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DECONVOLUTION METHODS FOR MULTI-DETECTORS

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DECONVOLUTION METHODS FOR MULTIPLE DETECTORS

INTRODUCTION

Deconvolution of a single convolution equation is usually an ill-posed problem. This has been sufficiently illustrated in the literature. The shortcomings of linear and of non-linear deconvolution methods can be found, for instance, in the very clear review paper [1].^c Advances in the theory of holomorphic functions of several complex variables led Berenstein, Taylor and Yger to realize that systems of convolution equations could be deconvolved exactly, thus avoiding the above ill-posedness. Their preliminary papers [2,3]^e eventually led to this project. The practical interest of this observation is that whenever such a set of convolution equations represents a set of physically realizable devices (e.g. transducers, sensors) then one has, by use of a digitally implemented inverse, essentially an arbitrary bandwidth device. Let us make these ideas exact by recalling the following definitions and deconvolution scheme.

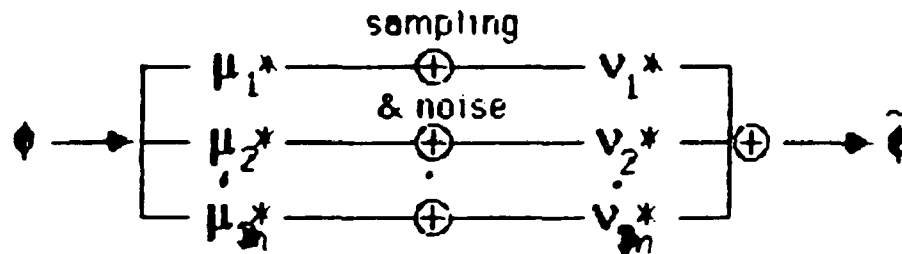
Let $\mu_1, \dots, \mu_m$ be a collection of distributions of compact support in $\mathbb{R}^n$ ("convolvers"). Their Fourier transforms $\hat{\mu}_j(z)$ are holomorphic functions of n complex variable in the Paley-Wiener class. We say that $\mu_1, \dots, \mu_m$ are strongly coprime if there are constants $\varepsilon > 0$, $C > 0$ such that for every $z \in \mathbb{C}^n$:

$$\sum_{j=1}^m |\hat{\mu}_j(z)| \geq \epsilon \exp(-C \{\log(1+|z|) + |\operatorname{Im} z|\}).$$

In that case, one can prove the existence of "deconvolvers" $v_1, \dots, v_m$. That is, distributions of compact support such that

$$\mu_1 * v_1 + \dots + \mu_m * v_m = \delta.$$

Once this v_j have been explicitly found, and herein lies the difficulty we set out to solve, one has the following scheme to determine an unknown signal ϕ



A very thorough discussion of the meaning of this scheme, and its implementation in the 1-d case, can be found in the final report to a previous ARO supported project [4].

In this project we set ourselves the following tasks:

- (i) find explicit and, relatively easy, formulas for the deconvolvers $v_1, \dots, v_m$ in the 1-d and 2-d situations.
- (ii) show that in the above deconvolution scheme the overall system, in the presence of noise, is entirely dependent on the sampling rate and noise characteristics, i.e. no "inherent" bandwidth

limitations. This means to show that the above block diagram behaves like

$$\diamond \longrightarrow \text{average over small detector} = \bar{\diamond},$$

- (iii) construct 1-d simulations of the deconvolution scheme to show its feasibility beyond purely theoretical considerations.

These three objectives have all been accomplished and documented as it will be explained now. Furthermore, the payoff of the powerful techniques developed for these questions has also taken place in two other areas: explicit solution of algebraic equations and their complexity, and the local Pompeiu problem, these will be discussed below.

First of all, we have two schemes to find deconvolvers. An analytic one is presented in Chapter 1 where everything depends on the following identity illustrated for the 2-d case with three convolvers:

$$\hat{u}(z) = \sum_{\zeta \in \mathbb{Z}} \frac{u(\zeta)}{J(\zeta) \hat{\mu}_3(\zeta)} \begin{vmatrix} g_1^1(z, \zeta) \cdots g_1^3(z, \zeta) \\ g_2^1(z, \zeta) \cdots g_2^3(z, \zeta) \\ \hat{\mu}_1(\zeta) \hat{\mu}_2(\zeta) \hat{\mu}_3(\zeta) \end{vmatrix} = \hat{v}_1(z) \cdot \hat{\mu}_1 \circ u(z) + \cdots + \hat{v}_3(z) \cdot \hat{\mu}_3 \circ u(z)$$

$$g_1^j = \frac{\hat{\mu}_j(z_1, \zeta_2) - \hat{\mu}_j(\zeta_1, \zeta_2)}{z_1 - \zeta_1}, \quad g_2^j = \frac{\hat{\mu}_j(z_1, z_2) - \hat{\mu}_j(z_1, \zeta_2)}{z_2 - \zeta_2}.$$

$J(\zeta)$ = Jacobian determinant of $\hat{\mu}_1, \hat{\mu}_2$ at ζ

Here u can be taken as a mollifier of small support and the estimate $\tilde{\phi}$ of the signal ϕ will then be

$$\tilde{\phi} = v_1 * \psi_1 + \dots + v_3 * \psi_3,$$

where $\psi_j = (\mu_j * \phi + \text{noise}) * u = \text{"smoothened" data}$. The resolution level will just correspond to the sampling rate. This chapter has appeared in print in [5].

In other words, we can say we construct analytically deconvolvers v_j , once for all, and then we implement a numerical integration scheme (choice of u and sampling).

There is a second method to find numerical deconvolvers directly, explained by Dr. Taylor in the recent West Point ARO Conference on Computational and Applied Mathematics. In this second method, for a fixed grid (resolution) we compute the deconvolvers numerically and the existence of the analytic deconvolvers guarantees that when the mesh of the grid decreases and the noise decrease the corresponding estimates $\tilde{\phi}$ converge to ϕ . We do not yet have sufficient numerical data to compare these two methods.

In Chapter 2, we present an evaluation of the effect of noise on deconvolution, background limited detectors, and amplifier limited detectors are considered, a criterion is given to evaluate and design different systems of convolvers. We prove that deconvolution of strongly coprime systems behave well with respect to noise. This is crucial for practicality and for the actual engineering design of such systems. This chapter constitutes the paper [6] which has

been submitted by invitation and accepted for publication in the forthcoming IEEE Proceedings in Multidimensional Signal Processing. It is an expansion of part of the Ph.D. thesis of E. Vincent Patrick written under the direction of Dr. Berenstein.

In some applications, we find ourselves with occluded near regions where data could be collected by sensors but no deconvolution in the above sense performed. This is the so called Local Pompeiu problem. Let us say a signal f is collected by averaging its values over any square of side a , but that this can be done only inside some disk of radius R . What is the precise relation between R and a (if there is any) such that the data collected determines f (this would allow to decide whether f is what we are looking for or not). Furthermore, can we reconstruct f from this data? Surprisingly, the answer to both questions is yes, at least when $R > \sqrt{a}$ for the first problem and $R > \frac{3}{2} \sqrt{a}$ for the second. This is the content of Chapters 3 and 4, respectively. It is not necessary to emphasize the importance of this work in the problem of Automated Target Recognition. Chapter 3 has been expanded and appeared in print as [8] and Chapter 4, [9], has been accepted for publication.

Let us conclude this introduction with the observation that the formulas from Chapter 1 have had an unexpected impact in problems about systems of algebraic equations, their complexity and properties of the solutions of the

algebraic Bezout equation. In this sense, this paper and its follow ups [10,11] have been cited in numerous papers on complexity theory and related algebraic problems. See [12] for an (already slightly outdated) introduction to this subject.

REFERENCES FOR THE INTRODUCTION

- [1]. A.R. Davies et al, "A comparison of statistical regularization and Fourier extrapolation methods for numerical deconvolutions", in Numerical Treatment of Inverse Problems in Differential and Integral Equations, P. Deuflhard and E. Hairer, eds., Boston, MA: Birkhauser, 1981, pp. 320-334.
- [2]. C.A. Berenstein et al., "Sur quelques formules explicites deconvolution", J.Optics 14, 1983, pp. 75-82, and in Technical Digest, Signal Recovery, 1983, Winter Meeting Optical Society of America, pp. WA4-1 to WA-4, 1983.
- [3]. C.A. Berenstein and A. Yger, "Le probleme de la deconvolution", J. Funct. Anal. 54, 1983, pp. 113-160.
- [4]. C.A. Berenstein et al., "Deconvolution methods for multi-sensors", DTIC #ADA 152351, 1984, 68 pages.
- [5]. C.A. Berenstein and A. Yger, "Analytic Bezout Identities", Advances in Applied Mathematics 10, pp. 51-74, 1989.
- [6]. C.A. Berenstein and E.V. Patrick, "Exact deconvolution for multiple convolution operators - an overview plus performance characterizations for imaging sensors", to appear in IEEE Proc. in Multidimensional Signal Processing.
- [7]. E.V. Patrick, "Some solutions to problems in depth vision from differential geometry and deconvolution methods," Ph.D. dissertation, Dept. Mathematics, University of Maryland, 1987.
- [8]. Carlos A. Berenstein and Roger Gay, "Le probleme de Pompeiu local," J. Analyse Mathematics, 52, pp. 133-166, 1989.

- [9]. Carlos A. Berenstein et al., "Inversion of the local Pompeiu transform" to appear in J. Analyse Mathematics.
- [10]. Carlos A. Berenstein and A. Yger, "Bounds for the degrees in polynomial equations," to appear in Michigan Mathematical J.
- [11]. Carlos A. Berenstein and A. Yger, "Effective Bounds in $\mathbb{Q}(z_1, \dots, z_n)$ ", preprint, 1988, 70 pages.
- [12]. Carlos A. Berenstein and D.C. Strappa, "Small Degree Solutions for the Polynomial Bezout Equations," Linear Algebra Applications 98, pp. 41-56, 1988.

CHAPTER 1

ANALYTIC BEZOUT IDENTITIES

by

Carlos A. Berenstein^{*} and Alain Yger^{*}

1. Introduction

In a number of papers [2,3,4,6,7,12] the problem of finding explicit solutions $h_1, \dots, h_m$ for the Bezout equation: $f_1 h_1 + \dots + f_m h_m = 1$ has been considered. If $f_1, \dots, f_m$ are complex polynomials in n variables and they have no common zeros in $\mathbb{C}^n$, the existence of explicit analytic expressions for the corresponding polynomials $h_1, \dots, h_m$ has a number of applications to control theory and commutative algebra. The most notable application being that of Brownawell [11] where the essentially best possible estimate for the degrees of the $h_1, \dots, h_m$ was obtained using such explicit analytic expressions. Up to date no purely algebraic proof of these bounds has been found.

Similarly, in the deconvolution problem, this time the functions $f_1, \dots, f_m$ being the Fourier transforms of a strongly coprime family $\mu_1, \dots, \mu_m$ of distributions of compact support, one searches for a procedure to compute explicitly distributions of compact support $\nu_1, \dots, \nu_m$ such that $\mu_1 * \nu_1 + \dots + \mu_m * \nu_m = \delta$. (Here $\hat{\nu}_j$ play the role of h_j in the equivalent formulation $\hat{\mu}_1 \hat{\nu}_1 + \dots + \hat{\mu}_m \hat{\nu}_m = 1$). This question arises in problems of robust filtering, image processing, etc. [10]. In [7] we wrote down formulas for a solution $\nu_1, \dots, \nu_m$ of the deconvolution problem in terms of interpolation series. The problem we have faced recently is that, while for the 1-d case these formulas can be easily implemented, in the higher dimensional case they are far too cumbersome, some of them seem to be beyond the range of symbolic languages like MACSYMA on which we had, perhaps too

optimistically, relied upon to carry through the computations involved. For that reason we present here a new version of our original deconvolution formulas which assumes extra conditions on the family $\mu_1, \dots, \mu_m$ but has as a payoff a very simple formula for the deconvolutors $\nu_1, \dots, \nu_m$. We give herein simple examples where these extra conditions are satisfied.

The problem of finding an efficient algorithm to compute the above mentioned solutions to the algebraic Bezout equation being still open, we also analyze here the particular case in which those polynomials can be computed in terms of interpolation formulas. Finding an algorithm with a low complexity for this problem will have many important applications in the theory of distributed parameter systems and in robotics.

We have also found that a language barrier prevented our work [7] to be more easily available to some engineers, and hope that the present paper will overcome those shortcomings.

The first author would like to express his gratitude to the Université de Bordeaux I for its hospitality while this work was carried on. His work on the algebraic aspects of this paper are inspired on the questions raised by the AFOSR-URI project at the University of Maryland. We will also like to thank our friends Dale Brownawell and B. Alan Taylor for many helpful remarks.

2. Analytic case

We will consider only entire functions f of n complex variables satisfying inequalities of the form

$$(1) \quad |f(z)| \leq A(1+|z|)^m e^{H(\operatorname{Im} z)}, \quad z \in \mathbb{C}^n,$$

$\operatorname{Im} z = (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n) \in \mathbb{R}^n$ where H is a convex continuous function in $\mathbb{R}^n$, homogeneous of degree 1 (i.e. $H(\lambda x) = \lambda H(x)$ when $\lambda > 0$). We call such a function H a *supporting function*. By the Paley-Wiener theorem [18] there is a distribution μ of compact support in $\mathbb{R}^n$ such that $f = \hat{\mu}$, the Fourier transform of μ . Furthermore, the supporting function H_0 of $\operatorname{cv} \operatorname{supp} \mu$ will satisfy $H_0 \leq H$ (Here cv denotes the convex hull).

Conversely, if $f = \hat{\mu}$ we can take $H = H_0$ in (1) and m is related to the order of μ in a simple manner. Hereafter we will just write $f \in \hat{\mathcal{E}}' (= \hat{\mathcal{E}}'(\mathbb{R}^n))$ if f satisfies (1).

For simplicity denote $\rho(z) := \log(2+|z|) + |\operatorname{Im} z|$. A family $f_1, \dots, f_m$ of functions in $\hat{\mathcal{E}}'$ is said to be *strongly coprime* if there is a constant c such that

$$(2) \quad \sum_{j=1}^m |f_j(z)|^2 \geq e^{-c\rho(z)}, \quad z \in \mathbb{C}^n.$$

It is well known [14] that (2) is a necessary and sufficient condition for the existence of functions $h_1, \dots, h_m \in \hat{\mathcal{E}}'$ such that

$$(3) \quad \sum_{j=1}^m f_j h_j = 1.$$

In other words, a strongly coprime family is precisely a family for which the analytic Bezout equation (3) has a solution. If we

consider (3) in terms of the distributions $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_m$ such that $\hat{\mu}_j = f_j$ and $\hat{\nu}_j = h_j$ then we have the identity

$$(4) \quad \mu_1 * \nu_1 + \dots + \mu_m * \nu_m = \delta,$$

i.e. $\nu_1, \dots, \nu_m$ solve the deconvolution problem stated in [7].

We will say sometimes that the family of distributions $\mu_1, \dots, \mu_m$ is strongly coprime.

It might be useful to explain why is (4) called a deconvolution problem. If we have an unknown signal (function or even distribution or random process) φ then the usual data one measures would be $\varphi_1, \dots, \varphi_m$ given by

$$(5) \quad \varphi_1 := \mu_1 * \varphi, \dots, \varphi_m := \mu_m * \varphi.$$

The way to recover φ is by deconvolution (which is still given here by convolution with distributions of compact support).

$$(6) \quad \varphi = \nu_1 * \varphi_1 + \dots + \nu_m * \varphi_m.$$

As we have mentioned in the Introduction our problem is to find easily computable functions g_j and corresponding distributions ν_j solving (3) and (4) respectively. We note that under the strongly coprime condition (2), or even under the weaker assumption that (2) is only satisfied for real values of z ($z \in \mathbb{R}^n$), there are readily available tempered distributions α_j solving the deconvolution problem, namely let

$$(7) \quad \hat{\alpha}_j(z) = \frac{\bar{f}_j(z)}{\sum_1^m |f_j(z)|^2}, \quad z \in \mathbb{R}^n.$$

The problem is that the a_j do not have compact support and furthermore, the a_j themselves are not so readily computable (except by inverting the Fourier transform). Nevertheless there are many situations where these a_j are still very useful, among other reasons because they minimize the noise amplification of the deconvolution process (6) (see [17] for an example of implementation in 2-d). On the other hand, in many applications it is often not necessary to obtain an exact solution to (4) but one is allowed to replace the Dirac δ in (4) by a (sufficiently) smooth function u with small support, i.e. an approximation to δ . It is this approximate deconvolution problem that is more readily solvable, even with very good knowledge on the support of the distributions ν_j , which will turn out to be (reasonably) smooth functions.

Since the method we use relies on Koppelman type formulas, like those developed in [1,9], we need the following explicit relation whose proof is an immediate verification.

Lemma 1. Let μ be a distribution of compact support in $\mathbb{R}^n$, $1 \leq k \leq n$ and $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$. The holomorphic function of $2n$ complex variables $g_k(z, \zeta)$ defined by

$$(8) \quad g_k(z, \zeta) := \frac{\hat{\mu}(z_1, \dots, z_k, \zeta_{k+1}, \dots, \zeta_n) - \hat{\mu}(z_1, \dots, z_{k-1}, \zeta_k, \dots, \zeta_n)}{z_k - \zeta_k}$$

is the Fourier transform (for ζ fixed) of the distribution denoted $I = I(\mu, \zeta, k)$, which evaluated at $\varphi \in C_0^\infty(\mathbb{R}^n)$ has the value

$$(9) \quad \langle I, \varphi \rangle := -1 \int \left[\int_0^{t_k} \varphi(t_1, \dots, t_{k-1}, u, 0, \dots, 0) e^{i\zeta_k(u-t_k)} du \right] \cdot e^{-i(t_{k+1}\zeta_{k+1} + \dots + t_n\zeta_n)} d\mu(t).$$

(By abuse of language we have written $\int \varphi(t) d\mu(t)$ to denote $\langle \mu, \varphi \rangle$).

We note that for the distribution I we have $\text{cv supp } I \subseteq \text{cv supp } \mu$. Furthermore the collection of functions $g_1, \dots, g_n$ satisfies

$$(10) \quad g_1(z, \zeta)(z_1 - \zeta_1) + \dots + g_n(z, \zeta)(z_n - \zeta_n) = \hat{\mu}(z) - \hat{\mu}(\zeta).$$

Associated to these holomorphic functions we have a $(1,0)$ differential form g in the variable ζ given by

$$(11) \quad g = g(z, \zeta, \mu) := \sum_{k=1}^n g_k(z, \zeta) d\zeta_k.$$

Given a family of m entire holomorphic functions $f_1, \dots, f_m$ its zero set Z is defined as

$$(12) \quad Z := \{z \in \mathbb{C}^n : f_1(z) = \dots = f_m(z) = 0\}.$$

In our applications we will only consider the case where the set Z is discrete. We say that Z is almost real if there is constant $A > 0$ such that

$$(13) \quad Z \subseteq \{z \in \mathbb{C}^n : |\text{Im } z| \leq A \log(2 + |z|)\}.$$

It is well-known that an almost real zero set Z is discrete [8], [15]. For a discrete set Z , $r > 0$, we can define a counting function $n(Z, r) := \#(Z \cap B_r)$, $B_r = \{z \in \mathbb{C}^n : |z| < r\}$ = Euclidean ball of center 0 and radius r . The distance function is $d(z, Z) :=$

$$\min(1, \min(|z - \zeta| : \zeta \in Z)).$$

Given a family of n distributions of compact support in $\mathbb{R}^n$, $\mu_1, \dots, \mu_n$, let us denote by H_1 the supporting function of $\text{cv} \bigcup_{j=1}^n \text{supp } \mu_j$, that is

$$(13) \quad H_1(\theta) := \max_{1 \leq j \leq n} \max(x \cdot \theta : x \in \text{supp } \mu_j), \quad (\theta \in \mathbb{R}^n),$$

$$x \cdot \theta = x_1 \theta_1 + \dots + x_n \theta_n.$$

Definition 1. A family of n distributions $\mu_1, \dots, \mu_n$ of compact support in $\mathbb{R}^n$ is well-behaved if there exists positive constants A, B, N, κ and a supporting function H_0 such that $0 \leq H_0 \leq H_1$, such that the zero set Z of the functions $f_1 = \hat{\mu}_1, \dots, f_n = \hat{\mu}_n$, is almost real,

$$(14) \quad n(Z, r) = O(r^A),$$

and, denoting

$$(15) \quad |f(z)| := \left[\sum_{j=1}^n |f_j(z)|^2 \right]^{1/2},$$

the following inequality holds:

$$(16) \quad |f(z)| \geq \frac{B d(z, Z)^\kappa e^{H_0(\text{Im } z)}}{(1+|z|)^N}.$$

Definition 2. A well-behaved family $\mu_1, \dots, \mu_n$ is very well-behaved if there are constants $c_1, M, c_1 > 0$, such that for every $\zeta \in Z$ we have

$$(17) \quad |J(\zeta)| := \left| \det \left[\frac{\partial f_j}{\partial z_1}(\zeta) \right]_{1,j} \right| \geq c_1 (1+|\zeta|)^{-M}.$$

This condition implies that the common zeros of $f_1, \dots, f_n$ are simple, that we can take $s = 1$ in (16) and that if $\zeta, \zeta' \in Z$, $\zeta \neq \zeta'$ then $|\zeta - \zeta'| \geq c_2(1 + |\zeta|)^{-M'}$ for some positive constants c_2, M' .

We will say also that functions $f_1, \dots, f_n$ are (very) well-behaved if the above properties hold.

Given a family $f_1, \dots, f_m$ in $\mathcal{E}'(\mathbb{R}^n)$, $m > n$, with no common zeros we introduce the following functions and differential forms. First, let $g^j = g^j(z, \zeta, \mu_j)$, $f_j = \hat{\mu}_j$, be the $(1,0)$ differential forms in ζ given by (11), we write $g^j = \sum_{k=1}^n g_k^j d\zeta_k$. Recall the coefficients g_k^j are holomorphic in both z and ζ . Let F be the vector valued holomorphic function $F := (f_1, \dots, f_m)$, we write

$$|F(\zeta)| := \left[\sum_{j=1}^m |f_j(\zeta)|^2 \right]^{1/2},$$

which is a nowhere vanishing C^∞ function of ζ . Let

$$(18) \quad \varphi = \varphi(z, \zeta) := \sum_{j=1}^m f_j(\zeta) f_j(z) / |F(\zeta)|^2,$$

$$(19) \quad Q = Q(z, \zeta) := \sum_{j=1}^m f_j(\zeta) g^j(z, \zeta) / |F(\zeta)|^2.$$

Therefore φ is a C^∞ function of (z, ζ) , $\varphi(\zeta, \zeta) = 1$ and, as a function of z , φ is a linear combination of the f_j . Q is a $(1,0)$ differential form in ζ , its coefficients are C^∞ in (z, ζ) and holomorphic in z . Finally, the $n+1$ functions Δ_j , C^∞ in (z, ζ) and holomorphic in z are defined by the identities

$$(20) \quad g^1 \wedge \dots \wedge g^j \wedge \dots \wedge g^n \wedge Q = \Delta_j d\zeta_1 \wedge \dots \wedge d\zeta_n, \quad 1 \leq j \leq n$$

$$(21) \quad g^1 \wedge \dots \wedge g^n = \Delta_{n+1} d\zeta_1 \wedge \dots \wedge d\zeta_n.$$

It is clear that the Δ_j are simply $n \times n$ determinants whose entries are obtained from the coefficients of $g^1, \dots, g^n, Q$.

Therefore, as functions of z , they are finite linear combinations of products of n among the functions g_k^j , $1 \leq k \leq m$. Note that these products are just Fourier transforms of convolutions of n distributions of the form $I(\mu_j, \zeta, k)$. (see (9)).

In order to obtain simple and easily computable deconvolution formulas we need to assume that a strongly coprime family of distributions $\mu_1, \dots, \mu_m$ contains a (very) well-behaved subfamily $\mu_1, \dots, \mu_m$. Furthermore, we need some control on the relation between the support of all the μ_j versus the supports of the first n . Let

$$(22) \quad H_2(\theta) = \max_{1 \leq j \leq n} \max(x \cdot \theta : x \in \text{supp } \mu_j), \quad (\theta \in \mathbb{R}^n).$$

One such relation between the supporting functions H_0, H_1, H_2 is given by

$$(23) \quad H_2 \leq 2H_1, \text{ and}$$

$$(24) \quad 2(n-1)H_1(\theta) + H_2(\theta) + H_2(\theta) < 2n H_0(\theta) \quad \text{if } \theta \neq 0.$$

The last condition is equivalent to

$$(25) \quad \exists r_0 > 0 \text{ such that } r_0 |\theta| \leq 2nH_0(\theta) - 2(n-1)H_1(\theta) - H_2(\theta).$$

With all this notation in place we are now ready to state the first deconvolution formula.

Theorem 1 Let $\mu_1, \dots, \mu_m$ be a strongly coprime family of distributions such that $\mu_1, \dots, \mu_n$ is a very well-behaved subfamily. Assume further that (23) and (25) hold. For any $u \in C_0^\omega(\mathbb{R}^n)$ with $\text{supp } u \subseteq \{x \in \mathbb{R}^n : |x| \leq r_0\}$ one can write

$$(26) \quad \hat{u}(z) = \sum_{\zeta \in \mathbb{Z}} \hat{u}(\zeta) \frac{\Delta_{n+1}(z, \zeta)}{J(\zeta)} \varphi(z, \zeta) + \sum_{j=1}^n (-1)^{N+1-j} f_j(z) \sum_{\zeta \in \mathbb{Z}} \frac{\Delta_j(z, \zeta)}{J(\zeta)} \hat{u}(\zeta).$$

Formula (26) can be rewritten as

$$\hat{u}(z) = \sum_{j=1}^m h_j(z) f_j(z),$$

where the h_j are given by explicit interpolation formulas and they are Fourier transforms of a series of distributions which are computable in terms of the original $\mu_1, \dots, \mu_m$. In the particular case where $m = n+1$ then formula (26) can be also written as

$$(27) \quad \hat{u}(z) = \sum_{\zeta \in \mathbb{Z}} \frac{\hat{u}(\zeta)}{J(\zeta) f_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z, \zeta) & \dots & g_1^{n+1}(z, \zeta) \\ g_n^1(z, \zeta) & \dots & g_n^{n+1}(z, \zeta) \\ f_1(z) & \dots & f_{n+1}(z) \end{vmatrix}.$$

Proof of Theorem 1. It follows the lines of Theorem 3 from our paper [7]. It uses the Koppelman type generalization of the Cauchy integral representation formula, specially in the version due to Berndtsson-Andersson [1]. One introduces first a parameter $\varepsilon > 0$, a function φ_ε and two (1,0) differential forms in ζ as follows:

$$(28) \quad \varphi_\varepsilon(z, \zeta) := \frac{\sum_{j=1}^n \bar{f}_j(\zeta) f_j(z) + \varepsilon}{|f(\zeta)|^2 + \varepsilon}$$

$$(29) \quad s(z, \zeta) := \sum_{j=1}^n (\zeta_j - z_j) d\zeta_j$$

$$(30) \quad Q_\varepsilon(z, \zeta) := \frac{\sum_{j=1}^n \bar{f}_j(\zeta) g_j(z, \zeta)}{|f(\zeta)|^2 + \varepsilon}.$$

where as before $f = (f_1, \dots, f_n)$, $|f(\zeta)|^2 = \sum_{j=1}^n |f_j(\zeta)|^2$. The

procedure from ([9], p.402 and p.409) gives two kernels $K_\varepsilon, P_\varepsilon$ (i.e. differential forms in the variable ζ of type $(n, n-1)$ and (n, n) respectively) such that if v is a holomorphic function in a neighborhood of $\bar{B}_R$, $z \in B_R$ then

$$(31) \quad v(z) = \frac{1}{(2\pi i)^n} \left\{ \int_{\partial B_R} v(\zeta) K_\varepsilon(z, \zeta) + \int_{B_R} v(\zeta) P_\varepsilon(z, \zeta) \right\}.$$

These two kernels are defined as follows. Let $G_1(t) = t^n$ and $G_2(t) = t$, we denote for any $a \in \mathbb{N}$

$$(32) \quad G_1^{(a)} = G_1^{(a)}(z, \zeta) := \frac{d^a}{dt^a} G_1 \Big|_{t=\varphi_\varepsilon(z, \zeta)}$$

$$(33) \quad G_2^{(a)} = G_2^{(a)}(z, \zeta) := \frac{d^a}{dt^a} G_2 \Big|_{t=\varphi(z, \zeta)}.$$

where φ is given by (18) and φ_ε by (28). With Q_ε defined by (30) and Q by (19), we define

$$(34) \quad K_{\epsilon}(z, \zeta) := \sum_{\alpha_0 + \alpha_1 + \alpha_2 = n-1} \frac{1}{\alpha_1! \alpha_2!} \frac{G_1^{(\alpha_1)} G_2^{(\alpha_2)} s \wedge (\bar{\partial} s)^{\alpha_0} \wedge (\bar{\partial} Q_{\epsilon})^{\alpha_1} \wedge (\bar{\partial} Q)^{\alpha_2}}{|z - \zeta|^{2(\alpha_0 + \alpha_1)}}.$$

$$(35) \quad P_{\epsilon}(z, \zeta) := \sum_{\alpha_1 + \alpha_2 = n} \frac{1}{\alpha_1! \alpha_2!} G_1^{(\alpha_1)} G_2^{(\alpha_2)} (\bar{\partial} Q_{\epsilon})^{\alpha_1} \wedge (\bar{\partial} Q)^{\alpha_2},$$

where $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{N}$. Everywhere the variable z is considered as a parameter and the $\bar{\partial}$ derivative is taken with respect to ζ . Due to our choice of function G_2 , the index α_2 can only take the values 0 and 1. For this reason the expression for P_{ϵ} becomes particularly simple

$$(36) \quad P_{\epsilon} = \varphi(\bar{\partial} Q_{\epsilon})^n + n \varphi_{\epsilon}(\bar{\partial} Q_{\epsilon})^{n-1} \wedge \bar{\partial} Q.$$

The terms $(\bar{\partial} Q_{\epsilon})^{n-1}$ and $(\bar{\partial} Q_{\epsilon})^n$ must be computed, for instance

$$(37) \quad (\bar{\partial} Q_{\epsilon})^n = \bar{\partial} Q_{\epsilon} \wedge \cdots \wedge \bar{\partial} Q_{\epsilon} \text{ (n times)} = (2i)^n n! \Delta_{n+1} \frac{\bar{J}(\zeta) \epsilon}{(|f(\zeta)|^{2+\epsilon})^{n+1}} d\lambda,$$

where $d\lambda = d\lambda(\zeta) = \text{Lebesgue measure in } \mathbb{C}^n$. (We have eliminated the variables (z, ζ) where they were evident, we will use this convention freely in the rest of the paper.)

It is clear that φ_{ϵ} and Q_{ϵ} are singular when $\epsilon = 0$ precisely at the points $\zeta \in Z$. The expression (37) shows what is the strength of this singularity in one of the terms of P_{ϵ} . The strategy of the proof is to try to get very singular terms so that when $\epsilon \rightarrow 0$ the volume integrals in (31) become sums, while the boundary integrals tend to zero when we set $v = \bar{u}$ and let $R \rightarrow \infty$ over a conveniently chosen sequence. The reason this idea works is the following lemma ([7], Corollary 4.1.1):

Lemma 2 Let σ_0 be the measure which is the sum of Dirac masses at the points of Z , i.e. for $\varphi \in C_0^\infty(\mathbb{C}^n)$ we have $\int \varphi d\sigma_0 = \sum_{\zeta \in Z} \varphi(\zeta)$.

Then, the family of measures σ_ϵ given by

$$(38) \quad d\sigma_\epsilon(\zeta) = \frac{\epsilon}{(|f(\zeta)|^2 + \epsilon)^{n+1}} d\lambda(\zeta)$$

converges, when $\epsilon \rightarrow 0$, to the measure

$$(39) \quad \frac{\pi^n}{n!} \frac{d\sigma_0}{|J(\zeta)|^2},$$

where, as always, J denotes the determinant Jacobian of $f_1, \dots, f_n$.

From (37) we see that the first term in (36) is amenable to Lemma 2. The second term is not singular enough, therefore it will be transformed using Stokes' formula in the corresponding integral of (31). Namely, due to type considerations, one obtains the first part of the following identity

$$\begin{aligned} & d_\zeta (v(\zeta) \varphi_\epsilon(z, \zeta) (\bar{\partial} Q_\epsilon(z, \zeta))^{n-1} \wedge Q(z, \zeta)) \\ &= \bar{\partial} (v \varphi_\epsilon (\bar{\partial} Q_\epsilon)^{n-1} \wedge Q) = v \varphi_\epsilon (\bar{\partial} Q_\epsilon)^{n-1} \wedge \bar{\partial} Q + v \bar{\partial} \varphi_\epsilon \wedge (\bar{\partial} Q_\epsilon)^{n-1} \wedge Q. \end{aligned}$$

The last identity follows from the fact that v is a holomorphic function in ζ and the $(2n-2)$ form $(\bar{\partial} Q_\epsilon)^{n-1}$ is $\bar{\partial}$ -closed.

Using this identity the representation formula (31) becomes

$$\begin{aligned} (40) \quad v(z) &= \frac{1}{(2\pi i)^n} \int_{\partial B_R} v(\zeta) (K_\epsilon + n \varphi_\epsilon (\bar{\partial} Q_\epsilon)^{n-1} \wedge Q) \\ &\quad + \frac{1}{(2\pi i)^n} \int_{B_R} v(\zeta) (\varphi (\bar{\partial} Q_\epsilon)^n - n \bar{\partial} \varphi_\epsilon \wedge (\bar{\partial} Q_\epsilon)^{n-1} \wedge Q), \end{aligned}$$

where the integration is in the variable ζ and we have suppressed the dependency on (z, ζ) of the kernels.

Lemma 3 The following identity holds

$$(41) \quad \bar{\partial} \varphi_\epsilon \wedge (\bar{\partial} Q_\epsilon)^{n-1} = (n-1)! (-1)^{\frac{(n+1)n}{2}} \frac{\epsilon}{(|f(\zeta)|^{2+\epsilon})^{n+1}} \cdot \left[\sum_{j=1}^n (-1)^j (f_j(z) - f_j(\zeta)) \bigwedge_{k \neq j}^n g^k \right] \wedge \bigwedge_{k=1}^n \bar{\partial} \bar{f}_j(\zeta),$$

where the wedge products in (41) are to be taken in their natural order, e.g. $\bigwedge_{k=1}^n g^k = g^1 \wedge \dots \wedge g^n$.

Proof of Lemma 3 We start by rewriting φ_ϵ

$$\begin{aligned} \varphi_\epsilon &= \frac{\sum_{j=1}^n f_j(z) \bar{f}_j(\zeta) + \epsilon}{|f(\zeta)|^{2+\epsilon}} = \frac{\sum_{j=1}^n \bar{f}_j(\zeta) (f_j(z) - f_j(\zeta)) + |f_j(\zeta)|^2 + \epsilon}{|f(\zeta)|^{2+\epsilon}} \\ &= 1 + \sum_{j=1}^n (f_j(z) - f_j(\zeta)) \frac{\bar{f}_j(\zeta)}{|f(\zeta)|^{2+\epsilon}}. \end{aligned}$$

Denote $f_j = f_j(\zeta)$ and $\varphi_j = \varphi_j(\zeta) := (|f(\zeta)|^{2+\epsilon})^{-1} \bar{f}_j$. Then we have

$$\varphi_\epsilon = 1 + \sum_{j=1}^n (f_j(z) - f_j) \varphi_j, \quad Q_\epsilon = \sum_{j=1}^n \varphi_j g^j.$$

Therefore,

$$\bar{\partial} \varphi_\epsilon \wedge (\bar{\partial} Q_\epsilon)^{n-1} = \left[\sum (f_j(z) - f_j) \bar{\partial} \varphi_j \right] \wedge \left[\sum \bar{\partial} \varphi_j \wedge g^j \right]^{n-1}$$

$$= \sum_{j=1}^n (f_j(z) - f_j) \left[\bar{\partial} \psi_j \wedge \left[\sum_{k \neq j} \bar{\partial} \psi_k \wedge g^k \right]^{n-1} \right],$$

since $\bar{\partial} \psi_j \wedge \bar{\partial} \psi_j = 0$. Using that the 2-forms $\bar{\partial} \psi_k \wedge g^k$ commute and that the product of two of them with the same index vanishes, we have

$$\begin{aligned} \left[\sum_{k \neq j} \bar{\partial} \psi_k \wedge g^k \right]^{n-1} &= (n-1)! \bigwedge_{k \neq j} (\bar{\partial} \psi_k \wedge g^k) \\ &= (n-1)! (-1)^{\frac{(n-1)n}{2}} \left(\bigwedge_{k \neq j} g^k \right) \wedge \left(\bigwedge_{k \neq j} \bar{\partial} \psi_k \right). \end{aligned}$$

Hence

$$\bar{\partial} \psi_j \wedge \left[\sum_{k \neq j} \bar{\partial} \psi_k \wedge g^k \right]^{n-1} = (n-1)! (-1)^{\frac{n(n+1)}{2}} \left(\bigwedge_{k \neq j} g^k \right) \wedge \left(\bigwedge_{k=1}^n \bar{\partial} \psi_k \right).$$

Now, we have $\bar{\partial} \psi_k = (|f|^2 + \epsilon)^{-1} \bar{\partial} \bar{f}_k - (|f|^2 + \epsilon)^{-2} \bar{f}_k \bar{\partial} |f|^2$. Therefore we can use that $\bar{\partial} |f|^2 \wedge \bar{\partial} |f|^2 = 0$ and obtain

$$\bigwedge_{k=1}^n \bar{\partial} \psi_k = \frac{\bigwedge_{k=1}^n \bar{\partial} \bar{f}_k}{(|f|^2 + \epsilon)^n} - \frac{\sum_{j=1}^n \left[f_j \bigwedge_{1 \leq k < j}^n \bar{\partial} \bar{f}_k \wedge \bar{\partial} |f|^2 \wedge \bigwedge_{j < k \leq n}^n \bar{\partial} \bar{f}_k \right]}{(|f|^2 + \epsilon)^{n+1}}.$$

If we now expand $\bar{\partial} |f|^2 = \sum_k f_k \bar{\partial} \bar{f}_k$, we see that only the term

$f_j \bar{\partial} \bar{f}_j$ remains in the triple product above. Hence

$$\bigwedge_{k=1}^n \bar{\partial} \psi_k = \frac{\bigwedge_{k=1}^n \bar{\partial} \bar{f}_k}{(|f|^2 + \epsilon)^n} - \frac{\left[\sum_{j=1}^n |f_j|^2 \right] \bigwedge_{k=1}^n \bar{\partial} \bar{f}_k}{(|f|^2 + \epsilon)^{n+1}} = \frac{\epsilon}{(|f|^2 + \epsilon)^{n+1}} \bigwedge_{k=1}^n \bar{\partial} \bar{f}_k.$$

This concludes the proof of Lemma 3. □

Lemma 3 tells us that $\bar{\partial}\varphi_\epsilon \wedge (\bar{\partial}Q_\epsilon)^{n-1} \wedge Q$ is the product of a measure with a smooth density, independent of ϵ , and the function $\epsilon(|f(\zeta)|^2 + \epsilon)^{-n-1}$. Lemma 2 can now be invoked to see that the volume integral in (40) reduces to a sum when $\epsilon \rightarrow 0$. In fact, let us choose R so that $|f(\zeta)|^2 = |f(\zeta)|^2 + \dots + |f(\zeta)|^2 = 0$ when $|\zeta| = R$. This choice is always possible since Z is a discrete set by assumption. In this case none of $\varphi_\epsilon, K_\epsilon, Q_\epsilon$ have singularities when $\epsilon = 0$ and $\zeta \in \partial B_R$. We set φ_0, K_0, Q_0 to be the correspondent quantities. Therefore

$$\begin{aligned} v(z) &= \frac{1}{(2\pi i)^n} \int_{\partial B_R} v(\zeta) (K_0 + n\varphi_0((\bar{\partial}Q_0)^{n-1} \wedge Q) \\ &\quad + \frac{1}{(2\pi i)^n} \lim_{\epsilon \rightarrow 0+} \int_{B_R} v(\zeta) \left[\varphi(\bar{\partial}Q_\epsilon)^n - n \bar{\partial}\varphi_\epsilon \wedge (\bar{\partial}Q_\epsilon)^{n-1} \wedge Q \right] \end{aligned}$$

Recall that $f_j(\zeta) = 0$ if $\zeta \in Z$ and $1 \leq j \leq n$. Using Lemma 2 and the definition (20) and (21) of the Δ_j we can compute explicitly the limit and obtain

$$\begin{aligned} (42) \quad v(z) &= \frac{1}{(2\pi i)^n} \int_{\partial B_R} v(\zeta) (K_0 + n\varphi_0((\bar{\partial}Q_0)^{n-1} \wedge Q) \\ &\quad + \sum_{\zeta \in Z \cap B_R} v(\zeta) \frac{\Delta_{n+1}(z, \zeta) \varphi(z, \zeta)}{J(\zeta)} + \sum_{j=1}^n (-1)^{n+1-j} f_j(z) \sum_{\zeta \in Z \cap B_R} v(\zeta) \frac{\Delta_j(z, \zeta)}{J(\zeta)}. \end{aligned}$$

Up to this moment we have only used that $Z \cap \partial B_R = \emptyset$ and that v is holomorphic in a neighborhood of $\bar{B}_R$. To let $R \rightarrow \infty$ we have to choose a sequence $R \rightarrow \infty$ judiciously. Recall that $u(Z, r) = \#(Z \cap B_r) \leq Cr^A$ for some positive constants A, C and all $r \geq 1$. Let M be the smallest integer $\geq C(R+1)^A + 1$, divide the

shell $\bar{B}_{R+1} \setminus B_R$ into M concentric subshells by choosing the boundaries to be $\partial B_{(R+j/M)}$, $0 \leq j \leq M$. There is at least one such subshell that is free from points of Z , choose R' to be the mid-radius of this subshell, then $d(\zeta, Z) \geq (2M)^{-1}$ if $|\zeta| = R'$. Starting from the sequence $R = q = 1, 2, \dots$ we construct a sequence R_q , $q < R_q < q + 1$, such that for some positive constants A_1, N_1

$$(43) \quad |f(\zeta)| \geq A_1 q^{-N_1} e^{H_0(\operatorname{Im} \zeta)} \quad \text{if } |\zeta| = R_q.$$

This follows from (16) and the choice of R_q .

