}{2} \right) \sin k_4 y \\ & + \sin k_5 \sqrt{3} \left(y - \frac{a}{2} \right) \sin k_6 y \end{aligned} \quad (100)$$

Using Equation 86 (with minus signs between the terms in Equations 86a, 86c, and 86e) and Equation 87 and substituting into E_z in Equation 100, we have

$$\begin{aligned} E_z = & \sin (2A - 2m\pi) \sin 2B + \sin [(A + 3B) - \pi(m + n)] \sin (B - A) \\ & + \sin [(A - 3B) - \pi(m - n)] \sin (B + A) \end{aligned} \quad (101)$$

Using Equation 89 and observing that (for m and n integer)

$$\sin [(A \pm 3B) - \pi(m \pm n)] = \sin (A \pm 3B) \cos \pi(m \pm n)$$

$$= \begin{cases} \sin (A \pm 3B) & \text{for } \begin{Bmatrix} m+n \\ m-n \end{Bmatrix} \text{ even} \\ -\sin (A \pm 3B) & \text{for } \begin{Bmatrix} m+n \\ m-n \end{Bmatrix} \text{ odd} \end{cases} \quad (102)$$

is just the same as Equation 90, then E_z will be zero provided both $(m + n)$ and $(m - n)$ are even (following Equations 91 through 94). [The same result obtains from $y = -(x/\sqrt{3} + a/2)$].

In the case of the trapezoid, we see that another solution is possible where the minus signs in Equation 102 are used. Thus,

$$E_z = \sin 2k_1x \sin 2k_2y - \sin k_3x \sin k_4y - \sin k_5x \sin k_6y \quad (103)$$

with both $(m + n)$ and $(m - n)$ odd, and the k_i given by Equation 84 is another solution. Furthermore, there are two more solutions that are maximum on $y = 0$ and these are

$$E_z = \sin 2k_1x \cos 2k_2y \pm \sin k_3x \cos k_4y \pm \sin k_5x \cos k_6y \quad (104)$$

with both $(m + n)$ and $(m - n)$ even for the minus signs in Equation 104 and both $(m + n)$ and $(m - n)$ odd for the plus signs in Equation 104. The k_i are still given by Equation 84.

The TE modes exhibit the same type of behavior. Without detail, these are

$$H_z^{(1)} = \cos 2k_1x \cos 2k_2y + \cos k_3x \cos k_4y + \cos k_5x \cos k_6y \quad (105)$$

with the k_i as in Equation 84 and both $(m + n)$ and $(m - n)$ even. Also, we have

$$H_z^{(2)} = \cos 2k_1x \cos 2k_2y - \cos k_3x \cos k_4y - \cos k_5x \cos k_6y \quad (106)$$

with both $(m + n)$ and $(m - n)$ odd.

Finally,

$$H_z^{(3)} = \cos 2k_1x \sin 2k_2y \pm \cos k_3x \sin k_4y \pm \cos k_5x \sin k_6y \quad (107)$$

with the k_i as in Equation 84 and both $(m + n)$ and $(m - n)$ even for the minus signs in Equation 107 and both $(m + n)$ and $(m - n)$ odd for the plus signs. The lowest TE mode found from the above four solutions is the TE_{01} mode (see Figures 21 and 22), i.e.,

$$H_z^{(3)}(TE_{01}) = \sin \frac{4\pi y}{3a} + 2 \cos \frac{2\pi x}{a\sqrt{3}} \sin \frac{2\pi y}{3a} \quad (108a)$$

which is zero on $y = 0$ and its counterpart

$$H_z^{(2)}(TE_{01}) = \cos \frac{4\pi y}{3a} - 2 \cos \frac{2\pi x}{a\sqrt{3}} \cos \frac{2\pi y}{3a} \quad (108b)$$

which is even about $y = 0$.

III. CUTOFF FORMULAS AND MODE PLOTS

A. ISOSCELES RIGHT TRIANGULAR WAVEGUIDE WITH TWO ELECTRIC AND ONE MAGNETIC WALL

$$(k_c)_{mn}^2 = \left(\frac{\pi}{2a} \right)^2 (m^2 + n^2) \quad (109)$$

is the cutoff wave number of this structure. We have claimed that the TE_{11} mode is the lowest order mode found by the methods of Section II. Thus,

$$(k_c)_{11}^2 = \frac{\pi^2}{2a^2} \quad (110)$$

The corresponding cutoff wavelength for this mode is

$$(\lambda_c)_{11} = \frac{2\pi}{k_{c11}} = 2\sqrt{2} a \quad (111)$$

which is larger than that of the corresponding square waveguide of x dimension, a (see Figure 9). Considering this fact along with the structure of the contour lines of constant H_z for the TE_{11} mode as shown in Figure 12, this mode is truly the lowest order mode of this guide. We see from a comparison of Figures 12 and 13 that we obtain the same result whether the magnetic wall is fixed at $x = a$ or at $y = 0$, as expected.

B. ISOSCELES RIGHT TRIANGULAR WAVEGUIDE WITH TWO ELECTRIC AND ONE MAGNETIC WALL AT $y = x$

For this geometry, the cutoff wave number is

$$(k_c)_{mn}^2 = \left(\frac{\pi}{a}\right)^2 (m^2 + n^2) \quad (112)$$

which is the same as that for the isosceles right triangular waveguide with perfectly conducting walls (also the same as for the square waveguide of side a). The lowest mode is the TE_{10} mode shown in Figure 14. Its cutoff is

$$(k_c)_{10}^2 = \left(\frac{\pi}{a}\right)^2 \quad (113)$$

and its cutoff wavelength is

$$(\lambda_c)_{10} = 2a \quad (114)$$

Thus, this is also a true lowest order mode.

C. 45- AND 135-DEGREE PARALLELOGRAM WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

This structure has the same cutoff formula as the isosceles right triangular waveguide (Reference 9) given by Equation 112. However, because of the integer restrictions necessary for satisfaction of the fourth boundary conditions, the lowest order mode found is the TE_{11} (see Figure 15). Thus, its cutoff wave number is

$$(k_c)_{11}^2 = 2 \left(\frac{\pi}{a}\right)^2 \quad (115)$$

and

$$(\lambda_c)_{11} = a\sqrt{2} \quad (116)$$

We know (see Section II and Reference 15) that there are two modes with cutoff wave numbers lower than the above TE_{11} mode, but we have been unable to determine them in closed form.

D. 90- AND 90- AND 135- AND 45-DEGREE WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

This geometry has the same general cutoff formula as the 45- and 135-degree parallelogram waveguide (see Equation 112). In this case, we do have a TE_{10} mode (see Figure 16), but it is not the lowest order mode (see Section II and Reference 16). Its cutoff wavelength is

$$(\lambda_c)_{10} = 2a \quad (117)$$

just as for the rectangular waveguide. However, this is not a true lowest order mode (see Figure 16).

E. 45- AND 135-DEGREE TRAPEZOIDAL WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

This trapezoidal waveguide contains a high degree of symmetry, i.e., not only are the four angles either 45 or 135 degrees but the length of its "top" section along $x = 0$ is equal to its width a (see Figure 6). Its cutoff formula is given by

$$(k_c)_{mn}^2 = \frac{4\pi^2}{a^2} (m^2 + n^2) \quad (118a)$$

for certain solutions, while

$$(k_c)_{mn}^2 = \frac{\pi^2}{a^2} [(2m-1)^2 + (2n-1)^2] \quad (118b)$$

for the other solutions (see Section II, Equations 79 and 80). The lowest order mode is the TE_{11} mode given by Equations 118b and 80a. Its cutoff wave number and wavelength are

$$(k_c)_{11}^2 = 2 \frac{\pi^2}{a^2} \quad (119)$$

or

$$(\lambda_c)_{11} = a\sqrt{2} \quad (120)$$

Since Equation 120 is smaller than the cutoff wavelength for the rectangular waveguide, it is likely that our TE_{11} mode in Equation 81 is not a true lowest order mode. Contour lines of constant H_z are shown in Figure 18 for this TE_{11} mode.

F. 60- AND 120-DEGREE PARALLELOGRAM WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

The cutoff formula for this geometry is the same as for the equilateral triangular waveguide (see Reference 9)

$$(k_c)_{mn}^2 = \frac{(4\pi)^2}{3a^2} \left(m^2 + \frac{n^2}{3} \right) \quad (121)$$

The TE_{11} mode (see Figures 19 and 20) is the lowest order mode found from Equation 96.

Thus,

$$(k_c)_{11}^2 = \frac{(4\pi)^2}{3a^2} \left(\frac{4}{3} \right) \quad (122)$$

and

$$(\lambda_c)_{11} = \frac{3a}{4} \quad (123)$$

If we compare the values in Equations 122 and 123 with those for the equilateral triangular guide (Reference 9), we see that although we have solutions that are even and odd about $y = 0$, these TE_{11} solutions are not the true lowest order modes (see Reference 16).

G. 60- AND 120-DEGREE TRAPEZOIDAL WAVEGUIDE WITH PERFECTLY CONDUCTING WALLS

This waveguide has the same general cutoff formula as given above by Equation 121. In this instance, however, we have a TE_{01} mode, which may be a true lowest order mode. It has a cutoff wave number given by

$$(k_c)_{01}^2 = \left(\frac{4\pi}{3a} \right)^2 \quad (124)$$

(the same as for the lowest order mode of an equilateral triangular waveguide), and its corresponding cutoff wavelength is

$$(\lambda_c)_{01} = \frac{3a}{2} \quad (125)$$

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Just as for the equilateral triangular waveguide, this mode has two distributions, one of which is even and the other odd about $y = 0$ (see Figures 21 and 22).

