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NETWORK SIMULATOR DEFINITION

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A white paper was delivered to RADC which provided a detailed analysis of a technique utilizing state transition probability matrices that provided a real~~istic model~~ of a digital communications channel. As such, the technique offers a computationally viable method for implementing a large number of realistic channel models. This work will feed or transition to efforts eventually implementing this technique to perform network survivability analyses. The results will be incorporated in more comprehensive reports resulting from this work and in the actual implementation of these techniques in RADC's Network Design Laboratory.

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A DECISION DIRECTED PHASE ESTIMATOR MODEL

INTRODUCTION

The objective of this effort was to develop techniques for modeling digital errors which occur in a communication system. The model was to be computationally efficient for applications which require a large number of replications of the model such as a communication network analysis. In this effort, an important component of a digital communication system, namely a digital phase-locked loop was analyzed. The analysis produced a stochastic model of the digital error producing process associated with the system. A state transition probability matrix formulation was used which allowed the calculation of a variety of digital error performance measures.

In this report, an analysis will be performed of a digital phase-locked loop. Using standard analytical procedures and assumptions, a set of discrete conditonal probability density functions (pdf's) for a phase state variable shall be developed. This will then form the basis for a state transition matrix formulation which will be used to calculate various probability of error statistics.

The standard cross-correlation demodulator described in the literature implies that a carrier with proper phase is present at the receiver. In practice, the correct phase must be determined at the receiver. One popular technique for performing this function is the decision directed phase estimate. It finds its place in this report because of the "memory" it imparts to the error producing process.

There are many underlying causes of phase noise that might be hypothesized. Here, only the presence of additive white Gaussian noise as the source of phase noise will be assumed and an analysis will be performed of a demodulator employing a phase-locked loop. For clarity of presentation, a binary phase shift keyed demodulator shall be assumed. The analysis is applicable to multiple phase demodulators by using an appropriate transformation of the decision boundaries as described previously. This analysis will prove useful for any demodulator which requires a phase reference to be generated at the receiver. That is, as before, the dimensionality of the system is being reduced by considering only the phase transformation technique for the received signal vector.

SYSTEM MODEL

A typical implementation of a decision directed phase estimator is shown in Figure 6. Consistent with the notation introduced earlier, the transmitted waveform, s(t), along with the additive noise waveform, z(t), is shown at the input to the demodulator. The additive noise is assumed to be white Gaussian noise with a two sided power spectral density of N_0. A reference sinusoidal waveform is used to generate an in-phase and quadrature signal, each of