We are now ready to estimate the terms in the boundary integral of (42) for $R=R_q$. We will assume $|z| \leq C_0 < \infty$ and consider those q such that $R_q \geq C_0 + 1$.

First, let us observe that the functions g_k^j , $1 \leq j \leq m$, $1 \leq k \leq n$, satisfy an estimate of the form

$$(44) \quad |g_k^j(z, \zeta)| \leq C_1 (1+|z|)^{M_1} (1+|\zeta|)^{M_1} e^{H_2(\operatorname{Im} z) + H_2(\operatorname{Im} \zeta)},$$

for some constants $M_1, C_1 > 0$. If $1 \leq j \leq n$ we can replace H_2 by H_1 . We can now estimate the coefficients of differential from Q . Denote $\|Q(z, \zeta)\|$ the largest absolute value of the coefficients of $d\zeta_k$ at the point (z, ζ) . We proceed as follows. First,

$$(45) \quad |F(\zeta)| \geq |f(\zeta)| \geq A_1 q^{-M_1} e^{H_0(\operatorname{Im} \zeta)} \quad \text{if } |\zeta| = R_q.$$

Therefore

$$\|Q(z, \zeta)\| \leq \frac{1}{|F(\zeta)|} \left(\sum_{j=1}^n \|q^j(z, \zeta)\|^2 \right)^{1/2}$$

$$\leq \frac{C_1 (1+|z|)^{M_1} e^{H_2(\operatorname{Im} z)} (1+R_q)^{M_1} e^{H_2(\operatorname{Im} \zeta)}}{A_1 q^{-N_1} e^{H_0(\operatorname{Im} \zeta)}}$$

which leads to

$$(46) \quad \|Q(z, \zeta)\| \leq C_3 q^{N_2} e^{H_2(\operatorname{Im} \zeta) - H_0(\operatorname{Im} \zeta)}.$$

The constant C_3 depends in fact on z , but $|z| \leq C_0$ and $N_2 = M_1 + N_1$ (In fact C_3 can be estimated in terms of $e^{|\operatorname{Im} z|}$ and polynomials in $|z|$). Similarly, with possibly different values for the constants C_3, N_2 appearing below we have

$$(47) \quad \|\bar{\partial}Q(z, \zeta)\| \leq C_3 q^{N_2} e^{H_2(\operatorname{Im} \zeta) - 2H_0(\operatorname{Im} \zeta)}$$

$$(48) \quad \|\bar{\partial}Q_0(z, \zeta)\| \leq C_3 q^{N_2} e^{2H_1(\operatorname{Im} \zeta) - 2H_0(\operatorname{Im} \zeta)}$$

$$(49) \quad |\varphi_0(z, \zeta)| \leq C_3 q^{N_1} e^{-H_0(\operatorname{Im} \zeta)}$$

$$(50) \quad |\varphi(z, \zeta)| \leq \frac{|F(z)|}{|F(\zeta)|} \leq C_3 q^{N_1} e^{-H_0(\operatorname{Im} \zeta)}.$$

To estimate K_0 we recall that α_2 can only take the values 0 and 1 in (34). In case $\alpha_2 = 0$, we have to estimate terms of the form $\varphi_0^{n-\alpha_1} (\bar{\partial}Q_0)^{\alpha_1}$, with $0 \leq \alpha_1 \leq n-1$. There are powers of q that we will disregard, the estimate is then the following functions of $\operatorname{Im} \zeta$

$$\|\varphi_0^{n-\alpha_1} (\bar{\partial} Q_0)^{\alpha_1}\| \ll e^{-(n+\alpha_1+1)H + 2\alpha_1 H}.$$

Since $H_1 \geq H_0$, the worst case estimate occurs when $\alpha_1 = n - 1$. Hence the terms corresponding to $\alpha_2 = 0$, $\alpha_0 + \alpha_1 = n-1$, in the definition of K_0 can all be estimated by

$$(51) \quad C_3 q^{\frac{N_3}{2}} e^{-2nH_0(\operatorname{Im} \zeta) + 2(n-1)H_1(\operatorname{Im} \zeta)}, \quad |\zeta| = R_q.$$

The terms with $\alpha_1 = 1$, $\alpha_0 + \alpha_1 = n - 2$ correspond to the estimate of $\|\varphi_0^{n-\alpha_1} (\bar{\partial} Q_0)^{\alpha_1} \wedge \bar{\partial} Q\|$. The worst case occurs this time when $\alpha_1 = n - 2$ and we obtain an estimate of the form

$$(52) \quad C_3 q^{\frac{N_3}{2}} e^{-2nH_0(\operatorname{Im} \zeta) + 2(n-2)H_1(\operatorname{Im} \zeta) + 2H_2(\operatorname{Im} \zeta)}, \quad |\zeta| = R_q.$$

In (42) we have one more term to estimate for $|\zeta| = R_q$,

$$(53) \quad \|\varphi_0 (\bar{\partial} Q_0)^{n-1} \wedge Q\| \leq C_3 q^{\frac{N_3}{2}} e^{-2nH_0(\operatorname{Im} \zeta) + 2(n-1)H_1(\operatorname{Im} \zeta) + H_2(\operatorname{Im} \zeta)}.$$

The conditions (23) and (25) imply that the largest exponential factor in (51), (52) and (53) is the one in (53) and it satisfies

$$(54) \quad -2nH_0(\operatorname{Im} \zeta) + 2(n-1)H_1(\operatorname{Im} \zeta) + H_2(\operatorname{Im} \zeta) \leq -r_0 |\operatorname{Im} \zeta|$$

since we have assumed that $u \in C_0^\infty(\bar{B}_{r_0})$ we have

$$(55) \quad |\hat{u}(\zeta)| \leq C_4 (1+|\zeta|)^{-N_3-2n} e^{r_0 |\operatorname{Im} \zeta|}, \quad \zeta \in \mathbb{C}^n,$$

which allows us to conclude that, if $v = \hat{u}$,

$$(56) \quad \lim_{q \rightarrow \infty} \int_{\partial B_{R_q}} \hat{u}(\zeta) \left[K_0(z, \zeta) + n\varphi_0(z, \zeta) (\bar{\partial} Q_0(z, \zeta))^{n-1} \wedge Q(z, \zeta) \right] = 0.$$

To conclude the proof of Theorem 1 we need to show that the series appearing in the representation formula (26) converge absolutely and uniformly in compact subsets of $\mathbb{C}^n$ to functions in $\hat{\mathcal{E}}'$. Since we have assumed that Z is almost real, the estimates of all the terms $\Delta_{j,\varphi,J}^{-1}$ are in terms of powers of $|\zeta|$. Using that $n(Z,r) = O(r^A)$ and $|\hat{u}(\zeta)|$ decreases as fast as $|\zeta|^{-N_4}$ for $\zeta \in Z$ we have the desired convergence once N_4 is chosen sufficiently large. The support of all the distributions thus obtained is contained in the convex set K whose supporting function is $(n+1)H_2$. $\square$

Remarks 1. One can see that the condition $2H_1 \geq H_2$ cannot be relaxed if the other conditions of theorem remain in the same, otherwise the exponent in (52) would become positive and we would not be able to prove (56).

2. A way to weaken the conditions in Theorem 1 is to impose some better lower bound on $|F|$ than (45) that depends only on the first n functions. We will do so in Theorem 2 below

3. It is clear that one needs only $u \in C_{O_{r_0}}^N(\bar{B}_{r_0})$ for N sufficiently large to obtain (26).

The following example shows how Theorem 1 simplifies enormously the computation of the deconvolution formula proposed in [7].

Let μ_1, μ_2, μ_3 be the characteristic functions of the squares centered at 0, of sides parallel to the axes and of length $2\sqrt{3}$, $2\sqrt{2}$, 2 respectively. One can easily show [7], [17] that

" μ_1, μ_2 " is a very well-behaved family with

$$H_0(\operatorname{Im} \zeta) = \sqrt{2}(|\operatorname{Im} \zeta_1| + |\operatorname{Im} \zeta_2|)$$

Here $H_1(\operatorname{Im} \zeta) = H_2(\operatorname{Im} \zeta) = \sqrt{3}(|\operatorname{Im} \zeta_1| + |\operatorname{Im} \zeta_2|)$. In this case the main hypotheses (24) reduces to verify that $4\sqrt{2} - 3\sqrt{3} > 0$!.

Since $|x_1| + |x_2| = \sqrt{x_1^2 + x_2^2}$ for $x \in \mathbb{R}^2$, we have

$r_0 = 4\sqrt{2} - 3\sqrt{3} \geq 0.2$. The variety Z in this case is given by

$$Z = \left\{ \left(\frac{j\pi}{\sqrt{3}}, \frac{k\pi}{\sqrt{2}} \right) : j, k \in \mathbb{Z}^* \right\} \cup \left\{ \left(\frac{j\pi}{\sqrt{2}}, \frac{k\pi}{\sqrt{3}} \right) : j, k \in \mathbb{Z}^* \right\}.$$

(There were about forty different types of terms in [7] compute.).

Before we proceed to state Theorem 2 we need to point out that the representation formula (31) does not depend on the particular choice of the differential forms g^j we have chosen, rather on the fact that (10) is satisfied. That is,

$$\sum_{k=1}^n g_k^j(z, \zeta)(z_k - \zeta_k) = f_j(z) - f_j(\zeta).$$

Now, let $f = \mu$ and $h(z) = \sin Bz_1$ for some $B > 0$, and denote by g the differential form associated to f by (8) and (11). Let us define a differential form γ by

$$(57) \quad \gamma(z, \zeta) := f(\zeta) \frac{\sin Bz_1 - \sin B\zeta_1}{z_1 - \zeta_1} d\zeta_1 + h(z)g(z, \zeta),$$

writing $\gamma = \sum \gamma_k d\zeta_k$ we have

$$\gamma_1(z, \zeta)(z_1 - \zeta_1) + \dots + \gamma_n(z, \zeta)(z_n - \zeta_n) = f(z)h(z) - f(\zeta)h(\zeta)$$

therefore we can associate γ to the product $f \cdot h$. It is also

clear that as a function of z the γ_k are Fourier transforms of distribution of compact support easily computable terms of μ and B . Obviously we can replace $\sin Bz_1$ by $\sin Bz_j$ without any problems, hence, given a family $f_1, \dots, f_m$ we can construct an augmented family $f_1, \dots, f_m, f_{m+1} := f_1 \cdot \sin Bz_1, \dots, f_{2m} := f_m \cdot \sin Bz_1, \dots, f_{(n+1)m} := f_m \cdot \sin Bz_n$. The corresponding g^j for $j \geq m+1$ are computed following the procedure (57). It is clear that if $f_1, \dots, f_m$ was strongly coprime, the augmented family remains strongly coprime. If $f_1, \dots, f_n$ form a very well-behaved family we will keep the notation H_0, H_1, H_2 to indicate the support functions corresponding to the m original members of the augmented family $f_1, \dots, f_{(n+1)m}$.

Theorem 2 Let $f_1, \dots, f_m$ be a strongly coprime family such that the subfamily $f_1, \dots, f_n$ is very well-behaved. There are constants $B_0 \geq 0$, $r_0 > 0$ such that for any $B \geq B_0$, and any $u \in C_0^\infty(\bar{B}_{r_0})$ the representation formula (26) is valid for the augmented family $f_1, \dots, f_m, \dots, f_{(n+1)m}$ defined above if either of the following two conditions holds:

$$(58) \quad H_2 \leq 2H_1 \quad \text{and} \quad 2(n-1)H_1 < (2n-1)H_0$$

$$(59) \quad 2H_1 < H_2 \quad \text{and} \quad 2(n-2)H_1 + H_2 < (2n-1)H_0.$$

Proof The proof is exactly the same as that of Theorem 1 except for improvements on the estimates (46), (47) and (50) for the new Q and φ . Recall that it is there where all the functions $f_1, \dots, f_{(n+1)m}$ appear. Let $F_1 := (f_1, \dots, f_{(n+1)m})$ and keep the

notation $F = (f_1, \dots, f_m)$ as before. We have

$$(60) |F_1(\zeta)| = |F(\zeta)| (1 + |\sin B\zeta_1|^2 + \dots + |\sin B\zeta_n|^2)^{1/2} = |F(\zeta)| \vartheta(\zeta)$$

It is clear that for some positive constants c_n, c'_n we have

$$\begin{aligned} \vartheta(\zeta) &= (1 + |\sin B\zeta_1|^2 + \dots + |\sin B\zeta_n|^2)^{1/2} \\ &\geq c'_n \left(e^{2B|\operatorname{Im} \zeta_1|} + \dots + e^{2B|\operatorname{Im} \zeta_n|} \right)^{1/2} \\ &\geq c_n e^{\frac{2B|\operatorname{Im} \zeta|_1}{n}} \geq c_n e^{\frac{2B|\operatorname{Im} \zeta|}{n}}, \end{aligned}$$

$$\text{where } |\operatorname{Im} \zeta|_1 = \sum_{j=1}^n |\operatorname{Im} \zeta_j|, \quad |\operatorname{Im} \zeta| = \left(\sum_{j=1}^n |\operatorname{Im} \zeta_j|^2 \right)^{1/2}.$$

We estimate first g^j , $j \geq m+1$, for $|z-\zeta| \geq 1$ since that is the only case that appears in the proof of (56). As it follows from (57) we have, for some i, k ($1 \leq i \leq n$, $1 \leq k \leq m$), the estimate

$$\begin{aligned} \|g^j(z, \zeta)\| &\leq C e^{B|\operatorname{Im} z_1|} \left[|f_k(\zeta)| (1 + |\sin B\zeta_1|^2)^{1/2} + \|g^k(z, \zeta)\| \right] \\ &\leq C (1 + |z|)^{N_1} (1 + |\zeta|)^{N_1} e^{B|\operatorname{Im} z_1|} \left[|f(\zeta)| \vartheta(\zeta) + e^{\frac{H_2(\operatorname{Im} z)}{2}} e^{\frac{H_2(\operatorname{Im} \zeta)}{2}} \right]. \end{aligned}$$

It follows that

$$\begin{aligned} \|Q(z, \zeta)\| &\leq \frac{1}{|F_1(\zeta)|} \left[\sum_{j=1}^{(n+1)m} \|g^j(z, \zeta)\|^2 \right]^{1/2} \\ &\leq C_1 (1 + |z|)^{N_1} (1 + |\zeta|)^{N_1} e^{B|\operatorname{Im} z| + H_2(\operatorname{Im} z)} \\ &\quad \cdot \left\{ \frac{e^{H_2(\operatorname{Im} \zeta)}}{|F_1(\zeta)|} + \frac{|F(\zeta)| \vartheta(\zeta)}{|F_1(\zeta)|} \right\}. \end{aligned}$$

Therefore, for a positive constant C_2 depending on z , we obtain

$$(61) \quad \|Q(z, \zeta)\| \leq C_2(1+|\zeta|)^{N_1} \left[1 + e^{H_2(\operatorname{Im} \zeta) - H_0(\operatorname{Im} \zeta) - \frac{B}{n} |\operatorname{Im} \zeta|_1} \right].$$

Similary,

$$(62) \quad |\varphi(z, \zeta)| \leq \frac{|F_1(z)|}{|F_1(\zeta)|} \leq C_2(1+|\zeta|)^{N_1} e^{H_0(\operatorname{Im} \zeta) - \frac{B}{n} |\operatorname{Im} \zeta|_1}.$$

Finally,

$$\|\bar{\partial}Q(z, \zeta)\| \leq C \max_{1 \leq j, k \leq (n+1)m} \frac{\|\partial f_j(\zeta)\| g^k(z, \zeta)}{|F_1(\zeta)|^2}.$$

Now, for $j \geq n+1$ we have for some $1 \leq i \leq n$, $1 \leq \ell \leq m$

$$\begin{aligned} \|\partial f_j(\zeta)\| &= \|\sin B\zeta_1 \cdot \partial f_\ell(\zeta) + f_\ell(\zeta) B \cdot \cos B\zeta_1 d\zeta_1\| \\ &\leq C_1(1+|\zeta|)^{N_2} (\partial(\zeta) e^{H_2(\operatorname{Im} \zeta)} + |F(\zeta)| \partial(\zeta)). \end{aligned}$$

It follows that

$$\begin{aligned} \|\bar{\partial}Q(z, \zeta)\| &\leq C_2 \frac{(1+|\zeta|)^{N_2}}{|F_1(\zeta)|^2} \left[|F(\zeta)|^2 \partial(\zeta)^2 + 2|F(\zeta)| \partial(\zeta)^2 e^{H_2(\operatorname{Im} \zeta)} + \partial(\zeta) e^{2H_2(\operatorname{Im} \zeta)} \right] \\ &\leq C_2(1+|\zeta|)^{N_2} \left[1 + 2e^{H_2(\operatorname{Im} \zeta) - H_0(\operatorname{Im} \zeta)} + e^{2H_2(\operatorname{Im} \zeta) - 2H_0(\operatorname{Im} \zeta) - \frac{B}{n} |\operatorname{Im} \zeta|_1} \right] \end{aligned}$$

Choose $B_0 \geq 0$ such that

$$(63) \quad 2H_2(\operatorname{Im} \zeta) - 2H_0(\operatorname{Im} \zeta) - \frac{B_0}{n} |\operatorname{Im} \zeta|_1 \leq 0.$$

When $B \geq B_0$ we will have $\|Q(z, \zeta)\| \leq C_2(1+|\zeta|)^{N_1}$ and

$$(64) \quad \|\bar{\partial}Q(z, \zeta)\| \leq C_2(1+|\zeta|)^{N_1} e^{H_2(\operatorname{Im} \zeta) - H_0(\operatorname{Im} \zeta)},$$

where C_2 still denotes a constant depending on z of the form

$$C_2 = \text{const.} (1+|z|)^{N_1} e^{H_2(\text{Im } z) + B|\text{Im } z|_1}.$$

We can now return to the proof of Theorem 1 at the point where we obtained the estimates (51), (52), (53). Ignoring powers of q the exponential factors are:

$$(51') \quad \exp(-2nH_0(\text{Im } \zeta) + 2(n-1)H_1(\text{Im } \zeta) - \frac{B}{n}|\text{Im } \zeta|_1)$$

$$(52') \quad \exp(-(2n-1)H_0(\text{Im } \zeta) + 2(n-2)H_1(\text{Im } \zeta) + H_2(\text{Im } \zeta))$$

$$(53') \quad \exp(-(2n-1)H_0(\text{Im } \zeta) + 2(n-2)H_1(\text{Im } \zeta)).$$

Under the hypothesis (58) the largest of these three is (53') and its exponent satisfies

$$(53'') \quad -(2n-1)H_0(\text{Im } \zeta) + 2(n-1)H_1(\text{Im } \zeta) \leq -r_0|\text{Im } \zeta|,$$

for some $r_0 > 0$. If the hypothesis (59) holds, then the largest exponent is (52') and we define $r_0 > 0$ by

$$(52'') \quad -(2n-1)H_0(\text{Im } \zeta) + 2(n-2)H_1(\text{Im } \zeta) + H_2(\text{Im } \zeta) \leq -r_0|\text{Im } \zeta|.$$

In either case the rest of the proof is the same as that of Theorem 1. □

Example As shown in [7] the family μ_1, μ_2, μ_3 obtained by taking μ_1 = characteristic function of the unit square = $\chi_{[-1,1] \times [-1,1]}$, μ_2 a rotation of μ_1 by 36° , and μ_3 a rotation of μ_1 by 45° , satisfies the first conditions of Theorem 1 and Theorem 2 with $H_0(\theta) = |\theta|$ since the squares contain the unit disk. One can easily convince oneself that the hypothesis (24) does not hold (e.g. take $\theta = (t, 0)$, $t > 0$.) On the other hand one can take

$H_1(\theta) = H_2(\theta) = \sqrt{2}|\theta|$ since all the squares are contained in the disk of radius $\sqrt{2}$. We are in the situation of hypothesis (58) and its verification reduces to the fact that

$$r_0 = 3 - 2\sqrt{2} > 0$$

Furthermore $B_0 = 4(\sqrt{2}-1)$ works in this case.

3. Polynomial case

The conditions on Theorem 1 and 2 imply that the convex set defined by H_0 contains a ball. If we want to prove an algebraic version of (26), the fact that this condition is not satisfied plays a role. Such a representation was stated in [3,4] without proof. We analyze here the conditions under which it is valid.

Theorem 3. Let $p_1, \dots, p_m$ be a family of polynomials in C^n without common zeros, suppose further that:

- a) $D := \max_{1 \leq j \leq m} \deg p_j = \deg p_i$ for $1 \leq i \leq n$.
- b) $Z = \{z \in C^n : p_i(z) = 0; 1 \leq i \leq n\}$ is discrete.
- c) $J(z) :=$ Jacobian determinant of $p_1, \dots, p_n$ at z is $\neq 0$ for all $z \in Z$
- d) $p_1, \dots, p_n$ have no zeros at infinity, i.e. $\#Z = D^n$.

Then

$$(65) \quad 1 = \sum_{\zeta \in Z} \frac{\Delta_{n+1}(z, \zeta)}{J(\zeta)} \varphi(z, \zeta) + \sum_{j=1}^n (-1)^{n+1-j} p_j(z) \sum_{\zeta \in Z} \frac{\Delta_j(z, \zeta)}{J(\zeta)},$$

where φ, Δ_j are defined as in (18) - (21) with respect to the polynomials $p_1, \dots, p_m$.

Remarks 1. The functions g_k^j defined by (8) are obviously polynomials of degree $D-1$. It follows that (66) has the form

$$(66) \quad p_1(z) A_1(z) + \dots + p_m(z) A_m(z) = 1$$

for some polynomials $A_j \in C[z_1, \dots, z_n]$ of degree at most $n(D-1)$. This follows from the fact they are given as $n \times n$ determinants involving the g_k^j .

2. As before, the case $m = n + 1$ leads to a particularly pleasing form of (66)

$$(67) \quad 1 = \sum \frac{1}{J(\zeta)} \frac{1}{p_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z, \zeta) & \cdots & g_1^{n+1}(z, \zeta) \\ \vdots & & \vdots \\ g_n^1(z, \zeta) & \cdots & g_n^{n+1}(z, \zeta) \\ p_1(z) & \cdots & p_{n+1}(z) \end{vmatrix}.$$

3. The two statements in condition (d) above are really a form of Bezout's theorem [19]. The meaning of the expression " $p_1, \dots, p_n$ have no common zeros at infinity" is that if we introduce homogeneous polynomials $H_j(z_0, \dots, z_n) := z_0^D p_j\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$ then the subset of $\mathbb{C}^{n+1}$ defined by $\{z_0=0, H_1=\dots=H_n=0\}$ is $\{0\}$. This is equivalent to the statement:

$$(68) \quad |p(z)| = \left(\sum_{j=1}^n |p_j(z)|^2 \right)^{1/2} \geq C |z|^D \quad \text{if } |z| \geq R_0 \geq 1.$$

If we call p_j^0 the leading homogeneous polynomial of p_j , then the statement (68) is also equivalent to:

$$(69) \quad \{z \in \mathbb{C}^n : p_1^0(z) = \dots = p_n^0(z) = 0\} = \{0\}.$$

We also note that (69) implies (b) above. That is, condition (d) above implies condition (b).

Proof of Theorem 3. The proof is the same as that of Theorem 1. This time we take $v=1$ and R arbitrary $\geq R_0$ (cf. (68)). One can estimate $\|Q(z, \zeta)\| \leq C/|\zeta|$, $\|\bar{\partial}Q(z, \zeta)\| \leq C/|\zeta|^2$, $|\varphi_0(z, \zeta)| \leq C/|\zeta|^D$, $\|\bar{\partial}\varphi_0(z, \zeta)\| \leq C/|\zeta|^2$ if $|\zeta| = R$ and $|z| \leq K \leq R-1$, with

$C = C(k) > 0$. These estimates imply that the boundary integral in (40) tends to zero when $R \rightarrow \infty$. D

If $p_1, \dots, p_n$ are such that their leading terms p_j^0 satisfy (69) but their degrees $\deg p_j = D_j$ are not all equal or $\max(D_j: 1 \leq j \leq n)$ is smaller than $D = \max(D_j: 1 \leq j \leq m)$, then we can still prove a version of Theorem 3. That corresponds to the analytic counterpart of Theorem 1, that is to Theorem 2. For the moment we continue to assume that $J(z) \neq 0 \quad \forall z \in Z = \{p_1 = \dots = p_n = 0\}$.

Let $L_1(z) = u_1 z_1 + \dots + u_n z_n$ be a linear homogeneous polynomial with generic coefficients. The condition that $\{z \in \mathbb{C}^n: L_1 = p_2^0 = \dots = p_n^0 = 0\} \neq \{0\}$ is an algebraic condition on the coefficients of L_1 . Therefore we can choose L_1 such that for any integer $d_1 \geq 0$ we have $\{p_1^0 L_1^{d_1} = p_2^0 = \dots = p_n^0 = 0\} = \{0\}$. Continuing in this fashion we can choose $L_1, \dots, L_n, d_1, \dots, d_n$ such that for any choice of constants $\varepsilon_1, \dots, \varepsilon_n$, if we define $\tilde{p}_j = p_j(L_j + \varepsilon_j)$, then $\{\tilde{p}_1^0 = \dots = \tilde{p}_n^0 = 0\} = \{0\}$ and $\deg \tilde{p}_j = D$ for $1 \leq j \leq n$. It is clear now that for most choices of ε_j we still have that all common zeros of $\tilde{p}_1^0, \dots, \tilde{p}_n^0$ are simple and $\tilde{p}_1^0, \dots, \tilde{p}_n^0, p_{n+1}, \dots, p_m$ have no common zeros. Theorem 3 can now be applied to this new family, one obtains polynomials A_j

$$(70) \quad \sum_{j=1}^m A_j p_j = 1, \quad \deg A_j \leq n(D-1)$$

and such that they have a representation of the type (65).

We remark that a representation such as (65) cannot be valid if the $p_1, \dots, p_n$ have common zeros at infinity. For instance, in the example of Masser-Philippon [11]:

$$p_1 = z_1^D, p_2 = z_1 - z_2^D, \dots, p_{n-1} = z_{n-2} - z_{n-1}^D, p_n = 1 - z_{n-1} z_n^{D-1}$$

one knows that $\delta = D^n - D^{n-1}$ is the best estimate possible for the degrees of A_j solving the polynomial Bezout equation. The polynomials $A_1 = z_n^\delta$, $A_2 = -z_n^\delta \frac{(z_1^D - z_2^{D^2})}{z_1 - z_2^D}$, $A_3 = -z_n^\delta \frac{(z_2^{D^2} - z_3^{D^3})}{z_2 - z_3^D}$, $\dots$,

$$A_{n-1} = -z_n^\delta \frac{(z_{n-2}^{D^{n-2}} - z_{n-1}^{D^{n-1}})}{z_{n-2} - z_{n-1}^D}, A_n = \frac{1 - z_{n-1}^{D^{n-1}} z_n^\delta}{1 - z_{n-1} z_n^{D-1}}, \text{ have exactly this}$$

degree. On the other hand if we had a representation like (65) we could conclude that there are solutions A_j of the polynomial Bezout equation with $\deg A_j \leq n(D-1)$ like in (70).

We would like now to show that the condition (c) of the simplicity of the zeros in Theorem 3 is not necessary. Regrettably, we only know how to do this in the case where $m = n + 1$.

Theorem 4. Let $p_1, \dots, p_{n+1}$ be a family of polynomials in $\mathbb{C}^n$ without any common zeros, $D = \deg p_1 = \dots = \deg p_n \geq \deg p_{n+1}$ and $(p_1^0 = \dots = p_n^0 = 0) = \{0\}$. Then we can find polynomials A_j of degree $\leq n(D-1)$ satisfying the identity $\sum_{j=1}^{n+1} A_j p_j = 1$. The coefficients of the A_j can be written in terms of the values of p_{n+1} and values of derivatives of p_{n+1} , and the coefficients of the $g_k^{n+1}(z, \zeta)$ (when considered as polynomials in z), all of

these evaluated at the points of $Z = \{\zeta: p_1(\zeta) = \dots = p_n(\zeta) = 0\}$.

Proof For $0 < \epsilon_j \ll 1$ the function $1/p_{n+1}$ is holomorphic in a neighborhood of $\Pi = \{|p_1| \leq \epsilon_1, \dots, |p_n| \leq \epsilon_n\}$. This set is a compact polynomially convex set. (The compactity follows from the condition (68)). By Sard's theorem one can choose the ϵ_j so that the sets $\{|p_j| = \epsilon_j\}$ are real analytic submanifolds of $\mathbb{C}^n$. (In fact, we only need it in a neighborhood of Π .) For any $v \in \mathcal{X}(\bar{\Pi})$ we have that integral

$$(71) \quad \text{Res}_Z(v d\zeta_1 \wedge \dots \wedge d\zeta_n) := \frac{1}{(2\pi i)^n} \int_{\substack{|p_1| = \sigma_1 \\ \vdots \\ |p_n| = \sigma_n}} v(\zeta) \frac{d\zeta_1 \wedge \dots \wedge d\zeta_n}{p_1(\zeta) \dots p_n(\zeta)}$$

is independent of choice of $\sigma_1, \dots, \sigma_n$ as long as $0 < \sigma_j \leq \epsilon_j$ and the $\{|p_j| = \sigma_j\}$ are smooth. Furthermore, if v is in the ideal generated by $p_1, \dots, p_n$ in $\mathcal{X}(\Pi)$ then this residue is zero. Therefore, it depends only the values of v at Z and a certain number of derivatives of v at Z (as it follows from the Nullstellensatz as presented e.g. in [13], [16]). In other words, the integral (71) can be considered as an operator defined by a certain linear combination of the Dirac masses δ_ζ and their derivatives $\frac{\partial |a|}{\partial \zeta^a} \delta_\zeta$, $\zeta \in Z$, applied to the holomorphic function v . This operator is very hard to compute explicitly except in very simple cases but it is perfectly defined as the common value of all the integrals (71). It is called the residue current of Z .

Let us consider now the polynomial $B(z)$ defined by:

$$(72) \quad B(z) = \text{Res}_z \frac{1}{p_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z, \zeta) & \cdots & g_1^{n+1}(z, \zeta) \\ \vdots & & \vdots \\ g_n^1(z, \zeta) & \cdots & g_n^{n+1}(z, \zeta) \\ p_1(z) & \cdots & p_{n+1}(z) \end{vmatrix} d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

This polynomial is in fact of the form $\sum_{j=1}^{n+1} A_j(z) p_j(z)$, with A_j polynomials of degree $\leq n(D-1)$ whose coefficients are computed in terms of the values of derivatives of p_{n+1} and the coefficients of g_k^j (as polynomials in z) evaluated over Z . The only problem is to show $B(z) \equiv 1$. We fix values $\sigma_j, 0 < \sigma_j < \epsilon_j$. Consider complex numbers $\alpha_1, \dots, \alpha_n$ sufficiently small and so chosen that:

- i) All the common zeros of $p_1 - \alpha_1, \dots, p_n - \alpha_n$ are simple and lie in $\{|p_1| < \frac{1}{2}\sigma_1, \dots, |p_n| < \frac{1}{2}\sigma_n\}$.
- ii) $p_1 - \alpha_1, \dots, p_n - \alpha_n, p_{n+1}$ have no common zeros.

Note that $p_1 + \alpha_1, \dots, p_n + \alpha_n$ still do not have any common zeros at ∞ . Let us denote $Z_\alpha = \{z \in \mathbb{C}^n : p_1 - \alpha_1 = \dots = p_n - \alpha_n = 0\}$.

Then

$$(73) \quad \text{Res}_{Z_\alpha} (v d\zeta_1 \wedge \cdots \wedge d\zeta_n) = \frac{1}{(2\pi i)^n} \int_{\substack{|p_1| = \sigma_1 \\ \vdots \\ |p_n| = \sigma_n}} v(\zeta) \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_n}{(p_1(\zeta) - \alpha_1) \cdots (p_n(\zeta) - \alpha_n)}$$

where $J(\zeta) = \frac{\partial(p_1 - \alpha_1, \dots, p_n - \alpha_n)}{\partial(\zeta_1, \dots, \zeta_n)} = \frac{\partial(p_1, \dots, p_n)}{\partial(\zeta_1, \dots, \zeta_n)}$ and the last

identity follows from Stokes' theorem. (Replace the contour by

little spheres about the distinct points of Z_a). This last identity is the effective computation of the residue current of Z_a , namely

$$\text{Res}_{Z_a} = \sum_{\zeta \in Z_a} J(\zeta)^{-1} \delta_\zeta.$$

The first identity in (73) shows that $\text{Res}_{Z_a} \rightarrow \text{Res}_Z$ as $\sigma \rightarrow 0$, i.e. the residue currents at $a = 0$ are continuous when acting on holomorphic $(n,0)$ forms. On the other hand, by Theorem 3 we have

$$(72) \quad \text{Res}_{Z_a} \frac{1}{P_{n+1}(\zeta)} \begin{vmatrix} g_1^1(z, \zeta) & \cdots & g_1^{n+1}(z, \zeta) \\ \vdots & & \vdots \\ g_n^1(z, \zeta) & \cdots & g_n^{n+1}(z, \zeta) \\ p_1(z) - a_1 & \cdots & p_{n+1}(z) \end{vmatrix} d\zeta_1 \wedge \cdots \wedge d\zeta_n = 1.$$

(Note that the g^j corresponding to p_j and to $p_j - a_j$ coincide). By continuity we obtain $B = 1$. This concludes the proof of Theorem 4. □

Remarks 1. The can obviously obtain the same result without assuming the degrees of $p_1, \dots, p_n$ coincide or that they are larger or equal than that of p_{n+1} .

2. The reasoning of Theorem 4 extends to a strongly coprime family of $n+1$ elements whose first n members form a well-behaved family. Under the other conditions of Theorem 1 or Theorem 2, we obtain a series representation of the solutions of the Bezout equation which we computed in terms of the residue current associated to Z . This time the series converges after grouping of terms.

3. The interest of the theorems in this section lies in the search for explicit algorithms to obtain solutions A_j for the algebraic Bezout equation $\sum_{j=1}^m A_j p_j = 1$ which satisfy Brownawell's estimate, $\deg A_j \leq 3 \mu n D^\mu$, $\mu = \min(n, m)$.

4. Conclusion

We have shown how explicit solutions to the analytic and algebraic Bezout equations can be obtained under natural restrictions on the original functions $f_1, \dots, f_m$. This work has applications to the implementation of deconvolution for multidetector systems.

5. References.

1. M. Andersson and B. Berndtsson, Henkin-Ramirez formulas with weight factors, Ann.Inst.Fourier 32(1982), 91-110.
2. C.A. Berenstein, P.S. Krishnaprasad and B.A. Taylor, Deconvolution methods for multisensors (1984), DTIC #ADA 152351.
3. C.A. Berenstein and D. Struppa, On explicit solutions to the Bezout equation, Systems & Control Letters 4(1984), 33-39.
4. C.A. Berenstein and D. Struppa, 1-Inverses for polynomial matrices of non-constant rank, Systems & Control Letters 6(1986), 309-314.
5. C.A. Berenstein and D. Struppa, Small degree solutions for the polynomial Bezout equation, to appear in Linear Algebra and its Applications.
6. C.A. Berenstein, B.A. Taylor and A. Yger, Sur quelques formules explicites de déconvolution, J.Optics 14(1983), 75-82.
7. C.A. Berenstein and A. Yger, Le problème de la déconvolution, J.Fcnal.Anal.54(1983), 113-160.

8. C.A. Berenstein and A. Yger, Ideals generated by exponential polynomials, Adv.in Mathematics 60(1986), 1-80.
9. B. Berndtsson, A formula for interpolation and division in C^n , Math.Anal.263(1983), 395-418.
10. N.K. Bose, Applied Multidimensional Systems Theory, Reidel 1984.
11. W.D. Brownawell, Bounds for the degrees in the Nullstellensatz, to appear in Annals of Math.
12. B. Buchberger, An algorithmic method in polynomial ideal theory, Multidimensional Systems Theory (ed.N.K. Bose), Reidel Publ., 1985.
13. L. Ehrenpreis, Fourier Analysis in Several Complex Variables, Wiley-Interscience, 1970.
14. L. Hörmander, Generators for some rings of analytic functions, Bull Amer.Math.Soc.73(1967), 943-949.
15. R.E.Molzon, B.Shiffman and N. Sibony, Average growth estimates for hyperplane sections of entire analytic sets, Math.Ann.257 (1981), 43-59.
16. V.P. Palamodov, Linear differential operators with constant coefficients, Springer Verlag, 1970.
17. E.V. Patrick, Deconvolution for the case of multiple characteristic functions of cubes in R^n , Technical Report Systems Research Center, University of Maryland, 1986.
18. L. Schwartz, Théorie des distributions, Hermann, 1950.
19. B.L. van der Waerden, Algebra, Springer Verlag, 1959.

CHAPTER 2

EXACT DECONVOLUTION FOR MULTIPLE CONVOLUTION OPERATORS - AN OVERVIEW PLUS PERFORMANCE CHARACTERIZATIONS FOR IMAGING SENSORS*

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1. INTRODUCTION

Throughout the last several years mathematical results have been presented which form the foundations for the use of multiple (parallel) linear operators, each given by convolution with a distinct kernel (or impulse response), in place of the use of a single such linear operator or, equivalently, in place of the use of multiple (parallel) operators each with the identical kernel [1] - [13]. See Fig. 1. The mathematical results cited above describe the conditions under which compactly supported distributions $\mu_1, \mu_2, \dots, \mu_m$ have associated to them compactly supported distributions $\nu_1, \nu_2, \dots, \nu_m$ such that

$$\sum_{i=1}^m \mu_i * \nu_i = \delta, \quad (1)$$

where δ is the Dirac distribution on $\mathbb{R}^n$ and where $*$ denotes convolution. We often refer to the μ_i as convolvers and to the ν_i as deconvolvers.

To introduce some explicit multiple operators and their role in deconvolution, let us look at an example and the results of a computer simulation [14]. Our example is for a case in which equation (1) holds for $m = 2$, and it is outlined in Fig. 2.

In the center of Fig. 2 the block diagram of Fig. 1(a) is reproduced for the case $m = 2$. (The operator ϕ in Fig. 1(a) will be discussed later.) We shall be considering in this example the one-dimensional case, that is, functions of one variable. To the left of the block diagram is the graph of the input signal, that is, the function f , which here consists of the sum of translated Gaussian pulses. (This function is one that is frequently used for the evaluation of deconvolution algorithms; see for example [15].)