IV. CONCLUSIONS

Closed-form mode solutions for eight uniform waveguides with unusual cross-sectional geometries/boundary conditions have been determined using finite sums of rectangular harmonics, symmetry, and the Riemann-Schwarz reflection principle. Although infinite sets of transverse electric and transverse magnetic modes are generated for these eight geometries, complete sets of modes (in the sense of the rectangular waveguide) have not been determined. In particular, the lowest order modes of certain shapes are "missing." We have speculated that these "missing" modes do not have the form of finite sums of rectangular harmonics and must be found using either an infinite sum of rectangular harmonics or a purely nonseparable type of solution.

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VI. FIGURES

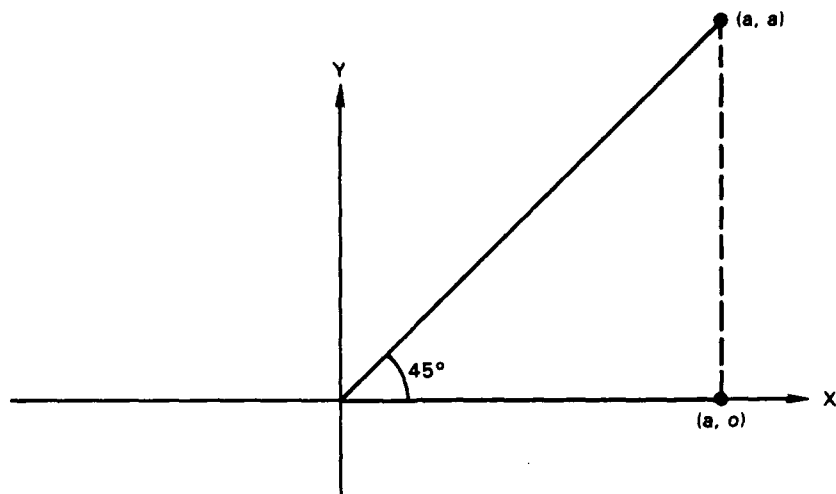


FIGURE 1. Isosceles Right Triangular Waveguide With Two Electric Walls and One Magnetic Wall at $x = a$ (Dashed Line).

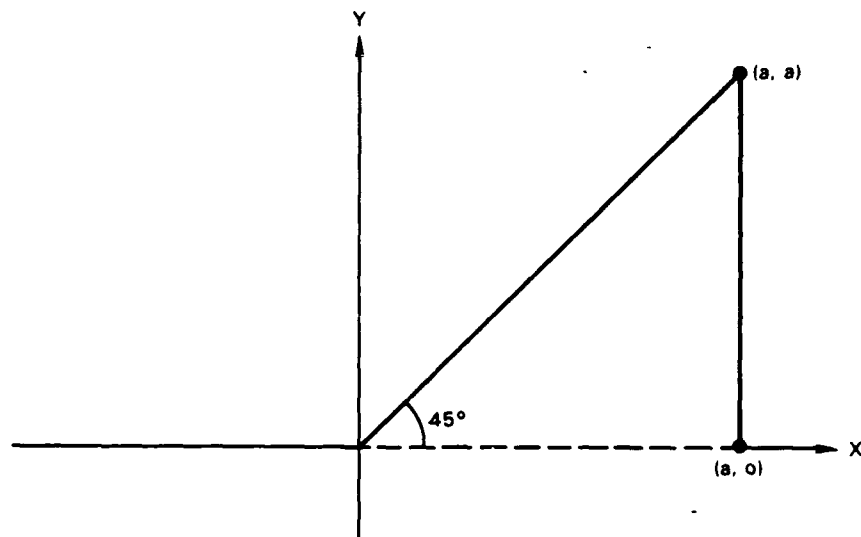


FIGURE 2. Isosceles Right Triangular Waveguide With Two Electric Walls and One Magnetic Wall at $y = 0$ (Dashed Line).

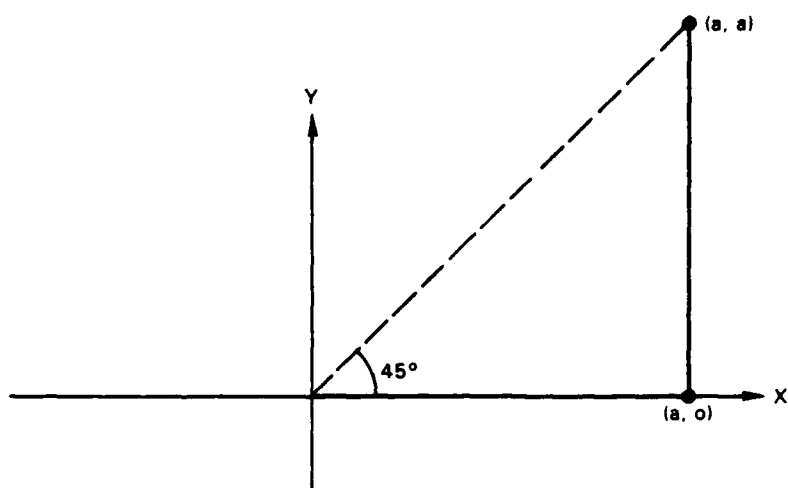


FIGURE 3. Isosceles Right Triangular Waveguide With Two Electric Walls and One Magnetic Wall at $y = x$ (Dashed Line).

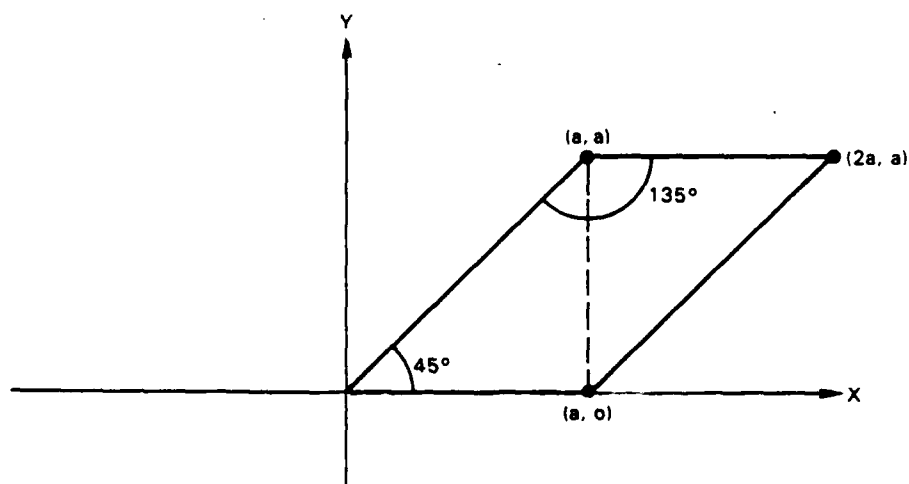


FIGURE 4. 45- and 135-Degree Parallelogram Waveguide With Perfectly Conducting Walls.

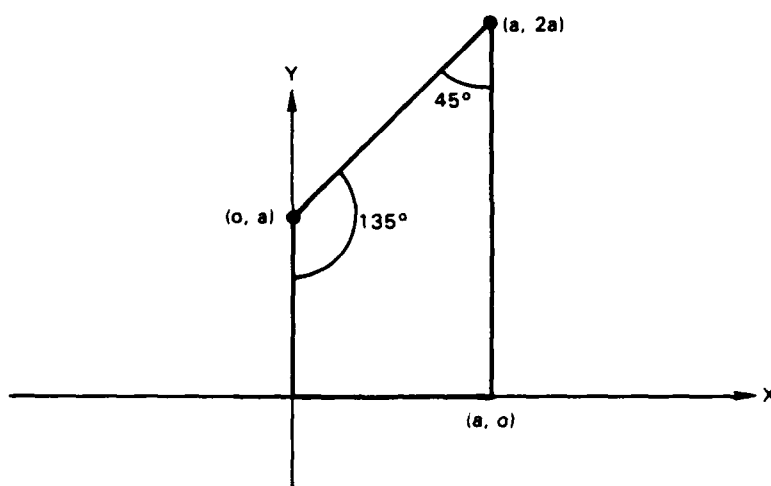


FIGURE 5. 90- and 90- and 135- and 45-Degree Parallelogram Waveguide With Perfectly Conducting Walls.

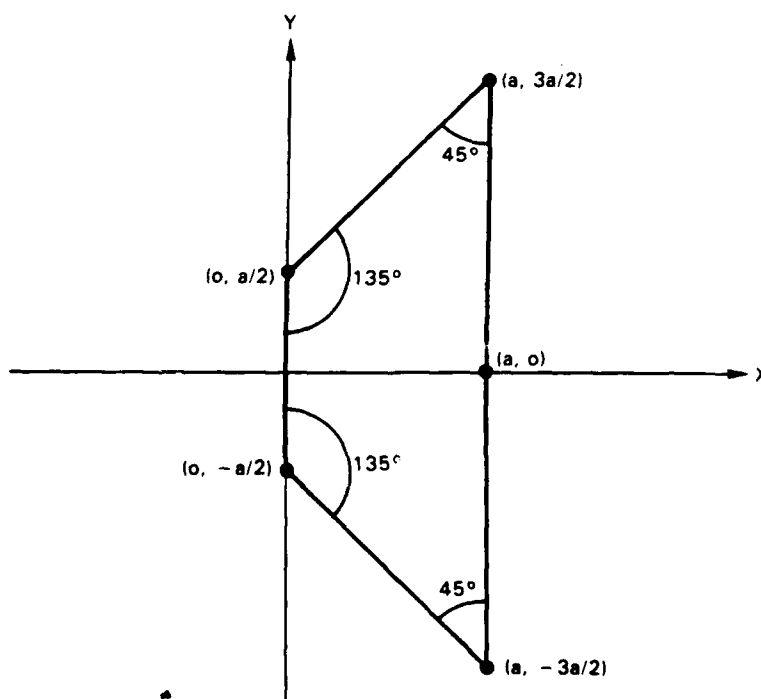


FIGURE 6. 45- and 135-Degree Trapezoidal Waveguide With Perfectly Conducting Walls.