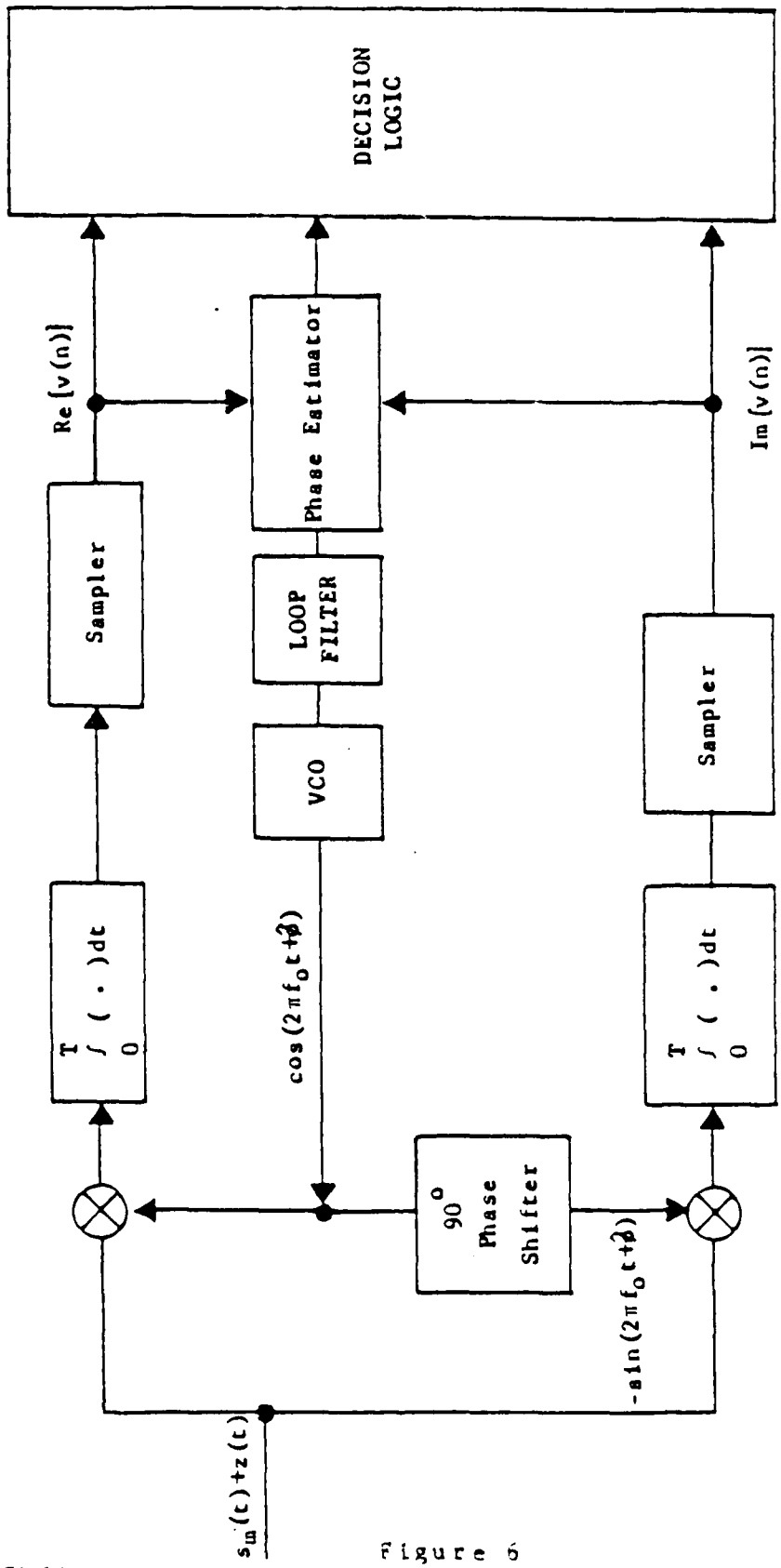


Figure 6
DEMODULATOR USING A DECISION DIRECTED PHASE ESTIMATOR

which is used in a cross-correlation integrator. A phase estimate, $\hat{\phi}$, is used in the argument of each of these reference waveforms. An accurate phase estimate is required to assure the correct phase of the received signal vector, v, which will ultimately be used in the decision process. This properly phased reference is generated by a voltage controlled oscillator (VCO). The control voltage for this VCO is generated using the phases of the previous received signal vectors. The phase of the received signal vector at time n is given by

$$\phi(n) = \tan^{-1}(\mathrm{Im}[v(n)]/\mathrm{Re}[v(n)])$$

Recall, from before, that due to the viewpoint being used in this dissertation, the received signal vector should ideally equal the real number 2aE. Thus, the phase, ϕ, should ideally equal zero.

In general, the phase reference, $\hat{\phi}(n)$, for the demodulator is based on more than one phase measurement of the received signal vector. An averaging technique is used to converge to the expected value of the received signal vector phases. Typically an analog or digital loop filter is used to process the sequence of received phases, $\phi(n)$, in order to produce a phase reference or estimate, $\hat{\phi}(n)$.

Here, the loop filter shall be modeled as a digital first-order network as shown in Figure 7. This, of course,

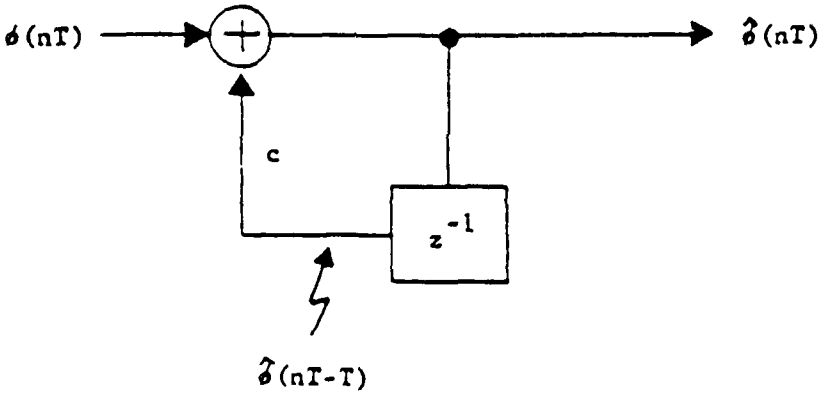


Figure 7
FIRST ORDER LOOP FILTER

computes values for the first-order difference equation:

$$\hat{\phi}(nT) = c\,\hat{\phi}(nT-T) + \phi(nT)$$

The transfer function for the system shown in Figure 7 has been derived in textbooks such as Gold [1969] and is given by

$$H(e^{jwT}) = (1+c-2c\cos wT)^{-1/2} \exp[jwT - j\tan^{-1}(\sin wT/(\cos wT - c))] \tag{4-1}$$

This function exhibits a periodic frequency selectivity in terms of the frequency variable w and the sampling period T. Note that for $wT < \pi$, that is, for frequencies less than one-half the sampling rate, or in this case the bit rate, the loop filter behaves as a low pass filter which is characterized by the parameter c. As such, this filter can either be interpreted as a digital model of an analog loop filter with the above frequency characteristic or as an infinite precision model of a digital loop filter implementation. For clarity, attention is restricted only to a first-order digital filter. Extension to higher order filters can be carried out in a similar manner.