The input signal f is acted on by two convolution operators, one with kernel μ_1 and one with kernel μ_2 . The resulting output functions are g_1

and g_2 , respectively:

$$g_i(x) = (f \circ \mu_i)(x) = \int_{\mathbb{R}} f(y) \mu_i(x-y) dy, \quad i = 1, 2, \quad x \in \mathbb{R}.$$

In this example, the kernels μ_1 and μ_2 are essentially the simplest possible kernels with compact support: the kernel functions are each constant over some bounded intervals and zero elsewhere. (The support of the kernel function refers to (the closure of) the set on which the function is non-zero; the support is called compact if this set is bounded.) The kernel functions are shown in Fig. 2: μ_1 is supported by the interval $[-1, 1]$, and μ_2 by the interval $[-\sqrt{2}, \sqrt{2}]$.

The output functions g_1 and g_2 are also shown in Fig. 2. We often refer to these as the data functions, for in applications g_1 and g_2 are typically the results from two distinct measurement operations which are modeled by the "convolvers" μ_1 and μ_2 . An application that is a physical realization of the situation presented in Fig. 2 is spectroscopy. In that case, the function f is an unknown density function with the wavelength of the optical radiation as the variable: $f(x)$ is the energy per unit wavelength at wavelength x . To make a measurement of nonzero energy from this density function it is necessary to integrate f over some range of wavelengths, for example, $g_1(x) = \int_{x-\Delta}^{x+\Delta} f(y) dy$. We could also make the second measurement $g_2(x) = \int_{x-\sqrt{2}\Delta}^{x+\sqrt{2}\Delta} f(y) dy$. A linear scaling of the wavelength variables converts these integrals into exactly the example in Fig. 2. In this interpretation, the peaks in the unknown spectrum f have a separation smaller than the lengths of wavelength intervals over which we integrate. Consequently, the data functions g_1 and g_2 retain no obvious indication of two peaks.

Of course, the classical deconvolution problem is to recover the unknown input function f from only one data function, say $g_0 = \mu_0 * f$. The term multiple operator deconvolution is used to emphasize that in recovering f we may use more than one data function. In fact, part of the problem is deciding how many are sufficient. More precisely, the issue is how many of which convolvers μ_i are sufficient for the recovery of f . As we shall see later, the two we have chosen for our example in Fig. 2 are in fact sufficient. They are in fact sufficient in a very strong manner. Not only is f uniquely determined by g_1 and g_2 , but in fact there exist linear operators given by convolution with distributions ν_1 and ν_2 , both with support no greater than the larger of the supports of μ_1 and μ_2 , such that

$$f = \nu_1 * g_1 + \nu_2 * g_2.$$

We will give the explicit formulas for ν_1 and ν_2 later. Here, to finish our example and discussion of Fig. 2, we have shown to the right in Fig. 2 the result of recovering f from g_1 and g_2 for a digital simulation [14]: discrete versions of "deconvolvers" ν_1 and ν_2 were constructed and, using discrete samples from the data functions g_1 and g_2 , a discrete approximation of f was constructed by the sum of discrete convolutions. A quite good reconstruction was obtained. (We attribute the asymmetrical ringing in the result to a combination of truncation error and the use of a loop in the simulation that recalculated a quantity that should have been calculated only once.)

We would like to emphasize one final point: the example is based on mathematical results regarding relationships between convolution operators and smooth functions on $\mathbb{R}^n$. Whenever we mention the use of discretization, it is always in the sense of a discrete approximation to a smooth function. When we

refer to discrete data, we mean samples of the underlying data functions. The reconstruction of the unknown input depends directly upon how well the discrete data can be used to approximate the data functions for a given choice of interpolating functions. In Fig. 2 it should be clear that if g_1 and g_2 were sampled at a rate of one sample per unit interval ($2\frac{1}{2}$ tick marks), one could hardly expect an accurate estimate of f . The sampling rate for the results on Fig. 2 was 50 samples per unit interval (20 samples per tick mark interval).

Thus we always view the theory, the applications, and the algorithms in the following order. First the problems and results are stated in terms of continuous domains that model the applications. The accuracy of the algorithmic implementation of the results then is understood to depend on the sampling rate, with convergence as sampling rates increase. The specification of a possible minimum sampling rate for a given reconstruction problem is typically a problem-specific task in interpolation error estimates. In fact, the Nyquist sampling rate is sufficient for band limited functions f .

The example in Fig. 2 illustrates why equation (1) is of interest for applications in which the convolver μ_i must correspond to a physical, analog device wherein the impulse response is dictated by a solid state or biological process. It is frequently possible to select such analog convolvers which satisfy approximately the multiple operator criteria such that equation (1) will hold. Then each associated deconvolver can be digitally implemented. The fact that the deconvolvers act linearly and have compact support means that their implementation is straightforward; their action as continuous linear operators implies stability. Most importantly, the evident high bandwidth of the overall operator is accomplished without any essential change in the response functions of the analog devices. The term overall operator

refers to the operator given by the kernel distribution $\sum_{i=1}^m \mu_i \circ \nu_i = \delta$. Of course, because of practical constraints such as analog and digital approximations and computation time, the design objective for the overall operator would not be the identity operator with impulse response δ but rather a high bandwidth approximation of the identity operator given by an impulse response ϕ . In terms of the distributions in equation (1), since convolutions commute,

$$\sum_{i=1}^m (\mu_i \circ \phi) \circ \nu_i = \sum_{i=1}^m \mu_i \circ (\nu_i \circ \phi) = \phi. \quad (2)$$

In a sense ϕ can be considered to be made up of "parts," each of which arises from one of the practical constraints just listed, along with a special part that is deliberately added to control the noise power spectrum of the output of the overall operator.

The publications on this subject have appeared primarily in the mathematical literature. The following issues regarding (1) have been addressed: sufficient conditions for the existence of solutions [1] - [4], [16]; examples of sets of distributions that satisfy the sufficient conditions [5] - [7]; construction of explicit solutions, that is, explicit formulas for the deconvolvers [7] - [9]; and construction and evaluation of approximate solutions [9], [11], [14].

Only recently have specific applications of (1) been mentioned. The work of Berenstein, Krishnaprasad, and Taylor [14] was one of the first times that (1) and contemporary mathematical methods for understanding the equation were applied to physical problems. This work also discussed the question of additive noise and the question of the continuity of the overall operator with respect to the distributions $\mu_1, \mu_2, \dots, \mu_m$. The noise question is in regard to noise added following the action of the operators defined by the μ_i .

while the continuity question is in regard to the dependence of the overall performance on either the actual analog approximations of the μ_1 or the digital approximations of the ν_1 .

The approximation methods of [11] were motivated by this work of Berenstein et al. These methods exploit the approximation in (2). In conjunction with the analysis, a computer simulation for R^2 was performed. This simulation dramatically illustrated (2) for imaging devices in which the analog convolvers were solid state photodetectors. With these results there was an increased interest in imaging applications. This led to the consideration of not just detectors but of linear systems consisting of sequences of operators with each operator of the multiple operator type. These activities led to the need to answer basic systems analysis questions.

In what follows three topics in multiple operator deconvolution are discussed. The first is that, in a sense that is relevant for applications, multiple operators are necessary for the deconvolution problem to be well-posed. We present examples to illustrate the ill-posedness of single operator deconvolution, and a theorem which shows that, except for an uninteresting case, m must be greater than one in our problem statement.

The second topic is that of identifying convolvers $\mu_1, \dots, \mu_m$ for which equation (1) holds and the construction of the deconvolvers.

The final topic is the major one here: measuring the utility of a multiple operator design. We describe the result of our application of standard methods of linear systems and random signals to the multiple operator type of system of equations (1) and (2). This analysis was necessary if one was to seriously consider multiple operator designs. While the extended bandwidth was well understood, analyzed, and even illustrated in simulations, the consequence of the introduction of noise and of design errors was not fully

understood. It was clear that since the operator was linear and continuous that there would be no instability due to noise (at least for smooth approximations), which is already an improvement over the case of single operator reconstruction methods [5], [14], [15]. However, the performance needed to be explicitly described so that standard tools such as resolution, equivalent bandwidth, and signal to noise ratio would be available for systems engineering design studies.

This investigation was motivated in large part by the potential application of these multiple operator methods to electro-optics, especially to imaging devices. We have in mind imaging devices that are for the detection, transformation, and display of electromagnetic radiation for a human observer as well as such devices for artificially intelligent "observers." The problems and the desired solutions have the flavor of this application. While the analysis and the results are in a sense general, much is framed and guided by the motivating problems.

This interest in electro-optics is made explicit in the last sections of this paper. There we examine convolvers and additive noises of the type encountered in imaging detector arrays. The method developed here for general performance comparisons is used to compare conventional detector arrays with arrays configured such that equation (1) holds (up to the resolution limitation imposed by a choice of optical sampling rate). In terms of the familiar parameters of resolution, modulation transfer function, and signal to noise, we show that the performance of the strongly coprime design exceeds that of the conventional design. Moreover, the comparison has a quantified, functional form suitable for system design trade-off studies.

2. THE NECESSITY OF MULTIPLE OPERATORS

Recall our mathematical problem statement from the Introduction: under what conditions do the compactly supported distributions $\mu_1, \mu_2, \dots, \mu_m$ on $\mathbb{R}^n$ have associated to them compactly supported distributions $\nu_1, \nu_2, \dots, \nu_m$ such that $\sum_{i=1}^m \mu_i \circ \nu_i = \delta$?

Here we offer some observations and a theorem which motivate this statement. A brief demystification of "compactly supported distributions" is given in the Appendix.

In Fig. 1(a) and in the problem statement, the integer m , the number of distributions μ_i , is not specified beforehand. The number of distributions as well as the distributions $\nu_1, \dots, \nu_m$ are parts of the problem. In the example above for $\mathbb{R}$ it was stated that $m = 2$ sufficed. When the μ_i are characteristic functions of cubes on $\mathbb{R}^n$, then there are choices for the μ_i such that $m = n+1$ is sufficient.

Fig. 1(b) represents a different, more restricted problem: it represents a restriction of the problem statement above to $m = 1$. We show below that there is only one trivial case in which a single compactly supported distribution μ_0 has an associated compactly supported distribution ν_0 such that $\mu_0 \circ \nu_0 = \delta$. Before presenting this general result we briefly review three well known observations which help to motivate the problem statement. For these remarks let μ_0 be the characteristic function of the interval $(-1, 1)$ in $\mathbb{R}$ (μ_0 takes the value one on the interval and zero elsewhere).

Observation 1. There can be no such ν_0 if we permit the input signals f to belong to any space that contains $C(\mathbb{R})$, the continuous functions on $\mathbb{R}$.

If such a ν_0 existed then, for any continuous f , $\nu_0 \circ (\mu_0 \circ f) = f$. But for f continuous, $\mu_0 \circ f$ is (represented by) a continuous function and,

hence, for $f_n(x) = \sin(n\pi x)$, $n = 1, 2, \dots$, it is easily checked that $\mu_0 * f_n$ is zero everywhere.

The problematic functions f_n in Observation 1 can be eliminated by considering only functions in $L^2(\mathbb{R})$. For in this case we can consider Fourier transforms of f , of μ_0 , (μ_0 is in $L^1(\mathbb{R})$), and of $\mu_0 * f$ (in $L^2(\mathbb{R})$). Let $\hat{\mu}_0$ denote the Fourier transform of μ_0 . Since $\hat{\mu}_0(\omega) = 2 \frac{\sin \omega}{\omega}$, $\omega \in \mathbb{R}$, then $(\mu_0 * f)^\wedge = \hat{\mu}_0 \hat{f}$ determines $\hat{f}$ almost everywhere. Since the Fourier transform is an isometry from $L^2(\mathbb{R})$ to $L^2(\hat{\mathbb{R}})$, f is uniquely determined by $\mu_0 * f$. But still:

Observation 2. There can be no such ν_0 even if we restrict the input signals f to belong in $L^2(\mathbb{R})$.

If such a ν_0 existed, then let $[-M, M]$ be the support of ν_0 . Let χ_{M+2} be the characteristic function $(-(M+2), M+2)$ and let $f_n(x) = \sin n\pi x$, $n = 1, 2, \dots$. Then $\mu_0 * (f_n \chi_{M+2})$ is zero on $(-(M+1), M+1)$, hence $\nu_0 * \mu_0 * (f_n \chi_{M+2})$ is zero on $(-1, 1)$.

The difficulty in Observation 2 is due to the fact that we seek ν_0 with compact support. What if we dropped this requirement? This would be problematic for applications, for then to get better estimates of f from $\mu_0 * f$ we would have to process larger and larger subsets of the domain of $\mu_0 * f$. But even if we could tolerate noncompactness, there is still a problem of boundedness or continuity.

Observation 3. There can be no (non-compactly supported) continuous linear operator L from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ such that $L(\mu_0 * f) = f$ for all f in $L^2(\mathbb{R})$.

The difficulty is that convolution with μ_0 does not carry $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. For example, let ψ_ϵ be the approximate identity defined by $\psi_\epsilon(x) = \frac{1}{2\epsilon} \mu_0\left(\frac{x}{\epsilon}\right)$, $x \in \mathbb{R}$. Then the L^1 norm of ψ_ϵ is $\|\psi_\epsilon\|_1 = 1$, while $\|\psi_\epsilon\|_2 = \frac{1}{2\epsilon}$

for the L^2 norm, and $\psi_\epsilon * \mu_0$ converges to μ_0 in L^2 . If L existed as described, then by continuity

$$L(\mu_0) = L(\lim_{\epsilon \rightarrow 0} \psi_\epsilon * \mu_0) = \lim_{\epsilon \rightarrow 0} L(\psi_\epsilon * \mu_0) = \lim_{\epsilon \rightarrow 0} \psi_\epsilon.$$

These observations illustrate ways in which the problem of inverting the convolution or solving the convolution equation $\mu_0 * f = g$ for a specific μ_0 fails to be well-posed (that is, that the solution f to $\mu_0 * f = g$ exists, is unique, and depends continuously on g) even with a variety of restrictions on f . Our problem statement in part seeks to find m and $\mu_1, \dots, \mu_m$ such that the inversion of the simultaneous distribution equations $\mu_1 * f = g_1, \dots, \mu_m * f = g_m$ is well-posed, along with an extra condition on the form of the inverse. But possibly our observations were due entirely to an unfortunate choice for the distribution μ_0 . Possibly there is a compactly supported μ_1 for which there exists a compactly supported distribution ν_1 such that $\mu_1 * \nu_1 = \delta$. This issue is settled by the following theorem.

Theorem. Let μ_1 and ν_1 be compactly supported distributions on $\mathbb{R}^n$ such that $\mu_1 * \nu_1 = \delta$, the Dirac delta distribution or unit impulse (at the origin in $\mathbb{R}^n$). Then there exist $a \in \mathbb{R}^n$ and $C \in \mathbb{C}$, $C \neq 0$, such that $\mu_1 = C\delta_a$, the Dirac delta distribution translated to $a \in \mathbb{R}^n$.

Proof. See Appendix.

These observations and this theorem are primary examples of the difficulties that are avoided whenever we can use multiple operators $\mu_1, \dots, \mu_m$ for which equation (1) holds. Of course, there is a vast literature and many approaches to address the difficulties of ill-posed problems such as the inversion of a single convolution equation [15], [17] and [18]. Our interest is in exploiting those cases in which using multiple operators we have a well-posed inverse problem. In the next section we identify some of those cases.

3. EXISTENCE AND CONSTRUCTION OF DECONVOLVERS

Our problem statement is typically viewed in two parts. The first part is existence, the identification of sets of convolvers $\mu_1, \dots, \mu_m$ such that deconvolvers $\nu_1, \dots, \nu_m$ exist. The second part is the explicit representation of the deconvolvers.

These problems are addressed using methods from the study of analytic functions of several complex variables. For if μ_1 is the distribution on $\mathbb{R}^n$ with compact support, then its Fourier-Laplace transform $\hat{\mu}_1$ is an analytic function on $\mathbb{C}^n$. Moreover, $|\hat{\mu}_1(z)|$, $z \in \mathbb{C}^n$ increases for large $|z| = |(z_1, z_2, \dots, z_n)| = \left(\sum_i |z_i|^2\right)^{1/2}$ in a manner which completely characterizes these analytic functions which are transforms of distribution of compact support (the Paley-Wiener-Schwartz theorem, see Appendix). Thus for a given set of convolvers $\mu_1, \dots, \mu_m$, by taking transforms the existence of the deconvolvers is equivalent to the existence of solutions $\hat{\nu}_1, \dots, \hat{\nu}_m$ of the analytic Bezout equation

$$\sum_{i=1}^m \hat{\mu}_i(z) \hat{\nu}_i(z) = 1, \quad z \in \mathbb{C}^n, \quad (3)$$

with $\hat{\nu}_1, \dots, \hat{\nu}_m$ each in the Paley-Wiener class of functions.

In this form one quickly sees that a necessary condition on the convolvers μ_i is that they have no common zeros. Moreover, because the $\hat{\nu}_i$ are to satisfy certain growth conditions for large $|z|$, one has a stronger condition which turns out to be both necessary and sufficient.

Theorem [1], [16]. For the compactly supported distributions $\mu_1, \dots, \mu_m$ on $\mathbb{R}^n$ there exists compactly supported distributions $\nu_1, \dots, \nu_m$ such that $\sum_{i=1}^m \mu_i \circ \nu_i = \delta$ if and only if there exist positive constants c_1 and c_2 and

a positive integer N such that

$$\left[\sum_1 |\hat{\mu}_1(z)|^2 \right]^{1/2} \geq c_1 e^{-c_2 |\operatorname{Im} z|} (1+|z|)^{-N}, \quad z \in \mathbb{C}^n. \quad (4)$$

A set of convolvers $\mu_1, \dots, \mu_m$ that satisfies the inequality in the theorem is often referred to as strongly coprime.

The following sets of convolvers are known to be strongly coprime:

- i) on $\mathbb{R}^2$, the characteristic functions of two disks wherein the ratio of the radii of the disks has any integer value from 2 to 200 [7];
- ii) on $\mathbb{R}^n$, the characteristic functions of $n+1$ cubes with side lengths $s_1, \dots, s_{n+1}$ such that $s_1^2, \dots, s_{n+1}^2$ are integers that are pairwise relatively prime and at most one of these integers is a perfect square of integers [7], [11].

The example of the introduction falls within case ii. Another case which is known is the characteristic functions of certain rotations of three squares in the plane [7].

Explicit formulas for the deconvolvers have been found for the disks on $\mathbb{R}^2$ (case i) and for $n = 1$, and $n = 2$ in case ii [5] - [9] and [14]. For example, the explicit solutions for the deconvolvers ν_1, ν_2 for $n = 1$ in case ii is the following. For $i = 1, 2$, $a_i > 0$, let χ_i be the characteristic function of the interval $(-a_i, a_i)$, with $a_2 = \sqrt{2}a_1$. Let ϕ be any smooth function. Then ν_1 and ν_2 such that $\chi_1 \circ \nu_1 + \chi_2 \circ \nu_2 = \delta$ are given by

$$\nu_1 \circ \phi = \frac{1}{8a_1 a_2} \chi_2 \circ \phi - \frac{1}{24a_1} a_2^2 \frac{d^2}{dx^2} (\chi_2 \circ \phi) + \frac{d^3}{dx^3} ((\chi_2 K_1) \circ \phi)$$

where

$$K_1(x) = \frac{a_2}{2\pi^2} \sum_{k=1}^{\infty} \frac{\sin\left(\frac{\pi k x}{a_2}\right)}{k^2 \sin\left(\frac{\pi a_1 k}{a_2}\right)},$$

and ν_2 and K_2 are given by the permutation of indices. Note that the support of ν_1 is contained in that of χ_2 , and the support of ν_2 is contained in that of χ_1 .

Throughout the foregoing we have emphasized that we seek deconvolvers ν_i which have compact support, and we have described necessary and sufficient conditions on the convolvers μ_i for these to exist. The explicit determination of whether the μ_i are strongly coprime and the explicit construction of the deconvolvers is typically too restrictive an approach for general engineering applications, and new numerical optimization approaches are being developed [10].

There is, however, a middle ground that we use when we have a set of strongly coprime convolvers but do not have explicit representations for compactly supported deconvolvers. These alternatives are due to the fact that the deconvolvers are not unique. To see this nonuniqueness, consider the one-dimensional example of the Introduction. Let μ_1 and μ_2 be compact supported and strongly coprime, and let ν_1 and ν_2 be compactly supported deconvolvers. Let λ be any compactly supported distribution. Then it is readily seen that both $\nu'_1 = \nu_1 + \lambda \cdot \mu_2$ and $\nu'_2 = \nu_2 - \lambda \cdot \mu_1$ are compactly supported and $\sum_{i=1}^2 \mu_i \cdot \nu'_i = \delta$. Going further, it was not necessary that λ be a compactly supported distribution. That is, we can construct deconvolvers that are not compactly supported.

We frequently use approximate deconvolvers even when they are not compactly supported. For example, let φ be any integrable, smooth function

with integral over $\mathbb{R}$ equal to 1. Then, for $\varphi_\tau(x) = \frac{1}{\tau} \varphi(\frac{x}{\tau})$, $\nu_1 \circ \varphi_\tau$ is an approximate deconvolver and $\nu_1 \circ \varphi_\tau$ converges to ν_1 as $\tau \rightarrow 0$. We can choose, also, that the $\nu_1 \circ \varphi_\tau$ be L^1 functions. Let $\hat{\varphi}$ have compact support and be sufficiently differentiable. (The function φ is usually called a mollifier.)

Finally, we can construct approximate, non-compactly supported deconvolvers on $\mathbb{R}^n$ solely from the knowledge that the convolvers are strongly coprime. Let $\omega = (\omega_1, \dots, \omega_n) = \text{Re } z = (\text{Re } z_1, \text{Re } z_2, \dots, \text{Re } z_n)$ and for what follows we will need to consider the Fourier transform only on the real subspace $\mathbb{R}^n$ of $\mathbb{C}^n$. Since $\hat{\mu}_i \in C^\infty(\mathbb{R}^n)$, and with $\varphi \in C^\infty(\mathbb{R}^n)$, with $\hat{\varphi}$ compactly supported, let

$$D_i(\omega) = \frac{\bar{\hat{\mu}}_i(\omega)}{\sum_{j=1}^m |\hat{\mu}_j(\omega)|^2}, \quad \hat{h}_i = D_i(\omega) \hat{\varphi}(\omega), \quad i = 1, \dots, m. \quad (5)$$

This defines $h_i \in L^1(\mathbb{R}^n)$ such that $\sum_{i=1}^n h_i \circ \mu_i = \varphi$ ($\bar{z}$ denotes the complex conjugate of z). This is the class of deconvolvers that are discussed in the last section, because from among these one can find optimal deconvolvers when there is additive noise.

4. PERFORMANCE OF OPTIMAL DECONVOLVERS.

While (5) is exhibited essentially by inspection, the result can be obtained in a more systematic fashion as well as in a more general form. We first recall some standard tools, apply these tools to a simple case, and then proceed to the more general form. The diagram in Fig. 3 represents an

operator L acting on a function f . Let (temporarily) f be bounded and in $C^\infty(\mathbb{R}^n)$. Let $\mu_1, \mu_2, \dots, \mu_m$ be an arbitrary set of m distributions with compact support. For each linear operator defined by μ_i let η_i be a sample function of a zero mean, wide-sense stationary random process that is added to the output of μ_i . Let $\eta_i \in L^\infty(\mathbb{R}^n)$, and let N_i^2 ($N_i \geq 0$) be the noise power spectral density of the process (see, for example, [19] Ch. 4, Ch. 6). For each distinct i and j let η_i be independent of η_j and let each η_j be independent of f . Let ν_i be defined by $(\nu_i \circ \phi)^\wedge = D_i \hat{\phi}$, where $\hat{\phi}, D_i \in C^\Gamma(\mathbb{R}^n)$, $\hat{\phi}$ has compact support, and r is sufficiently large so that $[D_i \hat{\phi}]^\wedge \in L^1(\mathbb{R}^n)$. Let $g \in L^\infty(\mathbb{R}^n)$ be defined by

$$g = Lf = \sum_{i=1}^m (\mu_i \circ f + \eta_i) \circ (\nu_i \circ \phi). \quad (6)$$

In the usual manner, with E denoting expectation,

$$E\{g\} = \sum_{i=1}^m \mu_i \circ f \circ (\nu_i \circ \phi). \quad (7)$$

Let T'_y denote translation by y , $T'_y(x) = x + y$, let $\sim$ denote inverse Fourier transform, and let $\|\cdot\|_p$ denote the L^p norm. Directly from the definition of wide-sense stationary and noise power spectral density it follows that

$$E\left\{(g - E\{g\}) \left[(g - E\{g\}) \circ T'_y\right]\right\} = \left[\sum_{i=1}^m N_i^2 |D_i|^2 |\hat{\phi}|^2 \right]^\sim(y), \quad (8)$$

and, for $y = 0$, that

$$E\left\{(g - E\{g\})^2\right\} = \frac{1}{(2\pi)^n} \left| \sum_{i=1}^m N_i^2 |D_i|^2 |\hat{\phi}|^2 \right|_1. \quad (9)$$

The simplest configuration for L is all distributions equal, all deconvolvers trivial, and all random processes identically distributed:

$$\mu_1 = \mu_0, \quad \nu_1 = \delta, \quad N_1^2 = N_0^2, \quad \text{for } i = 1, 2, \dots, m. \quad (10)$$

Then

$$E\{g\} = m\mu_0 * \phi * f, \quad E\{(g - E\{g\})^2\} = \frac{m}{(2\pi)^n} \|N_0 \hat{\phi}\|_2^2. \quad (11)$$

The utility of (7) - (9) or of (11) is that if L is followed by a linear operator u with kernel u (which could model a specific "end-use") then one can compare the function $(u(E\{g\}))^2$ with the constant function $E\{(u(g - E\{g\}))^2\}$. In the case of the simplest configuration, (10) and (11), there are the following formulas and bounds.

$$\begin{aligned} (u(E\{g\}))^2 &= E\{ug\}^2 = (u * (m\mu_0 * \phi * f))^2 = m^2 \left[(\hat{u}\hat{\mu}_0 \hat{\phi} \hat{f}) \right]^2 \\ &\leq \left[\frac{m}{(2\pi)^n} \|\hat{u}\hat{\mu}_0 \hat{\phi} \hat{f}\|_1 \right]^2 \\ &\leq \left[\frac{m}{(2\pi)^n} \right]^2 \|\hat{u}\hat{\mu}_0 \hat{\phi}\|_2^2 \|\hat{f}\|_2^2 \quad \text{when } f \in L^2(\mathbb{R}^n); \end{aligned} \quad (12)$$

and

$$E\{(u(g - E\{g\}))^2\} = \frac{m}{(2\pi)^n} \|\hat{u}N_0 \hat{\phi}\|_2^2. \quad (13)$$

The function $E\{ug\}$ is referred to as the signal, its square $E\{ug\}^2$ is referred to as the signal power or energy, and $E\{(u(g - E\{g\}))^2\}$ is referred to as the noise power. Typically the ratio of $E\{ug\}^2$ to $E\{(u(g - E\{g\}))^2\}$ is considered or, alternatively, the positive square root of the ratio. Here we shall consistently use the latter. If this ratio is evaluated at some distinguished point, the value defines a "signal to noise ratio." We denote by ρ the evaluation of the absolute value of the function at some distinguished

point. Given L and for a given choice of ϕ , f , u , and φ define the signal to noise ratio

$$\mathcal{SNR}(uL) = \frac{\varphi u(E\{g\})}{\left[E\left\{(u(g-E\{g\}))^2\right\}\right]^{1/2}} \quad (14)$$

For a fixed choice of ϕ , f , u , and φ , two operators L and L' can be compared and ordered by (14).

On the other hand, for a choice of ϕ , f , u , and φ , (14) is determined for the case of the trivial operator in (10) by the pair of functions

$$m\hat{\mu}_0 \text{ and } \sqrt{m} N_0. \quad (15)$$

In general, let operators L and L' (for example, as in Fig. 3) have transfer functions and noise power spectral densities $\hat{\mu}$, N^2 and $\hat{\mu}'$, N'^2 , respectively. For a choice of u we shall say that $uL|uL'$ (i.e., " uL divides uL' ") if there exists a "quotient" function $\hat{q} \in L^\infty(\mathbb{R}^n)$ such that $\hat{u}\hat{q}\hat{\mu} = \hat{u}\hat{\mu}'$. If $uL|uL'$ and $|\hat{q}|^2|\hat{u}|^2N^2 \leq |\hat{u}|^2N'^2$, we say that $uL \geq uL'$.

This definition is motivated by the following. As usual, let ϕ be such that a linear operator Ω with kernel q can be associated with $\hat{q}$ by considering $\hat{q}\phi$. Let $\mathcal{S}$ be any continuous, translation invariant, linear operator. For fixed u if $uL \geq uL'$, then $\frac{\mathcal{SNR}(u\mathcal{S}L)}{\mathcal{SNR}(u\mathcal{S}L')} \geq 1$. Consequently, $\sup_{\mathcal{S}} \mathcal{SNR}(u\mathcal{S}L) \geq \sup_{\mathcal{S}} \mathcal{SNR}(u\mathcal{S}L')$.

Next consider the operator L diagrammed in Fig. 3 for the case in which $\mu_1, \mu_2, \dots, \mu_m$ are distinct and strongly coprime (i.e., satisfy (4)). An obvious consequence is $\sum_{i=1}^m |\hat{\mu}_i(\omega)|^2 > 0$ and, equivalently,

$$0 \neq (\hat{\mu}_1(\omega), \hat{\mu}_2(\omega), \dots, \hat{\mu}_m(\omega)) \in \mathbb{C}^n, \quad \omega \in \mathbb{R}^n. \quad (16)$$

Consequently we can visualize (16) as is shown in Fig. 4a. A similar

illustration can be used to visualize $\hat{f}(\omega)(\hat{\mu}_1(\omega), \hat{\mu}_2(\omega), \dots, \hat{\mu}_m(\omega)) = (\hat{f}(\omega)\hat{\mu}_1(\omega), \hat{f}(\omega)\hat{\mu}_2(\omega), \dots, \hat{f}(\omega)\hat{\mu}_m(\omega))$, except the "curve" passes through the origin if and only if $\hat{f}(\omega) = 0$. The power spectral densities are real and nonnegative (thus we write N_i^2 and choose $N_i \geq 0$). Assume

$$N_i(\omega) > 0, \quad \omega \in \mathbb{R}^n, \quad i = 1, 2, \dots, m. \quad (17)$$

We can visualize (17) as is shown in Fig. 4b. The case of strongly coprime multiple operators has the useful feature that the consideration of (16) and (17) pointwise in conjunction with (7) - (9) uniquely determines an alternative choice for the D_i of (5). This choice will be optimal in the sense it has the smallest $E\{(g-E\{g\})^2\}$ among all sets of deconvolvers.

Proposition. Let $N_i \in L^\infty(\mathbb{R}^n)$, $N_i(\omega) > 0$ for $\omega \in \mathbb{R}^n$, $i = 1, 2, \dots, m$. Then there is a function $D : \mathbb{R}^n \rightarrow \mathbb{C}^m$ uniquely determined (almost everywhere) by the condition that for each $\omega \in \mathbb{R}^n$, $D(\omega) = (D_1(\omega), D_2(\omega), \dots, D_m(\omega))$ solves:

$$\text{Minimize } \sum_{i=1}^m |z_i|^2 N_i^2(\omega) \text{ on the set } \left\{ z \in \mathbb{C}^m : \sum_{i=1}^m z_i \hat{\mu}_i(\omega) = 1 \right\}. \quad (18)$$

In fact

$$D_i(\omega) = \frac{\frac{\bar{\mu}_i(\omega)}{N_i^2(\omega)}}{\sum_{j=1}^m \frac{|\hat{\mu}_j(\omega)|^2}{N_j^2(\omega)}} \quad (19)$$

Proof. Any z that satisfies (18) is clearly contained in the linear subspace of $\mathbb{C}^m$ determined by the span of

$$\{(\hat{\mu}_1(\omega), 0, \dots, 0), (0, \hat{\mu}_2(\omega), 0, \dots, 0), \dots, (0, 0, \dots, 0, \hat{\mu}_m(\omega))\}. \quad (20)$$

That is, $z_i = 0$ if $\hat{\mu}_i(\omega) = 0$. Equivalently, there exists $\lambda =$

$(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{C}^m$ such that

$$(z_1 N_1(\omega), z_2 N_2(\omega), \dots, z_m N_m(\omega)) = (\lambda_1 \hat{\mu}_1(\omega), \lambda_2 \hat{\mu}_2(\omega), \dots, \lambda_m \hat{\mu}_m(\omega)). \quad (21)$$

Let $\sum'_1$ denote $\sum_{\substack{i=1 \\ \hat{\mu}_i(\omega) \neq 0}}^m$. Then (18) reduces to

$$\text{Minimize } \sum'_1 |\lambda_i|^2 |\hat{\mu}_i(\omega)|^2 \text{ on } \left\{ \sum'_1 \lambda_i \frac{|\hat{\mu}_i(\omega)|^2}{N_i(\omega)} = 1 \right\}. \quad (22)$$

From this it follows that the λ_i are all real, so that (22) has the form

$$\text{Minimize } \sum'_1 (\lambda_i |\hat{\mu}_i(\omega)|)^2 \text{ on } \left\{ \sum'_1 \lambda_i |\hat{\mu}_i(\omega)| \frac{|\hat{\mu}_i(\omega)|}{N_i(\omega)} = 1 \right\}. \quad (23)$$

With the new variable $\lambda_i |\hat{\mu}_i(\omega)|$, it is elementary to see that (23) has the unique solution

$$\lambda_i |\hat{\mu}_i(\omega)| = \frac{\frac{|\hat{\mu}_i(\omega)|}{N_i(\omega)}}{\sum_{j=1}^m \frac{|\hat{\mu}_j(\omega)|^2}{N_j^2(\omega)}} \quad (\text{for } \hat{\mu}_i(\omega) \neq 0). \quad (24)$$

Consequently, from (21), the unique z corresponding to the minimum is $D(\omega)$ as in (19). ■

In addition to $N_i > 0$, $i = 1, 2, \dots, m$, we shall assume $N_0 > 0$.

Further, we shall assume that the N_i are sufficiently differentiable and that $\frac{1}{N_i} = O(|\omega|^p)$ for some integer p , $i = 0, 1, 2, \dots, m$ (that is, $1/N_i$ does not grow faster than $|\omega|^p$). With this we can find $\hat{\phi} = O(|\omega|^{-p'})$ so that $(D_1 \hat{\phi})^\sim \in L^2(\mathbb{R}^n)$ and for $\hat{\phi}$ sufficiently smooth and with compact support then $(D_1 \hat{\phi})^\sim \in L^1(\mathbb{R}^n)$.

Corollary. For the choice of D_1 from the Proposition,

$$\sum_{i=1}^m \hat{\mu}_i D_i = 1, \left[\sum_{i=1}^m |D_i|^2 N_i^2 \right]^{1/2} = \left[\frac{1}{\sum_{j=1}^m \frac{|\hat{\mu}_j|^2}{N_j^2}} \right]^{1/2}. \quad (25)$$

Let L_0 identify the trivial configuration of L in (10) and let L_s identify the strongly coprime configuration. Unless explicitly indicated to the contrary, L_s indicates that the deconvolvers D_i of (19) are used. The first of the functions in (25) is the transfer function for L_s and the second is the square root of the noise power spectral density. The corresponding functions for L_0 are (15). The mollifier $\hat{\phi}$ is suppressed but understood. From (25) obviously $L_s | L$ for any operator L . (With an abuse of notation we use L , L_0 and L_s to denote configurations consisting of linear operators and additive noises.) From (15) and (25) the quotient $\hat{q}$ for $L = L_0$ is $\hat{m}\hat{\mu}_0$. Let N_s^2 denote the noise power spectral density of L_s . In the sense discussed earlier let $\mathfrak{D}_0$ denote the linear operator associated with $\hat{m}\hat{\mu}_0$. That $L_s | L_0$ with quotient $\hat{m}\hat{\mu}_0$ means $L_0 = \mathfrak{D}_0 L_s$. Then $\mathfrak{D}_0 L_s$ has functions corresponding to (25) (transfer function, square root of noise power spectral density) given by

$$\hat{m}\hat{\mu}_0 \sum_{i=1}^m \hat{\mu}_i D_i = \hat{m}\hat{\mu}_0, \quad |\hat{m}\hat{\mu}_0|_{N_s} = \hat{m}|\hat{\mu}_0| \left[\sum_{i=1}^m |D_i|^2 N_i^2 \right]^{1/2} = \left[\frac{\hat{m} \frac{|\hat{\mu}_0|^2}{N_0^2}}{\sum_{j=1}^m \frac{|\hat{\mu}_j|^2}{N_j^2}} \right]^{1/2} \sqrt{\hat{m}N_0}. \quad (26)$$

By definition $\mathcal{U}L_s \geq \mathcal{U}L_0$ if $|\hat{m}\hat{\mu}_0|_{N_s}(\omega) \leq \sqrt{\hat{m}N_0}(\omega)$ on the support of $\hat{u}$, and $\mathcal{U}L_0 \geq \mathcal{U}L_s$ if $\mathcal{U}L_0 | \mathcal{U}L_s$ and $|\hat{m}\hat{\mu}_0|_{N_s}(\omega) \geq \sqrt{\hat{m}N_0}(\omega)$ on the support of $\hat{u}$. Thus, whether $\mathcal{U}L_0 \geq \mathcal{U}L_s$ or $\mathcal{U}L_s \geq \mathcal{U}L_0$ holds depends, in part, on whether one of the following inequalities holds on the support of $\hat{u}$: from (15) and (26)

$$|\hat{\mu}_0|_{N_S(\omega)} \stackrel{\leq}{\geq} \sqrt{m} N_0(\omega) \longleftrightarrow \left[\sum_{i=1}^m \frac{N_0^2(\omega)}{N_i^2(\omega)} |\hat{\mu}_i(\omega)|^2 \right]^{1/2} \stackrel{\geq}{\leq} \sqrt{m} |\hat{\mu}_0(\omega)|. \quad (27)$$

In (27) the notation means that the upper inequality symbol on the left is to be paired with the upper inequality symbol on the right and lower left with lower right.

The comparison in (27) can in special cases be viewed from a slightly different perspective. First, view the left side of the second inequality in (27) as the Fourier transform of a kernel. Define

$$\hat{e}(\omega) = \left[\sum_{i=1}^m \frac{N_0^2(\omega)}{N_i^2(\omega)} |\hat{\mu}_i(\omega)|^2 \right]^{1/2}. \quad (28)$$

We refer to $\hat{e}$ as the *envelope transfer function* corresponding to the *envelope operator* $\mathfrak{E}$ for a given strongly coprime L_S in comparison with a given L_0 . If $\sqrt{m}\mathfrak{E}$ acts on L_S , then the pair of functions associated with $\sqrt{m}\mathfrak{E}L_S$ is

$$\sqrt{m}\hat{e}, \sqrt{m}N_0. \quad (29)$$

Recall that the pair for L_0 is given by (15) (rewritten for convenience)

$$m\hat{\mu}_0, \sqrt{m}N_0. \quad (15)$$

That is, the composition of $\sqrt{m}\mathfrak{E}$ with L_S has a noise power spectral density equal to that of L_0 . If, for example, $\hat{\mu}_0$ is real and positive, then it makes sense to compare (29) with (15). It is easy to check that the condition $\hat{u}\sqrt{m}\hat{e} \geq \hat{u}m\hat{\mu}_0$ (on the support of $\hat{u}$) coincides with our definition $\mathfrak{U}L_S \geq \mathfrak{U}L_0$, and $\hat{u}m\hat{\mu}_0 \geq \hat{u}\sqrt{m}\hat{e}$ coincides with what we mean by $\mathfrak{U}L_0 \geq \mathfrak{U}L_S$. These two inequalities are precisely the content of the comparison of the right side of (27). One could say that $\sqrt{m}\mathfrak{E}$ is the normalization of L_S to the noise

power spectral density of L_0 .