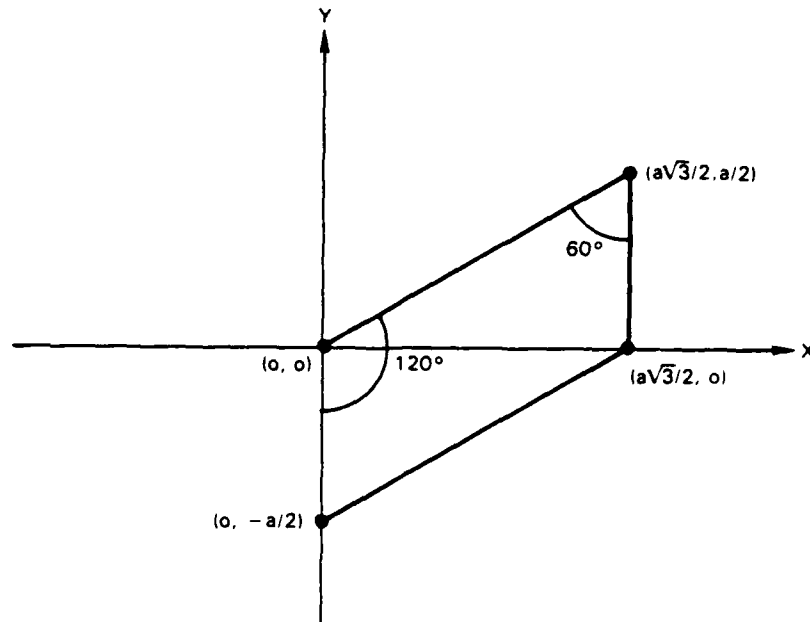


FIGURE 7. 60- and 120-Degree Parallelogram Waveguide With Perfectly Conducting Walls.

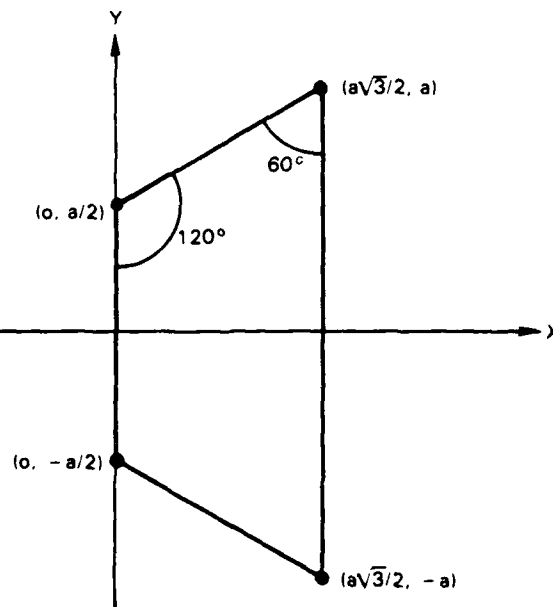


FIGURE 8. 60- and 120-Degree Trapezoidal Waveguide With Perfectly Conducting Walls.

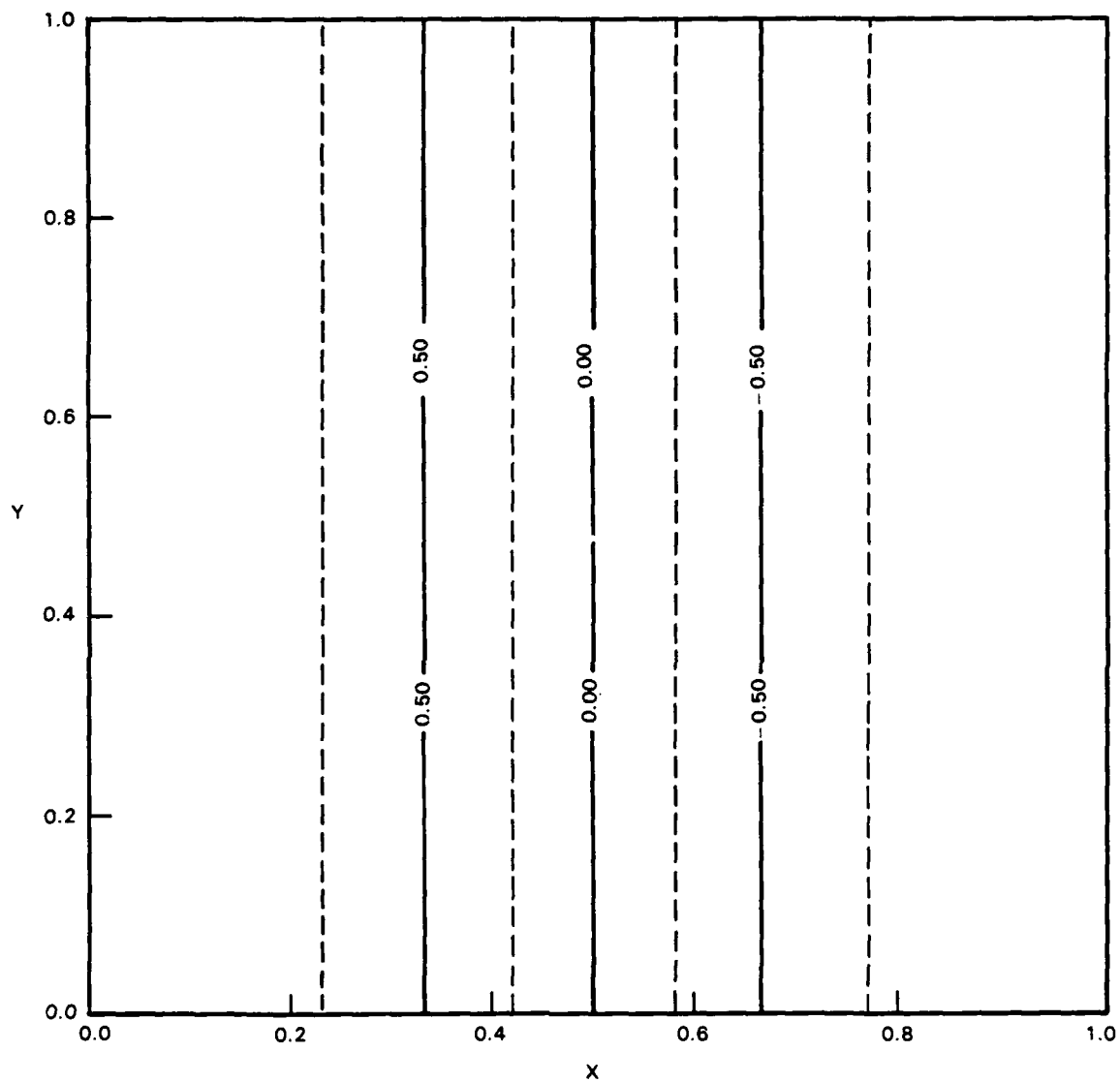


FIGURE 9. Contour Plot of TE_{10} Mode for Square Waveguide.