The system model from Figure 6 is redrawn in Figure 8 where the first-order loop filter is shown explicitly and the usual low pass notation is used. Note that the loop filter's coefficient, to be defined shortly, is now c_0 and scales the phase ϕ. This digital filter would, in general, be unstable due to the unity feedback gain. However, in the

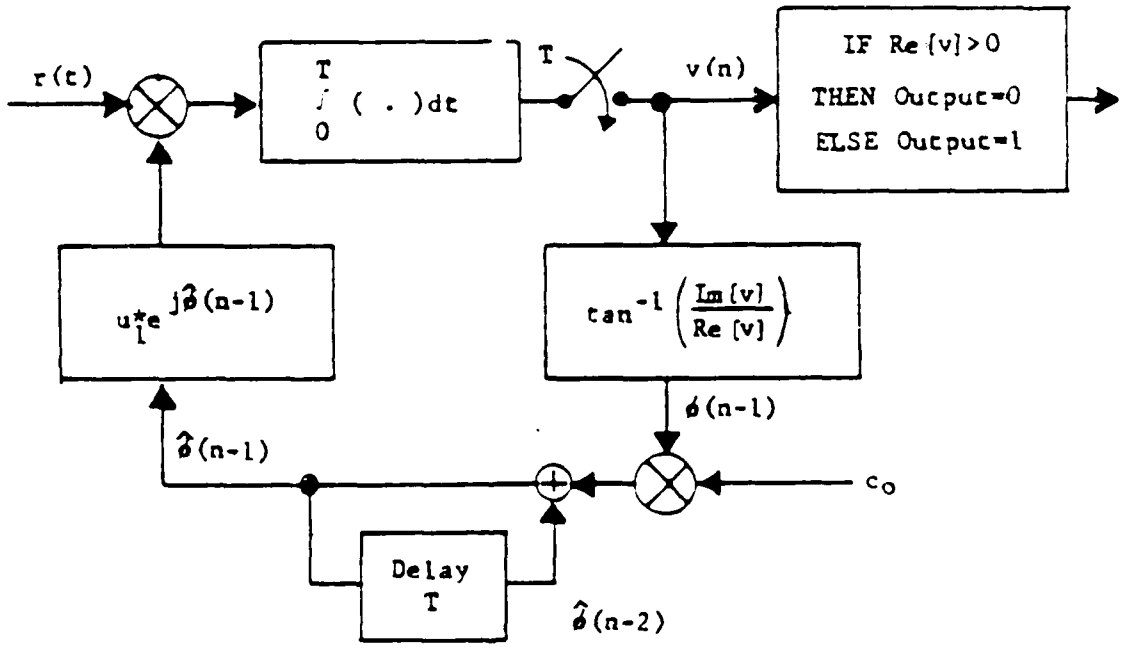


Figure 3
DEMODULATOR WITH DECISION DIRECTED PHASE ESTIMATION

present context, the modulo 2π arithmetic will assure that the output is always finite. (For example, see page 30, Lindsey [1973].) Furthermore, the distribution of the input random variables will produce a viable model. The output of the cross-correlation integrator is sampled every T seconds and the loop filter is a digital filter making a computation every T seconds. Again, $\hat{\phi}$ is the digital filter's estimate of the correct phase reference, which is dependent on past values of $\hat{\phi}$ and the current value of ϕ. Note that the phase reference estimate, $\hat{\phi}$, remains constant throughout the integration period and is based on the output of the digital filter up to the previous bit time.

DEVELOPMENT OF PHASE ESTIMATION MODEL

In this section, a stochastic model for a demodulator with a phase-locked loop will be developed. This model will be based on the phase reference estimate, $\hat{\phi}$. It will be shown that these phase estimates will completely determine the state of the system model shown in Figure 8.

Recall that the system model assumes additive white Gaussian noise. Nevertheless, the usual assumption of independence for the sequence of received signal vectors, v(n), cannot be made. This is due to the fact that the phase of the received signal vector, $\phi(n)$, is a function not only of the independent noise samples but is also a function of the phase estimate, $\hat{\phi}(n-1)$. As shown in Figure 8, these

phase estimates are a function of previous phase estimates and thus independence cannot be assumed. It will prove useful to introduce another sequence of random variables which are independent. Define $\psi(n)$ to be the phase of the cross-correlation demodulator output with a perfect phase reference when the equivalent low pass signal $r(t)$ is applied to the input. That is, if the phase reference is perfect, and the loop filter is open, the phase sequence at the output of the cross-correlator shall be equal to $\psi(n)$. The distribution of $\psi(n)$ shall be determined shortly. It is important to realize at this time that this is an independent sequence of random variables. There is no "memory" introduced by the white Gaussian noise itself. The memory is introduced by the feedback nature of the implementation of a practical receiver employing a phase-locked loop.