For either point of view we consider $W_> = \{\omega \in \mathbb{R}^n : \hat{e}(\omega) \geq \sqrt{m}|\hat{\mu}_0(\omega)|\}$ and $W_< = \{\omega \in \mathbb{R}^n : \hat{e}(\omega) \leq \sqrt{m}|\hat{\mu}_0(\omega)|\}$. For all u such that $\hat{u}$ has support in $W_>$ it follows from (27) and the definitions that $uL_S \geq uL_0$. Consequently,

$$\frac{\mathcal{P}NR(u\Omega_0 L_S)}{\mathcal{P}NR(uL_0)} = \frac{\left\| \hat{u} \frac{\hat{\phi}}{\hat{e}} N_0 \right\|_2}{\left\| \hat{u} \frac{\sqrt{m} \hat{\mu}_0}{\hat{e}} \hat{\phi} N_0 \right\|_2} \geq 1, \quad (30)$$

where Ω_0 is used to denote the linear operator corresponding to the transfer function $m\hat{\mu}_0$ of L_0 .

Assume $\hat{\mu}_0(0) \neq 0$ and define

$$\Omega_0 = \{\omega \in \mathbb{R}^n : \forall t \in [0,1) \quad |\hat{\mu}_0(t\omega)| > 0\}.$$

Note that for $\mathbb{R}^1$ the usual definition of limiting resolution is $\sup \Omega_0$. If $\text{supp}(\hat{u})$ is compact and $\text{supp}(\hat{u}) \subset \Omega_0$, then $u\Omega_0^{-1}$ makes sense, consequently $uL_0 \geq uL_S$. Hence, if $\text{supp}(\hat{u})$ is compact and $\text{supp}(\hat{u}) \subset W_< \cap \Omega_0$, then $uL_0 \geq uL_S$. Consequently,

$$\frac{\mathcal{P}NR(uL_S)}{\mathcal{P}NR(u\Omega_0^{-1}L_0)} = \frac{\left\| \hat{u} \frac{1}{\sqrt{m} \hat{\mu}_0} \hat{\phi} N_0 \right\|_2}{\left\| \hat{u} \frac{1}{\hat{e}} \hat{\phi} N_0 \right\|_2} \leq 1. \quad (31)$$

In general the inequality cannot be extended to all of $W_< \cap \Omega_0$ because of the behavior of $1/\hat{\mu}_0$ on the boundary.

There is no information regarding $\frac{\mathcal{P}NR(uL_S)}{\mathcal{P}NR(uL_0)}$ implied by either $uL_S \geq uL_0$ or $uL_0 \geq uL_S$. Additional information is needed. For example, it may be sufficient to know the effect of the so-called "boost" $uL_0 \mapsto u\Omega_0^{-1}L_0$. In

particular, if $\text{supp}(\hat{u})$ is compact and $\text{supp}(\hat{u}) \subset \Omega_0$ then

$$\text{supp}(\hat{u}) \subset W_> \text{ and } \frac{\mathcal{YNR}(\mathcal{U}_0^{-1}L_0)}{\mathcal{YNR}(\mathcal{U}_0)} \geq 1 \longrightarrow \frac{\mathcal{YNR}(\mathcal{U}_S)}{\mathcal{YNR}(\mathcal{U}_0)} \geq 1, \quad (32a)$$

and

$$\text{supp}(\hat{u}) \subset W_< \text{ and } \frac{\mathcal{YNR}(\mathcal{U}_0^{-1}L_0)}{\mathcal{YNR}(\mathcal{U}_0)} \leq 1 \longrightarrow \frac{\mathcal{YNR}(\mathcal{U}_S)}{\mathcal{YNR}(\mathcal{U}_0)} \leq 1. \quad (32b)$$

For $\text{supp}(\hat{u}) \subset \mathbb{R}^n - \Omega_0$, or even for $\text{supp}(\hat{u}) \cap (\mathbb{R}^n - \Omega_0) \neq \emptyset$, it is often the case in applications that $\mathcal{U}_0$ is not defined. Since $\mathcal{U}_S$ is defined for all $\mathcal{U}$ it makes sense in such cases to consider $\mathcal{U}_S \geq \mathcal{U}_0$.

5. EXAMPLES: OPTIMAL DECONVOLVERS FOR CHARACTERISTIC FUNCTIONS OF SETS IN $\mathbb{R}^n$

Collections of sets in $\mathbb{R}^n$ such that the characteristic functions of the sets in the collection are strongly coprime have been described in Section 3.

Let χ_S denote the characteristic function of a set S . Consider the case in which the noise power spectral density has the form $\|\chi_S\|_1 N_0^2$, where $\|\chi_S\|_1$ is the L^1 norm of χ_S (equivalently, the Lebesgue measure of the set). For such a case, let sets $S_1, S_2, \dots, S_m$ be chosen so that, for $\mu_i = \chi_{S_i}$, the $\mu_1, \mu_2, \dots, \mu_m$ are strongly coprime. Then, from the Proposition in the previous section,

$$D_1(\omega) = \frac{\frac{\bar{\mu}_1(\omega)}{|\mu_1|_1}}{\sum_{j=1}^m \frac{|\hat{\mu}_j(\omega)|^2}{|\mu_j|_1}} \quad (33)$$

and (25) becomes

$$\sum_{i=1}^m \hat{\mu}_i D_i = 1, \left[\sum_{i=1}^m |D_i|^2 N_i^2 \right]^{1/2} = \left[\frac{1}{\sum_{j=1}^m \frac{|\hat{\mu}_j|^2}{\|\mu_j\|_1}} \right]^{1/2} N_0. \quad (34)$$

Let S_0 be any set, let $\mu_0 = \chi_{S_0}$ be its characteristic function, and consider this to be the convolver in L_0 defined by (10) (i.e., m parallel, identical convolvers). Let the noise power spectral density have the same form as above, $N_0^2 = \|\mu_0\|_1 N_0^2$. From (27) and (28) one obtains an envelope transfer function $\hat{e}_d$ and the associated comparison for these two: a convenient renormalization by the constant $\|\mu_0\|_1^{1/2}$ is made in

$$\hat{e}_d(\omega) = \frac{\hat{e}(\omega)}{\|\mu_0\|_1^{1/2}} = \left[\sum_{i=1}^m \frac{|\hat{\mu}_i(\omega)|^2}{\|\mu_i\|_1} \right]^{1/2} \geq \sqrt{m} \frac{|\hat{\mu}_0(\omega)|}{\|\mu_0\|_1^{1/2}}. \quad (35)$$

For an explicit example let $S_i \subset \mathbb{R}^2$ be the region in a focal plane of an imaging device which corresponds to a single light sensitive detector. The exposure time interval is assumed fixed and the image is assumed constant. Then $\mu_i = \chi_{S_i}$ is the idealized response function of the detector. (The actual shape of the response function can be incorporated into the mollifier ϕ of Fig. 3. That is, if α_i is the actual detector response function, and if the deviation of α_i from μ_i is due to, say, a material diffusion process that is common to all detectors, then $\alpha_i = \mu_i * \psi$, where ψ models the diffusion. Such a common ψ would not be deconvolved; one would use $\sum_{i=1}^m \mu_i * \psi \nu_i = \psi$.) Then $\hat{\mu}_i$ is what is referred to as the "detector MTF." The form of the noise power spectral density above is that of many noise processes in electro-optical detectors, that is, that the noise power spectral density is proportional to the area of the detector. Such is the case for infrared detectors and this proportionality is contained in the definition of the fami-

liar "D". The above form is also valid for the so-called background limited case. It also has this form for $\mathbb{R}^3$ when the time interval is included as the third dimension. Further, a background limited slit detector corresponds to the above forms for $\mathbb{R}^1$ with the slit width as the coordinate. (The background limited case consists of a relatively small signal of interest superimposed on a relatively large constant signal so that the noise in the signal of interest is due to the "shot" noise of the constant signal.) (For detector characteristics discussed above see, for example, [20, Ch.2].)

In Fig. 5, Fig. 6, and Fig. 7 the transfer functions for such cases are shown. In Fig. 5, a comparison is shown for the example in $\mathbb{R}^1$ described in the Introduction. The characteristic functions μ_1 and μ_2 for the two intervals $(-1,1)$ and $(-\sqrt{2},\sqrt{2})$, respectively, are strongly coprime. The envelope transfer function $\hat{e}_d$ is shown and is compared with the transfer function for the two identical, parallel convolvers as in (35) where $\mu_0 = \mu_1$. The choice $\mu_0 = \mu_1$ is used rather than $\mu_0 = \mu_2$ in this comparison because μ_1 is "better" than μ_2 in the sense that the first zero of $\hat{\mu}_1$ (i.e., its bandwidth) is greater than the first zero of $\hat{\mu}_2$. Recall from the scaling property for Fourier transforms on $\mathbb{R}^1$ that $\mu_1(x) = \mu_2(\sqrt{2}x)$ for all $x \in \mathbb{R}^1$ if and only if $\sqrt{2}\hat{\mu}_1(\sqrt{2}\omega) = \hat{\mu}_2(\omega)$ for all $\omega \in \mathbb{R}^1$. In making the comparison we use two (i.e., $m = 2$) identical parallel convolvers rather than only one so as not to bias the comparison in favor of the strongly coprime case. In terms of the applications to electro-optics, the m strongly coprime detectors must make m sequential measurements of the optical signal, and the noise for each measurement is assumed independent. Consequently, there is an inherent noise averaging due to these m measurements, and we wish to have the number of independent measurements the same for both cases. The case of sequential measurement with identical detectors for noise averaging is exactly

what is done in so-called "time delay integration" for scanned infrared detector arrays [21]. Fig. 5 illustrates the consequence of the strongly coprime condition: the envelope response is approximately an envelope for the modulus of the other two responses and, correspondingly, is without zeroes. Also, it can be observed that the envelope response decreases approximately as $1/|\omega|$.

In Fig. 6 and Fig. 7 the envelope transfer function is shown for an example in $\mathbb{R}^2$, the case of three squares Q_1, Q_2, Q_3 of side length $1, \sqrt{2}, \sqrt{3}$, respectively. The characteristic functions of these three squares are strongly coprime. The comparison (35) is illustrated by graphing the modulus of the corresponding transfer functions for two subsets of $\mathbb{R}^2$: the ω_1 -axis $\{\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_2 = 0\}$ (see Fig. 6) and the diagonal $\{\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1 = \omega_2\}$ (see Fig. 7). All graphs use the Euclidean distance as abscissa, $|\omega| = (\omega_1^2 + \omega_2^2)^{1/2}$. The comparison illustrated in Fig. 6 and Fig. 7 is for $\hat{\mu}_0 = \hat{\chi}_{Q_1}$. (As before, $\hat{\chi}_{Q_1}$ has the greatest bandwidth and the scaling property for $\mathbb{R}^n$ has the form $\mu_1(x) = \mu_2(kx)$ for $k > 0$ and for all $x \in \mathbb{R}^n$ if and only if $k^n \hat{\mu}_1(k\omega) = \hat{\mu}_2(\omega)$ for all $\omega \in \mathbb{R}^n$.) The comparison is essentially the same as that for the two intervals in $\mathbb{R}^1$. The difference between the ω_1 -axis and the diagonal illustrates that approximately the envelope response decreases as $|\omega|^{-1}$ along the ω_1 -axis and as $|\omega|^{-2}$ along the diagonal.

From (35) (and as illustrated by the figures) the following statements can be made. These are stated as "observations" because the results can not be given in terms of explicit inequalities. Some notation is helpful. Define

$$\Omega_1 = \{\omega \in \mathbb{R}^n : \forall t \in [0, 1] \ |\mu_1(t\omega)| > 0\} \text{ and } \Omega = \bigcap_{i=1}^m \Omega_i. \quad (36)$$

Observations. Let $\mu_1, \mu_2, \dots, \mu_m \in L^1(\mathbb{R}^n)$ be strongly coprime characteristic

functions of sets in $\mathbb{R}^n$ as considered above. With each μ_1 let there be associated as in Fig. 3 an additive wide-sense stationary noise with noise power spectral density of the form $\|\mu_1\|_1 N_\emptyset^2$. Let L_s be the configuration in Fig. 3 with deconvolvers determined by the Proposition. Let L_0 be the trivial configuration as in (10) with $\mu_0 = \mu_1$, $N_0 = N_1$.

Observation 1: For u with $\text{supp}(\hat{u}) \subset \Omega$, $\mathcal{U}L_s \approx \mathcal{U}L_0$.

Observation 2: For u with $\text{supp}(\hat{u}) \subset \mathbb{R}^n - \bigcup_{i=2}^m \Omega_i$, $\mathcal{U}L_s \geq \mathcal{U}L_0$.

Observation 3: For u with $\text{supp}(\hat{u}) \subset \bigcup_{i=2}^m \Omega_i - \Omega$, $\mathcal{U}L_s \leq \mathcal{U}L_0$.

Observation 4: For u with $\text{supp}(\hat{u})$ compact, $\text{supp}(\hat{u}) \subset \Omega_1$, let Ω_0^{-1} be the boost on u , $\mathcal{U}L_0 \mapsto \mathcal{U}\Omega_0^{-1}L_0$ (see (31) and (32)). If Observation 3 can be neglected then

$$\frac{\mathcal{P}\mathcal{N}\mathcal{R}(\mathcal{U}\Omega_0^{-1}L_0)}{\mathcal{P}\mathcal{N}\mathcal{R}(\mathcal{U}L_0)} \geq 1 \implies \frac{\mathcal{P}\mathcal{N}\mathcal{R}(\mathcal{U}L_s)}{\mathcal{P}\mathcal{N}\mathcal{R}(\mathcal{U}L_0)} \geq 1.$$

As discussed at (31) it is not possible to extend this to all of Ω_1 , for $\hat{\mu}_1 = 0$ on the boundary of Ω_1 . However, it still is desirable to have a means to compare L_s with the more well known, more thoroughly studied trivial configurations. In the next section this is accomplished by pushing the troublesome set $\{\hat{\mu}_1 = 0\}$ out toward infinity.

6. MORE COMPARISONS: STRONGLY COPRIME VERSUS CHANGE OF SCALE

Let L_s be the same as above. In the above L_s was compared with L_0 , where L_0 was chosen to be μ_1 and $N_1^2 = \|\mu_1\|_1 N_\emptyset^2$. In these cases μ_1 was the "best" in the sense $\Omega_1 \subset \Omega_i$ $i = 1, 2, \dots, m$. Here L_s will be compared with a one parameter family of such L . Define L_0 by the trivial configura-

tion of m parallel, identical $\mu_{\langle\Delta\rangle}$ as in (10), where $N_{\Delta}^2 = \|\mu_{\langle\Delta\rangle}\|_1 N_0^2$ and $\mu_{\langle\Delta\rangle}(x) = \mu_1\left(\frac{x}{\Delta}\right)$, $\Delta > 0$.

The primary result of this section is

Observation for Fixed Number Of Channels. Fix the number of parallel convolvers in both L_S and L_{Δ} to be m . Let the convolvers be characteristic functions of cubes on $\mathbb{R}^n$ and let the additive noise be as above. Assume that $\mathcal{U}$ is such that $\text{supp}(\hat{\mathcal{U}}) \subset \bigcup_{j=1}^m \{\omega \in \mathbb{R}^n : \omega_j = 0, 1 \neq j\}$. Then for $n \geq 2$

$$\mathcal{U}L_S \geq \mathcal{U}L_{\Delta} \quad \text{for all } 0 < \Delta \leq 1. \quad (40)$$

Corollary to Observation. For the conditions in the Observation above, it is advantageous to construct L_S using sets that are as large as possible.

Application of the Corollary. In parallel scanned imaging systems with square detectors wherein the systems are ranked using some $\mathcal{U}$ meeting the conditions of the Observation (e.g., horizontal or vertical bars), the detector size should be sufficiently large so that the array of detectors fills the image, and the detector sizes in the array should constitute a strongly coprime collection. (This application depends on sufficiently high sampling rates. See Section 1.)

The Observation is illustrated in Fig. 8 for $n = 2$. For Fig. 8 L_S is as in Fig. 6: in the notation just above L_S is configured from the parallel convolvers $\mu_{\langle 1 \rangle}, \mu_{\langle \sqrt{2} \rangle}, \mu_{\langle \sqrt{3} \rangle}$, and μ_1 is the characteristic function of the unit square. For this L_S the envelope transfer function $\hat{\epsilon}_{\Delta}$ is compared with $\sqrt{3} \frac{|\hat{\mu}_{\langle\Delta\rangle}|}{\|\mu_{\langle\Delta\rangle}\|_1}^{1/2}$, as in (35), for $\Delta = 1, 0.5, 0.2$, and 0.1 . For $\Delta = 1$

see Fig. 6; for $\Delta = 0.5, 0.2$, and 0.1 , see Fig. 8. The observation in (40) is clearly evident. (Here we neglect Observation 3 of the last section by

means of a broad interpretation of $\approx$ in Observation 1.)

The Observation (40) depends on the following properties. The first, which is again an approximation, is that for $A_j = \{\omega \in \mathbb{R}^n : \omega_i = 0, i \neq j\}$, the ω_j -axis,

$$\hat{\epsilon}_d|_{A_j}(\omega) \approx C|\omega|^{-1}. \quad (41)$$

The second is that $\frac{|\hat{\mu}_{\langle \Delta \rangle}(\omega)|}{\|\mu_{\langle \Delta \rangle}\|_1^{1/2}} = \Delta^{n/2} \frac{|\hat{\mu}_1(\Delta\omega)|}{\|\mu_1\|_1^{1/2}}$. Hence, for $n \geq 2$, for $\Delta \leq 1$, and for $\omega \in A_j$, $\sqrt{m} \frac{|\hat{\mu}_1(\omega)|}{\|\mu_1\|_1^{1/2}} \leq C|\omega|^{-1} \Rightarrow \sqrt{m} \frac{|\hat{\mu}_{\langle \Delta \rangle}(\omega)|}{\|\mu_{\langle \Delta \rangle}\|_1^{1/2}} \leq C|\omega|^{-1}$.

Fig. 9 and Fig. 10 show two counterexamples for cases not addressed in the Observation. Fig. 9 is for the case of the diagonal in $\mathbb{R}^2$, and Fig. 10 is for $n = 1$. The Observation fails on the diagonal $\mathcal{D} = \{\omega = (\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1 = \omega_2\}$ because

$$\hat{\epsilon}_d|_{\mathcal{D}}(\omega) \approx C|\omega|^{-2}. \quad (42)$$

It fails for $\mathbb{R}$ because (41) holds.

If in place of characteristic functions of cubes one uses characteristic functions of disks on $\mathbb{R}^2$, then the relationship between $\hat{\epsilon}_d$ and $\sqrt{m} \frac{|\hat{\mu}_{\langle \Delta \rangle}|}{\|\mu_{\langle \Delta \rangle}\|_1^{1/2}}$ is intermediate between that of the ω_j -axis and that of the diagonal for

$$\hat{\epsilon}_d(\omega) \approx C|\omega|^{-3/2}. \quad (43)$$

The significance of Observation (40) is that it provides a qualitative lower bound for the performance of the strongly coprime configuration. To the extent performance is characterized for the $\mathcal{U}_{\Delta}$, the "envelope" consisting of the collection over all Δ is a lower bound for the performance of $\mathcal{U}_{\mathbb{S}}$.

All of the above has focused on performance away from the origin. If the

figures are rescaled so that the $\mu_{\langle \omega \rangle}$ appear fixed with a sequence of L_s constructed from convolvers of increasing support, the Observation indicates that nothing is sacrificed away from zero while the envelope transfer function near zero is substantially increased. That is, $\mathcal{M}_s \geq \mathcal{M}_\Delta$ represents a substantial enhancement near $\omega = 0$, not merely approximately identical performance. On the other hand, this uniform improvement is for the case of $\mathcal{U}$ supported by the axes. For the cases off the axes for cubes and for the case of disks there is a trade-off between some loss away from zero and the gain near zero.

APPENDIX: REVIEW OF DISTRIBUTIONS OF COMPACT SUPPORT

We present here a short review of some properties of distributions of compact support in $\mathbb{R}^n$ (sometimes called *generalized functions*). The implicit reference throughout this Appendix is the very clear monograph [22].

The simplest example of a distribution of compact support is obtained from a continuous function F of compact support. (That is, F is zero outside a bounded set in $\mathbb{R}^n$.) One associates to F a scalar valued linear operator T_F acting on infinitely differentiable functions f by the rule

$$\langle T_F, f \rangle := \int_{\mathbb{R}^n} F(x)f(x)dx.$$

The fact that $\text{supp } F$ is bounded is what makes the above integral finite and well defined for an arbitrary f . This linear operator T_F is called a *distribution of compact support*.

If the function F were not only continuous but also continuously differentiable, we can similarly associate a distribution to each of its partial

derivatives. Their relationship to T_F is given by integration by parts:

$$\langle T_{\frac{\partial F}{\partial x_j}}, f \rangle = \int_{\mathbb{R}^n} \frac{\partial F}{\partial x_j} f dx = - \int_{\mathbb{R}^n} F \frac{\partial f}{\partial x_j} dx = - \langle T_F, \frac{\partial f}{\partial x_j} \rangle.$$

This motivates defining the derivative $\frac{\partial}{\partial x_j} T_F$ of the distribution T_F by the formula

$$\langle \frac{\partial}{\partial x_j} T_F, f \rangle := - \langle T_F, \frac{\partial f}{\partial x_j} \rangle,$$

which does not require that F be differentiable. One defines in the same

way more distributions $D^\alpha T_F = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} T_F$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| =$

$\alpha_1 + \dots + \alpha_n$, by

$$\langle D^\alpha T_F, f \rangle := (-1)^{|\alpha|} \langle T_F, D^\alpha f \rangle.$$

Further, for any pair of continuous functions F, G of compact support, and any multiindices α, β , the operator $D^\alpha T_F + D^\beta T_G$

$$\langle D^\alpha T_F + D^\beta T_G, f \rangle := \langle D^\alpha T_F, f \rangle + \langle D^\beta T_G, f \rangle$$

is a linear operator on the space of infinitely differentiable functions.

The collection of all finite sums of this kind is precisely the space of distributions of compact support in $\mathbb{R}^n$. That is, for a distribution of compact support μ we mean a scalar valued linear operator which can be represented in the form

$$\mu = \sum_{|\alpha| \leq N} D^\alpha T_{F_\alpha},$$

for some integer N and continuous functions F_α of compact support ([22], p. 74)).

Let us indicate how the unit impulse δ_a at a point $a \in \mathbb{R}$ can be written in the above form. Let

$$L(x) = \begin{cases} 0 & \text{if } x < a \\ x-a & \text{if } x \geq a \end{cases}$$

and let ϕ be any twice continuously differentiable function of compact support such that for some $\epsilon > 0$ $\phi(x) \equiv 1$ for $a-\epsilon \leq x \leq a+\epsilon$. Then $(\phi L)'' = \phi''L + 2\phi'L' + \phi L''$, and the function $\phi'L'$ is continuous in spite of the discontinuity of L' at $x = a$ since $\phi' \equiv 0$ near a . Letting $F = \phi L$ and $G = -(\phi''L + 2\phi'L')$ we get

$$\delta_a = \frac{d^2}{dx^2} F + T_G.$$

The space of infinitely differentiable functions in $\mathbb{R}^n$ is usually denoted by $\mathcal{E}$ (or $\mathcal{E}(\mathbb{R}^n)$ or $C^\infty(\mathbb{R}^n)$) and the space of distributions of compact support by $\mathcal{E}'$ (or $\mathcal{E}'(\mathbb{R}^n)$).

We can essentially work with distributions of compact support as if they were ordinary continuous functions of compact support. In particular, for a distribution $T \in \mathcal{E}'$ it makes sense to compute its Fourier-Laplace transform $\hat{T}(\zeta)$ for $\zeta \in \mathbb{C}^n$. Namely, let $\zeta \cdot x = \zeta_1 x_1 + \dots + \zeta_n x_n$ ($x \in \mathbb{R}^n$, $\zeta \in \mathbb{C}^n$), then

$$\hat{T}(\zeta) := \langle T, e^{-i\zeta \cdot x} \rangle,$$

where $i = \sqrt{-1}$. For instance, for the unit impulse δ_a at a point $a \in \mathbb{R}^n$ we have $\hat{\delta}_a(\zeta) = e^{-i\zeta \cdot a}$.

From the above representation of the distributions of compact support one can see that the function $\zeta \mapsto \hat{T}(\zeta)$ has two properties. First, it is an analytic function in the whole of $\mathbb{C}^n$ (i.e., an entire function). Second, there are positive constants A, B, N such that $\hat{T}$ satisfies everywhere the estimate

$$|\hat{T}(\zeta)| \leq A(1+|\zeta|)^N e^{B|\operatorname{Im}\zeta|},$$

$\operatorname{Im} \zeta = (\operatorname{Im} \zeta_1, \dots, \operatorname{Im} \zeta_n)$. It is usual to call the space of functions satisfying these two properties the Paley-Wiener class, $PW(\mathbb{C}^n)$.

The classical Paley-Wiener theorem for functions of compact support in L^2 can be extended to the case of distributions of compact support as follows.

Theorem (Paley-Wiener-Schwartz). The Fourier-Laplace transform is a one-to-one correspondence between the spaces $\mathcal{E}'(\mathbb{R}^n)$ and $PW(\mathbb{C}^n)$.

Recall now that if F and G are continuous functions of compact support then their convolution $F * G$ is given by

$$F * G(x) = \int_{\mathbb{R}^n} F(x-y)G(y)dy = \int G(x-y)F(y)dy$$

and it is again a continuous function of compact support. Moreover, if one of them is continuously differentiable, say F , then $\frac{\partial}{\partial x_j}(F * G) = \left(\frac{\partial F}{\partial x_j}\right) * G$. This observation allows for the definition of convolution of distributions of compact support in such a way that if $T, S \in \mathcal{E}'$ then $T * S = S * T \in \mathcal{E}'$. The unit impulse at the origin δ acts as the unit of this product $T * \delta = T$. Furthermore, $(T * S)^{\hat{}}(\zeta) \equiv \hat{T}(\zeta)\hat{S}(\zeta)$, so that the convolution becomes ordinary product of the analytic functions $\hat{T}$ and $\hat{S}$.

In this paper we were interested in solving the equation

$$\mu_1 * \nu_1 + \dots + \mu_m * \nu_m = \delta,$$

where $\mu_1, \dots, \mu_m$ are given in $\mathcal{E}'$ and we need $\nu_1, \dots, \nu_m \in \mathcal{E}'$. Using the Fourier-Laplace transform this equation is equivalent to the analytic Bezout equation

$$\hat{\mu}_1(\zeta)f_1(\zeta) + \dots + \hat{\mu}_m(\zeta)f_m(\zeta) = 1,$$

with unknowns $f_1, \dots, f_m \in PW(\mathbb{C}^n)$. In particular, to solve this equation is necessary that the functions $\hat{\mu}_1, \dots, \hat{\mu}_m$ have no common zeros in $\mathbb{C}^n$.

In the text we raised the question of whether one could find a deconvolver $\nu \in \mathcal{E}'$ for a single convolutor $\mu \in \mathcal{E}'$, i.e.,

$$\mu * \nu = \delta \text{ or equivalently } \hat{\mu}(\zeta)\hat{\nu}(\zeta) \equiv 1.$$

We need that $\mu(\zeta) \neq 0$ for every $\zeta \in \mathbb{C}^n$. It follows that there is an analytic function h in $\mathbb{C}^n$ such that

$$\hat{\mu}(\zeta) = e^{h(\zeta)}.$$

Since $\hat{\mu}$ is in the Paley-Wiener class one can show without difficulty that

$$|h(\zeta)| \leq A + B|\operatorname{Im} \zeta|$$

for some positive constants A, B . Therefore, by the Liouville theorem, h must be of the form

$$h(\zeta) = -i(a_1\zeta_1 + \dots + a_n\zeta_n) + c$$

for some $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $c \in \mathbb{C}$. This proves, by the the Paley-Wiener-Schwartz theorem, that $\mu = C\delta_a$, for some constant $C \in \mathbb{C}$.

Having no common zeros is not enough to solve the analytic Bezout equation, the necessary and sufficient condition is the following.

Theorem [1], [16]. Given $\mu_1, \dots, \mu_m \in \mathcal{E}'(\mathbb{R}^n)$, the necessary and sufficient condition for the existence of deconvolvers $\nu_1, \dots, \nu_m \in \mathcal{E}'(\mathbb{R}^n)$ is that there are constants $\epsilon > 0$, $M > 0$, $C > 0$ such that

$$\left(\sum_{j=1}^m |\hat{\mu}_j(\zeta)|^2 \right)^{1/2} \geq \epsilon \frac{e^{-C|\operatorname{Im} \zeta|}}{(1+|\zeta|)^M} \text{ for all } \zeta \in \mathbb{C}^n$$

This is a purely existential theorem and the work in [5] - [10] consists in obtaining explicit choices of deconvolvers $\nu_1, \dots, \nu_m$. This is not trivial. A portion of the work [7] and [11] also consists in finding simple and practical examples of $\mu_1, \dots, \mu_m$ that satisfy the above necessary and sufficient condition for the existence of deconvolvers.

REFERENCES

- [1] J.J. Kelleher and B.A. Taylor, "Finitely generated ideals in rings of analytic functions," *Math. Annalen* 193, pp. 225-37, 1971.
- [2] C.A. Berenstein and B.A. Taylor, "A new look at interpolation theory for entire functions of one variable," *Advances in Mathematics* 33, pp. 109-43, 1979.
- [3] C.A. Berenstein and B.A. Taylor, "Interpolation problems in $\mathbb{C}^n$ with applications to harmonic analysis," *J. Analyse Math.* 38, pp. 188-254, 1980.
- [4] C.A. Berenstein and B.A. Taylor, "Mean periodic functions," *Intern. J. Math. and Math. Sciences* 3, pp. 199-236, 1980.
- [5] C.A. Berenstein, B.A. Taylor, and A. Yger, "On some explicit deconvolution formulas," *Technical Digest, Signal Recovery*, Optical Society of America, Winter Meeting 1983, pp. WA4-1 to WA4-4, 1983.
- [6] C.A. Berenstein, B.A. Taylor, and A. Yger, "Sur quelques formules explicites de deconvolution," *Journal of Optics (Paris)* 14, pp. 75-82, 1983.
- [7] C.A. Berenstein and A. Yger, "Le probleme de la deconvolution," *J. Funct. Anal.* 54, pp. 113-60, 1983.
- [8] C.A. Berenstein, "An application of the Andersson-Berndtsson integral representation formula," *Revue de l'Institut Elie Cartan* 8, pp. 113-60, 1983.
- [9] C.A. Berenstein and A. Yger, "Analytic Bezout Identities," to appear in *Advances in Applied Math.*, vol. 10: 1, 1989.
- [10] C.A. Berenstein and B.A. Taylor, "Overdetermined systems of convolution equations," to appear *Proc. Sixth Army Conf. Applied and Comput. Math.*, Boulder, CO, June 1988.
- [11] E.V. Patrick, "Deconvolution for the case of multiple characteristic functions of cubes in $\mathbb{R}^n$," Systems Research Center, Univ. of Maryland, TR86-46.
- [12] C.A. Berenstein and E.V. Patrick, "Multiple operator deconvolution with additive noise; the Envelope Operator," Systems Research Center, Univ. of Maryland, TR87-109.
- [13] E.V. Patrick, "Some solutions to problems in depth vision from differential geometry and deconvolution methods," Ph.D. dissertation, Dept. Math., Univ. of Maryland, December 1987.
- [14] C.A. Berenstein, P.S. Krishnaprasad, and B.A. Taylor, "Deconvolution methods for multi-sensors," ARO contract DAAG29-81-D- 0100, DTIC no. AD A152 351, 68 pages, 1984.

- [15] A.R. Davies, M. Iqbal, K. Maleknejad, and T.C. Redshaw, A comparison of statistical regularization and Fourier extrapolation methods for numerical deconvolution," in *Numerical treatment of inverse problems in differential and integral equations*, P. Deuffhard and E. Hairer, eds., Boston, MA: Birkhäuser, 1981, pp. 320-334.
- [16] L. Hörmander, "Generators for some rings of analytic functions," *Bulletin Amer. Math. Soc.* 73, pp. 943-9, 1967.
- [17] A. Mórozov, *Methods for solving incorrectly posed problems*, New York, NY: Springer-Verlag, 1984.
- [18] C.W. Groetsch, *The theory of Tikhonov regularization for Fredholm equations of the first kind*, Boston, MA: Pitman Publ., 1984.
- [19] W.B. Davenport and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*, New York, NY: McGraw-Hill, 1958.
- [20] R.H. Kingston, *Detection of optical and infrared radiation*, New York, NY: Springer-Verlag, 1978.
- [21] K.H. Norsworthy and M.W. Pollock, "Parsimonious sampling for scanning mosaic sensors," in SPIE vol. 750 *Infrared Systems and Components*, pp. 73-78, 1987.
- [22] J. Barros-Neto, *An introduction to the theory of distributions*, Malabar, FL: Krieger Publ. Co., 1981.

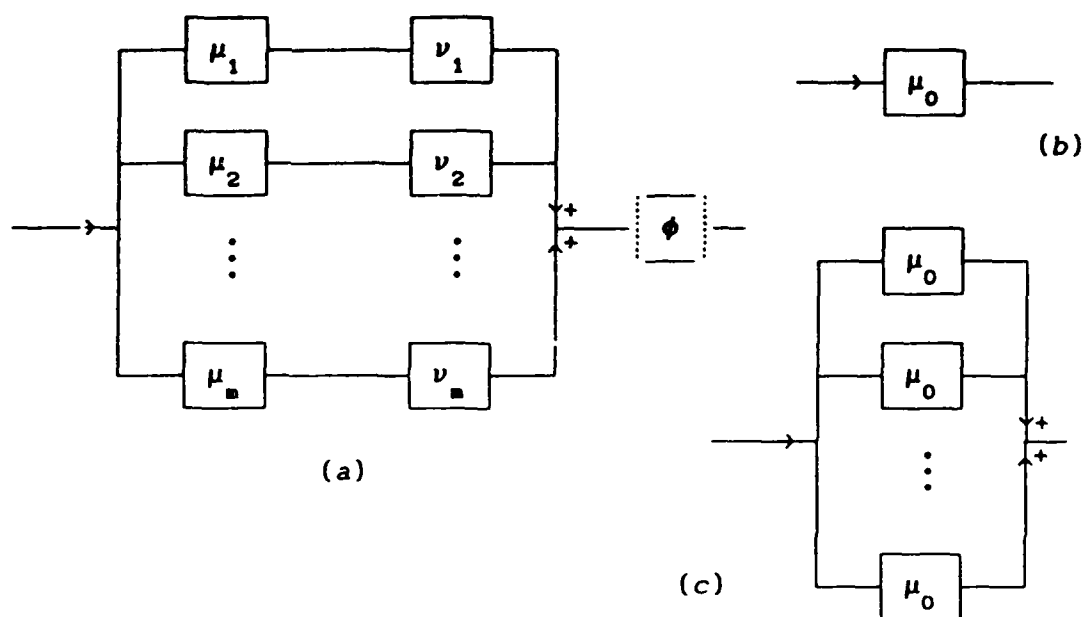


Figure 1. (a) Multiple parallel linear operators with distinct distributions μ_i . Single operator (b) and the multiple parallel operators with identical distributions μ_0 (c).

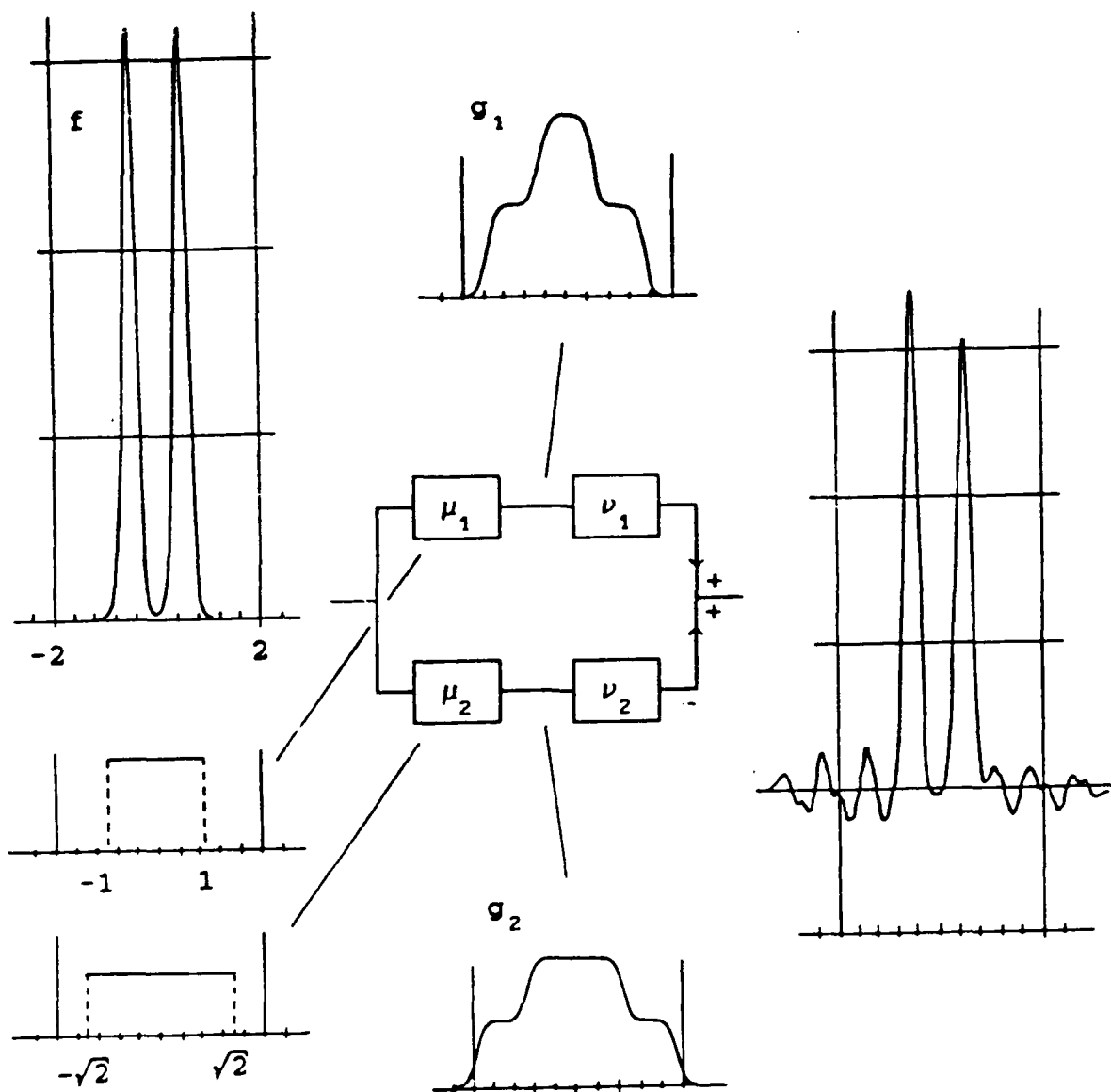


Figure 2

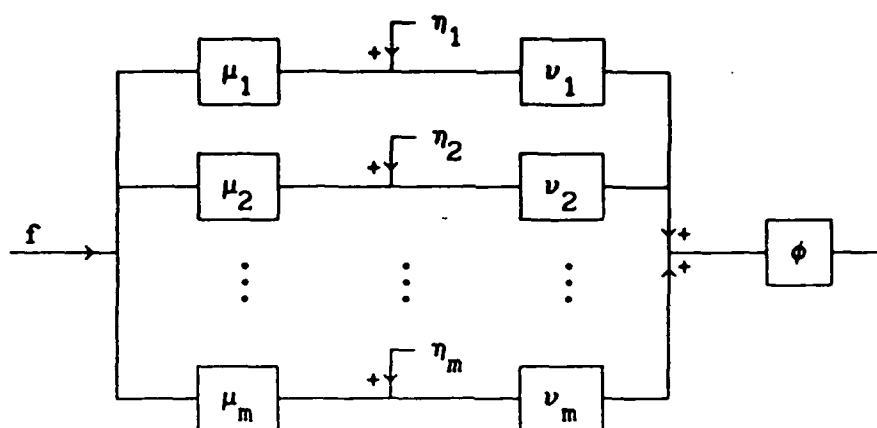


Fig. 3. Multiple operator configuration consisting of m parallel convolvers μ_i , m noise signals η_i , and m deconvolvers ν_i .