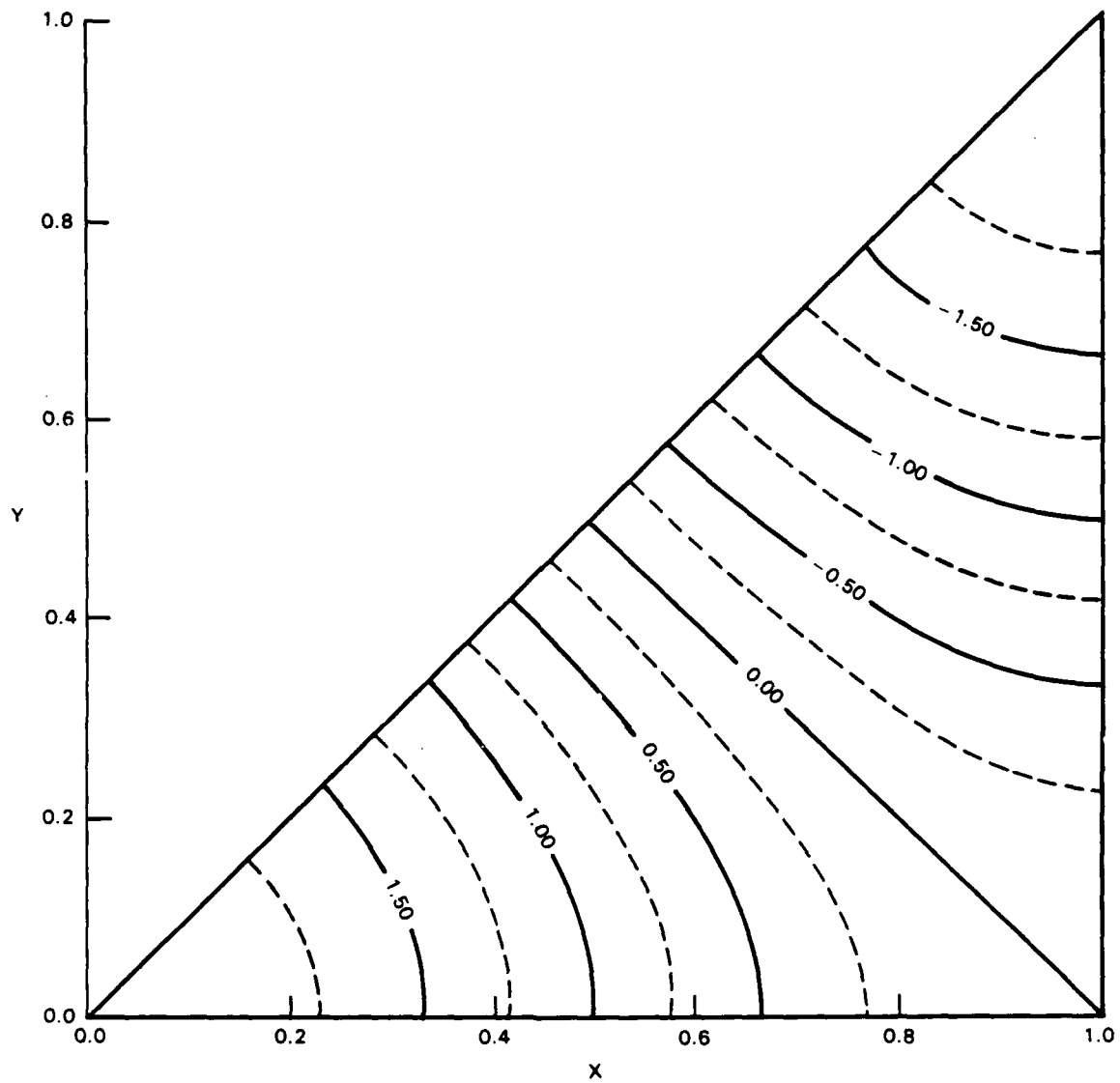


FIGURE 10. Contour Plot of TE_{10} Mode for Isosceles Right Triangular Waveguide With Perfectly Conducting Walls.

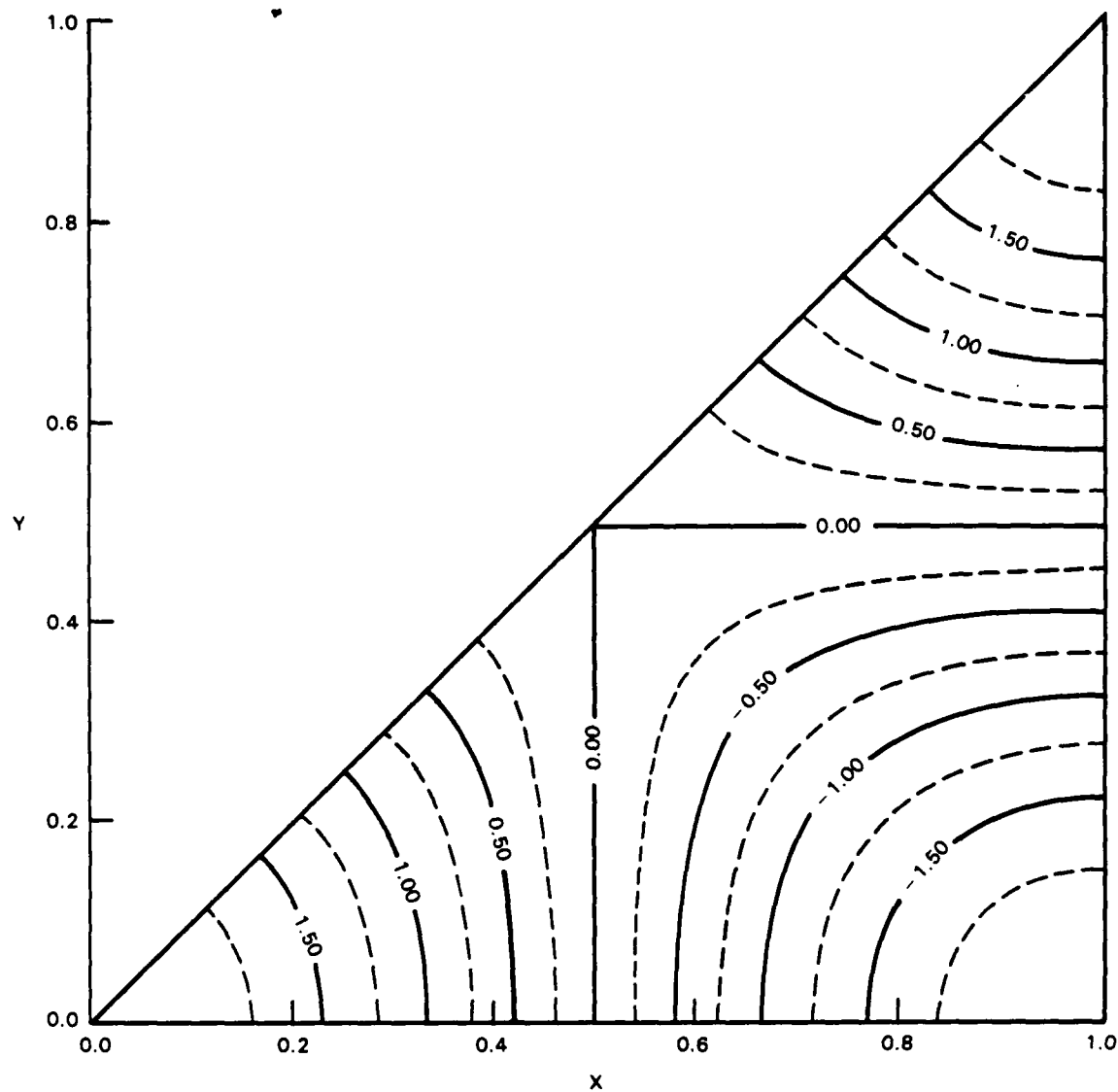


FIGURE 11. Contour Plot of TE_{11} Mode for Isosceles Right Triangular Waveguide With Perfectly Conducting Walls.

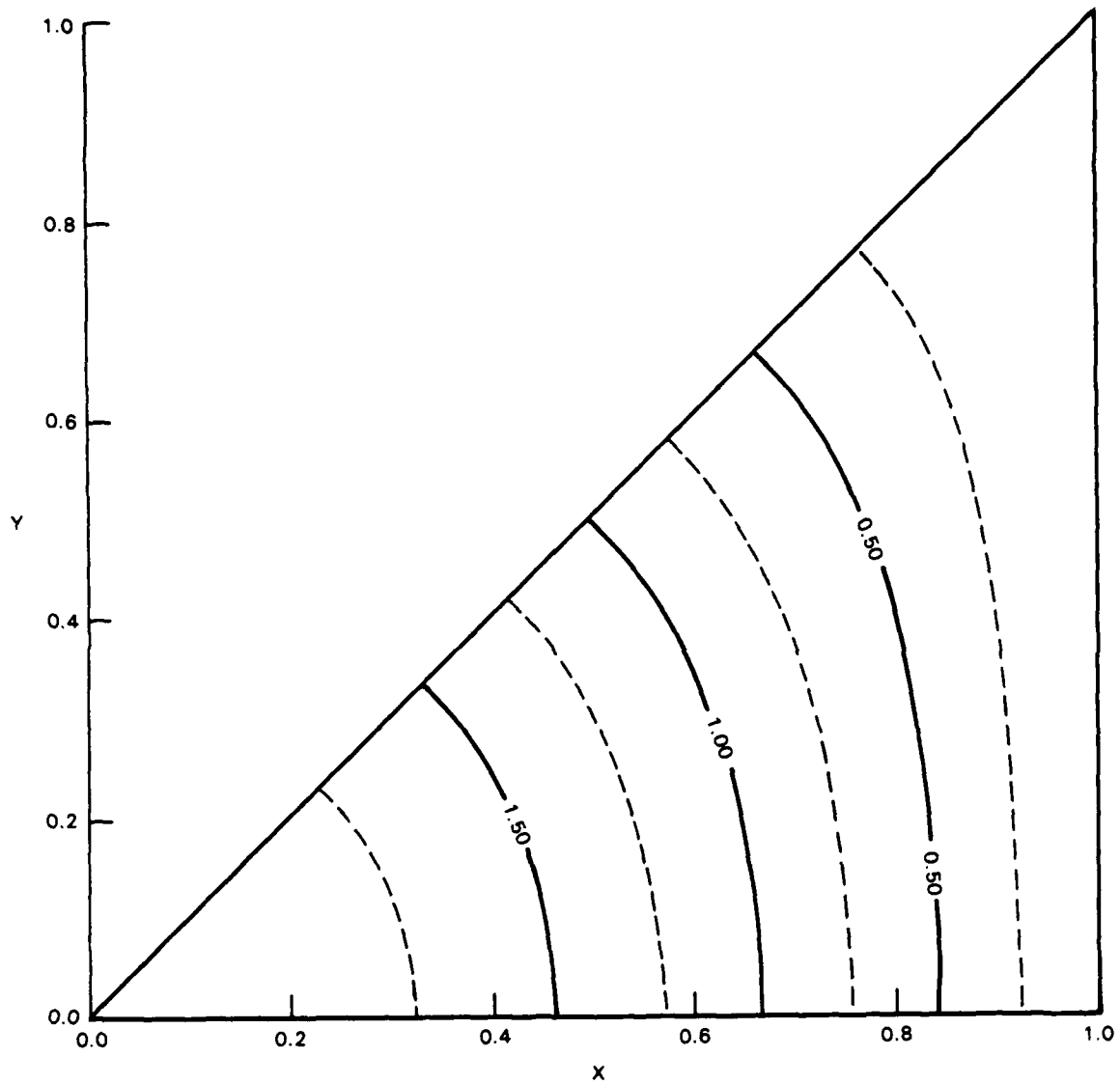


FIGURE 12. Contour Plot of TE_{11} Mode for Isosceles Right Triangular Waveguide With a Magnetic Wall at $x = 1$.