Now consider the closed loop implementation in Figure 8. In general, the phase reference $\hat{\phi}(n-1)$ is not perfect due to the noise. Ironically, if a perfect received signal waveform, i.e., $z(n)=0$, arrived by chance, the phase of the received signal vector, $\phi(n)$, would not, in general, be zero. As a result of the previous noisy phase estimates, the phase reference, $\hat{\phi}(n-1)$, may be incorrect. Hence, $\phi(n)$ does exhibit memory due to the recursive estimation of the phase reference $\hat{\phi}(n-1)$.

Now, a useful representation for the current received signal vector's phase, $\phi(n)$, is presented based on the previous discussion. Assume that $\phi(n)$ is composed of two terms as follows:

$$\phi(n) = \psi(n) - \hat{\phi}(n-1) \quad ; \text{mod } 2\pi \tag{4-2}$$

where $\psi(n)$ is the aforementioned sequence of independent random variables and $\hat{\phi}(n-1)$ is the previous phase reference estimate.

Now consider the digital implementation of the loop filter in terms of the sequence of random variables, $\psi(n)$. The loop filter is a first-order recursive filter which converges to the correct phase reference. The correct phase reference must be equal to the expected value of the sequence of the random input phases $\psi(n)$. An implementation of the first-order loop filter using the random sequence, $\psi(n)$, is expressed as

$$\hat{\phi}(n) = c_0 \psi(n) + c_1 \hat{\phi}(n-1) \tag{4-3}$$

The coefficient c_1 controls the roll-off frequency for the model of the low pass filter, as indicated earlier in the discussion of the generic first order digital filter. The coefficient c_0 is a normalization factor which assures that

each phase estimate satisfies $-\pi < \hat{\phi}(n) \leq \pi$. Since $\psi(n)$ also is defined in the region $(-\pi,\pi]$, the following relationship exists

$$c_0 + c_1 = 1 \tag{4-4}$$

Now, some very interesting limiting properties of the coefficients are considered. From digital filtering theory, it is known that as c_1 approaches 1, the roll-off frequency for the low pass filter approaches zero. Therefore, in this limiting case, i.e., $c_1=1$ and $c_0=0$, Equations 4-3 and 4-2, respectively, yield

$$\hat{\phi}(n) = \hat{\phi}(n-1) = \text{constant}$$

$$\phi(n) = \psi(n) - \text{constant}$$

That is, the phase reference remains constant. It follows that if the initial value chosen for this phase reference is correct then we have the case of perfect coherent demodulation for all time. Conversely, as c_1 approaches zero, the phase reference is determined only by the phase of the previous received signal vector. In this case, $c_1=0$ and $c_0=1$, and Equations 4-3 and 4-2, respectively, yield

$$\hat{\phi}(n) = \psi(n)$$

$$\phi(n) = \psi(n) - \psi(n-1)$$

It then follows that if the modulation is binary phase shift keying and c_1 equals zero, the results obtained here are equivalent to differential phase shift keying.

Equations 4-2, 4-3, and 4-4 represent the defining equations of the system model considered in this chapter. Despite the fact that the independent random phase sequence $\psi(n)$ is very useful for a theoretical analysis, it does not provide the most efficient implementation for the system model. A more efficient implementation was included in Figure 8. This implementation readily follows from the defining equations. Substituting equation 4-2 into 4-3:

$$\begin{aligned}
\hat{\phi}(n) &= c_0 [\phi(n) + \hat{\phi}(n-1)] + c_1 \hat{\phi}(n-1) \\
&= c_0 \phi(n) + (c_0 + c_1) \hat{\phi}(n-1) \\
&= c_0 \phi(n) + \hat{\phi}(n-1) \qquad (4\text{-}5)
\end{aligned}$$

The last relation results from Equation 4-4. Equation 4-5 is the implementation shown in Figure 8 which obviously is a simpler implementation but does not provide sufficient insight for the analysis in the next section.