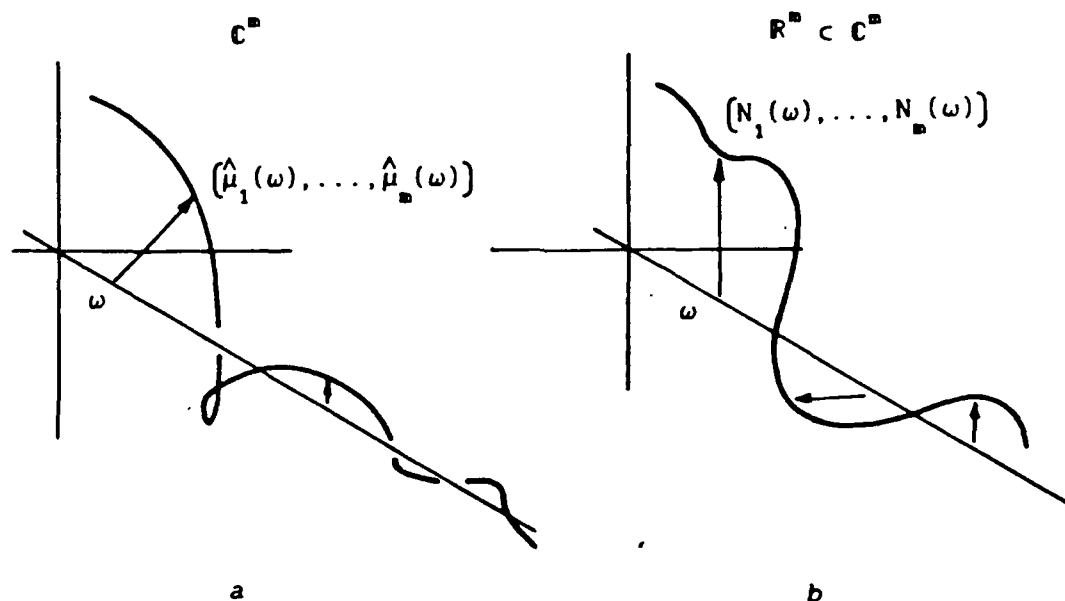


Figure 4

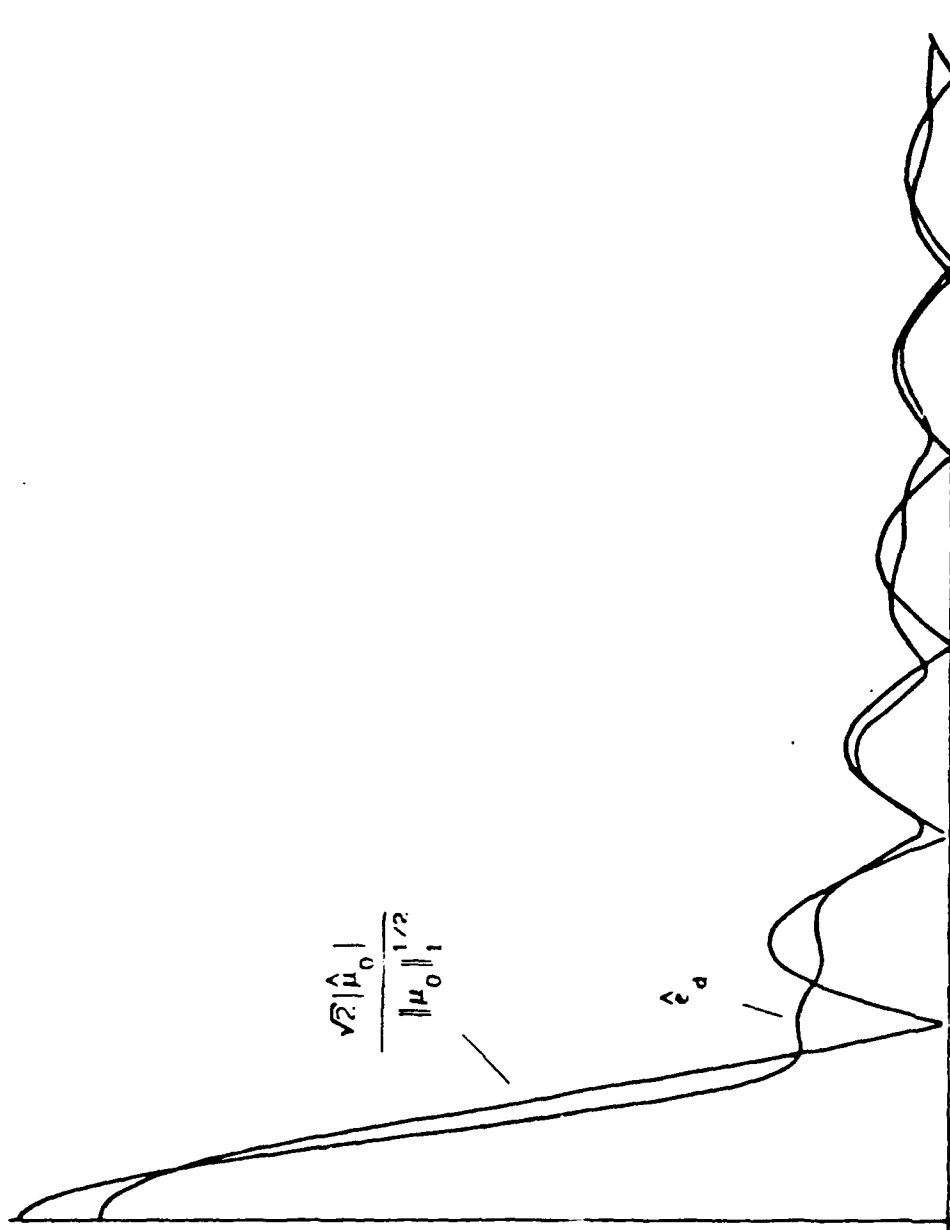


Figure 5 R^1

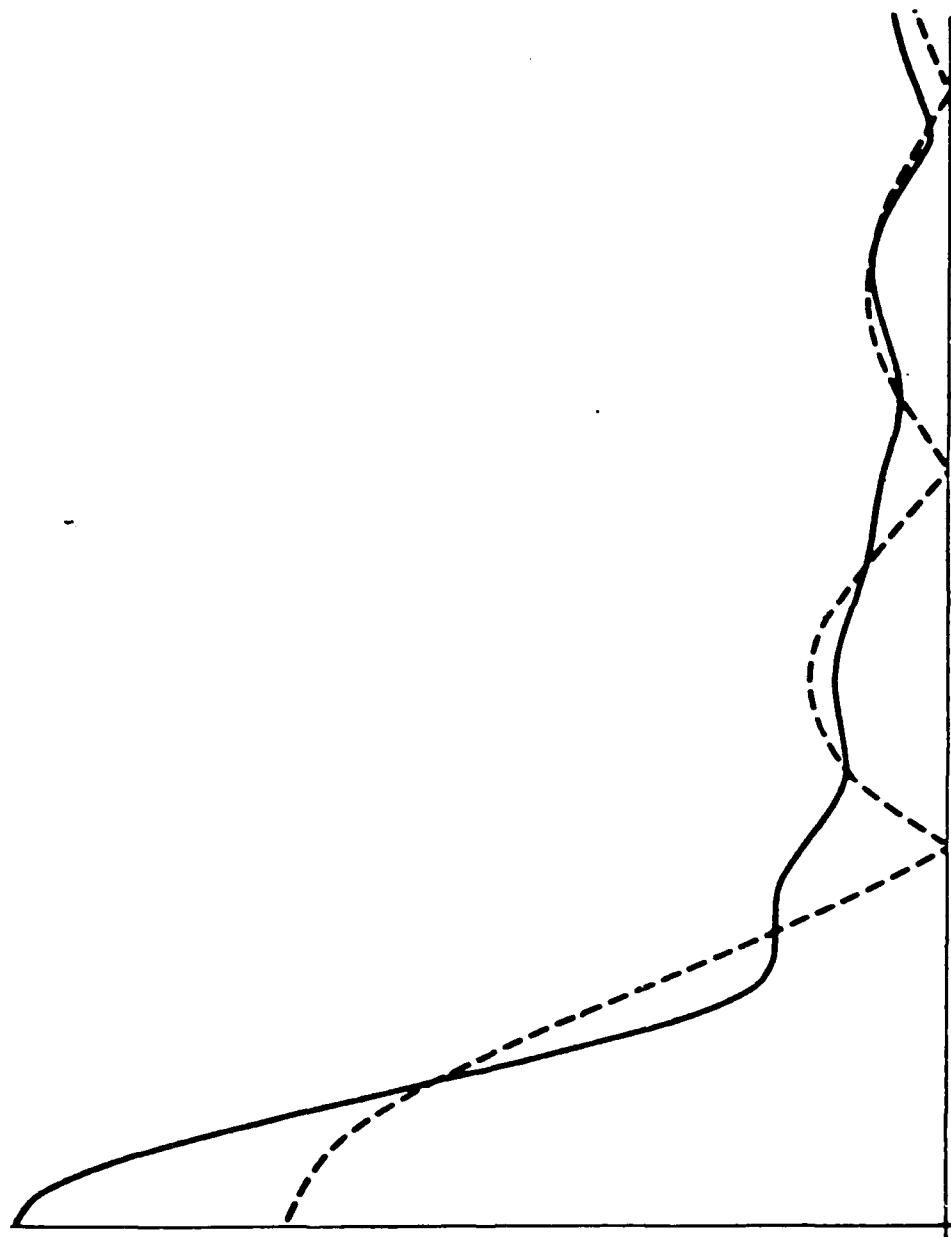


Figure 6 R^2 ; abscissa = ω_1 -axis solid = $\hat{\epsilon}_d$; broken: $\sqrt{3} \frac{|\hat{\mu}_0|}{|\mu_0|^{1/2}} \cdot \hat{\mu}_0 = \hat{\chi}_0$

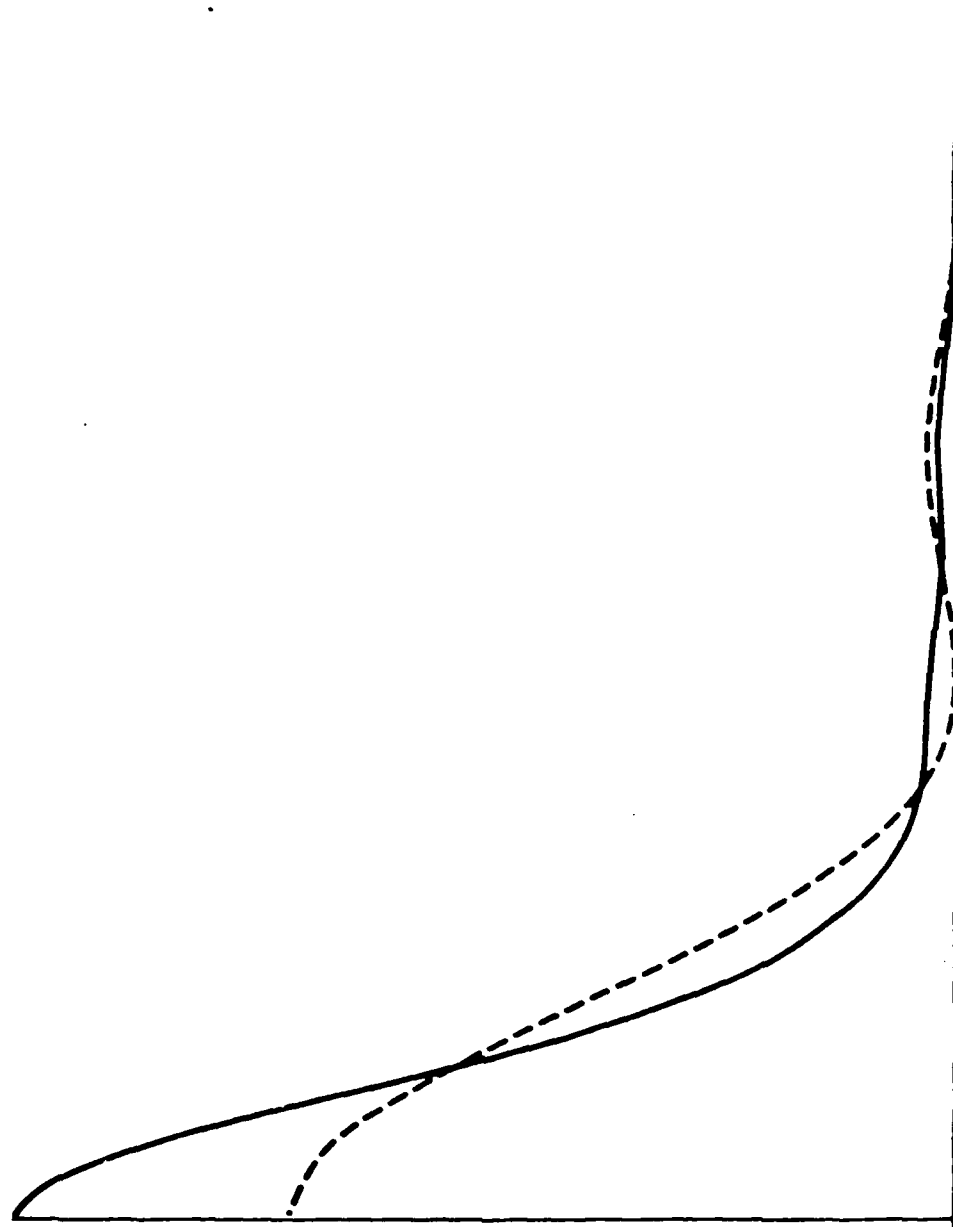


Figure 7 $\hat{\mu}_0^2$; abscissa = $\hat{\epsilon}_d$; broken: $\sqrt{3} \frac{|\hat{\mu}_0|}{|\mu_0|^{1/2}} \cdot \hat{\mu}_0 = \hat{\lambda}_{0,1}$

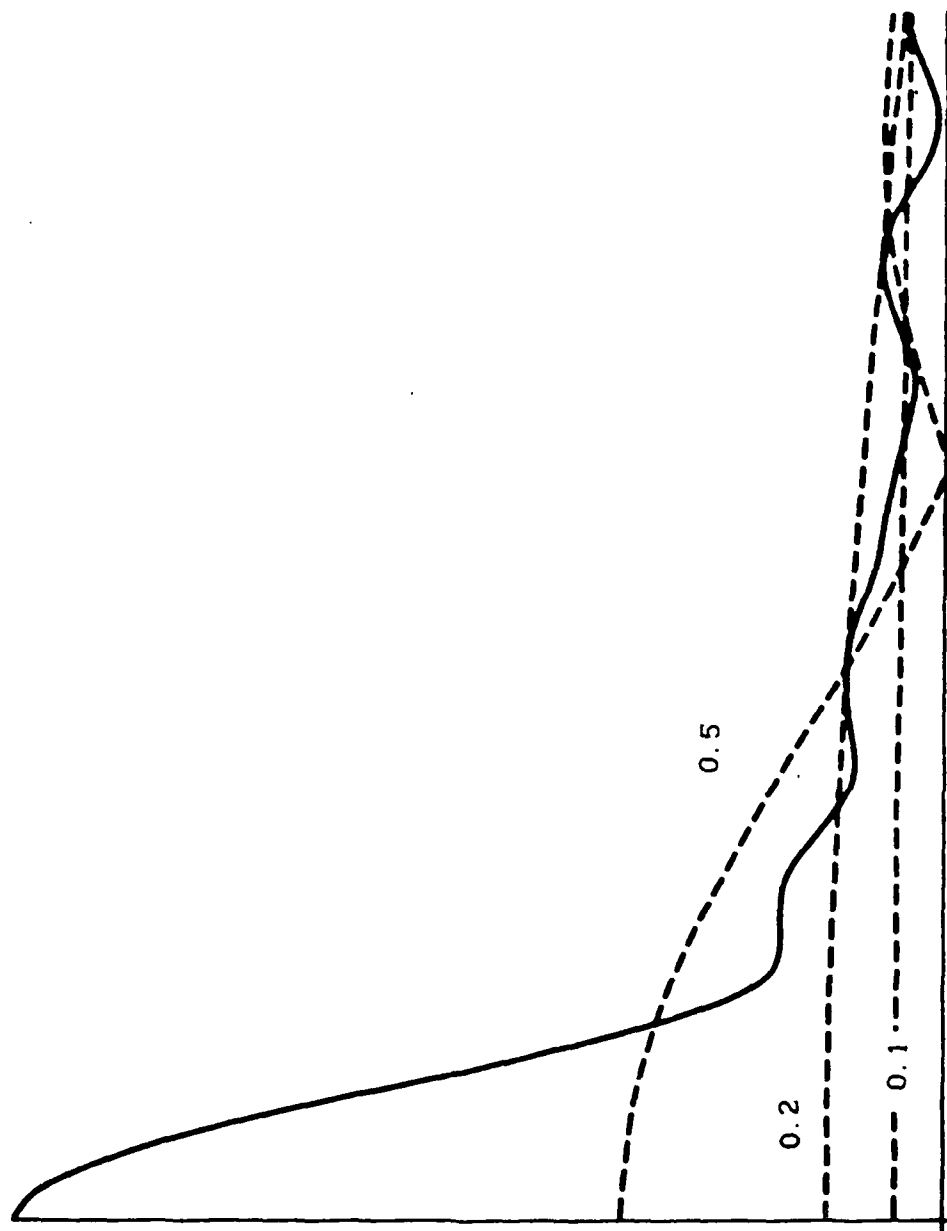


Figure 8 R^2 ; abscissa = w_1 -axis solid: $\hat{c}_d$; broken: $\sqrt{3} \frac{|\hat{\mu}_{<0>}|}{\|\mu_{<0>}\|^{1/2}}$, $\delta = 0.5, 0.2, 0.1$

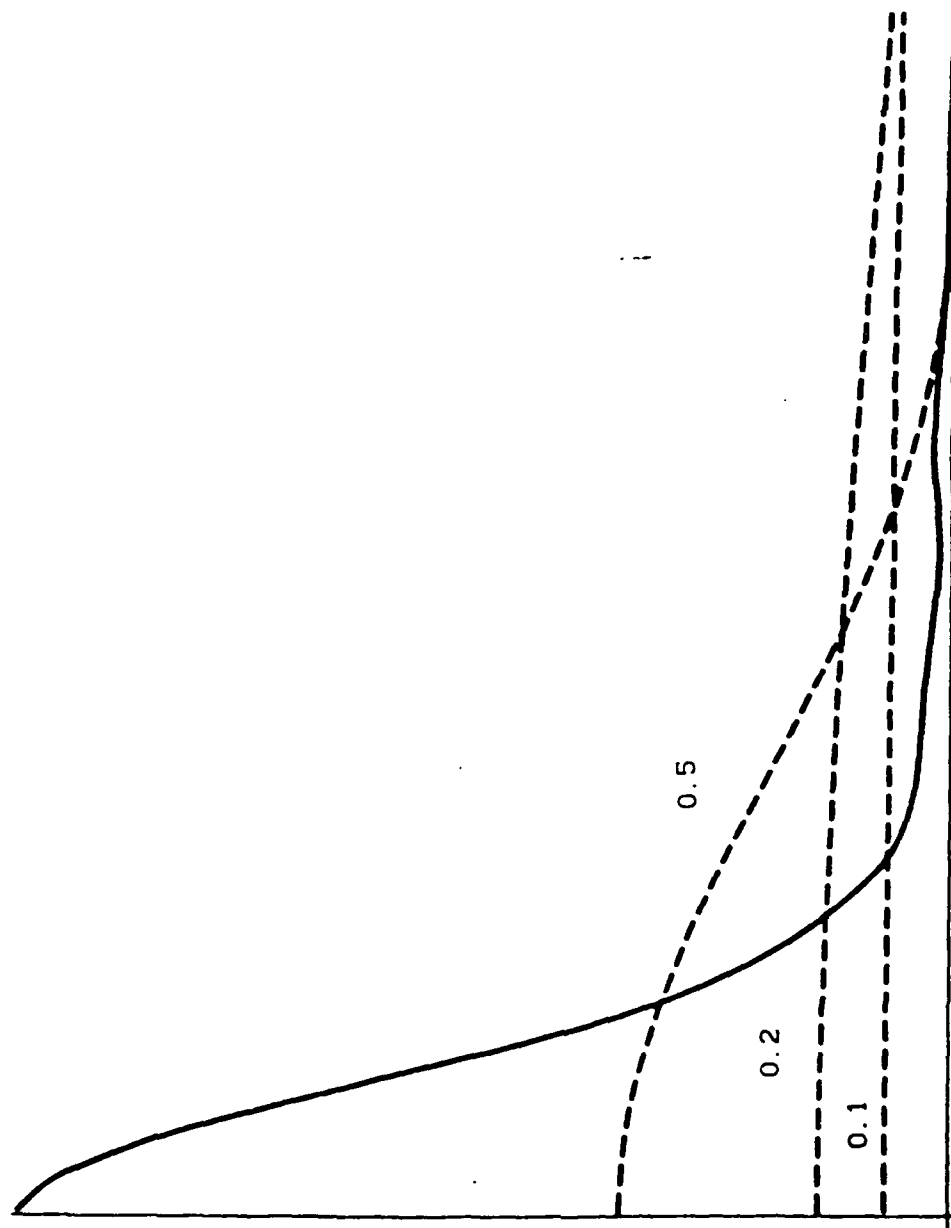


Figure 9 R^2 ; abscissa = diagonal solid: $\hat{c}_d$; broken: $\sqrt{3} \frac{|\hat{\mu}_{<\omega}|}{\|\mu_{<\omega}\|^{1/2}}$, $\delta = 0.5, 0.2, 0.1$

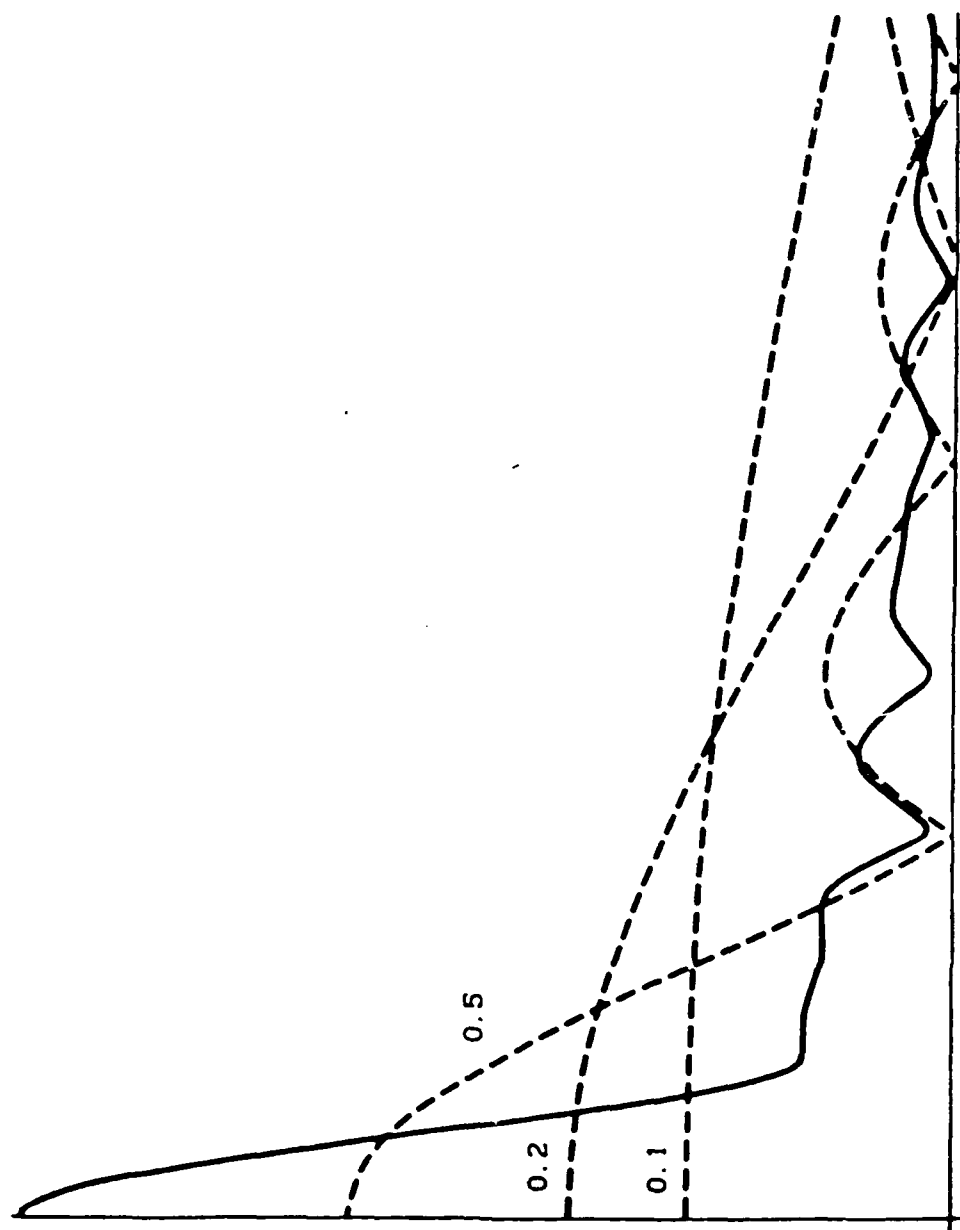


Figure 10 R^1 ; solid: $\hat{e}_d$; broken: $\sqrt{2} \frac{|\hat{\mu}_{<O>}|}{\|\mu_{<O>}\|_1^{1/2}}$, $\Delta = 0.5, 0.2, 0.1$

CHAPTER 3

A local version of the two-circles theorem

by

C.A. Berenstein and R. Gay

1. Introduction. -

One of the oldest questions in integral geometry has been that of recovering a function f in R^n from the knowledge of its average over balls. It is easy to see that unless f decays sufficiently fast at infinity the average over all balls of a fixed radius could vanish without f being identically zero. It is not always possible to assume such decay but a very elegant result of Zalcman [20] and, independently, Brown-Schreiber-Taylor [10], describes explicitly a countable set E_n such that averages over all balls of radii r_1, r_2 suffice as long as $r_1/r_2 \notin E_n$. This-"two circles" theorem can be described as saying that the map

$$C(R^n) \rightarrow C(R^n) \oplus C(R^n)$$

$$f \rightarrow \left(\int_{B(x,r_1)} f(y)dy, \int_{B(x,r_2)} f(y)dy \right)$$

is injective if and only if $r_1/r_2 \notin E_n$.

($B(x,r) = \{y \in R^n : |x-y| < r\}$). Under slightly stronger conditions on the quotient r_1/r_2 this map has also a continuous and explicit inverse [8]. This result and other variants of the so-called Pompeiu problem have been generalized to symmetric spaces (see the surveys [21], [1] for positive results and their limitations)

In practical situations of a tomographic nature one is limited to balls that fit into a fixed region Ω . One could take smaller and smaller balls when approaching the boundary $\partial\Omega$ of Ω , this is roughly the situation when we consider the case

Ω = unit ball of $\mathbb{R}^n$ as the hyperbolic space, but it is clear that it might be hard to accomplish if we are dealing with physical devices whose size cannot be made infinitesimally small or cannot even be changed at will. It is this kind of problem that we call a local version of the two-circles theorem. The main difference with the above mentioned results is that we do not have any longer the whole group of Euclidean motions at our disposal which was the crucial ingredient lying behind the two circles theorem and its generalizations. The inversion formula of [8] would allow us to reconstruct f away from $\partial\Omega$ but gives no indication of whether we could change the values of f in a collar-like region near $\partial\Omega$ without affecting its average. There is some recent work on systems of convolution equations in convex domains which deals with this type of question [4] but the hypotheses required are far too restrictive to be satisfied by our simple looking problems. Nevertheless, using a combination of ideas from classical harmonic analysis and results of Cormack-Quinto on the Radon transform on spheres [12] we are able to prove the following.

Theorem. - Let r_1, r_2 be positive numbers, $r_1/r_2 \notin \mathbb{E}_n$, Ω an open subset of $\mathbb{R}^n$ such that every point lies in an open ball contained in Ω of radius strictly larger than $r_1 + r_2$. If $f \in C(\Omega)$ satisfies

$$\int_{B(x, r_j)} f(y) dy = 0 \quad \text{for every } \overline{B}(x, r_j) \subseteq \Omega, \quad j = 1, 2$$

then $f \equiv 0$. Furthermore, this statement does not hold if Ω fails the above geometrical restriction.

The method of proof allows us to generalize this theorem greatly, providing in particular new local mean-value theorems for harmonic functions.

We will like to express our appreciation to Professor L. Zalcman who called our attention to these problems.

The second author wishes to thank the Mathematics Department of the University of Maryland for its hospitality while this work was carried out.

2. Preliminaries. -

We will follow the standard notation for distributions found in [14]. We denote $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ ($r > 0$), $\bar{B}(x, r)$ its closure and χ_r the characteristic function of $B(0, r)$. Let Ω be an open set in $\mathbb{R}^n$, $\Omega_r = \{x \in \Omega : d(x, \Omega^c) < r\}$. For a locally-integrable function f in an open set Ω the average

$$(1) \quad A_r(f, x) = \frac{1}{\omega_n r^n} \int_{B(x, r)} f(y) dy$$

is defined for $x \in \Omega_r$. Here ω_n is the volume of $B(0, 1)$. If we let $\mu_r = \chi_r / \omega_n r^n$, we can interpret this average as a convolution and hence it makes sense to define it for $f \in \mathcal{D}'(\Omega)$ giving a distribution $A_r(f)$ in $\mathcal{D}'(\Omega_r)$, namely $A_r(f) = f * \mu_r$. Therefore, for uniqueness questions, if the averages of f are zero, by restriction ourselves to Ω_ε , $\varepsilon > 0$ small, we can assume $f \in C^\infty$. Henceforth, all distributions with vanishing averages will be assumed to be C^∞ functions in Ω .

For $r > 0$, we denote by σ_r the distributions defining the spherical average

$$(2) \quad \lambda_r(f, x) = \int_{S^{n-1}} f(x+ry) d\sigma(y) = (\sigma_r * f)(x)$$

$d\sigma$ is the normalized Lebesgue measure on S^{n-1} .

For $T \in \mathcal{E}'$ the Fourier transform

$$\hat{T}(\zeta) = \langle T_x, e^{-i(x|\zeta)} \rangle, \quad (x|\zeta) = \sum_j x_j \zeta_j,$$

is an entire function in $\mathbb{C}^n$ which satisfies, for some $A, N > 0$, the estimates

$$(3) \quad |\hat{T}(\zeta)| < A(1+|\zeta|)^N \exp(H(\operatorname{Im}\zeta)).$$

where $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}^n$, $\operatorname{Im}\zeta = \eta$ and H is the supporting function of the support of T , i.e.:

$$H(\eta) = \operatorname{Max}\{(x|\eta) : x \in \operatorname{supp} T\}$$

Note that H is also the supporting function of $\operatorname{cv}(\operatorname{supp} T)$, the convex hull of $\operatorname{supp} T$. The Fourier transform is an isomorphism between the convolution algebra $\mathcal{E}'(\mathbb{R}^n)$ and $\hat{\mathcal{E}}'(\mathbb{R}^n)$, the algebra of entire functions of exponential type and polynomial growth on the real axis.

A distribution T will be called invertible (or $\hat{T}$ is slowing decreasing) if whenever $S \in \mathcal{E}'(\mathbb{R}^n)$ and $\hat{S}/\hat{T}$ is an

entire function, then there is a distribution $U \in E'(\mathbb{R}^n)$ such that $\hat{U} = \hat{S}/\hat{T}$, that is

$$(4) \quad S = T * U$$

and we have the identity

$$(5) \quad H_S = H_T + H_U$$

or, what amounts to the same thing

$$(6) \quad cv(\text{supp} U * T) = cv(\text{supp} U) + cv(\text{supp} T),$$

where, for two sets $A, B \subseteq \mathbb{R}^n$ we have $A \pm B = \{x \pm y; x \in A, y \in B\}$.

We will need to use that μ_r is an invertible distribution.

This will follow from the explicit formula for $\hat{\mu}_r$ given below and the characterization of invertible distributions: T is invertible if and only if there is a positive constant a such that for all $\xi \in \mathbb{R}^n$

$$(7) \quad \text{Max} \{ |T(\xi + \eta)|; \eta \in \mathbb{R}^n, |\eta| < a \cdot \log(2 + |\xi|) \} > (a + |\xi|)^{-a}$$

The Fourier transform of a radial distribution T is a radial function, i.e.: if:

$$\langle T, f \circ A^{-1} \rangle = \langle T, f \rangle$$

for every $A \in O(n)$ then

$$\hat{T}(\zeta) = \hat{T}(A \cdot \zeta)$$

for every $A \in O(n)$, $\zeta \in \mathbb{C}^n$, and depends, for $\xi \in \mathbb{R}^n$, only on $|\xi|$. Hence we consider the associated even entire function $\tilde{T}$ of one variable by

$$(8) \quad \tilde{T}(|\xi|) = \hat{T}(\xi) \quad \text{and} \quad \hat{T}(\zeta) = \tilde{T}((\zeta_1^2 + \dots + \zeta_n^2)^{1/2})$$

Let us call $\mathcal{E}'_0(\mathbb{R}^n)$, the space of radial distributions. This correspondence establishes an isomorphism between the algebras $\hat{\mathcal{E}}'_0(\mathbb{R}^n)$ and $\hat{\mathcal{E}}'_0(\mathbb{R})$. Using this notation we have

$$(9) \quad \tilde{\mu}_r(t) = n 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) J_{\frac{n}{2}}(rt) / (rt)^{n/2}$$

$$(10) \quad \tilde{\sigma}_r(t) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) J_{\frac{n-2}{2}}(rt) / (rt)^{\frac{n-2}{2}}$$

and, more generally, if $f(x) = \phi(|x|)$ is a radial function of compact support

$$(11) \quad \hat{f}(\zeta) = \tilde{f}(t) = \frac{(2\pi)^{n/2}}{t^{\frac{n-2}{2}}} \int_0^\infty \phi(\rho) J_{\frac{n-2}{2}}(\rho t) \rho^{n/2} d\rho \quad (|\zeta|=t)$$

To show that μ_r is invertible it is now sufficient to recall the asymptotic development of the Bessel functions [19] on the positive real axis

$$(12) \quad J_\nu(t) = \sqrt{\frac{2}{\pi}} t^{-1/2} \cos(t - \frac{\pi}{4} - \frac{\nu\pi}{2}) + O(t^{-3/2})$$

It follows, for $|\xi| > 1$ and some $C > 0$

$$\text{Max}\{|\hat{\mu}_r(\xi + \eta)| : \eta \in \mathbb{R}^n, |\eta| < \pi\} > C|\xi|^{-\frac{n+1}{2}}$$

which is the condition of invertibility.

From (12) we also obtain MacMahon's asymptotic development of the positive zeros $\alpha_{k,\nu}$ of J_ν

$$0 < \alpha_{1,\nu} < \alpha_{2,\nu} < \dots$$

$$(13) \quad \alpha_{k,\nu} = (2k+1)\frac{\pi}{2} + (2\nu+1)\frac{\pi}{4} + O(1/k)$$

which will be used further on.

3. Series development of mean-periodic functions.

Let Ω be an open convex set in $\mathbb{R}^n$ and $K = \text{cv}(\text{supp } \mu)$, $\mu \in E'(\mathbb{R}^n)$. We say that a function $f \in C^\infty(\Omega - K)$ is mean-periodic with respect to μ if

$$(14) \quad \mu * f(x) = \langle \mu_y, f(x-y) \rangle = 0 \quad \text{for all } x \in \Omega$$

If an exponential-polynomial, that is a finite linear combination of terms of the form $x^j e^{i(x|\zeta)}$

$(x^j = x_1^{j_1} \dots x_n^{j_n}, j_k \in \mathbb{N}, 1 \leq k \leq n)$, is mean-periodic with respect to μ then the frequencies ζ must satisfy $\hat{\mu}(\zeta) = 0$ since

$$(15) \quad (\mu * e^{i(\cdot|\zeta)})(x) = \hat{\mu}(\zeta) e^{i(x|\zeta)}$$

When the zeros of $\hat{\mu}$ are simple no non constant monomials can appear. More generally if a monomial x^j appears with non zero coefficient then

$$\left(\frac{\partial^{|j|}}{\partial \zeta^j} \right) \hat{\mu}(\zeta) = 0$$

for the corresponding frequency ζ .

For $n = 1$ there is a well-known series development for such functions in terms of the exponential polynomial solution of the same convolution equation (14) due to L. Schwartz [18], [15], [13]. The case of interest for us is $n \geq 2$, μ invertible. In this case, a development in terms of integrals over the zero set of $\hat{\mu}$ has been proved when $\Omega = \mathbb{R}^n$ [6]. For Ω arbitrary convex set, a similar development has been proved in [4] but only for a very restrictive class of invertible distributions. In all these cases one obtains also some knowledge of the behavior of the terms involved in this development. Unfortunately, the distributions μ_r , though invertible, do not satisfy the conditions required in [4], as was shown (for a different reason) in [3]; moreover we are interested in $\Omega = B(0, R)$. Therefore we cannot depend on any of the previously known

results. We obtain here a series development without additional information on the coefficients that appear in it; nevertheless the existence of this development is all we need later.

Proposition 1 - Let Ω be an open convex subset in $\mathbb{R}^n$ ($n > 2$), $\mu \in E'(\mathbb{R}^n)$ an invertible distribution, $K = \text{cv}(\text{supp } \mu)$. Any function $f \in C^\infty(\Omega - K)$, mean periodic with respect to μ can be written as

$$(16) \quad f(x) = \sum_{j=1}^{\infty} P_j(x) \quad (x \in \Omega - K)$$

with P_j exponential-polynomials also mean-periodic with respect to μ , and the series is convergent in the C^∞ -topology of $\Omega - K$. Furthermore, given a sequence $(s_j)_{j \geq 1}$ of positive numbers, letting $P_0 = 0$, we can choose the P_j so that the absolute value of all frequencies in P_{j+1} exceeds the largest absolute value of the frequencies in P_j by at least s_{j+1} .

Proof. Let us show first that, for any $s > 0$, the exponential polynomials which are mean-periodic with respect to μ and whose frequencies lie outside the ball of center 0 and radius s in $\mathbb{C}^n$ are dense in the space $N = \{f \in C^\infty(\Omega - K) : \mu * f = 0 \text{ in } \Omega\}$. N is a closed subspace of a Frechet space and we only need to show that if $v \in E'(\Omega - K)$ is orthogonal to the above exponential-polynomials then v is orthogonal to N . Hence $(\hat{v})^\sim$ is divisible by $\hat{\mu}$ at every point of $\mathbb{C}^n \setminus \overline{B}(0, s)$. Since $n > 2$, by Hartogs' theorem, $(\hat{v})^\sim / \hat{\mu}$ is an entire function. Since μ is invertible there is a distribution $T \in E'(\mathbb{R}^n)$ such that

$$\check{\nu} = \mu * \check{T}$$

We need to know where is the support of T . By (6)

$$cv(\text{supp}\check{\nu}) = cv(\text{supp}\mu) + cv(\text{supp}\check{T})$$

or

$$cv(\text{supp}T) - K = cv(\text{supp}\nu) \subseteq \Omega - K$$

By the Hahn - Banach theorem one concludes that

$$cv(\text{supp}T) \subseteq \Omega$$

Hence $\langle \nu, f \rangle = (\check{\nu} * f)(0) = (\check{T} * \mu * f)(0) = \langle T, \mu * f \rangle = 0$ for $f \in N$.

To end the proof of the proposition, we pick an exhaustion of $\Omega - K$ by convex compact sets K_j , hence we can find P_1 , exponential-polynomial with frequencies lying in $\{\zeta \in \mathbb{C}^n : \hat{\mu}(\zeta) = 0, |\zeta| > s_1\}$ such that

$$\sup_{K_1} |f - P_1| < 1.$$

Let $\sigma_1 =$ maximum of the absolute values of frequencies in P_1 . We can find P_2 with frequencies in $\{\zeta \in \mathbb{C}^n : \hat{\mu}(\zeta) = 0, |\zeta| > s_2 + \sigma_1\}$ such that

$$\max_{|a| < 1} \sup_{K_2} |D^a(f - P_1 - P_2)| < 1/2$$

Continuing in this fashion we obtain the desired expansion. $\square$

Remark. One can eliminate the requirement of μ being invertible by using [14, 16.4.1].