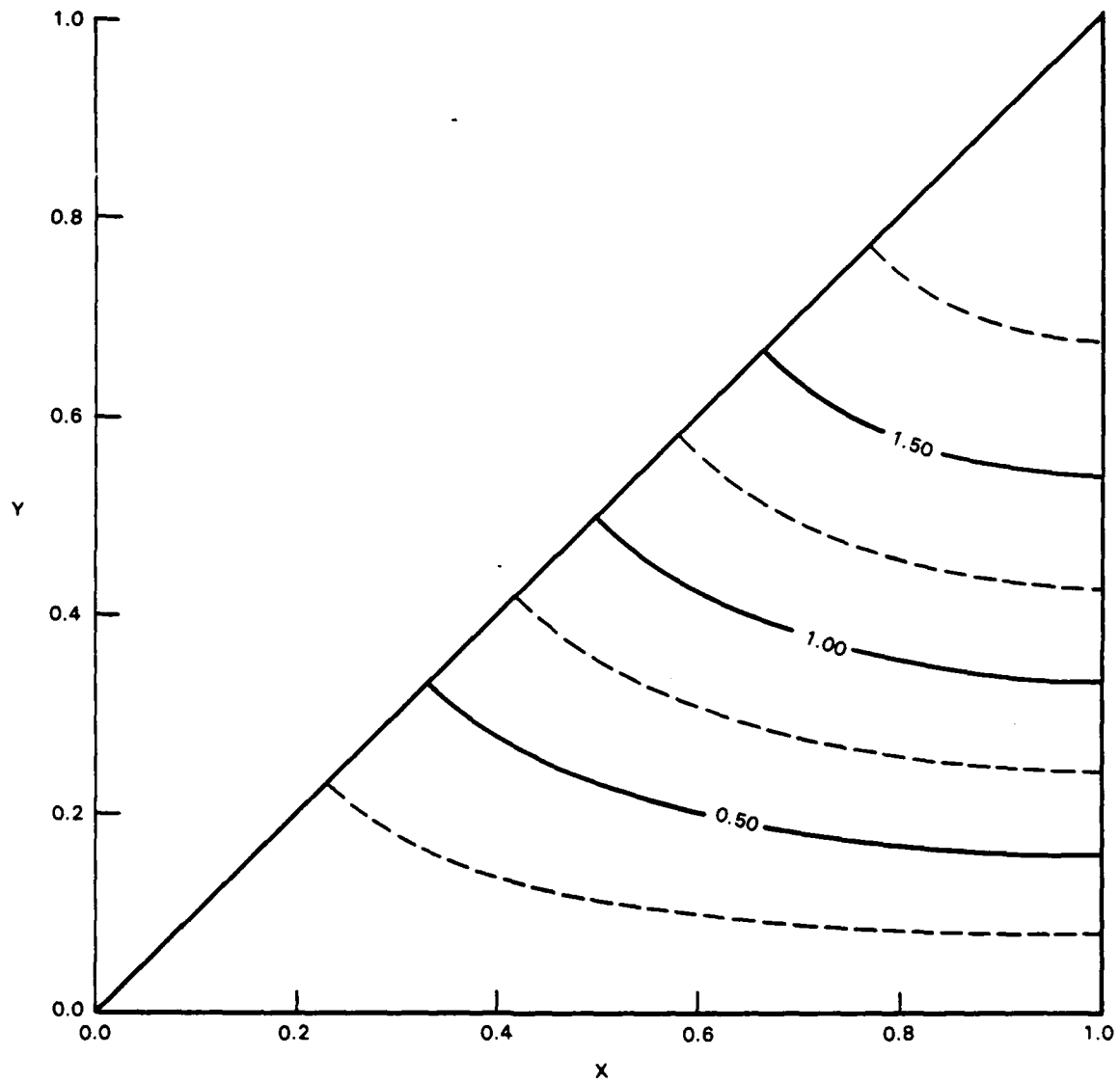


FIGURE 13. Contour Plot of TE_{11} Mode for Isosceles Right Triangular Waveguide With a Magnetic Wall at $y = 0$.

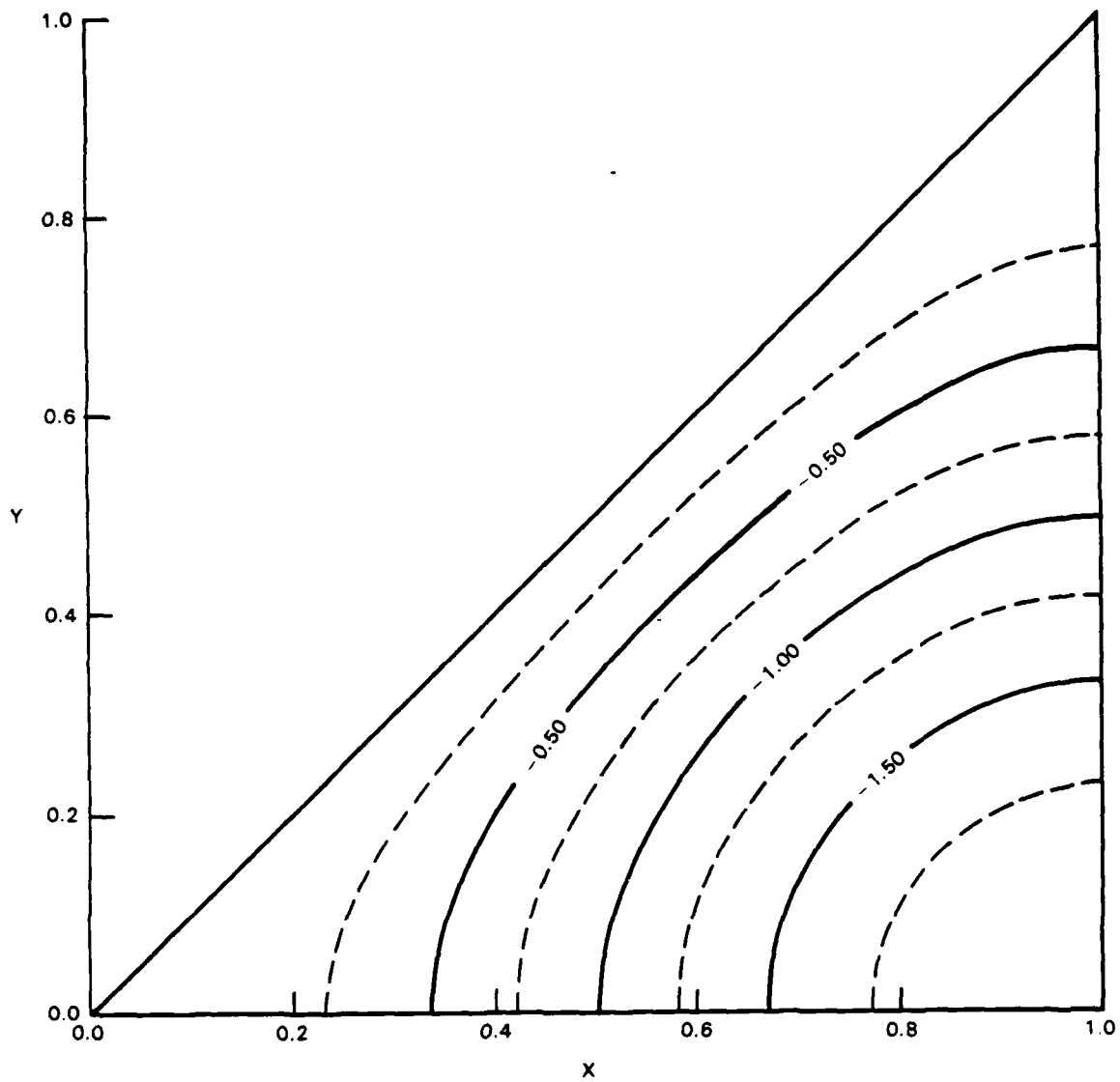


FIGURE 14. Contour Plot of TE_{10} Mode for Isosceles Right Triangular Waveguide With a Magnetic Wall at $y = x$.