EVALUATION OF CONDITIONAL PHASE DISTRIBUTIONS

In this section, the distribution of the phase estimates, $\phi(n)$, shall be determined. This distribution shall be conditioned on the previous phase estimate. It will be shown that this phase estimate is sufficient for completely characterizing the system model. The first step is to determine the probability density function of $\phi(n)$. Let X and Y correspond to the real and imaginary components, respectively, of the received signal vector v(n). As usual, assume the waveform corresponding to the received vector 2aE is transmitted. For additive white Gaussian noise, it is known that X and Y are distributed according to the joint Gaussian pdf:

$$p(x,y) = \frac{1}{2\pi V} e^{-[(x-2aE)^2 + y^2]/2V}$$

where $V = 2EN_0$. Next, use the change of variables:

$$R = (X^2 + Y^2)^{1/2}$$

$$\phi = \tan^{-1}(Y/X)$$

which yields

$$p(r,\phi) = \frac{r}{2\pi V} e^{-(r^2 + 4a^2E^2 - 4aEr\cos\phi)/2V}$$

Integration of $p(r,\phi)$ over the range of r yields $p(\phi)$,

$$p(\phi) = \int_0^{inf} p(r,\phi)\, dr$$

$$= \frac{1}{2\pi} e^{-q} \left(1 + (4\pi q)^{1/2} \cos\phi\, e^{q\cos^2\phi} \frac{1}{(2\pi)^{1/2}} \int_{-inf}^{(2q)^{1/2}\cos\phi} e^{-x^2/2}\, dx\right) \qquad (4\text{-}6)$$

where $q = a^2E/N_0$ (See Papoulis [1965], page 501 for a detailed derivation). Note that the probability density function of ϕ is in terms of q which is the signal to noise ratio at the receiver input.

Next, as motivation for the work in the remainder of this chapter, some observations will be made about the model up to this point. First note that the sequence defined by Equation 4-3 is Markovian-1, i.e.,

$$P\{\phi(n)|\phi(n-1),\phi(n-2),\ldots,\phi(0)\} = P\{\phi(n)|\phi(n-1)\} \qquad (4\text{-}7)$$

This fact makes this model perfectly suitable for representation in a state transition probability matrix. This powerful tool for dynamic modeling will be developed shortly.

Next, note that the dynamic, step-by-step, behavior of this state variable, $\phi(n)$, will completely determine the error modeling process. This observation is justified as follows: An error occurs, as in Example 3-2, whenever $\phi(n)$ is not in the interval $(-\pi/2, \pi/2]$. But by virtue of equation 4-5, the error decision can be placed in terms of the state variable, $\phi(n)$. No error occurs provided that

$$-\pi/2 < (\phi(n)-\phi(n-1)/c_0 \leq \pi/2$$

or,

$$-\pi c_0/2 < \phi(n)-\phi(n-1) \leq \pi c_0/2 \qquad (4\text{-}8)$$

As a result, the occurrence of an error is dependent on the amount of change in $\phi(n)$ from step to step. The dynamic and average properties of the error process can thus be completely determined from the conditional probability distribution for $\phi(n)$. The next task in this chapter, then, is to compute the conditional probability distributions for this state variable.

To compute the conditional pdf, $f(\phi(n)|\phi(n-1))$, it is noted from Equation 4-3 that, given $\phi(n-1)$, $\phi(n)$ is simply a linear function of $\flat(n)$. The pdf of $\flat(n)$ has already been computed and is given in Equation 4-6. Therefore, using standard techniques,

$$f(\phi(n)|\phi(n-1)) = \left. \frac{p(\Phi(n))}{\left| D_{\Phi}(c_o \Phi(n) + c_1 \phi(n-1)) \right|} \right|_{\Phi(n) = \frac{\phi(n) - c1\, \phi(n-1)}{c_0}}$$

$$= (1/c_0)\, p((\phi(n) - c_1 \phi(n-1))/c_0) \qquad (4\text{-}9)$$

where D_{Φ} is the first derivative with respect to Φ and $p(.)$ is the probability density function given in Equation 4-6.

To allow a computer implementation of the model, the state variable, $\phi(n)$, will now be discretized. The range of $\phi(n)$ is divided into N equal regions. It will be assumed that the discrete random variable takes values equal to the midpoints of these regions denoted by

$$\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_i, \ldots, \hat{\phi}_N$$

The probability associated with each of these N possibilities is given by integrating the continuous probability density functions about these midpoints as follows

$$P\{\phi(n)=\phi_i\} = \int_{\frac{\phi_i - \phi_{i-1}}{2}}^{\frac{\phi_{i+1} + \phi_i}{2}} p(\phi)\,d\phi$$

Similarly, use equation 4-9 to compute the discrete conditional transition probabilities:

$$P\{\phi(n)=\phi_j \mid \phi(n-1)=\phi_i\}$$

$$= c_0^{-1} \int_{\frac{\phi_j - \phi_{j-1}}{2}}^{\frac{\phi_{j+1} - \phi_j}{2}} p(c_0^{-1}\phi(n) - c_0^{-1} c_1 \phi_i)\, d\phi(n) \qquad (4\text{-}10)$$

where the pdf, p(.) is given in equation 4-6.

The stochastic error model for the phase-locked loop is now completely specified in terms of the discrete conditional transition probabilities of the state variable ϕ. These transition probabilities will now be used in the state transition matrix introduced in Chapter 3.