From (9) we know that the zero variety of $\hat{\mu}_r$ is the union of the hypersurfaces

$$(17) \quad \zeta^2 = \zeta_1^2 + \dots + \zeta_n^2 = \lambda_k^2 \quad k = 1, 2, \dots$$

where $\lambda_k = \alpha_{k, n/2}/r$. We disregard temporarily the dependence on r though it will play a role later on. Furthermore the function $\tilde{\mu}_r(t)$ vanishes at $t = \lambda_k$ with multiplicity one, in fact

$$(18) \quad \frac{d}{dt} \tilde{\mu}_r(t) = -n 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) r J_{(n/2)+1}(rt) / (rt)^{n/2}$$

and well known properties of Bessel functions show that this expression does not vanish for $t = \lambda_k$. Using the asymptotic expressions (12) and (13) we obtain

$$(19) \quad 0 \neq \frac{d\tilde{\mu}_r}{dt}(\lambda_k) = r n 2^{(n-1)/2} \Gamma(n/2) (-1)^{k+1} / (\lambda_k r)^{\frac{n+1}{2}} + o\left(k^{-\frac{n+3}{2}}\right)$$

We introduce some auxiliary radial distributions $T_{r,k}$ by the formula

$$(20) \quad \tilde{T}_{r,k}(t) = \frac{\tilde{\mu}_r(t)}{t^2 - \lambda_k^2}.$$

They are even and entire since $\tilde{\mu}_r(\pm\lambda_k) = 0$. Hence they correspond to radial distributions (in fact $C^{1,1}$ functions) whose supports are contained in the support of μ_r , i.e. $\overline{B}(0,r)$.

Furthermore they satisfy

$$(21) \quad (\Delta + \lambda_k^2) T_{r,k} = -\mu_r \quad \text{and}$$

$$(22) \quad \tilde{T}_{r,k}(\lambda_k) = \frac{\tilde{\mu}_r'(\lambda_k)}{2\lambda_k} = \text{const.} (-1)^{k+1} \lambda_k^{-\frac{n+3}{2}} + O(k^{-\frac{n+5}{2}}).$$

We remark that these distributions have conspicuously appeared in previous work on the Pompeiu problem [2], [7].

Proposition 2 - Let $r > 0$ be fixed. For any ρ , $0 < \rho < \infty$, we can decompose σ_ρ in the following form

$$(23) \quad \sigma_\rho = \nu_\rho + \mu_r * S_\rho,$$

where S_ρ is a radial distribution, whose support satisfies

$$(24) \quad \text{supp } S_\rho \subseteq \overline{B}(0, \text{Max}(r, \rho) - r)$$

and ν_ρ is given explicitly by:

$$(25) \quad \nu_\rho = - \sum_{k \geq 1} \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 \tilde{T}_{r,k}(\lambda_k)} \Delta T_{r,k}$$

hence $\text{supp } \nu_\rho \subseteq \overline{B}(0, r)$.

Proof. We consider the series

$$(26) \quad g(t) = \sum_{k \geq 1} \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 \tilde{T}_{r,k}(\lambda_k)} t^2 \tilde{T}_{r,k}(t)$$

The coefficients $\lambda_k^{-2} \tilde{\sigma}_\rho(\lambda_k) / \tilde{T}_{r,k}(\lambda_k)$ are uniformly bounded by a constant depending only on ρ as it can be seen from (10), (13) and (22), since $\lambda_k \sim \text{const. } k$. Therefore, if $|t| < R$, $\eta_k > 2R$ we have $|t^2 \tilde{T}_{r,k}(t)| < \text{const. } k^{-2}$ which guarantees the convergence of the series, and shows g is an even entire function. We can obtain more precise estimates by picking a sequence of circles of center 0 and radii

$$R_j = (4j+n+5)^\pi / 4r, \quad j = 1, 2, \dots$$

Decomposing the sum into those terms where

$\lambda_k < 2R_j$ and $\lambda_k > 2R_j$ one can estimate the second sum over $|t| = R_j$ by

$$\max_{|t|=R_j} |t^2 \tilde{\mu}_r(t)| \cdot C_0(\rho)$$

The first (finite) sum can be estimated by

$$C_1(\rho) \left(\max_{|t|=R_j} |t^2 \tilde{\mu}_r(t)| \right) \left(\max_{\Omega_{j,\varepsilon}} \sum_{0 < \lambda_k < 2R_j} \frac{1}{|\lambda_k^2 - t^2|} \right)$$

where $\Omega_{j,\varepsilon}$ is the region obtained from $|t| < R_j$ by removing disks of radius ε , $0 < \varepsilon$ very small, about $\pm \lambda_k$. One can then see, without difficulty, that the last sum is estimated by $\text{const. } \varepsilon^{-1}$. In any case we obtain as a final estimate

$$\max_{|t| < R_j} |g(t)| \leq C(\rho) \max_{|t| < R_j} |t^{2\tilde{\mu}_r}(t)|.$$

Thus g defines a radial distribution of order 2, v_ρ , by $\tilde{v}_\rho = g$, one can see v_ρ is given explicitly by (25). We also have

$$\tilde{\sigma}_\rho - \tilde{v}_\rho = \tilde{\mu}_r h,$$

with h even entire function since $g(\pm\lambda_k) = v_\rho(\pm\lambda_k) = \tilde{\sigma}_\rho(\pm\lambda_k)$ by (26). Since μ_r is an invertible distribution it follows $h = \tilde{S}_\rho$ for some $S_\rho \in \mathcal{E}'_0(\mathbb{R}^n)$. The identity (6) gives

$$(27) \quad \text{cv}(\text{supp}(\sigma_\rho - v_\rho)) = \text{cv}(\text{supp} S_\rho) + \text{cv}(\text{supp} \mu_r).$$

There are two cases to consider. If $\rho < r$, then the support on the left hand side of (27) is contained in $\overline{B}(0, r)$ and $\text{cv}(\text{supp} S_\rho) = \{0\}$, which says S_ρ is a polynomial in the Laplace operator; if $\rho > r$ then the left hand side of (27) is contained in $\overline{B}(0, \rho)$, which says $\text{cv}(\text{supp} S_\rho) \subseteq \overline{B}(0, \rho - r)$. $\square$

Remark The decomposition we have just given in proposition 2 works also if we replace σ_ρ by any radial distribution. We need only to change $(t/\lambda_k)^2$ by $(t/\lambda_k)^{2q}$ with q convenient non negative integer. In particular there is such a decomposition with $\sigma_0 = \delta$, the Dirac mass at the origin (take $q > \frac{n+1}{4}$)

Corollary 3 - Let f be a μ_r -mean-periodic function in $C^\infty(B(0,R))$ ($R > r$). Let $|x_0| < R - r$. Then, for any ρ , $0 < \rho < R - |x_0|$ we have

$$(28) \quad \lambda_\rho(f, x_0) = (v_\rho * f)(0) = - \sum_{k \geq 1} \frac{\tilde{\sigma}_\rho(\lambda_k)}{\lambda_k^2 \tilde{T}_{r,k}(\lambda_k)} \Delta(T_{r,k} * f)(x_0)$$

Proof. It suffices to use (2) and (23). $\square$

4. Local two-circles theorem.

Let r_1, r_2 be two positive numbers and consider the distributions μ_{r_1}, μ_{r_2} . They will have no common, mean-periodic, exponential-polynomials if and only if $\hat{\mu}_{r_1}$ and $\hat{\mu}_{r_2}$ have no common zeros. By (17) this occurs if and only if

$$r_1/r_2 \neq \text{quotient of two zeros of } J_{n/2}$$

The set

$$E_n = \{a_{k,n/2}/a_{j,n/2} : 1 \leq j, k < \infty\}$$

is the exceptional set described in the two-circles theorem.

Proposition 4 Let $R > r_1 + r_2$, $r_1/r_2 \notin E_n$. The only function in $C^\infty(B(0,R))$ which is mean-periodic with respect to both μ_{r_1} and μ_{r_2} is the zero function.

Proof. We assume $r_1 < r_2$. Let $f \in C^\infty(B(0,R))$ be μ_{r_1} -mean periodic. By proposition 1 we have

$$f(x) = \sum_{j \geq 1} p_j(x) \quad (|x| < R)$$

where the frequencies appearing in the exponential sums p_j lie in

$$\{\zeta \in \mathbb{C}^n : \hat{\mu}_{r_1}(\zeta) = 0\} = \bigcup_{k \geq 1} \{\zeta \in \mathbb{C}^n : \zeta^2 = (a_{k,n/2}/r_1)^2\} = \bigcup_{k \geq 1} v_k.$$

We fix now $k \geq 1$, and consider $T_{r_1,k} * f$ which is in $C^\infty(B(0, R-r_1))$, furthermore

$$(29) \quad T_{r_1,k} * f = \sum_{j \geq 1} T_{r_1,k} * p_j$$

If $p_j(x) = \sum_{\ell} c_{j,\ell} e^{i(x|\zeta_{j,\ell})}$ then

$$T_{r_1,k} * p_j = \sum_{\ell} c_{j,\ell} T_{r_1,k}(\zeta_{j,\ell}) e^{i(x|\zeta_{j,\ell})}$$

but $\hat{T}_{r_1,k}(\zeta_{j,\ell}) \neq 0$ only if $\zeta_{j,\ell} \in v_k$ in which case we obtain the value $\tilde{T}_{r_1,k}(\lambda_k) = 0$ (where λ_k is computed with respect to r_1). Therefore

$$(30) \quad T_{r_1,k} * f = \tilde{T}_{r_1,k}(\lambda_k) \sum_{j \geq 1} p_{j,k}$$

where $p_{j,k}$ is the sum of the terms in p_j whose frequencies lie in v_k . This series is convergent in $C^\infty(B(0, R-r_1))$. We convolve now with μ_{r_2} . We obtain

$$(31) \quad \mu_{r_2} * (T_{r_1,k} * f) = \tilde{T}_{r_1,k}(\lambda_k) \tilde{\mu}_{r_2}(\lambda_k) \sum_{j \geq 1} p_{j,k}$$

since μ_{r_2} is also a radial distribution. The expansion (31) is valid in $C^\infty(B(0, R-r_1-r_2))$. Since f is also μ_{r_2} -mean-periodic we have

$$0 = (T_{r_1, k} * \mu_{r_2} * f)(x) = \tilde{\mu}_{r_2}(\lambda_k)(T_{r_1, k} * f)(x)$$

for $|x| < R - r_1 - r_2$. The hypothesis $r_1/r_2 \notin E_n$ now implies that $\tilde{\mu}_{r_2}(\lambda_k) \neq 0$. Hence

$$(32) \quad (T_{r_1, k} * f)(x) = 0 \quad \text{for } |x| < R - r_1 - r_2$$

On the other hand we have (by (22))

$$(\Delta + \lambda_k^2)(T_{r_1, k} * f) = -(f * \mu_{r_1}) = 0 \quad \text{in } |x| < R - r_1$$

hence $T_{r_1, k} * f$ is a real analytic function in $|x| < R - r_1$. We conclude that

$$(33) \quad (T_{r_1, k} * f)(x) = 0 \quad \text{for } |x| < R - r_1$$

Applying now corollary 3, formula (28), we have

$$(34) \quad \lambda_\rho(f, x) = 0 \quad \text{whenever } |x| < R - r_1, \quad 0 < \rho < R - |x|$$

(We are allowed to take $\rho = 0$ by continuity). In particular

$$f(x) = 0 \text{ for } |x| < R - r_1$$

To finish the proof of the proposition we need to show f is zero in the remaining annulus, we do that using (34). It is at this point that we use Cormack-Quinto [12]. For any $y \in B(0, R)$, consider $R(f)(y) = \lambda_{|y|/2}(f, y/2)$. This is the Radon transform on spheres through the origin discussed in [12]. We want to show $Rf(y) = 0$. We only need to verify that the conditions stated in (34) are valid. Here

$$\rho = |y|/2, \quad x = y/2, \text{ hence}$$

$$R - |x| = R - \frac{|y|}{2} = R - \rho > R/2 > \rho$$

The only condition left to see is that $|x| < R - r_1$. We have $2r_1 < r_1 + r_2 < R$ hence $r_1 < R/2$ and $R - r_1 > R/2$, therefore $|x| < R - r_1$ holds.

By [12, corollary 2] $f(y) = 0$. (We note that in [12], they require that $f \in C^\infty(\mathbb{R}^n)$ while we only have $f \in C^\infty(B(0, R))$ but the proof of corollary 2 depends on an explicit inversion formula for the Radon transform on spheres which uses, for each y , values of f in a neighborhood of $\overline{B}(0, |y|)$.) $\square$

Remark. The crucial point of the proof above is (32). One does not really need the whole strength of Proposition 1 to obtain it. One can get by using the density of the exponential polynomial solutions in the sub-space N introduced in Proposition 1. Nevertheless, we feel that the proof is clearer using the expansion (16) as we have done.

We want to show that the condition $R > r_1 + r_2$ is sharp. It is easier to show this under the slight restriction that r_2/r_1 is not too well approximated by elements in E_n .

Definition. - For $N > 0$, we say that a positive number is N -well approximated by points in E_n if, for every $l > 1$, there are indices j, k such that

$$(35) \quad |r - \alpha_k / \alpha_j| < \frac{1}{l j^N}$$

where $\alpha_k = \alpha_{k, n/2}$

Proposition 5 For any $N > 2$, the set of numbers N -well approximated by E_n has zero measure in $(0, \infty)$

Proof. Given p, q , $0 < p < r < q$ and $v > 0$, from (13) we have

$$\alpha_{k, v} = \alpha_k = (2k+1)\pi/2 + (2v+1)\pi/4 + O(1/k)$$

Therefore, if r satisfies (35), for $l > 1$, we have

$$(36) \quad |r \cdot j - k + Ar + B| < C$$

for some constants A, B, C . Hence

$$pj - C_1 < k < qj + C_2$$

for some constant $C_1, C_2 > 0$. Hence the cardinal of the set of k satisfying (36) is bounded by $(q-p)j + L$, L constant > 0 .

Now, the set of N -well approximated numbers in $[p, q]$ is

$$(37) \quad \bigcap_{l \geq 1} \bigcup_{j, k \geq 1} \{r: p \leq r \leq q, |r - \alpha_k / \alpha_j| < 1/l j^N\}$$

For l fixed, the Lebesgue measure

$$|\bigcup_{j, k} \{r: p \leq r \leq q, |r - \alpha_k / \alpha_j| < 1/l j^N\}| < \frac{2}{l} \sum_{j \geq 1} \frac{(q-p)j + L}{j^N} < \frac{C_3}{l}$$

($C_3 > 0$) since $N > 2$. Therefore the set (37) has zero measure and by letting $q = p + 1$, $p \in \mathbb{N}$ we obtain the proposition.

It is interesting to compare proposition 5 with [8, Lemma 2.1] where examples of numbers which are not 2-well approximated by $E_n(n=2)$ are discussed. It might be that these include all rationals $\neq 1$ or all quadratic irrationals $\neq 1$, but no such theorem seems to be known. Also, it is easy to see that, for $N < 1$, every positive number is N -well approximated by E_n .

Proposition 6. - Let r_1, r_2 be two positive numbers such that r_2/r_1 is not N -well approximated by E_n . Denote by λ_k the positive zeros of $\tilde{\mu}_{r_1}$. There is a positive constant C such that

$$(38) \quad |\tilde{\mu}_{r_2}(\lambda_k)| > \frac{C}{k^{\frac{N-1}{2}}}$$

Proof. Let us denote $\alpha_k = \alpha_{k, n/2}$. Recall that $\lambda_k = \alpha_k / r_1$ and that

$$\tilde{\mu}_{r_2}(t) = \text{const.} \frac{J_{n/2}(r_2 t)}{(r_2 t)^{n/2}}$$

From the asymptotic development (13) we have

$$\alpha_{k+1} - \alpha_k = \pi + O(1/k)$$

Hence, if k is fixed and j_k is chosen such that $|r_2 \lambda_k - \alpha_{j_k}|$ is minimal we have

$$(39) \quad \varepsilon_k = |r_2 \lambda_k - \alpha_{j_k}| < \pi/2 + O(1/k).$$

Let us distinguish two cases: $\varepsilon_k < \pi/4$ or not. In the second case we have

$$\begin{aligned} |\cos(r_2 \lambda_k - \frac{(n+1)}{4} \pi)| &= |\cos(\pm \varepsilon_k + (2j_k + 1) \frac{\pi}{2} + O(\frac{1}{k}))| \\ &= |\sin(\varepsilon_k + O(1/k))| > \frac{\sqrt{2}}{2} + O(\frac{1}{k}) > c_0 > 0 \end{aligned}$$

for large k . In this case the asymptotic development (12) gives the estimate

$$|\tilde{u}_{r_2}(\lambda_k)| > c_1 k^{-\frac{n+1}{2}}$$

for some $c_1 > 0$ and all large k .

By hypothesis we have that for all j, k

$$|\frac{r_2}{r_1} - \frac{\alpha_1}{\alpha_k}| > \frac{c_2}{k^N} \quad (c_2 > 0)$$

Therefore $\varepsilon_k > C_3/k^{N-1}$ ($C_3 > 0$). Suppose also $\varepsilon_k < \pi/4$. By the mean-value theorem there is a ξ between α_{j_k} and $r_2\lambda_k$ such that

$$\frac{J_{n/2}(r_2\lambda_k)}{(r_2\lambda_k)^{n/2}} = -r_2 \frac{J_{n/2+1}(\xi)}{\xi^{n/2}} \cdot (r_2\lambda_k - \alpha_{j_k})$$

(Recall $J_{n/2}(\alpha_{j_k}) = 0$.) Note that $\delta_k = |\xi - \alpha_{j_k}| < \varepsilon_k < \pi/4$. Again by (12) we have to estimate

$$\begin{aligned} \cos\left(\xi - \frac{\pi}{4} - \frac{(n+2)\pi}{4}\right) &= \cos\left(\pm\delta_k + (2j_k+1)\frac{\pi}{2} + \frac{(n+1)\pi}{4} - \frac{(n+2)\pi}{4} + O(1/k)\right) \\ &= \cos(\pm\delta_k + j_k\pi + O(1/k)) \\ &= \pm \cos \delta_k + O(1/k). \end{aligned}$$

Then

$$|\tilde{u}_{r_2}(\lambda_k)| > \frac{C_4}{k^{N-1+\frac{n+1}{2}}} \quad (C_4 > 0).$$

Since $N > 1$ the estimate (38) holds in both cases. $\square$

Proposition 7 let f be a function in

$L^1_{loc}(B(0,R))$, $g \in L^1_{loc}(B(0,R))$, $\text{supp } g \subseteq B(0,r)$, g radial.

For $|x_0| < R - r$ and $\rho < R - r - |x_0|$ we have

$$(40) \quad \lambda_\rho(f * g, x_0) = (\lambda_{|\cdot|}(f, x_0) * g(\cdot))(y) \quad (|y| = \rho)$$

(The notation indicates that we are convolving in the variable denoted by a dot).

Proof. Recall that the average $\lambda_\rho(f, x_0)$ can also be computed by

$$\lambda_\rho(f, x_0) = \int_{O(n)} f(x_0 + Ay) dA$$

where y is any point with $|y| = \rho$, $O(n)$ is the orthogonal group and dA is the normalized Haar measure. Let

$$\phi(y) = (\lambda_{|\cdot|}(f, x_0) * g(\cdot))(y)$$

we have

$$\begin{aligned} \phi(y) &= \int_{\mathbb{R}^n} \left(\int_{O(n)} f(x_0 + A(y-x)) dA \right) g(x) dx \\ &= \int_{O(n)} \left(\int_{\mathbb{R}^n} f(x_0 + A(y-x)) g(x) dx \right) dA \end{aligned}$$

Set $u = Ax$ then $g(x) = g(u)$ and $dx = du$ Hence

$$\begin{aligned} \phi(y) &= \int_{O(n)} \left(\int_{\mathbb{R}^n} f(x_0 + Ay - u) g(u) du \right) dA \\ &= \lambda_\rho(f * g, x_0). \quad \square \end{aligned}$$

Corollary 8. Let g be radial integrable function of compact support and α a positive number. Then

$$(41) \quad \left(g(x) * \frac{J_{\frac{n-2}{2}}(a|x|)}{(a|x|)^{\frac{n-2}{2}}} \right)(y) = \tilde{g}(a) \frac{J_{\frac{n-2}{2}}(a|y|)}{(a|y|)^{\frac{n-2}{2}}}$$

Proof. Let $\xi \in \mathbb{R}^n$ be any vector with $|\xi| = a$, then

$$(42) \quad \begin{aligned} (g(\cdot) * e^{i(\xi|\cdot)})(y) &= \hat{g}(\xi) e^{i(\xi|y)} \\ &= \tilde{g}(a) e^{i(\xi|y)}. \end{aligned}$$

On the other hand

$$\begin{aligned} \lambda_\rho(e^{i(\xi|\cdot)}, 0) &= \tilde{\sigma}_\rho(a) \\ &= 2^{\frac{n-2}{2}} \Gamma(n/2) \frac{J_{\frac{n-2}{2}}(a\rho)}{(a\rho)^{\frac{n-2}{2}}}. \end{aligned}$$

Applying now to (42) Proposition 7 we obtain the desired formula (41). $\square$

Proposition 9. Let r_1, r_2 be two positive numbers such that r_2/r_1 is not N -well approximated by E_n . Let R be any number, $\max(r_1, r_2) < R < r_1 + r_2$. Then there is a non zero radial function $f \in C^\infty(B(0, R))$ which is mean periodic with respect to μ_{r_1} and μ_{r_2} .

Proof. Let $\phi \in \mathcal{D}([0, r_1]), \phi \not\equiv 0$ such that $\text{supp } \phi \subseteq [R - r_2, r_1]$. It follows from [16, theorem 2.1 page 247] that ϕ admits a series development of the form

$$(43) \quad \phi(t) = \sum_{k \geq 1} a_k \frac{J_{\frac{n-2}{2}}(\lambda_k t)}{(\lambda_k t)^{\frac{n-2}{2}}}$$

where $\lambda_k = \alpha_{k,n/2}/r_1$. This is the Sturm-Liouville expansion for a boundary value problem singular at $t = 0$ and derivative equal to zero at $t = r_1$. It can be seen by successive integrations of parts that

$$(44) \quad |a_k| = O(k^{-p}) \text{ for every } p > 0.$$

Since r_2/r_1 is not N -well approximated by E_n we see that

$$b_k = a_k / \tilde{\mu}_{r_2}(\lambda_k)$$

satisfies the same estimates as a_k (Proposition 6) Hence the function

$$(45) \quad f(x) = \sum_{k \geq 1} b_k \frac{J_{\frac{n-2}{2}}(\lambda_k |x|)}{(\lambda_k |x|)^{\frac{n-2}{2}}}$$

is a C^∞ radial function in $\mathbb{R}^n$, $f \not\equiv 0$. And, from corollary 8, it follows that f is μ_{r_1} mean-periodic. Furthermore

$$\begin{aligned} (\mu_{r_2} * f)(x) &= \sum_{k \geq 1} b_k \tilde{\mu}_{r_2}(\lambda_k) \frac{J_{\frac{n-2}{2}}(\lambda_k |x|)}{(\lambda_k |x|)^{\frac{n-2}{2}}} \\ &= \phi(|x|) \end{aligned}$$

which is zero in $B(0, R-r_2)$ and therefore the function f restricted to $B(0, R)$ is μ_{r_1} and μ_{r_2} mean-periodic. $\square$

The above propositions can be summarized by the following:

Theorem 10. Let $r_1 > 0$ and $r_2 > 0$ be such that $r_2/r_1 \in E_n$. The necessary and sufficient condition on a open set Ω of R^n so that the only distribution $T \in \mathcal{D}'(\Omega)$ which can be mean-periodic with respect to both μ_{r_1} and μ_{r_2} is $T = 0$, is that Ω is the reunion of balls of radii strictly larger than $r_1 + r_2$.

An amusing corollary of theorem 10 is the following:

Corollary 11 If $r_2/r_1 \in E_2$, $r_1 + r_2 < R$ and $f \in C(B(0, R))$ then the conditions

$$\int_{\partial B(z, r_j)} f(\zeta) d\zeta = 0 \text{ for every } z, |z| < R - r_j (j=1,2),$$

imply that f is holomorphic in the disk $B(0, R)$.

5. Generalizations

After the paper was written, we became aware of the work of J.D. Smith [17], in which local versions of certain two-circle theorems are also proved. Smith's results, which require $R > 2r_1 + r_2$ are less sharp than Proposition 4; nor does the method of proof seem to generalize to the other problems discussed in [20], e.g. the converse of the mean value property for harmonic functions. The aim of this section is to show that the methods used above do generalize.

Definition. We say that a radial distribution μ of compact support is hyperbolic if:

- (i) μ is invertible, and
- (ii) there is a constant C such that every zero λ of $\tilde{\mu}$ satisfies

$$|\operatorname{Im} \lambda| \leq C \log(2+|\lambda|).$$

Theorem 12 Let $\mu_1, \mu_2, \dots$ be a (possibly infinite) family of radial distributions of compact support, $\operatorname{cv}(\operatorname{supp} \mu_j) = \overline{B}(0, r_j)$. Suppose $\{z \in \mathbb{C}^n : \mu_j(z) = 0 \ \forall j\} = \emptyset$, μ_1 is hyperbolic, and $R - r_1 > \sup_j r_j$. Then $\{f \in \mathcal{D}(B(0, R)) : \mu_j * f = 0 \ \forall j\} = \{0\}$.

Proof: Due to the condition on R we can assume

$f \in C^\infty(B(0, R))$ as done before. The proof that leads to (32) can be repeated almost verbatim just using for each λ_k , zero of $\tilde{\mu}_1$, a convenient $\mu_j (j > 2)$ with $\tilde{\mu}_j(\lambda_k) \neq 0$. We obtain

$$(47) \quad T_{k,s} * f(x) = 0 \quad \text{for } |x| \leq R - r_1 - \sup_j r_j,$$

where $\tilde{T}_{k,s}(t) = \tilde{\mu}_1(t) (t^2 - \lambda_k^2)^{-s}$, $1 \leq s \leq m_k$, m_k = multiplicity of λ_k as a root of $\tilde{\mu}_1$.

On the other hand

$$(48) \quad (-1)^s (\Delta + \lambda_k^2)^s (T_{k,s} * f) = \mu_1 * f = 0 \quad \text{in } B(0, R - r_1),$$

therefore, $T_{k,s} * f$ is real analytic, and hence

$$(49) \quad T_{k,s}^* f = 0 \text{ in } B(0, R-r_1),$$

as before. It is at this point we have to be more careful to prove the correct version of Proposition 2. It will be replaced by the following:

Lemma 13 Let $\Lambda = \{\lambda_k\}$ = set of distinct zeros of $\tilde{u}_1$, then $\Lambda = \bigcup_{j=0}^{\infty} \Lambda_j$, where the Λ_j are finite and mutually disjoint sets. There is also a positive integer q such that for any ρ , $0 < \rho < \infty$ we can write

$$(50) \quad \sigma_\rho = v_\rho + u_1^* S_\rho,$$

where v_ρ, S_ρ are radial distributions satisfying

$$(51) \quad \text{supp } v_\rho \subseteq \overline{B}(0, r_1) \quad \text{and}$$

$$(52) \quad \text{supp } S_\rho \subseteq \overline{B}(0, \text{Max}(r_1, \rho) - r_1).$$

Furthermore,

$$(53) \quad v_\rho = \sum_{j=0}^{\infty} \Delta^q v_{\rho,j},$$

a convergent series in $E'_0(\mathbb{R}^n)$, each $v_{\rho,j}$ a finite linear combination of the distributions $T_{k,s}, \lambda_k \in \Lambda_j, 1 \leq s \leq m_k$ (if $m_0 > 1$ then one denotes by $\Delta^q v_{\rho,0}$ not only a finite linear combination of $\Delta^q T_{k,s}, \lambda_k \in \Lambda_0$ but also of $T_{0,m_0}, \Delta T_{0,m_0}, \dots, \Delta^{q-1} T_{0,m_0}$.)

Once this lemma has been proved, the proof of Theorem 12 is achieved the same way as it was done in Proposition 4 and we note that the hypotheses imply $2r_1 < R$.

Proof of Lemma 13 The proof of this lemma proceeds as in Proposition 2 by interpolating the values of $\tilde{\sigma}_\rho$ on the variety of zeros of $\tilde{u}_1$ (counted with multiplicities). We have to repeat with due care the procedure used in [13], [15], [18] since we need the precise statement (51), (52), and (53).

First we note that as in [5, lemma 4] (cf. also [11, p. 50]), the condition of hyperbolicity and the minimum modulus theorem allow us to construct a family of a Jordan quadrilaterals Γ_k , $k \in \mathbb{Z}$ symmetric with respect to the real axis and enjoying the following properties:

(54) for some $d > 0$ the horizontal sides lie on the curves

$$\operatorname{Im} z = \pm \log(d + |\operatorname{Re} z|),$$

and the vertical sides are arcs of circles.

(55) $0 \in \operatorname{int} \Gamma_0$ which is symmetric with respect to the origin (i.e. if $z \in \Gamma_0$ then $-z \in \Gamma_0$ also).

(56) for $k \neq 0$, Γ_{-k} is the symmetric of Γ_k with respect to the origin.

(57) for $j \neq k$, $\overline{\text{int } \Gamma_j} \cap \overline{\text{int } \Gamma_k} = \emptyset$, furthermore, for some positive number a we have that if $z \in \Gamma_j$,
 $\text{dist}(z, \Gamma_k) > (a+|z|)^{-a}$ for any $k \neq j$.

(58) for some positive constant b we have:

$$\text{diam } \Gamma_j < b(1+|z|)^b$$

and

$$\text{length } \Gamma_j < b(1+|z|)^b,$$

for any $z \in \overline{\text{int } \Gamma_j}$, any j .

(59) there is a constant $c > 0$ such that for any j , and any $z \in \Gamma_j$ we have

$$|\tilde{u}_j(z)| > (c+|z|)^{-c},$$

and this inequality is valid even for those z such that
 $\text{dist}(z, \Gamma_j) < 1/2 (a+|z|)^{-a}$ (the same a as in (57))

$$(60) \quad \Lambda \subset \bigcup_{j=-\infty}^{\infty} (\text{int } \Gamma_j).$$

(61) for some $d > 0$: if $j > 1$, $z \in \Gamma_j$, then $|z| > j/d$.

$$(62) \quad \Lambda_0 = \Lambda \cap \text{int } \Gamma_0, \Lambda_j = \Lambda \cap (\text{int } \Gamma_j \cap \text{int } \Gamma_{-j}), j > 1.$$

For the sake of definiteness we will index the points in Λ so that $\lambda_0 = 0$, and, for $k \geq 1$, either $\operatorname{Re} \lambda_k > 0$ or $\operatorname{Re} \lambda_k = 0$ and $\operatorname{Im} \lambda_k > 0$, and, finally, $\lambda_{-k} = -\lambda_k$.

Now consider the even entire function

$$(63) \quad f(t) = t^{2q} \tilde{\mu}_1(t),$$

for q a positive integer to be chosen conveniently later on.

We note that if $t \notin \overline{\operatorname{int} \Gamma_j} \cup \overline{\operatorname{int} \Gamma_{-j}}$ then

$$(64) \quad \phi_j(t) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{\tilde{\sigma}_\rho(s)}{f(s)} \frac{ds}{s-t} + \frac{1}{2\pi i} \int_{\Gamma_{-j}} \frac{\tilde{\sigma}_\rho(s)}{f(s)} \frac{ds}{s-t}$$

(where we disregard the second term if $j = 0$) is an even function which is a linear combination of terms of the form $(t^2 - \lambda_k^2)^{-s}$, for $\lambda_k \in \Lambda_j$ and $1 \leq s \leq m_k$ if $k \geq 1$, $1 \leq s \leq m_0 + 2q$ if $k = 0$. Hence ϕ_j can be defined as a rational function throughout $\mathbb{C}$ and the function $f(t) \phi_j(t)$ is an even entire function. We want to show now that q can be chosen so that

$$(65) \quad g(t) = \sum_{j=0}^{\infty} f(t) \phi_j(t)$$

is in $\tilde{E}'_0(\mathbb{R}^n)$ and the series converges in the topology of $\tilde{E}'_0(\mathbb{R}^n)$.

In fact, we have that for $|\operatorname{Im} t| < \log(d + |\operatorname{Re} t|)$ there is some $N > 0$ such that

$$(66) \quad |\tilde{\sigma}_\rho(t)| \leq C(\rho) (1+|t|)^N$$

and also

$$(67) \quad |\tilde{\mu}_1(t)| \leq C_0 (1+|t|)^N.$$

Therefore, for some $N_1 > 0$ sufficiently large, if $\text{dist}(t, \overline{\text{int } \Gamma_j} \cup \overline{\text{int } \Gamma_{-j}}) > 1$ we have by (66), (59) and (58), that with respect to an arbitrary point $z \in \overline{\text{int } \Gamma_j}$, which we can take it to be the point in the positive real axis closest to the origin,

$$|\phi_j(t)| \leq (N_1 + |z|)^{N_1} |z|^{-2q} < \text{const. } j^{-2}$$

by (61) (just take $2q > N_1 + 2$). Therefore, under the same condition on t we have

$$(68) \quad |f(t) \phi_j(t)| \leq C_1 j^{-2} (1+|t|)^M e^{r_1 |\text{Im} t|}$$

Using the condition (58) on the diameter of Γ_j and (67), this estimate remains valid throughout $\mathbb{C}$, after possibly increasing C_1 , M . This shows that the $\nu_\rho \in E'_0(\mathbb{R}^n)$ defined by

$$\tilde{\nu}_\rho(t) = g(t)$$

satisfies (51). It is also clear that the distributions $\nu_{\rho,j}$ such that $t^{2q} \nu_{\rho,j} = f(t) \phi_j(t)$ have the properties required

by (53) (with special care taken if $m_0 \neq 0$). To end the proof of the lemma we only have to show that $g(t) - \tilde{\sigma}_\rho(t)$ is divisibly by $\tilde{\mu}_1(t)$, the rest is the same as in Proposition 2. Note that if $t \in \text{int } \Gamma_j \cup \text{int } \Gamma_{-j}$ then we have

$$\psi_j(t) = \frac{1}{2\pi i} \int_{\Gamma_j + \Gamma_{-j}} \frac{\sigma(s)}{f(s)} \frac{ds}{s-t}$$

is holomorphic and, we pick a new residue at $s = t$ yielding

$$f(t) \psi_j(t) = f(t) \phi_j(t) + \sigma(t).$$

This concludes the proof of Lemma 13 and Theorem 12. $\square$

We give here the local version of Delsarte's theorem for harmonic functions.

Corollary 14 Let $H_n = \{\xi/\eta : \xi/\eta \in (0, \infty), \tilde{\sigma}_1(\xi) = \tilde{\sigma}_1(\eta) = 1\}$. If $R > r_1 + r_2$, $r_1/r_2 \notin H_n$, and u is a continuous function in $B(0, R)$ satisfying

$$u(x) = \lambda_{r_1}(u, x), \quad \text{for all } x, |x| < R - r_1,$$

and

$$u(x) = \lambda_{r_2}(u, x), \quad \text{for all } x, |x| < R - r_2,$$

then u is harmonic in $B(0, R)$.

Proof: From the asymptotic development of the Bessel functions and the formula (10), it follows that the radial distributions μ_j

defined by

$$\tilde{u}_j(t) = t^{-2} \widetilde{(\sigma_{r_j} - \delta)}(t) = (\tilde{\sigma}_{r_j}(t) - 1)/t^2$$

are hyperbolic. The hypothesis on r_1/r_2 guarantees these two entire functions have no common zeros. Theorem 13 shows now that the distribution Δu is zero in $B(0, R)$.

Remark As mentioned in [20], Delsarte proved this theorem in $\mathbb{R}^n$. He also showed that H_n is finite and $H_3 = \{1\}$. Hence, at least for dimension 3, any pair of distinct positive value r_1, r_2 would work in the above corollary.

The several other results in [20] can now be carried over to the local case without difficulty. It remains as an open question for the moment the elimination of the invertibility condition on μ_1 , which could probably be done following the Euclidean summation method of [6]. More interesting, in our view, is to try to extend this theorem to non-compact symmetric spaces of rank 1 or even to the Euclidean group thus obtaining a local version of the Pompeiu problem considered in [9].

As an example of this let us mention the following corollary of Theorem 13.

Corollary 15 Let $R > \sqrt{n} a$, if $f \in L^1_{loc}(B(0, R))$ has zero integral over any n -cube of side a contained in $B(0, R)$, then $f = 0$ a.e.

Proof Following the ideas from [9] we see we can consider all radial distributions μ whose Fourier transforms are of the form

$$(69) \quad \int_{O(n)} \hat{\chi}_Q(k\zeta) \hat{T}(k\zeta) dk = \hat{\mu}(\zeta)$$

where Q is the cube $[-a/2, a/2]^n$ and T is a distribution of compact support in the ball $B(0, \epsilon)$, $\epsilon + \sqrt{n} a < R$. Then, for any such μ , $\text{cv}(\text{Supp } \mu) \subset \bar{B}(0, r)$, and f will satisfy the equations:

$$\mu * f = 0 \quad \text{in } B(0, R-r).$$

Since this set of distributions generates the same closed ideal in $E'(R^n)$ as those are considered in [9, p. 602], then their Fourier transforms have no common zeros [9, section 9]. It only remains to find a distribution that plays the role of μ_1 in Theorem 13. The easiest one is obtained when

$$T = \frac{\partial^{2n} \chi_Q}{\partial x_1^2 \dots \partial x_n^2}$$

An easy computation shows that in this case, for

$$\mu_1 = \text{average over } O(\mu) \text{ of } \frac{\partial^{2n}}{\partial x_1^2 \dots \partial x_n^2} \chi_Q, \quad \text{we have}$$

$$(70) \quad \tilde{\mu}_1(t) = \text{const. } t^{(n/2)+1} J_{\frac{3n-2}{2}}(\sqrt{n} at/2)$$

which is clearly hyperbolic. (For $n = 2$, this can be obtained from Sonine second finite integral [19, p.376].)

6. References

- [1] C.A. Berenstein, Spectral synthesis on symmetric spaces, to appear in Contemporary Mathematics.
- [2] C.A. Berenstein, An inverse Spectral theorem and its relation to the Pompeiu problem, J. Analyse. Math 37 (1980) 128-144.
- [3] C.A. Berenstein and M.A. Dostal, On convolution equations I, L^1 anal. harm. dans le domaine complexe, Springer Lecture Notes in Math. 336(1973), 79-94.
- [4] C.A. Berenstein and D.C. Struppa, Solutions of convolution equations in convex sets, to appear in Amer. J. Math.
- [5] C.A. Berenstein and B.A. Taylor, A new look at interpolation theory for entire functions of one variable, Advances in Mathematics 33(1979), 109-143.
- [6] C.A. Berenstein and B.A. Taylor, Interpolation problems in C^n with applications to harmonic analysis, J. Analyse Math. 38(1980), 188-254.
- [7] C.A. Berenstein and P.C. Yang, An inverse Neumann problem, to appear.
- [8] C.A. Berenstein and A. Yger, Le problème de la deconvolution, J. Func. Analysis, 54(1983), 113-160.
- [9] C.A. Berenstein and L. Zalcman, Pompeiu's problem on symmetric spaces, Comment. Math. Helvetici 55(1980), 593-621.
- [10] L. Brown, B.M. Schreiber, and B.A. Taylor, Spectral synthesis and the Pompeiu problem, Ann. Inst. Fourier 23(1973), 125-154.
- [11] C.-C. Chow, La transformation de Fourier complexe et l'équation de convolution, Springer-Verlag (1973).
- [12] A.M. Cormack and E.T. Quinto, A Radon transform on spheres through the origin in R^n and applications to the classical Darboux equation, Trans. American Math. Soc. 260 (1980), 575-581.
- [13] L. Ehrenpreis, Solution of some problems of division, part IV. Amer. J. Math. 82 (1960), 522-588.
- [14] L. Hörmander, The analysis of linear partial differential operators, I and II, Springer-Verlag (1983).
- [15] J.P. Kahane, Sur quelques problèmes d'unicité et de prolongement, Ann. Inst. Fourier 5 (1953-54), 39-130.

- [16] B.M. Levitan and I.S. Sargsjan, Introduction to spectral theory, Translations of Math. Monograph AMS 39(1975).
- [17] J.D. Smith, Harmonic analysis of scalar and vector fields in R^n , Proc. Camb. Phil. Soc. 72(1973), 403-416.
- [18] L. Schwartz, Théorie générale des fonctions moyenne-périodiques, Ann. of Math. 48(1947) 857-929.
- [19] G. Watson, Theory of Bessel Functions, Cambridge Univ. Press, London/New York (1966).
- [20] L. Zalcman, Analyticity and the Pompeiu Problem, Arch. Rat. Mech. Anal 47(1972), 237-254.
- [21] L. Zalcman, Offbeat integral geometry, Amer. Math. Monthly 87(1980), 161-175.

CHAPTER 4

INVERSION OF THE LOCAL POMPEIU TRANSFORM

by

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1. Introduction.