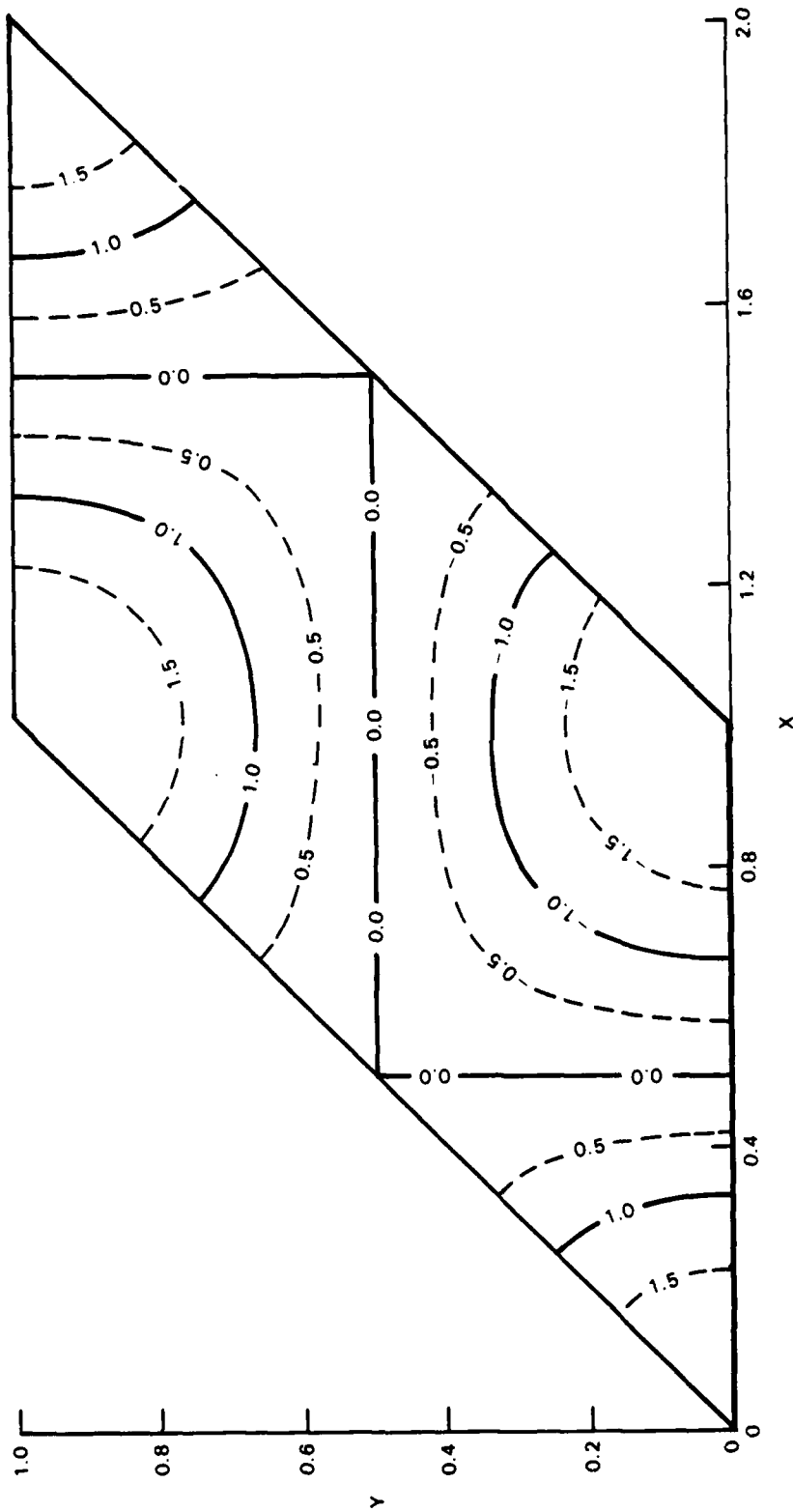


FIGURE 15. Contour Plot of TE₁₁ Mode for 45- and 135-Degree Parallelogram Waveguide.

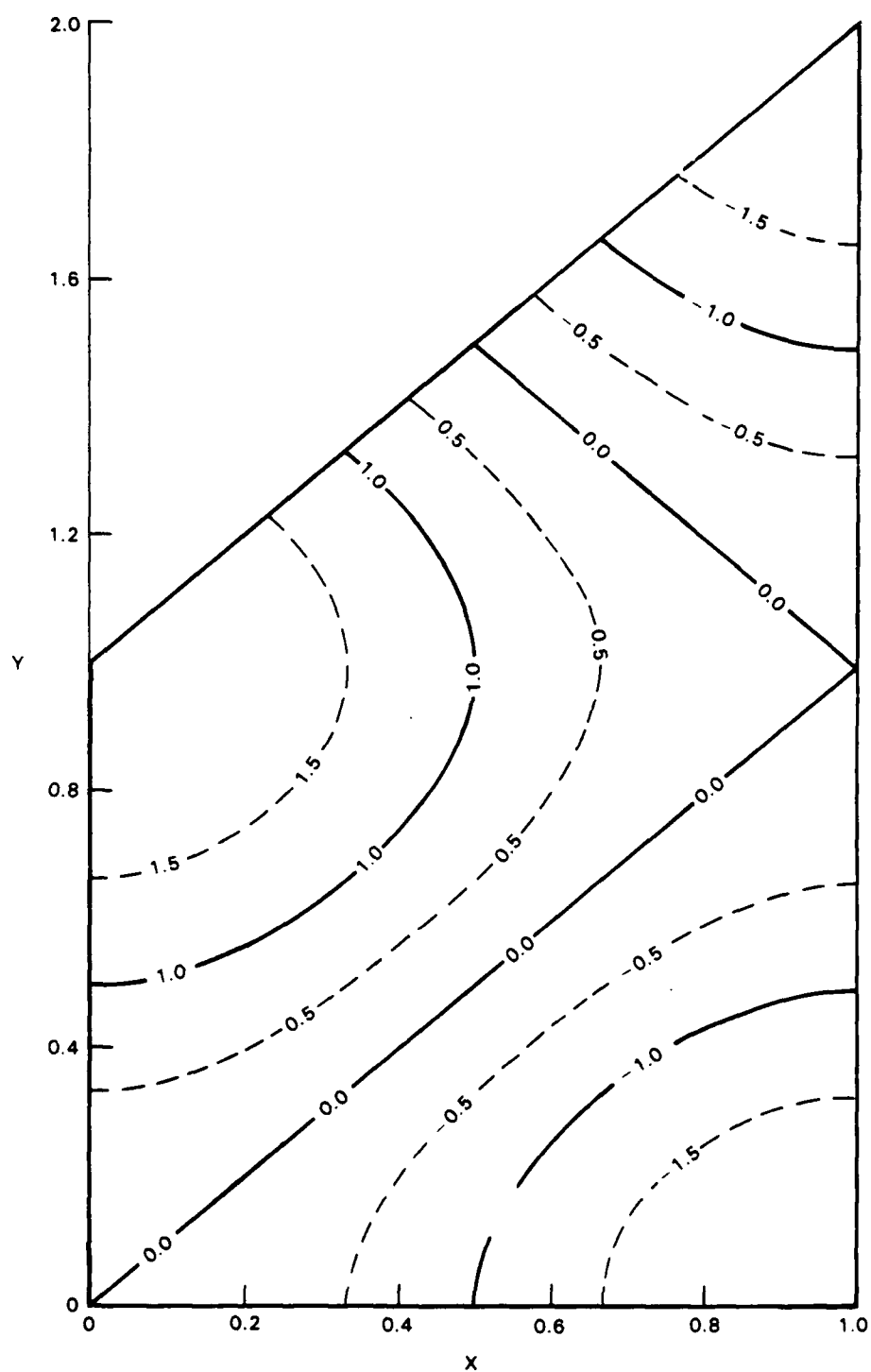


FIGURE 16. Contour Plot of TE₁₀ for 90- and 90- and 135- and 45-Degree Waveguide.

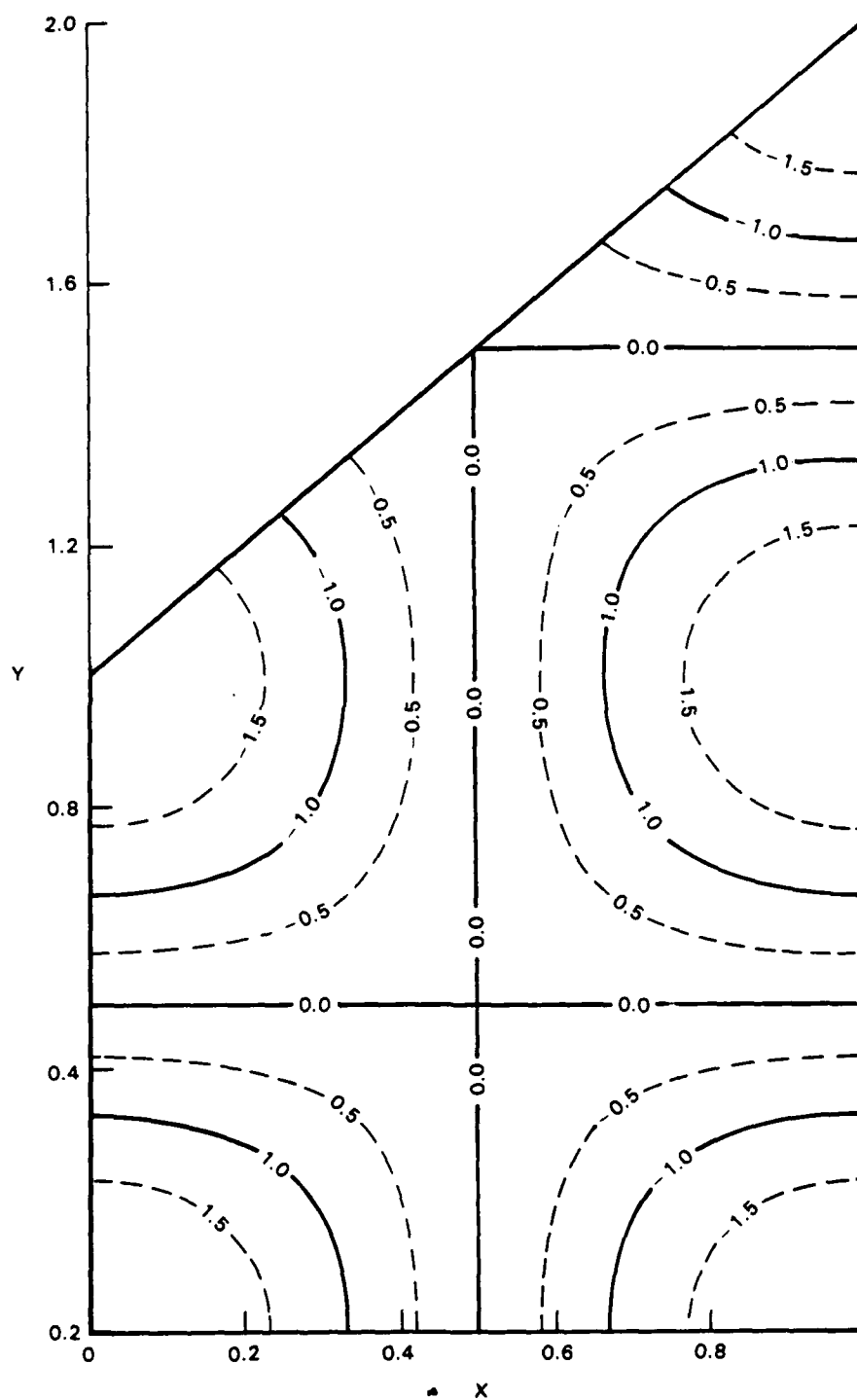


FIGURE 17. Contour Plot of TE_{11} Mode for 90- and 90- and 135- and 45-Degree Waveguide.