STATE TRANSITION MATRIX

In Chapter 3, it was shown how a sequence of Markov random variables of the discrete type could be conveniently represented in terms of a state probability vector and a state transition probability matrix. In that chapter, it was shown how a steady-state state probability vector, P[inf], could be obtained from the single-step state transition probability matrix, Π. An examination of Equation 3-7 shows that the components of this single-step transition matrix for the system considered here are completely determined by Equation 4-10. In anticipation of the work in this chapter, the same notation was used in Section 3.4. Thus, the model developed to this point fits exactly the framework of the matrix methods presented in in Section 3.4 and all results are applicable. Therefore, P[inf] is the steady-state state probability vector for the phase reference estimates.

It will now be shown how this steady-state state probability vector, P[inf], and the transition matrix, Π, can be used to calculate a variety of average error statistics. It has been shown that the occurrence of an error is related to the difference between the current phase estimate and the previous phase estimate. That is, the probability of a symbol error is given by:

$$P\{error(n)\} = P\{|\hat{\phi}(n) - \hat{\phi}(n-1)| > c_0 \pi/2\} \qquad (4\text{-}11)$$

This expression can be placed in terms of the N discrete phase possibilities positioned around 2π radians with equal spacing. These N phase possibilities represent different states in the Markov chain formulation.

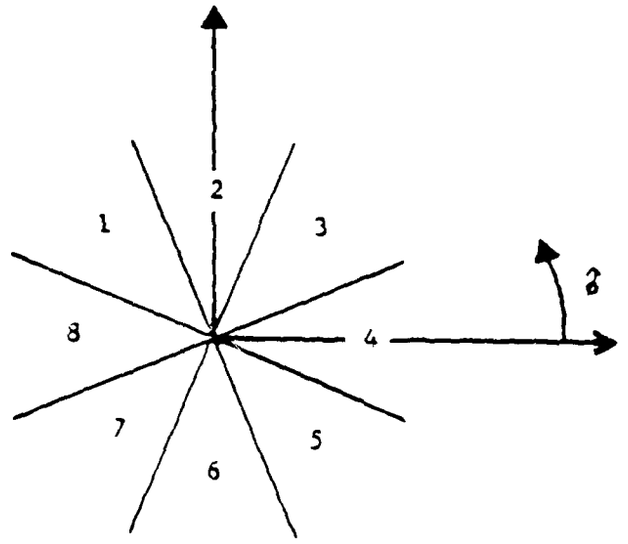
$$P\{error(n)\} = P\{|\hat{\phi}(n) - \hat{\phi}(n-1)| > c_0 N/4\} \qquad (4\text{-}12)$$

To simplify the presentation, it is assumed that c_0 is chosen such that $c_0 N/4$ is an integer. A multi-part example will be used to illustrate each part of the current deve.opment.

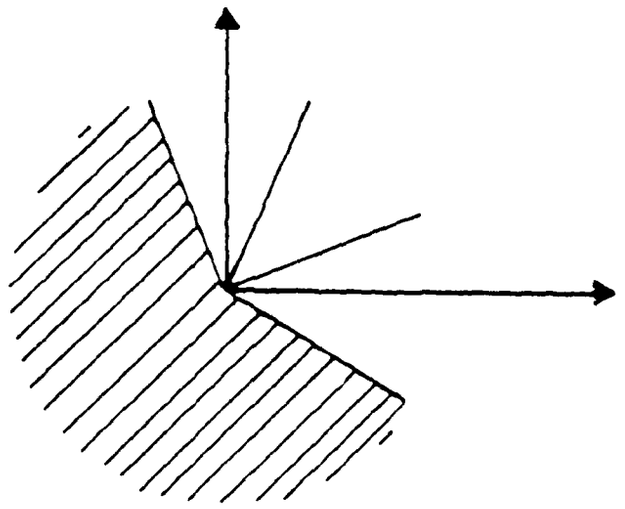
<u>Example</u> <u>4.1</u> Consider a binary phase-shift keyed demodulator as shown in Figure 8 and an additive white Gaussian noise channel. Let the loop coefficient, c_0, be equal to 0.5 and choose N to be equal to 8. A convenient set of discrete phase estimates and the associated boundaries for the transition probability computations in Equation 4-10 is shown in Figure 9a. Assume that the ph:se estimate at time n-1 is in state 3, i.e.,

$$\hat{\phi}(n-1) = \hat{\phi}_3 \qquad (4\text{-}13)$$

For this particular set of circumstances, the probability of



(a) STATE BOUNDARIES; N=8



(b) ERROR REGION FOR TIME n GIVEN $\hat{\phi}(n-1)=\hat{\phi}_3$

Figure 9
STATE REPRESENTATION FOR EXAMPLE 4.1

an error at time n is given by

$$P\{error(n)|\hat{\phi}(n-1)=\phi_3\} = P\{|\hat{\phi}(n) - \phi_3| > 1\}$$

$$= P\{\hat{\phi}(n)=\phi_1|\hat{\phi}(n-1)=\phi_3\} + \sum_{i=5}^{8} P\{\hat{\phi}(n)=\phi_i|\hat{\phi}(n-1)=\phi_3\} \qquad (4\text{-}14)$$

This error region is composed of five segments as shown in Figure 9b. The probability of falling in any one of these five segments is given by Equation 4-10.