The prototype of the Pompeiu transform and of the problems we will consider here is the Two-Disks problem we discuss presently. Let E be the collection of positive quotients of zeros of the Bessel function J_1 . This countable set is precisely the exceptional set for the following theorem (proved independently in [12] and [21]).

Theorem. Let $r_1, r_2 > 0$, $r_1/r_2 \notin E$. Then a continuous function in the plane is identically zero if and only if all its averages over any disk of radius either r_1 or r_2 vanish.

If we let χ_1, χ_2 be the characteristic functions of the disks $B(0, r_1)$, $B(0, r_2)$ we are saying that the map

$$(1.1) \quad P : f \mapsto (\chi_1 * f, \chi_2 * f)$$

is injective. In [7], [8] it was shown that a stronger arithmetical condition was necessary and sufficient for the existence (and the explicit construction) of two distributions ν_1, ν_2 of compact support such that

$$(1.2) \quad \nu_1 * (\chi_1 * f) + \nu_2 * (\chi_2 * f) = f.$$

Because ν_1, ν_2 might be distributions (the map P is smoothing), it is more natural to consider P in the Fréchet space $\mathcal{E} = \mathcal{E}(\mathbb{R}^2)$ of C^∞ functions. Then we have that $r_1/r_2 \notin E$ if and only if

$$P : \mathcal{E} \rightarrow \mathcal{E}^2$$

is injective. The existence of ν_1, ν_2 solving (1.2) is equivalent to $\text{Im } P$ being closed, hence P is right invertible. One can in this case also show that

$$(1.3) \quad \text{Im } P = \{(g, h) \in (\mathcal{E}(\mathbb{R}^2))^2 : \mu_2 * g = \mu_1 * h\}$$

and

$$P^{-1} : \text{Im } P \rightarrow \mathcal{E}$$

$$(g, h) \mapsto \nu_1 * g + \nu_2 * h.$$

The exact arithmetical condition is just the existence of a constant $A > 0$ such that

$$(1.4) \quad |r_1/r_2 - \xi/\eta| \geq \frac{1}{A} \eta^{-A},$$

for any pair $\xi, \eta > 0$, $J_1(\xi) = J_1(\eta) = 0$.

Since (1.2) is an explicit deconvolution formula, L. Zalcman asked in [20] whether some sort of explicit reconstruction formula would also exist under the sole condition $r_1/r_2 \notin E$. The theorem below answers this question in the affirmative. Naturally coupled to this question is the following local problem. Suppose we know the averages of f on disks of radii r_1 and r_2 , only when those disks lie in some fixed disk $B(0, R)$, could these data determine f in $B(0, R)$? A priori, formula (1.2), if valid, would only determine f in $B(0, R')$ for some convenient $R' < R$. This is just a consequence of the fact that (1.2) is given by convolutions, and ν_1, ν_2 do not have support at the origin. This is the nature of what was called elsewhere the local Pompeiu problem. It is shown in [2] that if $r_1 + r_2 < R$, $r_1/r_2 \notin E$, then f is determined in the whole disk $B(0, R)$ by those local averages. We will show here that one has even a local reconstruction formula.

These local theorems are also valid for other sets instead of disks, when one takes all rigid motions. The injectivity was shown in [3]. We give here a method to construct a local inverse in this more general situation.

Finally, we would like to mention that even for the injectivity question in $\mathbb{R}^2$, it is very hard to see when one can work just with translations of a

finite family of sets. The antecedent of several such results is the Three-Squares Theorem [5] where it is shown that if the averages of a continuous function f in the plane vanish on every square of sides either r_1, r_2 or r_3 (with sides parallel to the axes), then f is identically zero if and only if r_1, r_2, r_3 are $\mathbb{Q}$ -linearly independent. In [8], [9], explicit deconvolution formulas were found for this case under extra arithmetical assumptions on the triple r_1, r_2, r_3 . The local version of the Three-Squares Theorem requires that we place ourselves in a square of side R . In [4] it is shown that if $R > r_1 + r_2 + r_3$, then the uniqueness still holds. Its proof is akin to the methods used in this paper.

The motivation for this work lies in trying to find an algorithmic deconvolution approach (with due care for error bounds and noise behavior) in a situation where part of the scene is obstructed from our view but the object we are looking for lies very close to this obstruction. In the case of the Radon transform this is sometimes called the Hole problem or the Bagel problem [15]. It is interesting to note that these two problems are related, since one way to find an inversion formula is to use Cormack-Quinto's Spherical Radon Transform and its explicit inverse.

2. Preliminaries.

We recall some notation and basic properties of the Pompeiu transform. Let $E_1, \dots, E_N$ be a collection of compact sets in $\mathbb{R}^n$ of positive measure and let $M(n)$ be the group of Euclidean motions in $\mathbb{R}^n$, then the (global) Pompeiu transform (associated to $E_1, \dots, E_N$) is the map [10]

$$(2.1) \quad P : C(\mathbb{R}^n) \rightarrow C(M(n))^N$$

given by

$$(2.2) \quad (Pf)(g) = \left[\int_{gE_1} f dx, \dots, \int_{gE_N} f dx \right].$$

Given an open set $U \subseteq \mathbb{R}^n$ we can define open sets $G_j \subseteq M(n)$ by

$$(2.3) \quad G_j = \{g \in M(n) : gE_j \subseteq U\}$$

(these sets could be empty). We define the local Pompeiu transform

$$(2.4) \quad P : C(U) \rightarrow \bigoplus_{j=1}^N C(G_j)$$

defined exactly by the same formula (2.2). The family $E_1, \dots, E_N$ is said to have the Pompeiu property (respectively, the local Pompeiu property with respect to U) whenever the map (2.1) (respectively (2.4)) is injective. The most interesting case is the case of a single set with relatively nice boundary. One can prove the following theorem.

Theorem 1 [1], [6]. Let Ω be an open bounded set in $\mathbb{R}^n$, $E = \bar{\Omega}$, E^c connected and ∂E Lipschitz. Then E (or Ω) has the Pompeiu property if and only if there is no positive eigenvalue for the overdetermined Neumann problem

$$(N) \quad \begin{cases} \Delta u + \alpha u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, \quad u = 1 & \text{in } \partial\Omega. \end{cases}$$

As a corollary, one obtains two classes of examples:

- (1) A single ball never has the Pompeiu property;
- (2) A set Ω satisfying the hypotheses of Theorem 1 whose boundary is not a real analytic hypersurface has the Pompeiu property.

Examples of sets having the Pompeiu property and also having real analytic boundary exist, for instance, any ellipsoid which is not a ball. Since a

single ball does not have the Pompeiu property, the following theorem gives the best possible answer.

Theorem 2 [12], [21]. Let $Z_n = \{\xi/\eta : \xi > 0, \eta > 0, J_{n/2}(\xi) = J_{n/2}(\eta) = 0\}$. A pair of balls $B(0, r_1), B(0, r_2)$ has the Pompeiu property if and only if $r_1/r_2 \notin Z_n$.

To consider the local Pompeiu property it is clear we have to start with a family that satisfies the global Pompeiu property. We restrict ourselves to open sets U that can be covered by balls of a fixed radius R . In this case we might as well suppose that U is the ball $B(0, R)$. In this setting it is easy to state the local version of Theorem 2.

Theorem 3 [2]. Let $r_1, r_2 > 0, r_1/r_2 \notin Z_n$ and $R > r_1 + r_2$. Then the pair $B(0, r_1), B(0, r_2)$ has the local Pompeiu property with respect to the ball $B(0, R)$. Under the additional restriction that r_1/r_2 is badly approximated by elements of Z_n , the condition $R > r_1 + r_2$ is also necessary.

It is quite possible that $R > r_1 + r_2$ is always a necessary condition in Theorem 3. When considering a single set E having the global Pompeiu property, in order to decide whether E has the local Pompeiu property with respect to $B(0, R)$, it is natural to try to measure R against a value r such that $E \subseteq \bar{B}(x_0, r)$ for some x_0 . The condition that replaces $r_1 + r_2 < R$ is $2r < R$. We are compelled to introduce an extra technical condition due to the fact that we know nothing about ∂E . This condition, which we will explain below, is called hyperbolicity. There are a number of simple conditions that imply it (see [3]), for instance,

(1) Near an extreme point $x_1, x_1 \in E \cap \partial B(x_0, r)$, E coincides with a polyhedral angle with vertex x_1 . The fact that the walls or edges are straight plays no role. For instance, E is a cube in $\mathbb{R}^n$.

(2) There is an extreme point $x_1, x_1 \in E \cap \partial B(x_0, r)$, such that x_1 is

a point of strict convexity for ∂E and ∂E is of class C^3 near x_1 .

(3) In $\mathbb{R}^2$, near a point x_1 , $x_1 \in E \cap \partial B(x_0, r)$, ∂E is a Jordan curve, sufficiently smooth.

Theorem 4 [3]. Assume $E \subseteq \bar{B}(x_0, r)$ has a hyperbolic point

$x_1 \in E \cap \partial B(x_0, r)$. Assume further that E has the global Pompeiu property and $2r < R$. Then E has the local Pompeiu property with respect to $B(0, R)$.

We remark that the conditions $r_1 + r_2 < R$ in Theorem 3 and the corresponding $2r < R$ in Theorem 4 mean that the open sets G_j (respectively g) of $M(n)$ are not too small. On the other hand, they impose no restriction, when U is the complement of a closed convex set K . This is akin to say that the Pompeiu transform possesses the Hole property, as is the case for the Radon transform (in that case K must be compact).

The proofs of Theorems 1 and 4 depend on the introduction of a countable family of radial distributions associated to the set E . For that purpose we need to introduce the concept of the radialized version of a distribution (or circular symmetrization) with respect to a point x_0 . Namely, let $T \in \mathcal{D}'(\mathbb{R}^n)$ then $\mathcal{R}_{x_0} T = \mathcal{R}T$ is defined by

$$(2.5) \quad \langle \mathcal{R}_{x_0} T, \psi \rangle = \langle T(x), \int_{SO(n)} \psi(k(x-x_0)+x_0) dk \rangle,$$

where $\psi \in \mathcal{D}(\mathbb{R}^n)$, dk is the Haar measure on $SO(n)$. In case $T = \chi_E$, E a compact set in $\mathbb{R}^n$, then $\mathcal{R}_{x_0} T$ is a function of compact support with values in $[0, 1]$ which is a radial function with respect to x_0 , $\mathcal{R}_{x_0} T(x) = \varphi(|x-x_0|)$,

$$(2.6) \quad \varphi(t) = \sigma(E \cap \partial B(x_0, t)),$$

where σ is the normalized surface area. We consider φ as a radial Borel function in $\mathbb{R}^n$, of compact support, and therefore its Fourier transform $\mathcal{F}\varphi$

is an entire function in $\mathbb{C}^n$ in the Paley-Wiener class, which is radial, i.e., if we let

$$(2.7) \quad \Phi(\zeta) = \mathcal{F}\varphi(\zeta, 0, \dots, 0)$$

then Φ is an entire function in $\mathbb{C}$, even, satisfying the inequality

$$(2.8) \quad |\Phi(\zeta)| \leq A \exp(r|\operatorname{Im} \zeta|),$$

whenever $E \subseteq \bar{B}(x_0, r)$. In fact, Φ is the Fourier-Bessel transform of the radial function φ

$$\Phi(\zeta) = (2\pi)^{n/2} \int_0^\infty \varphi(t) J_{(n-2)/2}(\zeta t) t^{n-1} dt,$$

where $j_\ell(z) = J_\ell(z)/z^\ell$, J_ℓ being the Bessel function of order ℓ . We also denote $\Phi = \tilde{\varphi}$. Conversely, Φ determines $\mathcal{F}\varphi$ by

$$(2.10) \quad \mathcal{F}\varphi(\zeta_1, \dots, \zeta_n) = \Phi(\sqrt{\zeta_1^2 + \dots + \zeta_n^2}).$$

The smallest r which appears in (2.8) determines the smallest disk of center x_0 such that $E \subseteq \bar{B}(x_0, r)$. We call exterior radius of E , $r^* = r^*(E)$, the minimum value of r , with respect to all x_0 , such that this inclusion holds.

For fixed x_0 and any $\alpha \in \mathbb{N}^n$, we can construct a corresponding entire function of one complex variable Φ_α by

$$(2.11) \quad \Phi_\alpha(\zeta) := (\mathcal{R}_{x_0}(D^\alpha \chi_E))^\sim(\zeta),$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha}$. It is convenient to introduce the distributions $\mu_\alpha =$

$\mathcal{R}_{x_0} D^\alpha \chi_E$, which are radial with respect to x_0 . These definitions are justified by the following.

Lemma 4 [12]. E has the global Pompeiu property if and only if the functions

$\phi_\alpha, \alpha \in \mathbb{N}^n$, have no common zeros.

An even entire function ϕ of a complex variable is said to be hyperbolic if the following three conditions are satisfied.

(i) For some constant $A > 0$, it satisfies the estimate

$$|\phi(\zeta)| \leq A(1+|\zeta|)^A \exp(A|\operatorname{Im} \zeta|).$$

(ii) For some constant $a > 0$ and any real value ξ

$$\max\{|\phi(\xi+\eta)| : |\eta| \leq a \log(2+|\xi|)\} \geq (a+|\xi|)^{-a}.$$

(iii) For some constant $c > 0$ we have that

$$\phi(\zeta) = 0 \Rightarrow |\operatorname{Im} \zeta| \leq c \log(2+|\zeta|).$$

The first condition is simply that $\phi \in \mathcal{F}(\mathcal{E}'(\mathbb{R}))$. The second is usually called invertibility, it means that the principal ideal $\phi\mathcal{F}(\mathcal{E}'(\mathbb{R}))$ is closed. It coincides with the property that $\phi\mathcal{H}(\mathbb{C}) \cap \mathcal{F}(\mathcal{E}'(\mathbb{R})) = \phi\mathcal{F}(\mathcal{E}'(\mathbb{R}))$. The third one says that all the zeros of ϕ are almost real.

In [3], it is shown that a number of natural conditions on E imply that ϕ_0 , the radialized function of χ_E , is hyperbolic.

The idea of Lemma 4, and a fortiori Theorems 1 and 4, is that everything is reduced to the study of the radial case, as follows. Let us fix x_0 , which we take to be the origin for simplicity. Then the condition

$$\int_{gE} f dx = 0 \quad \forall g \in M(n) \quad (\text{or } \forall g \in G)$$

can be rewritten as an infinite system of convolution equations

$$(2.12) \quad \forall \rho \in SO(n): \chi_{\rho E} * f(x) = 0 \quad \forall x \in \mathbb{R}^n \quad (\text{or } |x| < R-r).$$

The proof of the injectivity of the (local) Pompeiu transform is based on the principle that if there is a non-zero solution of (2.12), we can assume it

to be C^∞ and show there must be a non-zero radial C^∞ function g in $\mathbb{R}^n$ (respectively in $B(0,R)$) solution of the denumerable system of convolution equations

$$(2.13) \quad \forall \alpha \in \mathbb{N}^n \quad \mu_\alpha * g(x) = 0 \quad \forall x \in \mathbb{R}^n \quad (\text{or } |x| < R-r),$$

where the μ_α are the distributions defined in (2.11). In particular, they involve derivatives (see [12], [3] for the details).

In order to find an inversion formula for the local Pompeiu transform we must make this principle a bit more precise. This depends on the following lemma.

Lemma 5. Let $E \subseteq \bar{B}(0,r)$ and $U = B(0,R)$, $R > 2r$. Then for every x such that $|x| < R-r$, $\alpha \in \mathbb{N}^n$, and any $f \in \mathcal{E}(B(0,R))$ we have

$$(2.14) \quad \mu_\alpha * f(x) = \int_{SO(n)} \langle \delta^{(\alpha)}(y), P(f) \left(\begin{bmatrix} -k^{-1} & x-ky \\ 0 & 1 \end{bmatrix} \right) \rangle dk,$$

where $M(n)$ is considered as the group of $(n+1) \times (n+1)$ matrices of the form $\begin{bmatrix} k & x \\ 0 & 1 \end{bmatrix}$, $k \in SO(n)$, $x \in \mathbb{R}^n$, and $\mathbb{R}^n$ is identified to the affine subspace $\{x_{n+1} = 1\}$ of $\mathbb{R}^{n+1}$.

Proof. The identity (2.14) is an easy computation and the only thing to verify is that for $|x| < R-r$, the image of E by $\begin{bmatrix} -k^{-1} & x-ky \\ 0 & 1 \end{bmatrix}$ is contained in $B(0,R)$ (recall that $\text{supp } \delta^{(\alpha)}(y) = \{0\}$). This is evident since $(-k^{-1})E \subseteq \bar{B}(0,r)$. Hence, translation by x keeps it inside $B(0,R)$.

Given a compact set E , let $E_\tau = \{x+\tau : x \in E\}$ the translate of E by $\tau \in \mathbb{R}^n$. Consider the functions

$$\psi_\tau(t) := (\mathcal{R}_{x_0} E_\tau)^\sim(t).$$

It is easy to see they are real analytic with respect to the parameter τ . In

analogy with Lemma 4 we have

Lemma 4'. E has the global Pompeiu property if and only if there is an open set $\mathcal{J} \subseteq \mathbb{R}^n$ such that the family of functions $\{\psi_\tau\}_{\tau \in \mathcal{J}}$ has no common zero.

Another minor technical lemma will be useful to us.

Lemma 6. Let P be a homogeneous harmonic polynomial of degree k and σ_ρ the normalized measure on the sphere $\partial B(0, \rho)$ ($0 < \rho$), considered as a radial distribution in $\mathbb{R}^n$. Then, with $|x| = r$, we have

$$(2.15) \quad P(x)\sigma_\rho(r) = \frac{(-1)^k}{2^{k-1}(k-1)!} \frac{\rho^{2-n}}{\Omega_n} P\left[\frac{\partial}{\partial x}\right] [(\rho^2 - |x|^2)^{k-1} \chi_{B(0, \rho)}(x)],$$

where $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface of the unit sphere in $\mathbb{R}^n$.

Proof. If we consider the Fourier transform of $P(x)\sigma_\rho(r)$, we have

$$\begin{aligned} \mathcal{F}(P(x)\sigma_\rho(r))(\zeta) &= P\left[i\frac{\partial}{\partial \zeta_1}, \dots, i\frac{\partial}{\partial \zeta_n}\right] \hat{\sigma}_\rho(\zeta) \\ &= i^k 2^{(n-2)/2} \Gamma\left(\frac{n}{2}\right) P\left[\frac{\partial}{\partial \zeta}\right] j_{(n-2)/2}(\rho\sqrt{\zeta_1^2 + \dots + \zeta_n^2}), \end{aligned}$$

where $j_\nu(z) = J_\nu(z)z^{-\nu}$. We can now apply an identity from [16, p. 126] to obtain

$$\begin{aligned} P\left[\frac{\partial}{\partial \zeta}\right] j_{(n-2)/2}(\rho\sqrt{\zeta_1^2 + \dots + \zeta_n^2}) &= P(\zeta) \left[\frac{1}{u} \frac{\partial}{\partial u}\right]^k j_{(n-2)/2}(\rho u) \Big|_{u=\sqrt{\zeta_1^2 + \dots + \zeta_n^2}} \\ &= (-1)^k \rho^{2k} P(\zeta) j_{(n/2)+k-1}(\rho\sqrt{\zeta_1^2 + \dots + \zeta_n^2}). \end{aligned}$$

On the other hand, the function $j_{(n/2)+k-1}(\rho\sqrt{\zeta_1^2 + \dots + \zeta_n^2})$ is the Fourier transform of a rather simple function. Namely, if $t = |\zeta|$,

$$\int_{\mathbb{R}^n} (\rho^2 - |x|^2)^{k-1} \chi_{B(0, \rho)}(x) e^{-ix \cdot \zeta} dx$$

$$\begin{aligned}
&= \frac{(2\pi)^{n/2}}{t^{(n-2)/2}} \int_0^\rho (\rho^2 - r^2)^{k-1} J_{(n-2)/2}(rt) r^{n/2} dr \\
&= (2\pi)^{n/2} \frac{\rho^{(n/2)+2k-1}}{t^{(n-2)/2}} \int_0^1 (1-s^2)^{k-1} J_{(n-2)/2}(\rho ts) s^{n/2} ds \\
&= (2\pi)^{n/2} 2^{k-1} (k-1)! \rho^{n+2k-2} J_{(n/2)+k-1}(\rho t),
\end{aligned}$$

as can be seen from [14, p. 688, formula 6.567.1]. Hence if we let

$$T(x) = T_{\rho, k}(x) = (\rho^2 - |x|^2)^{k-1} \chi_{B(0, \rho)}(x),$$

we have

$$\begin{aligned}
\mathcal{F}(P(x) \sigma_\rho(r))(\zeta) &= (-1)^k i^k \frac{\Gamma(n/2)}{\pi^{n/2} 2^k (k-1)!} \rho^{2-n} P(\zeta) \hat{T}(\zeta) \\
&= (-1)^k \frac{\Gamma(n/2) \rho^{2-n}}{\pi^{n/2} 2^k (k-1)!} \mathcal{F}\left(P\left[\frac{\partial}{\partial x}\right] T(r)\right)(\zeta).
\end{aligned}$$

Therefore

$$P(x) \sigma_\rho(r) = \frac{\rho^{2-n}}{\Omega_n} \frac{(-1)^k}{2^{k-1} (k-1)!} P\left[\frac{\partial}{\partial x}\right] T_{\rho, k}(r),$$

which is the formula (2.15). □

In what follows we will need to determine the radial distribution in $\mathbb{R}^n$, $\mu = \mu_{n, k, \alpha, \rho}$, that has as Fourier-Bessel transform

$$(2.16) \quad \tilde{\mu}(t) = j_{(n/2)+k-1}(\rho t) / (t^2 - \alpha^2),$$

where α is a zero of the numerator. It is easy to find the even function φ in $\mathbb{R}$ whose Fourier transform coincides with $\tilde{\mu}$. Therefore, what we need is to have a systematic way of obtaining μ from φ . In fact, this type of relation underlies all the proofs about the local properties of the Pompeiu

transform.

We have the following setup. Let $\mathcal{E}'_0(\mathbb{R}^n)$ be the space of radial distributions in $\mathbb{R}^n$ and $(\mathcal{E}'(\mathbb{R}^n))_0^\wedge$ the space of entire functions in the Paley-Wiener class which are radial when restricted to $\mathbb{R}^n$. Then, if $\mathcal{F}_n$ denotes the Fourier transform in $\mathbb{R}^n$, $\mathcal{FB}_n$ the Fourier-Bessel transform in $\mathbb{R}^n$, we have

$$(2.17) \quad \begin{array}{ccc} \mathcal{E}'_0(\mathbb{R}^n) & \xrightarrow[\cong]{\mathcal{F}_n} & (\mathcal{E}'(\mathbb{R}^n))_0^\wedge \\ \uparrow B_n & \mathcal{FB}_n & \downarrow S \cong \\ \mathcal{E}'_0(\mathbb{R}) & \xrightarrow[\cong]{\mathcal{F}_1} & (\mathcal{E}'(\mathbb{R}))_0^\wedge \end{array}$$

where $SF(t) = F(t, 0, \dots, 0)$ ($= F(\zeta)$ for any $\zeta \in \mathbb{C}^n$ such that

$\zeta_1^2 + \dots + \zeta_n^2 = t$), $S^{-1}f(\zeta) = f(\sqrt{\zeta_1^2 + \dots + \zeta_n^2})$. The isomorphism $B = B_n$ is defined by the commutative diagram and it is called a transmutation operator.

We remark that B is an isomorphism of convolution algebras, as such it has been originally studied by Bochner, Leray, Levitan [11], [17].

It is easier to find first B^{-1} explicitly. Namely, let μ be a smooth radial function of compact support in $\mathbb{R}^n$, then if $r = \sqrt{x_1^2 + \dots + x_n^2}$

$$S\mathcal{F}_n(t) = \int_{-\infty}^{\infty} e^{-itx_1} dx_1 \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mu(r) dx_2 \dots dx_n.$$

Comparing with the above diagram we see immediately that

$$\begin{aligned} \varphi(x_1) &= B^{-1}\mu(x_1) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \mu(r) dx_2 \dots dx_n \\ &= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^{\infty} \mu(\sqrt{x_1^2 + y^2}) y^{n-2} dy \end{aligned}$$

$$= \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_{|x_1|}^{\infty} \mu(s) s (s^2 - x_1^2)^{(n-3)/2} ds.$$

We introduce a change of variable, so that

$$\varphi(x_1) = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_{x_1^2}^{\infty} \mu(\sqrt{s}) (s^2 - x_1^2)^{(n-3)/2} ds.$$

Let $\varphi(x_1) = \Phi(x_1^2)$, $v = x_1^2$, then

$$\Phi(v) = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_v^{\infty} \mu(\sqrt{s}) (s-v)^{(n-3)/2} ds.$$

To invert this expression, we distinguish two cases. First, the case n is odd, $n = 2\ell + 1$, $\ell \geq 1$. Then

$$\mu(\sqrt{s}) = \frac{1}{\pi \ell} \left[\left[-\frac{d}{dv} \right]^\ell \Phi \right]_{v=s}.$$

Hence

$$(2.18) \quad \mu(s) = (B_{2\ell+1}\varphi)(s) = \frac{1}{\pi \ell} \left[\left[-\frac{1}{2x} \frac{d}{dx} \right]^\ell \varphi \right]_{x=s^2}.$$

The case of $n = 2\ell$, $\ell \geq 1$, is a bit harder. First we find, using Laplace transforms, that

$$\mu(\sqrt{s}) = \frac{1}{\pi \ell} \left[-\frac{d}{dv} \right]^\ell \left[\int_v^{\infty} (t-v)^{-1/2} \Phi(t) dt \right]_{v=s}.$$

Returning to the original function φ we get

$$(2.19) \quad \mu(s) = (B_{2\ell}\varphi)(s) = \frac{1}{\pi \ell} \left[-\frac{1}{2x} \frac{d}{dx} \right]^\ell \left[\int_x^{\infty} (\tau^2 - x^2)^{-1/2} \varphi(\tau) \tau d\tau \right]_{x=s}.$$

We can now apply these formulas to find the distributions $\mu = \mu_{n,k,\alpha,\rho}$

defined by (2.16).

Let $\nu = \frac{n-2}{2} + k$, $1/c_\nu := 2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2})$, then

$$j_\nu(\rho t) = c_\nu \rho^{-2\nu} \int_{-\rho}^{\rho} e^{-its} (\rho^2 - s^2)^{\nu-(1/2)} ds.$$

Hence, if $j_\nu(\rho\alpha) = 0$, we rewrite

$$\frac{j_\nu(\rho t)}{t^2 - \alpha^2} = \frac{1}{2\alpha} \left[\frac{j_\nu(\rho t)}{t - \alpha} - \frac{j_\nu(\rho t)}{t + \alpha} \right].$$

$$\frac{j_\nu(\rho t)}{t - \alpha} = c_\nu \rho^{-2\nu} \int_{-\rho}^{\rho} \frac{e^{-its} - e^{-ias}}{t - \alpha} (\rho^2 - s^2)^{\nu-(1/2)} ds,$$

and

$$\frac{j_\nu(\rho t)}{t + \alpha} = c_\nu \rho^{-2\nu} \int_{-\rho}^{\rho} \frac{e^{-its} - e^{-ias}}{t + \alpha} (\rho^2 - s^2)^{\nu-(1/2)} ds.$$

Therefore

$$\frac{j_\nu(\rho t)}{t^2 - \alpha^2} = \frac{c_\nu \rho^{-2\nu}}{2\alpha} \int_{-\rho}^{\rho} e^{-its} \psi_{\alpha, \nu}(s) ds,$$

where $\psi_{\alpha, \nu}(s)$ is the even function which, for $s \geq 0$, is given by

$$(2.20) \quad \psi_{\alpha, \nu}(s) = 2 \int_s^{\rho} \sin \alpha(s-u) (\rho^2 - u^2)^{\nu-(1/2)} du.$$

To find μ we apply the transmutation operator B_n to the function

$$\varphi_{\alpha, \nu} = \frac{c_\nu \rho^{-2\nu}}{2\alpha} \psi_{\alpha, \nu}.$$

For instance, for $n = 2$,

(2.21)

$$\mu_{2, k, \alpha, \rho}(r) = -\frac{1}{2^k \pi} \chi_{[0, \rho]}(r) \int_r^{\rho} (\tau^2 - r^2)^{-1/2} \left[\int_{\tau/\rho}^1 (1 - \xi^2)^{k-(1/2)} \cos \alpha(\tau - \rho\xi) d\xi \right] d\tau.$$

Finally, we recall a simple property of the Bessel functions.

Lemma 7. Let $a_1, a_2, a_3 > 0$ and $m \in \mathbb{Z}^+$. Let

$$\theta(z) = J_{n/2}(a_1 z) J_{n/2}(a_2 z) J_{(n/2)+m-1}(a_3 z).$$

Then there are constants $L_m, A_m > 0$, such that for any integer $\ell \geq L_m$ there exists $\rho_\ell \in]\ell, \ell+1[$ so chosen that if either $|z| = \rho_\ell$ or $|\operatorname{Im} z| \geq 1$ then

$$(2.22) \quad |\theta(z)| \geq A_m |z|^{-(1/2)(3n+1+2m)} e^{(a_1+a_2+a_3)|\operatorname{Im} z|}.$$

Here $n \geq 2$ is considered fixed throughout. If we are given $\delta > 0$ in such a way that $a_j \in [\delta, \delta^{-1}]$ then the constants A_m, L_m can be made to depend only on δ and m . They actually become explicit in the proof.

Proof. The function θ is an even entire function, therefore to prove (2.22) we can assume we have $\operatorname{Re} z \geq 0$. In this case we can use the following asymptotic development of the Bessel function J_ν ($\nu \geq 1$) (see [13])

$$(2.23) \quad |J_\nu(z) - \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{4}(2\nu+1))| \leq 3\sqrt{\frac{\pi}{2}} \frac{e^{|\operatorname{Im} z|}}{|z|^{3/2}} \left[\frac{|4\nu^2-1|}{8} \exp\left(\frac{\pi|4\nu^2-1|}{8|z|}\right) \right].$$

Clearly, when $|z| \geq \frac{\pi}{8}|4\nu^2-1|$ we obtain

$$|J_\nu(z) - \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{4}(2\nu+1))| \leq \frac{3e}{8} \sqrt{\frac{\pi}{2}} |4\nu^2-1| \frac{e^{|\operatorname{Im} z|}}{|z|^{3/2}}.$$

On the other hand, if $V = \{(2\ell+1)\pi/2 : \ell \in \mathbb{Z}\}$, $d(z, V) = \min\{1, \operatorname{dist}(z, V)\}$, then the cosine satisfies the Lojasiewicz inequality

$$|\cos z| \geq \frac{1}{\pi e} d(z, V) e^{|\operatorname{Im} z|}$$

Considering the expression (2.23) we see that we need to know how many zeros does the product $\cos(a_1 z - \frac{\pi}{4}(n+1)) \cos(a_2 z - \frac{\pi}{4}(n+1)) \cos(a_3 z - \frac{\pi}{4}(n+2m-1))$ has

in an interval $[\ell, \ell+1]$; as we have at most $\frac{a_1+a_2+a_3}{\pi}$ such zeros, we can find a value $\rho_\ell \in [\ell, \ell+1]$ such that the distance from ρ_ℓ to any of these zeros is at least $\frac{1}{2} \frac{\pi}{\pi+a_1+a_2+a_3}$. Hence, on the circle $|z| = \rho_\ell$, we have

$$|\cos(a_1 z - \frac{\pi}{4}(n+1))| \geq \frac{\min(1, a_1)}{2e(\pi+a_1+a_2+a_3)} e^{a_1 |\operatorname{Im} z|}.$$

On the other hand, if $|\operatorname{Im} z| \geq 1$, then if $\alpha_j = \min(1, a_j)$

$$|\cos(a_1 z - \frac{\pi}{4}(n+1))| \geq \frac{\alpha_1}{\pi e} e^{a_1 |\operatorname{Im} z|},$$

that is, the previous inequality is valid both for $|z| = \rho_\ell$ and for $|\operatorname{Im} z| \geq 1$. Therefore, on every such circle as well as on $|\operatorname{Im} z| \geq 1$ we have that

$$|J_{n/2}(a_1 z)| \geq \frac{\alpha_1 e^{a_1 |\operatorname{Im} z|}}{2e\sqrt{2\pi}(\pi+a_1+a_2+a_3)|a_1 z|^{1/2}}$$

if we assume $|z| \geq \frac{3}{4} \frac{e^{2\pi(\pi+a_1+a_2+a_3)n^2}}{a_1 \alpha_1}$. The same kind of estimates hold for the other two factors in θ . Let

$$(2.24) \quad L_m = \frac{3}{4} e^{2\pi(\pi+a_1+a_2+a_3)(n+2m)^2 / \min^2\{\alpha_1, \alpha_2, \alpha_3\}},$$

then for $\ell \geq L_m$, $|z| = \rho_\ell$ or for $|\operatorname{Im} z| \geq 1$, $|z| \geq L_m$ we have

$$(2.25) \quad |\theta(z)| \geq \frac{a_1^{-(n+1)/2} a_2^{-(n+1)/2} a_3^{-((n+1)/2)-m} \alpha_1 \alpha_2 \alpha_3 e^{(a_1+a_2+a_3)|\operatorname{Im} z|}}{[2e\sqrt{2\pi}(\pi+a_1+a_2+a_3)]^3 |z|^{1/2(3n+2m+1)}}$$

Remarks. 1) The sequences ρ_ℓ depends both on the a_j and on m .

2) We do not really use the full strength of the estimate (2.23). It would be enough to use Weber's estimates [19, § 7.33].

We conclude this preliminary section with upper bounds on the Bessel functions, which depend on the fact that the order ν is either an integer or half an integer. From [19, §2.2 and §3.32] we have for $\operatorname{Re} z > 0$ and q an

integer ≥ 0

$$J_q(z) = \frac{1}{\pi} \int_0^\pi \cos(q\theta - z \sin \theta) d\theta$$

$$J_{q+(1/2)}(z) = (-1)^q \left(\frac{z}{2\pi}\right)^q \int_0^\pi e^{iz \cos \theta} P_q(\cos \theta) \sin \theta d\theta,$$

where P_q is the Legendre polynomial of degree q . From [14, 8.917] we know that

$$|P_q(\cos \theta)| \leq 1.$$

Therefore

$$(2.26) \quad |J_q(z)| \leq e^{|\operatorname{Im} z|}, \quad |J_{q+(1/2)}(z)| \leq \sqrt{\frac{2}{\pi}} |z|^{1/2} e^{|\operatorname{Im} z|},$$

which has the advantage of being a global inequality, but loses $|z|^{1/2}$ and $|z|$ respectively, from the asymptotic expansions. This is a minor loss for us, and we will use this estimate below.

3. Reconstruction of a function in the case of two disks.

Let $Z_n = \{\xi/\eta : \xi > 0, \eta > 0, J_{n/2}(\xi) = J_{n/2}(\eta) = 0\}$. It is known that if $r_1 > 0, r_2 > 0$ and $r_1/r_2 \notin Z_n$, then the Pompeiu transform

$$P : f \mapsto (\chi_{B(0,r_1)})^{*f}, \chi_{B(0,r_2)}^{*f})$$

is injective. Moreover, this condition on r_1/r_2 is also necessary for the injectivity [12]. In case r_1/r_2 is badly approximated by elements of Z_n , that is, in case there are constants $\epsilon > 0, N > 0$ such that

$$(3.1) \quad \left| \frac{r_1}{r_2} - \frac{\xi}{\eta} \right| \geq \frac{c}{(1+|\eta|)^N}$$

one can construct explicitly distributions ν_1, ν_2 with $\text{supp } \nu_1 \subseteq \bar{B}(0, r_2)$, $\text{supp } \nu_2 \subseteq \bar{B}(0, r_1)$, such that [8]

$$(3.2) \quad \chi_{B(0, r_1)} * \nu_1 + \chi_{B(0, r_2)} * \nu_2 = \delta.$$

Therefore, under the arithmetic assumption (3.1) any function (or distribution) f in $\mathbb{R}^n$ can be reconstructed from the knowledge of its Pompeiu transform $Pf = (\chi_{B(0, r_1)} * f, \chi_{B(0, r_2)} * f)$. This process (3.2) is known as deconvolution. In case we only have a function f defined in a ball $B(0, R)$, with $R > r_1 + r_2$, even if we use the deconvolvers ν_1, ν_2 , we will only be able to reconstruct f in $B(0, R - (r_1 + r_2))$. For this reason, we will give a different kind of explicit inversion process for this local problem. It turns out that the arithmetic condition (3.1) will not be necessary, only $r_1/r_2 \notin \mathbb{Z}_n$. In particular, this answers on the affirmative a question posed by Zalzman [21] for the case $R = \infty$. In fact, our procedure is basically a method to approximate f with an estimate of the error. In this sense, we are dealing with several possible procedures that depend on the way we want to represent a given unknown signal f . For the sake of concreteness we will describe this in detail when we represent f in terms of the Fourier expansion in $B(0, R)$ by spherical harmonics. In a forthcoming paper we will explain the modifications necessary if one wants to use wavelets or the Radon transform on spheres.

To begin, let us point out that if f is not C^∞ in $B(0, R)$, then we can first consider $f * \varphi_\varepsilon$, $\text{supp } \varphi_\varepsilon \subseteq B(0, \varepsilon)$, $0 < \varepsilon \ll 1$, and since Pf is given by convolutions we will have $P(f * \varphi_\varepsilon) = Pf * \varphi_\varepsilon$. The first component of Pf , $\chi_{B(0, r_1)} * f$, is known in $B(0, R - r_1)$, hence that of $P(f * \varphi_\varepsilon)$.

$\chi_{B(0,r_1)} \circ f \circ \varphi_c$, will be known in $B(0, R-c-r_1)$. Similarly for the second component. Hence, replacing R by $R-c$, and then taking c so small that $R-c > r_1+r_2$, we can assume f is C^∞ .

Given a function $f \in C^\infty(B(0,R))$, we have a Fourier expansion of the form

$$(3.3) \quad f(x) = \sum_{m=0}^{\infty} \left[\sum_k a_{m,k}(r) S_{m,k}(\omega) \right], \quad x = r\omega, \quad \omega \in S^{n-1},$$

where $\{S_{m,k}\}_k$ is an orthonormal basis (with respect to the normalized measure of the unit sphere) of the spherical harmonics of degree m . That is, $r^m S_{m,k}(\omega) = H_{m,k}(x)$ is a harmonic homogeneous polynomial of degree m . The dimension of the vector space $\text{Span}(\{S_{m,k}\}_k)$ is $O(m^{n-2})$ [18]. The coefficients

$$a_{m,k}(r) = \int_{S^{n-1}} f(r\omega) S_{m,k}(\omega) d\sigma_1(\omega)$$

satisfy an inequality of the form

$$|a_{m,k}(r)| \leq C_N \frac{r^N}{m^N}$$

for any $N > 0$. Another way to write down this formula is as the action of the distribution $r^{-m} \sigma_r(x)$ on a smooth function

$$(3.4) \quad \begin{aligned} a_{m,k}(r) &= \langle r^{-m} \sigma_r(x), H_{m,k}(x) f(x) \rangle = \langle r^{-m} H_{m,k}(x) \sigma_r(x), f(x) \rangle \\ &= \frac{(-1)^m r^{2-n-m}}{2^{m-1} (m-1)! \Omega_n} \langle H_{m,k} \left(\frac{\partial}{\partial x} \right) T_{r,m}(|x|), f(x) \rangle \end{aligned}$$

where, as given by Lemma 6,

$$T_{r,m}(|x|) = (r^2 - |x|^2)^{m-1} \chi_{B(0,r)}(x).$$

the idea now will be to write $T_{r,m}$ as

$$T_{r,m} = g_1 * \chi_{B(0,r_1)} + g_2 * \chi_{B(0,r_2)},$$

with control on the supports of g_1 and g_2 . In this way we reduce the problem to a decomposition of radial functions. We fix some notation first.