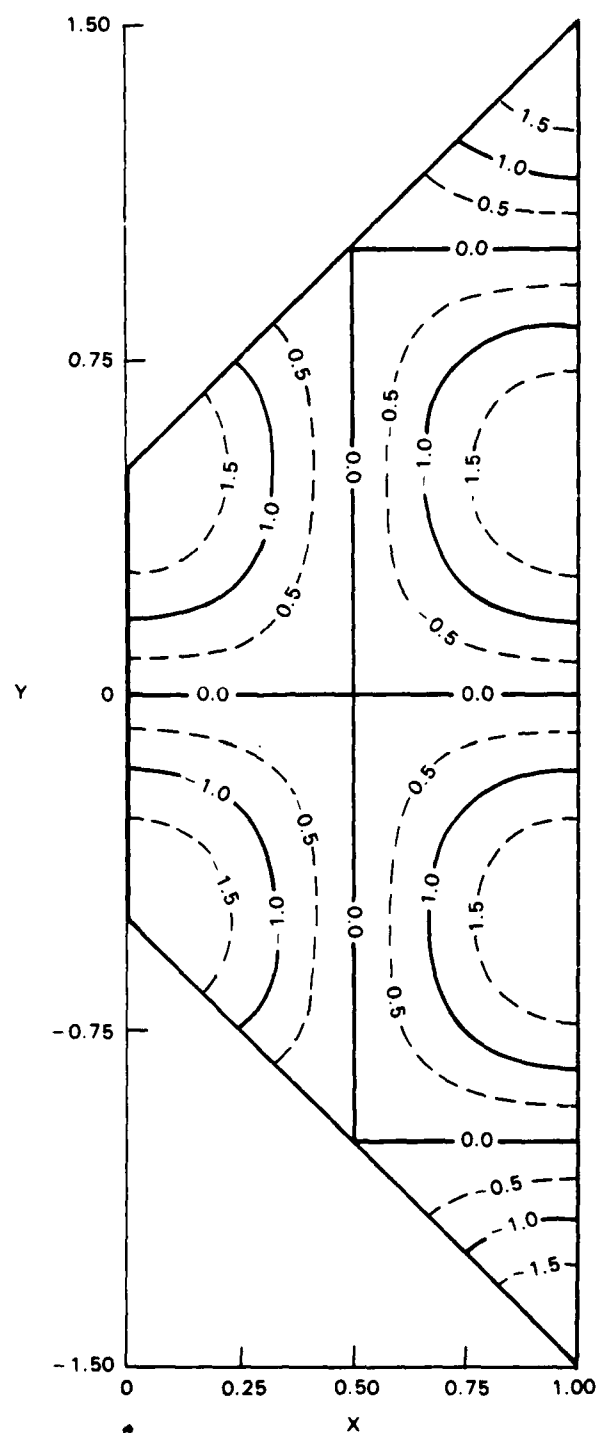


FIGURE 18. Contour Plot of TE_{11} Mode for 45- and 135-Degree Trapezoidal Waveguide.

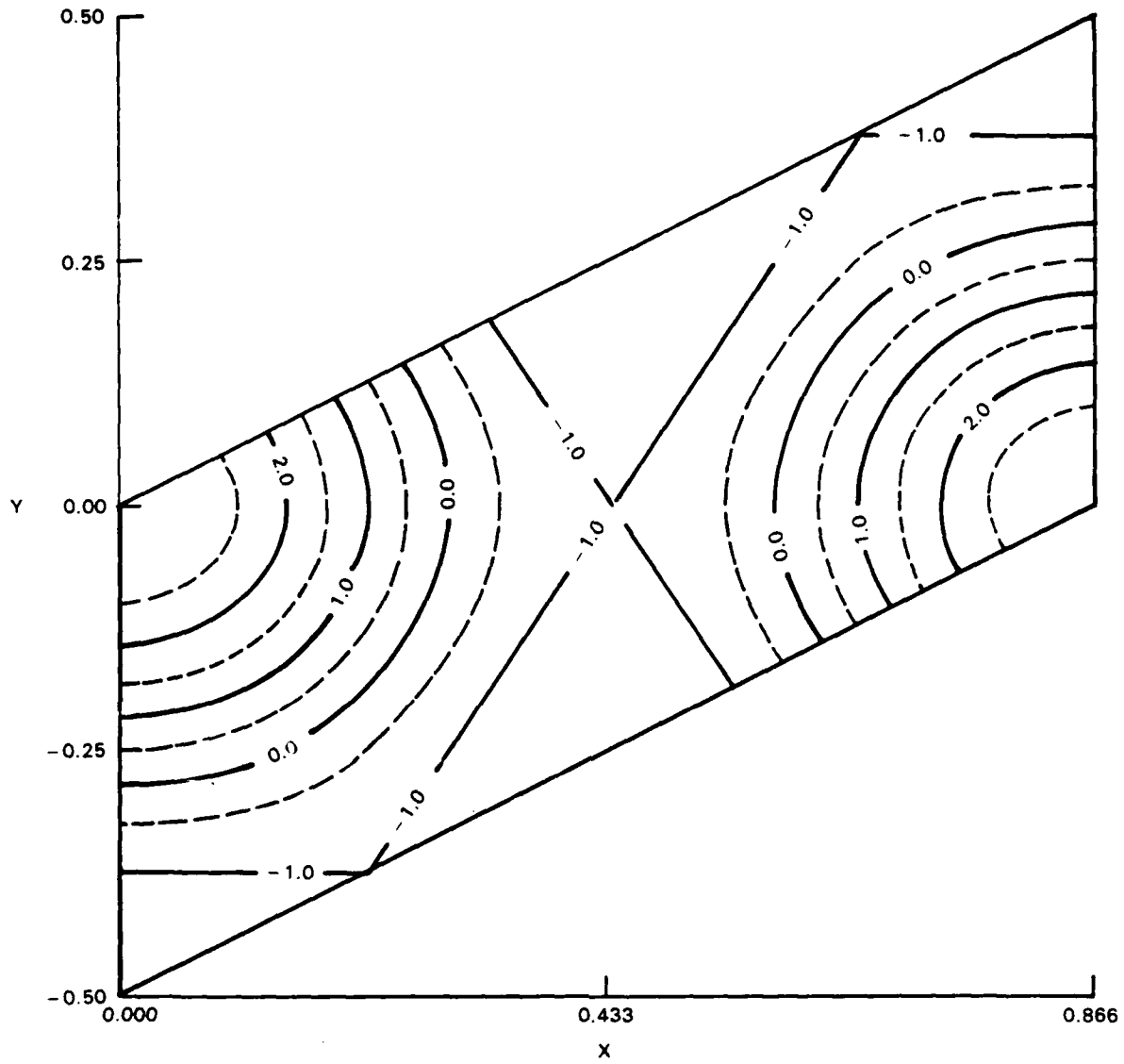


FIGURE 19. Contour Plot of TE_{11} Mode for 60- and 120-Degree Parallelogram Waveguide (Even Solution).