<u>End of Example</u>

It is noted that the probability of a symbol error defined in Equation 4-12 is a conditional probability. This is a result of the fact that this probability is a function of the random variable, $\hat{\phi}(n-1)$. For a particular value of $\hat{\phi}(n-1)$, (In Example 4.1, this particular value was ϕ_3) the probability of an error is a number calculated using Equation 4-10. Express the probability of a symbol error given a particular previous phase estimate as follows:

$$P\{error(n)|\hat{\phi}(n-1)=\phi_i\} = P\{|\hat{\phi}(n) - \phi_i| > c_0 N/4\} \qquad (4\text{-}15)$$

Equation 4-15 states that the probability of a symbol error is equal to the probability of a transition to a state more than $c_0 N/4$ states away. Recall that the components of the previously computed single step transition matrix, $\Pi = \{\pi_{ij}\}$ provide the probability of state transitions. However, the

computation in Equation 4-15 uses the elements of this single step transition matrix that produce a transition of more than $c_0 N/4$ states, modulo N. For a convenient representation of these elements, define the set $S_j(i)$ as the set of integers j for each i such that

$$S_j(i) = \{j,\ j=1,\ldots,N;\ \text{such that } |i-j| > c_0 N/4;\ \text{modulo } N\} \tag{4-16}$$

Now Equation 4-15 can be expressed in terms of the components of the single step transition matrix

$$P\{\text{error}(n) \mid \phi(n-1)=\phi_i\} = \sum_{S_j(i)} \pi_{ij} \tag{4-17}$$

The average probability of a symbol error, P_e, at an arbitrary time n is given by

$$P_e = \sum_{i=1}^{N} P\{\text{error}(n) \mid \phi(n-1)=\phi_i\}\ P\{\phi(n-1)=\phi_i\} \tag{4-18}$$

Substituting the components of the steady state probability vector, $P[\inf] = \{p_i(\inf)\}$, for the distribution of the previous phase estimate $\phi(n-1)$, and the results in Equation 4-17, into Equation 4-18 yields

$$P_e = \sum_{i=1}^{N} p_i(\inf)\Big[\sum_{S_j(i)} \tau_{ij}\Big] \qquad (4\text{-}19)$$

This equation relates that the probability of error is equal to the sum of probabilities of transition from state i to a state more than $c_0 N/4$ units away, weighted by the probability of being in state i. In order to express this relationship in a matrix form, define the matrix, S, as follows:

$$S = \langle s_{ij} \rangle \text{ where } s_{ij} = \begin{cases} 1 & \text{for all } |i-j| > c_0 N/4 \\ 0 & \text{otherwise} \end{cases} \qquad (4\text{-}20)$$

Thus, for those values of i and j for which an error occurs, the corresponding element of the matrix is equal to one.

<u>Example 4.1 Continued</u>

The matrix S for this example is given by

$$S = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{vmatrix} ; \; N = 8, \; c_0 = .5, \; c_0 N/4 = 1 \qquad (4\text{-}21)$$

<u>End of Example</u>

In order to use the matrix, S, a relatively unorthodox product operation, referred to as congruent matrix

multiplication is used. This operation does not follow the rules of standard matrix algebra. It is simply a device used for sifting or masking some of the components of a matrix. It is a term by term product of two identical order matrices, A and B, defined by:

$$C = A*B \; ; \quad c_{ij} = a_{ij} b_{ij} \; ; \; i,j = 1,..,N \qquad (4\text{-}22)$$

Let J be the Nx1 column vector with all elements equal to 1. The average probability of error derived in Equation 4-19 can now be expressed in matrix form as

$$P_e = P[inf] \; (\Pi * S) \; J \qquad (4\text{-}23)$$

As alluded to earlier, the major focus of this dissertation is on error models with memory. This is certainly the case for the phase-locked loop model in this chapter. An excellent measure of the error dependence resulting from this memory is the average probability of a bit error given that an error occurred in the previous bit time. This probability will be calculated next. As in Equation 4-18, the probability of an error at an arbitrary time n given an error at the previous time n-1 is given by

$$P_{e|e} = P\{error(n) \mid error(n-1)\}$$

$$= \sum_{i=1}^{N} P\{error(n) \mid \hat{\phi}(n-1)=\hat{\phi}_i\} \; P\{\hat{\phi}(n-1)=\hat{\phi}_i \mid error(n-1)\} \tag{4-24}$$

The last expression is the result of applying the discrete version of the Chapman-Kolmogorov equation for a Markov-1 process. That is, the probability of error at time n is dependent only on the previous phase estimate. The first probability in Equation 4-24 is given by Equation 4-17. The second probability requires more development. It is observed that this is the probability that the phase estimate ends, rather than begins, at $\hat{\phi}_i$ when an error occurs. This is just the inverse of the last problem solved in which a phase state was given and the transition probability matrix was used to compute the probability of jumping to the next phase state. In the current problem, the probability that the system was in each state of the process at a time in the past is calculated given that its present state is known. This problem is the inference problem. Before Equation 4-24 can be evaluated, some theory concerning these phase transitions to the past must be developed. In this development, the following indices shall be used for each of the phase estimates $\hat{\phi}$ at the following corresponding times:

$$\ldots\ldots\ \hat{\phi}(n-2)=\hat{\phi}_k\ ,\ \hat{\phi}(n-1)=\hat{\phi}_i\ ,\ \hat{\phi}(n)=\hat{\phi}_j\ ,\ \ldots\ldots \qquad (4\text{-}25)$$

Let

$$\psi_{ik}(n-1,n-2)=P\{\hat{\phi}(n-2)=\hat{\phi}_k \mid \hat{\phi}(n-1)=\hat{\phi}_i\}$$

Then,

$$\psi_{ik}(n-1,n-2)=\frac{p_k(n-2)\ \pi_{ki}(n-2,n-1)}{p_i(n-1)} \qquad (4\text{-}26)$$

As usual, the argument is dropped for the single-step forward and backward transition probabilities. Also, assume that the process is in the steady state so that Equation 3-19 applies and the limiting state probabilities are substituted in Equation 4-26, yielding

$$\psi_{ik}=\frac{p_k(\inf)}{p_i(\inf)}\ \pi_{ki} \qquad (4\text{-}27)$$

Now return to Equation 4-24. Using Bayes' theorem on the second conditional discrete probability distribution:

$$P\{\hat{\phi}(n-1)=\hat{\phi}_i \mid \text{error}(n-1)\}=$$

$$\frac{P\{\text{error}(n-1) \mid \hat{\phi}(n-1)=\hat{\phi}_i\}\ P\{\hat{\phi}(n-1)=\hat{\phi}_i\}}{P\{\text{error}(n-1)\}} \qquad (4\text{-}28)$$

Considering each of the terms on the right hand side:

$$P\{\hat{\phi}(n-1) = \hat{\phi}_i\} = p_i(\text{inf}) \tag{4-29}$$

That is, this is the steady state probability distribution of the phase estimates. Also, in the steady state:

$$P\{\text{error}(n-1)\} = P_e = \sum_{k=1}^{N} p_k(\text{inf}) \sum_{S_i(k)} \pi_{ki} \tag{4-30}$$

Finally, the last term:

$$P\{\text{error}(n-1) \mid \hat{\phi}(n-1) = \hat{\phi}_i\} = \sum_{S_k(i)} \psi_{ik}(n-1, n-2) \tag{4-31}$$

That is, an error results if the phase jumped "to" the conditioned value "from" a phase estimate $c_0 N/4$ units away. This probability uses the inverse single step transition probability discussed earlier. Therefore, substituting first Equation 4-26, then Equation 4-27 into Equation 4-31, in the steady state:

$$P\{error(n-1) \mid \phi(n-1)=\phi_i\} = \sum_{S_k(i)} \frac{p_k(n-2)}{p_i(n-1)} \pi_{ki}(n-2,n-1)$$

$$= \sum_{S_k(i)} \frac{p_k(inf)}{p_i(inf)} \pi_{ki} \qquad (4\text{-}32)$$

Now, substituting Equations 4-17 and 4-32 into Equation 4-24 yields:

$$P_{e|e} = \frac{\sum_{i=1}^{N} \left[\sum_{S_j(i)} \pi_{ij} \right] \left[\sum_{S_k(i)} \frac{p_k(inf)}{p_i(inf)} \pi_{ki} \right] p_i(inf)}{\sum_{k=1}^{N} p_k(inf) \sum_{S_i(k)} \pi_{ki}}$$

$$= \frac{\sum_{i=1}^{N} \left[\sum_{S_j(i)} \pi_{ij} \right] \left[\sum_{S_k(i)} p_k(inf)\, \pi_{ki} \right]}{P_e} \qquad (4\text{-}33)$$

Thus, the average probability of an error given an error in the previous bit has been calculated using a transition probability matrix formulation. Other measures for dependent errors can be computed in a similar fashion.

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