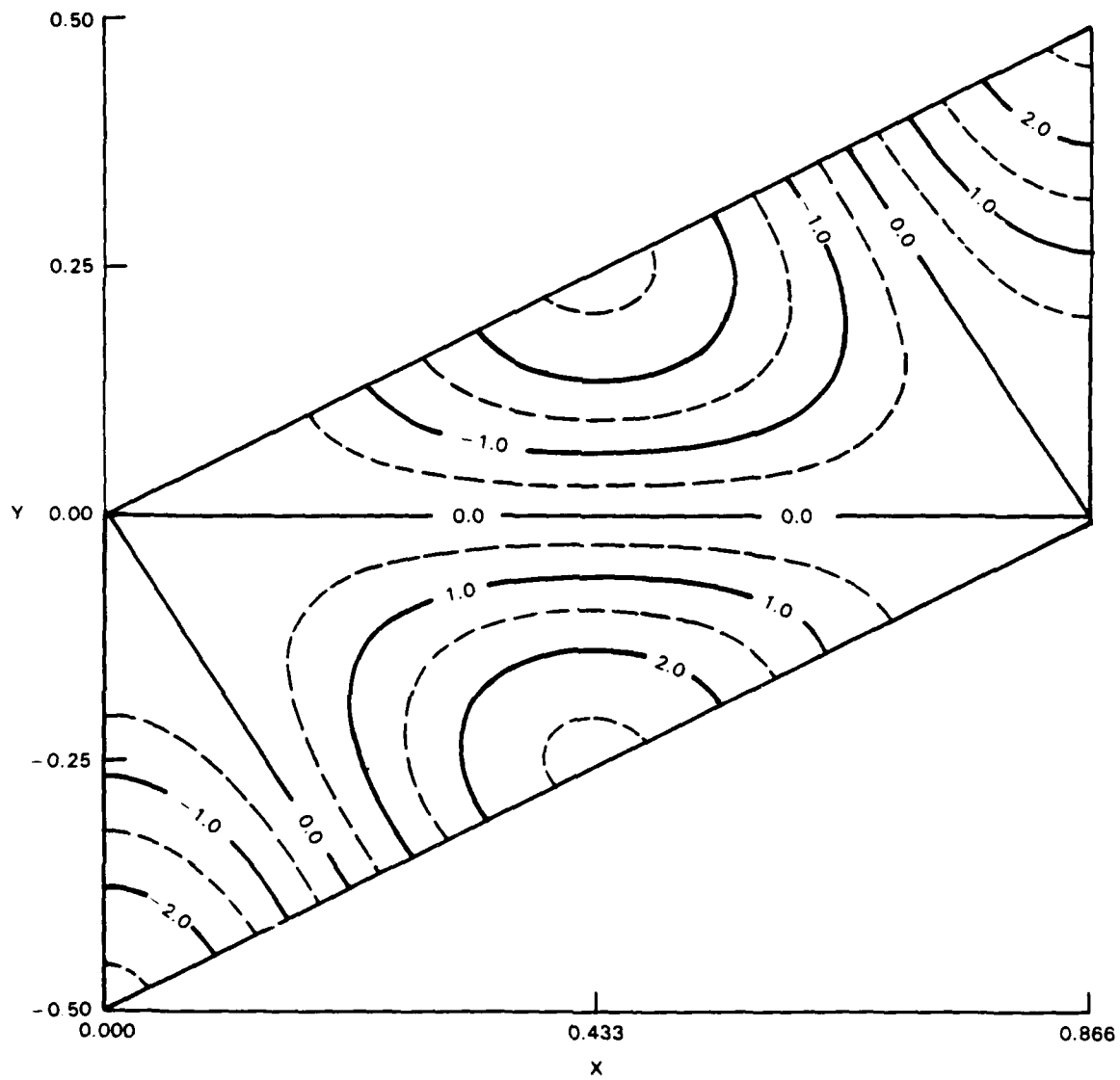


FIGURE 20. Contour Plot of TE_{11} Mode for 60- and 120-Degree Parallelogram Waveguide (Odd Solution).

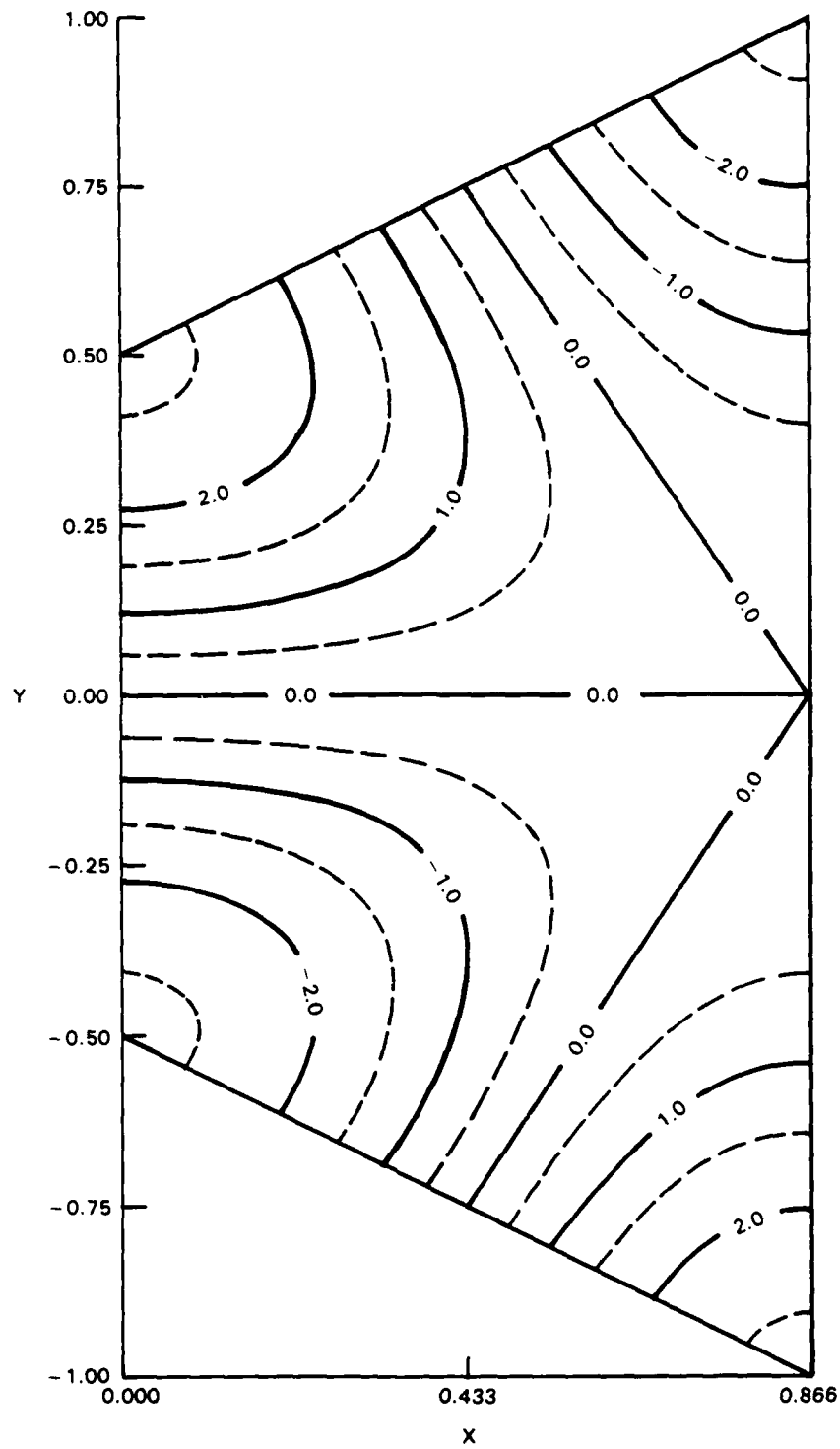


FIGURE 21. Contour Plot of TE_{01} Mode for 60- and 120-Degree Trapezoidal Waveguide (Odd Solution).

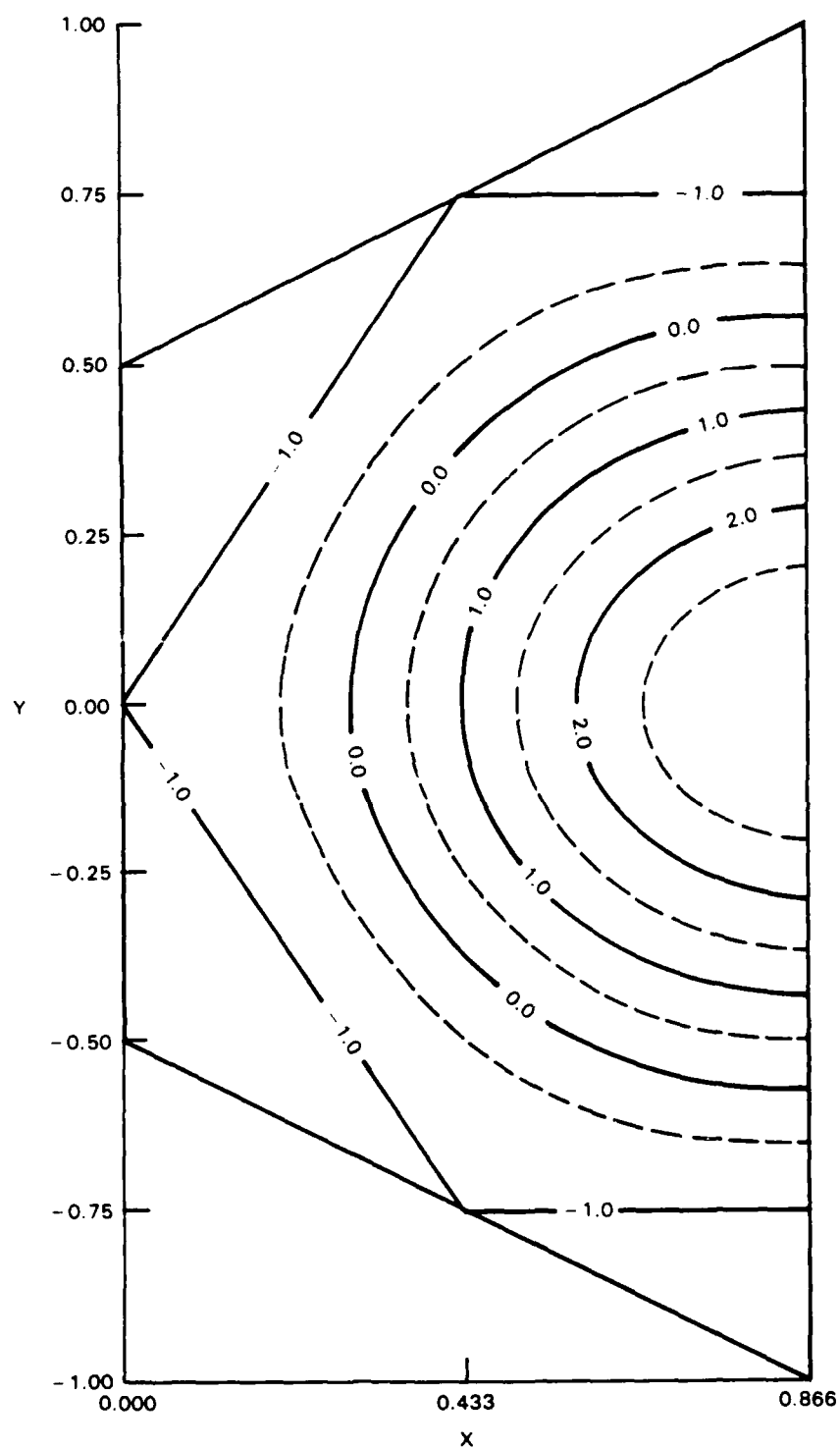


FIGURE 22. Contour Plot of TE_{01} Mode for 60- and 120-Degree Trapezoidal Waveguide (Even Solution).

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