Choose a strictly increasing sequence of positive numbers ε_k with limit $\frac{R}{r_1+r_2} - 1$, and a corresponding increasing sequence of radii

$$R_k = (r_1+r_2)(1+\varepsilon_k), \quad k \geq 1, \quad R_0 = 0.$$

We want to decompose the radial distribution $T_{r,m}$, and the decomposition will depend on the value k such that $r \in [R_{k-1}, R_k]$. We recall that the Fourier-Bessel transform of $T_{r,m}$ is given by

$$\tilde{T}_{r,m}(t) = (2\pi)^{n/2} 2^{m-1} (m-1)! r^{n+2m-2} j_{(n/2)+m-1}(rt).$$

Proposition 8. Let $r_1, r_2 > 0$, $r_1/r_2 \notin \mathbb{Z}_n$, $r_1+r_2 < R$, $\{\varepsilon_k\}$ fixed as above. For every r, m with $R_{k-1} \leq r < R_k$ there are two sequences of radial distributions μ_ℓ, ν_ℓ whose Fourier-Bessel transforms satisfy

$$(3.5) \quad |j_{(n/2)+m-1}(rz) - (j_{n/2}(r_1 z) \tilde{\mu}_\ell(z) + j_{n/2}(r_2 z) \tilde{\nu}_\ell(z))| \leq \frac{c'}{\ell} r^{-(n/2)-m+(3/2)} \|z\|^{-(n/2)-m+(13/2)} e^{R_k} |\operatorname{Im} z|$$

for $\ell \geq c m^2$

$$(3.6) \quad |\tilde{\mu}_\ell(z)| \leq c'' r^{-(n/2)-m+(3/2)} \|z\|^{-m+2} e^{(r_2+\beta_k)} |\operatorname{Im} z|$$

$$(3.7) \quad |\tilde{\nu}_\ell(z)| \leq c'' r^{-(n/2)-m+(3/2)} \|z\|^{-m+2} e^{(r_1+\beta_k)} |\operatorname{Im} z|.$$

In this statement, we use the notation $\|z\| = \max\{1, |z|\}$, c, c' being strictly positive constants depending only on $r_1, r_2, R, \varepsilon_1$ and n , while c''

depends also on ℓ and m . The quantity β_k denotes $c_k(r_1+r_2)$.

Proof. We are going to apply Lemma 7 with $a_1 = r_1$, $a_2 = r_2$, $a_3 = \beta_k$. The corresponding value $L_m \leq \frac{3}{4} \frac{e^{2\pi(\pi+R)(n+2m)^2}}{c_k^2(r_1+r_2)^2} \leq cm^2/c_k^2$. We let q be an even integer to be fixed later on. To simplify the notation we denote $f(z) = J_{(n/2)+m-1}(rz)$.

The idea of the proof is to profit from the simple fact

$$(3.8) \quad \frac{1}{2\pi i} \int_{|\zeta|=\rho_\ell} \frac{f(\zeta)}{\zeta-z} d\zeta = \begin{cases} f(z) & \text{if } |z| < \rho_\ell \\ 0 & \text{if } |z| > \rho_\ell. \end{cases}$$

The usual method of interpolation theory consists in rewriting (3.8) as follows

$$(3.9) \quad f(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho_\ell} \frac{f(\zeta)}{\zeta^{q_\theta(\zeta)}} \frac{(\zeta^{q_\theta(\zeta)} - z^{q_\theta(z)})}{\zeta - z} d\zeta \\ + \frac{1}{2\pi i} z^{q_\theta(z)} \int_{|\zeta|=\rho_\ell} \frac{f(\zeta)}{\zeta^{q_\theta(\zeta)}} \frac{d\zeta}{\zeta - z} \quad \text{for } |z| < \rho_\ell$$

and

$$(3.10) \quad \frac{1}{2\pi i} \int_{|\zeta|=\rho_\ell} \frac{f(\zeta)}{\zeta^{q_\theta(\zeta)}} \frac{(\zeta^{q_\theta(\zeta)} - z^{q_\theta(z)})}{\zeta - z} d\zeta \\ = - \frac{z^{q_\theta(z)}}{2\pi i} \int_{|\zeta|=\rho_\ell} \frac{f(\zeta)}{\zeta^{q_\theta(\zeta)}} \frac{d\zeta}{\zeta - z} \quad \text{for } |z| > \rho_\ell.$$

We define an even entire function H_ℓ by

$$(3.11) \quad H_\ell(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho_\ell} \frac{f(\zeta)}{\zeta^{q_\theta(\zeta)}} \frac{(\zeta^{q_\theta(\zeta)} - z^{q_\theta(z)})}{\zeta - z} d\zeta,$$

that is, the interpolation of f on the zeros of $z^{q_\theta(z)}$. Note that H_ℓ is

even independently of the parity of the integer q ; we took q even to simplify the computations below, but this is not necessary.

We are going to show the family $\{H_\ell\}_{\ell \geq L_m}$ is a bounded family in $\hat{\mathcal{S}}'(R)$. For that purpose we recall the upper bounds for f and θ that we obtain from (2.26),

$$(3.12) \quad |f(z)| \leq e^r r^{-((n-3)/2)-m} \|z\|^{-((n-3)/2)-m} e^r \|z\|$$

$$(3.13) \quad |\theta(z)| \leq e^{R_k} (r_1 r_2)^{-(n-1)/2} \beta_k^{-((n-3)/2)-m} \|z\|^{-(3(n-1)/2)-m+1} e^{R_k} \|z\|$$

We note that for n even these two estimates are too big by a power $\|z\|^{1/2}$ and $\|z\|^{3/2}$ respectively. Also the constants e^r and e^{R_k} can be replaced by e^R . Choose $\ell \geq cm^2/c_k^2 \geq L_m \geq 2$, then for $|\zeta| = \rho_\ell$ we have:

$$|\theta(\zeta)| \geq c_1 \beta_k^{-((n-1)/2)-m} \rho_\ell^{-(1/2)(3n+2m+1)} e^{R_k} |\operatorname{Im} \zeta|,$$

where $c_1 = c_1(r_1, r_2, R)$ is given by

$$c_1 = \frac{(r_1 r_2)^{-(n+1)/2} \min\{1, r_1, r_2, \beta_k\}^3}{(2e\sqrt{2\pi}(\pi+R))^3}.$$

For $|z| > \rho_\ell$ we can use (3.10) to estimate H_ℓ . We let $q = n+3$ if n is odd, $q = n+4$ if n is even, then

$$\begin{aligned} |H_\ell(z)| &\leq \frac{e^r \beta_k^{((n-1)/2)+m}}{c_1} r^{-((n-3)/2)-m} \rho_\ell^{-q+n+3} \frac{|z|^q |\theta(z)|}{|z|-\rho_\ell} \\ &\leq c_2 r^{-((n-3)/2)-m} \frac{|z|^{-(n/2)-m+(11/2)}}{|z|-\rho_\ell} e^{R_k} \|z\|, \end{aligned}$$

where

$$c_2 \leq \frac{e^{2R} r_1 r_2 \beta_k (2e\sqrt{2\pi}(\pi+R))^3}{\min\{1, r_1, r_2, \beta_k\}^3} \leq c'_2(r_1, r_2, R) = c'_2$$

(we have taken into account here the fact that the upper estimate for θ can be improved by $\|z\|^{3/2}$ when $n = \text{even}$). For $|z| < \rho_\ell$ we use (3.9), so that

$$(3.14) \quad |H_\ell(z)| \leq |f(z)| + c_2 r^{-((n-3)/2)-m} \frac{\|z\|^{-(n/2)-m+(11/2)} e^{R_k} |\operatorname{Im} z|}{\rho_\ell^{-|z|}}.$$

These estimates are fine when $|z| \leq \ell-1$ or $|z| \geq \ell+1$, since then $||z| - \rho_\ell| \geq 1$. In this case we have

$$(3.15) \quad |H_\ell(z)| \leq 2c'_2 r^{-((n-3)/2)-m} \|z\|^{-(n/2)-m+(11/2)} e^{R_k} |\operatorname{Im} z|.$$

In the remaining annulus, $\ell-1 \leq |z| \leq \ell+1$, we can use the maximum principle to obtain

$$(3.16) \quad |H_\ell(z)| \leq c''_2 r^{-((n-3)/2)-m} \|z\|^{-(n/2)-m+(11/2)} e^{R_k} |z|.$$

The new constant c''_2 is not substantially different from c'_2 , and it is independent of m since if $m + \frac{n}{2} \geq \frac{11}{2}$, $\left(\frac{\|z\|}{\ell-1}\right)^{(n/2)+m-(11/2)} \leq \left(\frac{\ell+1}{\ell-1}\right)^{(n/2)+m-(11/2)} \leq c_3$, independent of m since $\ell \geq cm^2/c_1^2$. The other values of m are handled by the same reasoning. We conclude that $\{H_\ell\}_\ell$ is bounded in $\operatorname{Exp}(\mathbb{C}) = \{\text{entire functions of exponential type}\}$. To conclude that $\{H_\ell\}_\ell$ is bounded in $\hat{\mathcal{E}}'(\mathbb{R})$ we need estimates on the real axis. From (3.14) we know that if $x \in \mathbb{R} \setminus [\ell-1, \ell+2]$ then

$$(3.17) \quad |H_\ell(x)| \leq 2c'_2 r^{-((n-3)/2)-m} \|x\|^{-(n/2)-m+(11/2)}.$$

As it was done in [2] we write

$$H_\ell(x) = H_\ell(x) - H_{\ell+3}(x) + H_{\ell+3}(x),$$

hence, by (3.17), the last term satisfies the correct inequality when $x \in [\ell-1, \ell+2]$, and we only need to estimate the difference $(H_\ell - H_{\ell+3})(x)$. Let Γ_ℓ be the Jordan curve which is the boundary of the region $\{\operatorname{Re} z \geq 0\}$.

$\rho_\ell \leq |z| \leq \rho_{\ell+3}$, $|\operatorname{Im} z| \leq 1$ and Γ'_ℓ its symmetric with respect to the imaginary axis. They are chosen with a convenient choice of orientation so that

$$(3.18) \quad H_\ell(x) - H_{\ell+3}(x) = \frac{1}{2\pi i} \left(\int_{\Gamma_\ell} + \int_{\Gamma'_\ell} \right) \frac{f(\zeta)(\zeta^q \theta(\zeta) - x^q \theta(x))}{\zeta^q \theta(\zeta)(\zeta - x)} d\zeta.$$

For $\zeta \in \Gamma_\ell$ and $x \in [\ell-1, \ell+2]$, we have

$$\begin{aligned} \left| \frac{\zeta^q \theta(\zeta) - x^q \theta(x)}{\zeta - x} \right| &\leq \max_{|w-x| \leq 5} \left| \frac{d}{dw} (w^q \theta(w)) \right| \\ &\leq c'_3 e^{6R} (r_1 r_2)^{-(n-1)/2} \beta_k^{-((n-3)/2)-m} |x|^{-(n/2)-m+(11/2)}; \end{aligned}$$

by the same reasoning that led to (3.16) the constant c'_3 is independent of m . For $\zeta \in \Gamma'_\ell$ and $x \in [\ell-1, \ell+2]$, we can use

$$\begin{aligned} |\zeta^q \theta(\zeta) - x^q \theta(x)| &\leq |\zeta^q \theta(\zeta)| + |x^q \theta(x)| \\ &\leq 2c'_3 e^{6R} (r_1 r_2)^{-(n-1)/2} \beta_k^{-((n-3)/2)-m} |x|^{-(n/2)-m+(11/2)}. \end{aligned}$$

We conclude that for $x \in [\ell-1, \ell+2]$ we have

$$|H_\ell(x)| \leq c'_3 r^{-(n-3)/2-m} \|x\|^{-(n/2)-m+(11/2)},$$

which is therefore valid everywhere. By applying the proof of the Phragmén-Lindelöf theorem to the function $(z + \frac{i}{m})^{(n/2)+m-(11/2)} e^{iR_k z} H_\ell(z)$ on the quadrant $\operatorname{Re} z \geq 0$, $\operatorname{Im} z \geq 0$ we see that

$$(3.19) \quad |H_\ell(z)| \leq c_4 r^{-((n-3)/2)-m} \|z\|^{-(n/2)-m+(11/2)} e^{R_k |\operatorname{Im} z|},$$

where c_4 is independent of m . This argument works in the other three quadrants. This shows that this sequence $\{H_\ell\}_\ell$ is bounded in $\hat{\mathcal{E}}'(\mathbb{R})$. Moreover, we can show its limit is f and one can estimate the error by the previous

procedure.

If $|z| < \rho_\ell$, formula (3.10) yields (see (3.14)):

$$(3.20) \quad |H_\ell(z) - f(z)| \leq c_2 r^{-((n-3)/2)-m} \frac{\|z\|^{-(n/2)-m+(11/2)} e^{R_k |Im z|}}{\rho_\ell - |z|} \\ \leq 2 \frac{c_2}{\ell} r^{-((n-3)/2)-m} \|z\|^{-(n/2)-m+(13/2)} e^{R_k |Im z|}$$

if $|z| \leq \ell/2$. For $|z| \geq \ell/2$ we have from (3.19) and the estimates for f that

$$(3.21) \quad |H_\ell(z) - f(z)| \leq |H_\ell(z)| + |f(z)| \\ \leq \frac{c'_4}{\ell} r^{-((n-3)/2)-m} \|z\|^{-(n/2)-m+(13/2)} e^{R_k |Im z|},$$

c'_4 is independent of m and bigger than $2c_2$, hence the last estimate is valid everywhere as stated in (3.5).

We can describe $\tilde{\mu}_\ell$ and $\tilde{\nu}_\ell$ explicitly using the Residue Theorem. Let us do it for the case the sequence $\{\epsilon_k\}$ has been chosen with the additional requirement that the pairs $j_{n/2}(r_j z)$ and $j_{(n/2)+m-1}(\beta_k z)$ ($j = 1, 2$) have no common zeros. We note that if there were a common zero the multiplicity of this zero for the product of the two functions would be exactly two and the formula below can be modified easily. From the definition of H_ℓ we can write for example when n is odd

$$H_\ell(z) = z^{n+3} j_{n/2}(r_1 z) j_{n/2}(r_2 z) g_{3,\ell}(z) + z^{n+3} j_{n/2}(r_1 z) j_{(n/2)+m-1}(\beta_k z) g_{2,\ell}(z) \\ + z^{n+3} j_{n/2}(r_2 z) j_{(n/2)+m-1}(\beta_k z) g_{1,\ell}(z) + \theta(z) P(z),$$

where

$$g_{1,\ell}(z) = \sum_{\substack{|\alpha| < \rho_\ell \\ j_{n/2}(r_1 \alpha) = 0}} \frac{f(\alpha)}{\alpha^{n+3} j_{n/2}(r_2 \alpha) j_{(n/2)+m-1}(\beta_k \alpha)} \frac{j_{n/2}(r_1 z)}{r_1 j'_{n/2}(r_1 \alpha) (z - \alpha)},$$

and analogous formulas for $g_{2,\ell}$ and $g_{3,\ell}$. P is the polynomial of degree $\leq n+2$ (in fact, P must be an even function) given by

$$P(z) = \operatorname{Res}_{\zeta=0} \left[f(\zeta) \frac{z^{n+2} + \zeta z^{n+1} + \dots + \zeta^{n+2}}{\zeta^{n+3} \theta(\zeta)} \right].$$

We define

$$\tilde{\mu}_\ell(z) = z^{n+3} j_{n/2}(r_2 z) g_{3,\ell}(z) + z^{n+3} j_{(n/2)+m-1}(\beta_k z) g_{2,\ell}(z)$$

$$\tilde{\nu}_\ell(z) = z^{n+3} j_{(n/2)+m-1}(\beta_k z) g_{1,\ell}(z) + P(z) j_{(n/2)}(r_1 z) j_{(n/2)+m-1}(\beta_k z).$$

Hence

$$(3.22) \quad H_\ell(z) = \tilde{\mu}_\ell(z) j_{n/2}(r_1 z) + \tilde{\nu}_\ell(z) j_{n/2}(r_2 z),$$

and the estimates (3.6) and (3.7) follow from the asymptotic estimates of the Bessel functions. Even though we need to replace $n+3$ by $n+4$ in case n is even, the estimates remain the same by the remark made after (3.13). $\square$

The previous proposition leads to the following

Theorem 9. Let $r_1, r_2 > 0$, $r_1/r_2 \notin \mathbb{Z}_n$, $r_1 + r_2 < R$, $\{c_k\}_{k \geq 1}$ a strictly increasing sequence of positive numbers with limit $\frac{R}{r_1+r_2} - 1$, $R_k = (r_1+r_2)(1+c_k)$ ($k \geq 1$), $R_0 = 0$. For any $r > 0$, $r \in [R_{k-1}, R_k[$, and S a normalized spherical harmonic of degree m there are two sequences of distributions $\mathfrak{u}_\ell, \mathfrak{s}_\ell$ of order $\leq n+3$ with compact support in $B(0, R-r_1)$ (respectively $B(0, R-r_2)$) such that for $\ell \geq cm^2$ and any function f in $C^\infty(B(0, R))$

$$(3.23) \quad \left| \int_{S^{n-1}} f(r\omega) S(\omega) d\sigma(\omega) - \langle \mathfrak{u}_\ell, \chi_{B(0, r_1)} \cdot f \rangle - \langle \mathfrak{s}_\ell, \chi_{B(0, r_2)} \cdot f \rangle \right|$$

$$\leq \frac{\gamma}{\ell} (R-r)^{-N} r^{-(n-3)/2} \max_{\substack{|\alpha| \leq N \\ |x| \leq R'_k}} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right|,$$

where $N = [\frac{1}{2}(n+13)] + 1$, $R'_k = \frac{2}{3}R + \frac{1}{3}R_k$ and $\gamma > 0$ is a constant depending on r_1, r_2, R, n, c_1 .

Proof. Let $m \geq 1$ and let H be the harmonic polynomial homogeneous of degree m such that $H(\rho\omega) = \rho^m S(\omega)$, $\omega \in S^{n-1}$, $\rho > 0$. From (3.4) we have

$$\int_{S^{n-1}} f(r\omega) S(\omega) d\sigma(\omega) = \frac{r^{2-n-m}}{2^{m-1}(m-1)! \Omega_n} \langle T(|x|), H\left(\frac{\partial}{\partial x}\right) f(x) \rangle,$$

where $T(|x|) = (r^2 - |x|^2)^{m-1} \chi_{B(0,r)}(x)$. We have seen that the Fourier-Bessel transform of $\frac{r^{2-n-m}}{2^{m-1}(m-1)! \Omega_n} T(|x|)$ is given by

$$t \mapsto 2^{(n-2)/2} \Gamma\left(\frac{n}{2}\right) r^m j_{(n/2)+m-1}(rt).$$

We can apply Proposition 8 and find explicit radial distributions μ_ℓ, ν_ℓ with $\text{supp } \mu_\ell \subseteq B(0, R_k - r_1)$, $\text{supp } \nu_\ell \subseteq B(0, R_k - r_2)$ such that the radial distribution τ_ℓ whose Fourier Bessel transform is given by

$$\tilde{\tau}_\ell(z) = 2^{(n-2)/2} \Gamma\left(\frac{n}{2}\right) r^m \left[j_{(n/2)+m-1}(rz) - j_{n/2}(r_1 z) \tilde{\mu}_\ell(z) - j_{(n/2)}(r_2 z) \tilde{\nu}_\ell(z) \right]$$

satisfies

$$|\tilde{\tau}_\ell(z)| \leq \frac{c}{\ell} r^{-(n-3)/2} \|z\|^{-(n/2)-m+(13/2)} e^{R_k \|Im z\|} \quad (\ell \geq \text{const. } m^2).$$

If we define the distributions

$$u_\ell(x) = \frac{(-1)^m r^m}{r_1^n \Omega_n} H\left(\frac{\partial}{\partial x}\right) \mu_\ell(|x|)$$

$$v_\ell(x) = \frac{(-1)^m r^m}{r_2^n \Omega_n} H\left(\frac{\partial}{\partial x}\right) \nu_\ell(|x|)$$

then

$$\int_{S^{n-1}} f(r\omega) S(\omega) d\sigma(\omega) - \langle \tau_{\ell}, \chi_{B(0, r_1)} \circ f \rangle - \langle \tau_{\ell}, \chi_{B(0, r_2)} \circ f \rangle = \langle \tau_{\ell}, H\left[\frac{\partial}{\partial x}\right] f \rangle.$$

We will now estimate $\langle \tau_{\ell}, H\left[\frac{\partial}{\partial x}\right] f \rangle$ using Plancherel's formula. Let ψ_k be a smooth radial function which is identically one on the ball $B(0, \frac{1}{3}R + \frac{2}{3}R_k)$, with support in $\bar{B}(0, \frac{2}{3}R + \frac{1}{3}R_k)$. One can find them in such a way that

$$\max_{x, |\alpha|=q} \left| \frac{\partial^\alpha}{\partial x^\alpha} \psi_k(x) \right| \leq \gamma_q (R - R_k)^{-q} \leq \gamma_q (R - r)^{-q}.$$

The constants γ_q are independent of R and k . Then

$$\begin{aligned} |\langle \tau_{\ell}, H\left[\frac{\partial}{\partial x}\right] f \rangle| &= |\langle \tau_{\ell}, H\left[\frac{\partial}{\partial x}\right] (\psi_k f) \rangle| \leq \int_{\mathbb{R}^n} \tilde{\tau}_{\ell}(|\xi|) |H(\xi)| |(\psi_k f)^{\wedge}(\xi)| d\xi \\ &\leq \frac{c}{\ell} r^{-(n-3)/2} \int_0^{\infty} \|\rho\|^{(n/2)+(11/2)} \int_{S^{n-1}} |S(\omega)| |(\psi_k f)^{\wedge}(\rho\omega)| d\sigma(\omega) d\rho \\ &\leq \frac{c'}{\ell} r^{-(n-3)/2} \left(\int_{S^{n-1}} |S(\omega)|^2 d\sigma(\omega) \right)^{1/2} \int_0^{\infty} \|\rho\|^{(n/2)+(11/2)} \max_{\omega \in S^{n-1}} |(\psi_k f)^{\wedge}(\rho\omega)| d\rho. \end{aligned}$$

On the other hand

$$|(\psi_k f)^{\wedge}(\xi)| \leq \gamma'_N \|\xi\|^{-N} (R - r)^{-N} \max_{\substack{|x| \leq R' \\ |\alpha| \leq N^k}} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right|,$$

where $N \geq \frac{n+13}{2}$ and $R'_k = \frac{2}{3}R + \frac{1}{3}R_k$.

This concludes the proof of Theorem 9 for $m \geq 0$. For $m = 0$ everything works the same way, except that T is now the normalized measure of the sphere of radius r , with Fourier-Bessel transform equal to $j_{(n-2)/2}(rt)$ up to a constant. $\square$

The estimate (3.23) leaves us with a little problem for small values of

r. In the case $0 \leq r < R - (r_1 + r_2)$, we can use an explicit formula to approximate δ in the form

$$\delta_0 \approx \chi_{B(0, r_1)} \cdot \alpha_1 + \chi_{B(0, r_2)} \cdot \alpha_2.$$

α_j radial distributions of fixed order, $\text{supp } \alpha_1 \subseteq \bar{B}(0, r_2)$, $\text{supp } \alpha_2 \subseteq B(0, r_1)$. Hence we compute directly

$$f(x) \approx [\alpha_1 \cdot \chi_{B(0, r_1)} \cdot f] + [\alpha_2 \cdot \chi_{B(0, r_2)} \cdot f](x)$$

for $|x| < R - (r_1 + r_2)$, the precise estimate of the error being of the form as in (3.21) (without the dependence on r and $R - r$); we use as a function θ just the product $j_{n/2}(r_1 z) j_{n/2}(r_2 z)$.

If we choose the sequence ϵ_k in Theorem 9 so that every zero of $\theta(z) = j_{n/2}(r_1 z) j_{n/2}(r_2 z) j_{(n/2)+m-1}(\beta_k z)$ is simple, the distributions μ_ℓ, ν_ℓ , and a fortiori η_ℓ and ξ_ℓ can be made completely explicit. We only need to know the explicit form of a radial distribution whose Fourier-Bessel transform has the form

$$\frac{j_{(n/2)+m-1}(\rho z)}{z^{2-\alpha}}, \quad (m \geq 1)$$

when α is a zero of $j_{(n/2)+m-1}(\rho z)$.

These distributions have been computed in section 2; recall for example that when $n = 2$, $\tilde{\mu}_{m, \alpha, \rho}(t) = \frac{j_{(n/2)+m-1}(\rho t)}{t^{2-\alpha}}$, with

$$\mu_{m, \alpha, \rho}(r) = -\frac{1}{2^m \pi^2} \chi_{[0, \rho]}(r) \int_r^\rho (\tau^2 - r^2)^{-1/2} \left[\int_{\tau/\rho}^1 (1 - \xi^2)^{m-(1/2)} \cos \alpha(\tau - \rho \xi) d\xi \right] d\tau.$$

The sequence of approximations for the Fourier coefficients

$$\frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{\pm im\theta} d\theta, \quad m \geq 1, \quad r \in [R_{k-1}, R_k] \quad \text{can be written as}$$

$$\left\{ \langle u_{r,m,\ell}, \chi_{B(0,r_1)} \rangle + \langle s_{r,m,\ell}, \chi_{B(0,r_2)} \rangle \right\}_{\ell \in \mathbb{N}},$$

where

$$u_{r,m,\ell} = \frac{(-1)^m r^m}{2\pi r_1^2} \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right)^m \mu_\ell$$

$$s_{r,m,\ell} = \frac{(-1)^m r^m}{2\pi r_2^2} \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right)^m \nu_\ell$$

and

$$\begin{aligned} \mu_\ell = & -\Delta^3 \left[\frac{\chi_{B(0,r_2)}}{\pi r_2^2} \cdot \sum_{\substack{0 < \alpha < \rho_\ell \\ j_m(\beta_k \alpha) = 0}} \frac{j_m(r\alpha)}{\alpha^5 \beta_k j_1(r_2 \alpha) j_1(r_2 \alpha) j'_m(\beta_k \alpha)} \mu_{m,\alpha,\beta_k} \right. \\ & + \frac{1}{2^{m-1} (m-1)! \pi} \frac{(\beta_k^2 - |\cdot|^2)^{m-1}}{\beta_k^{2m}} \chi_{B(0,\beta_k)} \cdot \\ & \left. \sum_{\substack{0 < \alpha < \rho_\ell \\ j_1(r_2 \alpha) = 0}} \frac{j_m(r\alpha)}{\alpha^5 r_2 j_1(r_1 \alpha) j_m(\beta_k \alpha) j'_1(r_2 \alpha)} \mu_{1,\alpha,r_2} \right], \\ \nu_\ell = & -\Delta^3 \left[\frac{1}{2^{m-1} (m-1)! \pi} \frac{(\beta_k^2 - |\cdot|^2)^{m-1}}{\beta_k^{2m}} \chi_{B(0,\beta_k)} \cdot \right. \\ & \left. \sum_{\substack{0 < \alpha < \rho_\ell \\ j_1(r_1 \alpha) = 0}} \frac{j_m(r\alpha)}{\alpha^5 r_1 j_1(r_2 \alpha) j_m(\beta_k \alpha) j'_1(r_1 \alpha)} \mu_{1,\alpha,r_1} \right] \\ & + \left[S_k - T_k \Delta + U_k \Delta^2 \right] \left[\frac{\chi_{B(0,r_1)}}{2\pi r_1^2} \cdot \frac{(\beta_k^2 - |\cdot|^2)^{m-1}}{2^{m-1} \pi (m-1)! \beta_k^{2m}} \chi_{B(0,\beta_k)} \right]. \end{aligned}$$

The polynomial $S_k + T_k z^2 + U_k z^4$ is defined as

$$S_k + T_k z + U_k z^2 = \operatorname{Res}_{\zeta=0} \left[J_m(r\zeta) \frac{z^5 + \zeta z^4 + \dots + \zeta^5}{\zeta^6 J_1(r_1 \zeta) J_1(r_2 \zeta) J_m(\beta_k \zeta)} \right].$$

4. Local inversion of the Pompeiu transform for the case of a square.

In this section we will show how to reconstruct a function f defined in the disk $B = B(0, R)$ of $\mathbb{R}^2$ in terms of its averages over squares of side $2a$, with $2\sqrt{2}a < R$. We will also see that this inversion procedure generalizes to practically any set E with the Pompeiu property.

The idea is to reduce ourselves to the situation of the previous section. Let Q be the square $[-a, a] \times [-a, a]$, χ_Q its characteristic function. Then

$$(4.1) \quad T = \frac{\partial^2 \chi_Q}{\partial x \partial y} = \delta_{(a, a)} - \delta_{(-a, a)} + \delta_{(-a, -a)} - \delta_{(a, -a)}.$$

It is immediate that $\mathcal{R}_0 T = 0$. On the other hand, this is not true for some derivatives of T . We compute directly the Fourier-Bessel transform of $\mathcal{R}_0 \left(\frac{\partial^k}{\partial z^k} T \right)$. Let t be a positive number, choose $\zeta \in \mathbb{C}^2$, $|\zeta| = t$, and identify $z = x + iy$ with $(x, y) \in \mathbb{R}^2$, then

$$\begin{aligned} \left(\mathcal{R}_0 \left(\frac{\partial^k}{\partial z^k} T \right) \right)^\sim(t) &= \langle \mathcal{R}_0 \left(\frac{\partial^k}{\partial z^k} T \right)(z), e^{-iz \cdot \zeta} \rangle \\ &= \langle \frac{\partial^k}{\partial z^k} T(z), \mathcal{R}_0(e^{-iz \cdot \zeta}) \rangle \\ &= \langle \frac{\partial^k}{\partial z^k} T(z), \tilde{\sigma}_{|z|}(t) \rangle \\ &= \langle \frac{\partial^k}{\partial z^k} T(z), J_0(t|z|) \rangle \\ &= (-1)^k \langle T(z), \frac{\partial^k}{\partial z^k} J_0(t|z|) \rangle \end{aligned}$$

(as shown in Lemma 6)

$$= t^{2k} \langle T(z), \left(\frac{z}{2}\right)^k J_k(t|z|) \rangle.$$

That is,

$$(4.2) \quad \left[\mathcal{R}_0 \frac{\partial^k}{\partial z^k} T \right]^\sim(t) = \begin{cases} 0 & \text{if } k \not\equiv 2 \pmod{4} \\ 4 \left(\frac{a}{2}\right)^k t^{2k} e^{-i(\pi/4)k} J_k(\sqrt{2}ta) & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

In particular,

$$(4.3) \quad \left[\mathcal{R}_0 \left[\frac{\partial^2}{\partial z^2} T \right] \right]^\sim(t) = -ia^2 t^4 J_2(\sqrt{2}at)$$

$$(4.4) \quad \left[\mathcal{R}_0 \left[\frac{\partial^6}{\partial z^6} T \right] \right]^\sim(t) = \frac{ia^6}{16} t^{12} J_6(\sqrt{2}at).$$

The two functions above have only the origin as a common zero of order 4, by a well known property of the Bessel functions (see [19, p. 485]). On the other hand, the function $(\mathcal{R}_0 \chi_Q)^\sim(0) \neq 0$. If we use these three functions with a reasoning similar to that in the last section we would need $R > 3\sqrt{2}a$ to obtain an inversion formula which will be even a bit more complicated. (We will return to this point below.) We sidestep this question by working only with the radial distributions $\mathcal{R}_0 \left[\frac{\partial^2}{\partial z^2} T \right]$ and $\mathcal{R}_0 \left[\frac{\partial^6}{\partial z^6} T \right]$ to first reconstruct $\Delta^2 f$. Namely, consider the two radial distributions μ_1, μ_2 such that

$$\tilde{\mu}_1(t) = -ia^2 J_2(\sqrt{2}at)$$

$$\tilde{\mu}_2(t) = \frac{ia^6}{16} t^8 J_6(\sqrt{2}at),$$

whose Fourier Bessel transforms have no common zeros and which are supported in $\bar{B}(0, \sqrt{2}a)$. One can modify Lemma 7 by considering the function

$$\theta(t) = \tilde{\mu}_1(t) \tilde{\mu}_2(t) J_m(\varepsilon 2\sqrt{a}t),$$

with $0 < \varepsilon < R - 2\sqrt{a}$. The inequality (2.22) is replaced by

$$(4.5) \quad |\theta(z)| \geq |z|^{-(1/2)(3+2m)} e^{2\sqrt{a}(1+\varepsilon)} |1m z|.$$

Whenever $|z| = \rho'_\ell$ or $|1m z| \geq 1$, $\ell \geq L'_m$, for some $\nu'_m, L'_m > 0$.

The proof of Theorem 9 furnishes distributions $\nu'_\ell, \mathfrak{S}'_\ell$ such that for any function $f \in C^\infty(B(0, R))$

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-im\theta} d\theta - \langle \nu'_\ell, \mu_1 \circ f \rangle - \langle \mathfrak{S}'_\ell, \mu_2 \circ f \rangle \right| \\ & \leq \frac{\gamma'}{\ell} (R-r)^{-N} \max_{\substack{|\alpha| \leq N \\ |z| \leq R'_k}} \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} f(z) \right|, \end{aligned}$$

where, as before, the distributions $\nu'_\ell, \mathfrak{S}'_\ell$ depend also on r, m , have support in $B(R\sqrt{2}a)$, their order and the constant N do not depend on r or m ;

R'_k has the same meaning as in Theorem 9, that is $R'_k = \frac{2}{3}R + \frac{1}{3}\sqrt{2a(1+\varepsilon)}$.

Applying (4.6) to $\Delta^2 f$, we can compute its Fourier coefficients in $B(0, R)$ in terms of the functions $\varphi_1 = \left[\mathcal{R}_0 \left(\frac{\partial^2}{\partial z^2} T \right) \right] \circ f$ and $\varphi_2 = \left[\mathcal{R}_0 \left(\frac{\partial^6}{\partial z^6} T \right) \right] \circ f$ in $B(R\sqrt{2}a)$, since $\Delta^2 \mu_1 = \mathcal{R}_0 \left(\frac{\partial^2}{\partial z^2} T \right)$ and $\Delta^2 \mu_2 = \mathcal{R}_0 \left(\frac{\partial^6}{\partial z^6} T \right)$. By Lemma 5, we can compute the functions φ_1, φ_2 in terms of the Pompeiu transform of f . In other words, we can explicitly recover the Fourier coefficients of $\Delta^2 f$ in terms of $P(f)$.

We need now to show how to compute the Fourier coefficients of f in terms of $P(f)$ and $\Delta^2 f$. This can be accomplished by observing that

$$(\mathcal{R}_0 \chi_Q)^\sim(0) = \text{area}(Q) = 4a^2$$

and that $(\mathcal{R}_0 \chi_Q)^\sim(z)$ has an asymptotic behavior of the same type as that of the Bessel functions, as it was shown in [3, Section 6]. Hence we can apply the same procedure as above to the two radial distributions $\mathcal{R}_0 \chi_Q$ and $\Delta^2 \delta_0$

and in this way, we can approximate the Fourier coefficients of f in $B(0,R)$ as closely as we want in terms of the "data" $P(f)$.

Remark 1. The procedure for the local inversion of the Pompeiu transform associated to a square is valid for a very general class of domains having the Pompeiu property. In fact, let E be a compact subset of $\mathbb{R}^n$ having the global Pompeiu property. Let us assume that for some base point x_0 , the boundary of E contains a hyperbolic point p in the sense of [3, section 2]. One can assume that $\partial B(x_0, r) \cap E = \{p\}$, where $r = \text{dist}(x_0, p)$ and $E \subseteq \bar{B}(x_0, r)$. This is just a minimal regularity condition on the extremal point p of E . By Proposition 5 in [3], we have that near the real axis we have the asymptotic development

$$(4.7) \quad (\mathcal{R}_{x_0} \chi_E)^\sim(t) \sim \frac{\text{const}}{t^{\nu_0}} \cos(tr - \nu_1),$$

where ν_0, ν_1 are some positive constants related to the geometry of ∂E near p . Outside a horizontal strip there is a corresponding good lower bound.

This behavior is the same as that of the Bessel functions.

Let μ_1 be $\mathcal{R}_{x_0} \chi_E$. If we assume that $R > 3r$, let $\varepsilon_0 > 0$ be such that $3r + 3\varepsilon_0 < R$. For any $\varepsilon > 0$ we can consider the set E_ε which is the translate of E along the unit direction $\overrightarrow{x_0 p}$ by ε , then

$$(4.8) \quad (\mathcal{R}_{x_0} \chi_{E_\varepsilon})^\sim(t) \sim \frac{\text{const}}{t^{\nu_0}} \cos(t(r+\varepsilon) - \nu_1),$$

since the only extremal point of E_ε of interest is just $p + \varepsilon \overrightarrow{x_0 p}$. It is then immediate that we can choose $\varepsilon \leq \varepsilon_0$ so that $\tilde{\mu}_1$ and $(\mathcal{R}_{x_0} \chi_{E_\varepsilon})^\sim$ have only finitely many common zeros. Let us choose ε'_0 with such a property, $\mu_2 = \mathcal{R}_{x_0} \chi_{E_{\varepsilon'_0}}$, and $z_1, \dots, z_N$ the common zeros of $\tilde{\mu}_1$ and $\tilde{\mu}_2$. The argument

in [3, Proposition 5] shows that one cannot only translate along the ray $\overrightarrow{x_0 p}$

but even along a small open cone $\mathcal{I}$ with vertex 0 in such a way that the translate of p remains the only extremal point in $E_\tau \cap \partial B(x_0, |p+\tau-x_0|)$ for $\tau \in \mathcal{I}$. By Lemma 4' the corresponding functions ψ_τ have no common zero if $|\tau| \leq \varepsilon_0$.

For each z_j we can choose distinct τ_j such that $(\mathcal{R}_{x_0} \chi_{E_{\tau_j}})^\sim(z_j) \neq 0$, hence there is a linear combination of these finitely many functions which will not vanish at any of the points $z_1, \dots, z_N$. This linear combination $\tilde{\mu}_3$ will automatically have a good asymptotic development. The radial distributions (with respect to x_0) μ_1, μ_2, μ_3 are supported by the balls $\bar{B}(x_0, r_1)$, $\bar{B}(x_0, r_2)$, $\bar{B}(x_0, r_3)$ respectively, $r_1 + r_2 + r_3 < R$ and the method of the previous section applies with small modifications. Lemma 7 allows us to compute $\mu_j * f$ in terms of $P(f)$, in this way one obtains an inversion formula in any ball of radius $R > 3r$.

Remark 2. In comparing the previous remark with the construction for the square one notes that the construction for the square could be further simplified if we knew that $(\mathcal{R}_0 \chi_Q)^\sim$ and $\mathcal{R}_0 \left(\frac{\partial^2}{\partial z^2} T \right)^\sim$ have no common zeros. Though we believe this is true we have not been able to prove it.

References

- [1] C.A. Berenstein, An inverse spectral theorem and its relation to the Pompeiu problem, *J. Analyse Math.* 37 (1980), 128-144.
- [2] C.A. Berenstein, R. Gay, A local version of the two circles theorem, *Israel J. Math.* 55 (1986), 267-288.
- [3] C.A. Berenstein, R. Gay, Le problème de Pompeiu local, *J. Analyse Math.* 52 (1988), 133-166.
- [4] C.A. Berenstein, R. Gay, A. Yger, The three squares theorem, a local version, to appear in the *Festschrift for M. Cotlar* (ed. C. Sadosky).
- [5] C.A. Berenstein, B.A. Taylor, The "three squares" theorem for continuous functions, *Arch. Rat. Mech. Anal.* 63 (1977), 253-259.
- [6] C.A. Berenstein, P. Yang, An inverse Neumann problem, *J. Reine Angew Math.* 382 (1987), 1-21.
- [7] C.A. Berenstein, A. Yger, Sur quelques formules explicites de déconvolution, *J. Optics* (Paris), 1983, vol. 14, 2, 75-82.
- [8] C.A. Berenstein, A. Yger, Le problème de la déconvolution, *J. Funct. Anal.* 54 (1983), 113-160.
- [9] C.A. Berenstein, A. Yger, Analytic Bezout Identities, *Advances in Applied Mathematics* 10 (1989), 51-74.
- [10] C.A. Berenstein, L. Zalcman, The Pompeiu problem in symmetric spaces, *Comment. Math. Helvetici* 55, 1980, 593-621.
- [11] S. Bochner, K. Chandrasekharan, Fourier Transforms, *Annals of Mathematics Studies* 19, Princeton Univ. Press, 1949.
- [12] L. Brown, B.M. Schreiber, B.A. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* 23 (1973), 125-154.
- [13] L. Gatteschi, On the zeros of certain functions with application to Bessel functions, *Indag. Math.* 14 (1952), 224-229.
- [14] I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, series and products, New York, Academic Press, 1980.
- [15] S. Helgason, Groups and Geometric Analysis, Academic Press, New York, 1984.
- [16] E.W. Hobson, The theory of spherical and ellipsoidal harmonics, Cambridge University Press, 1931.
- [17] J. Leray, Hyperbolic differential equations, The Institute for Advanced Study, Princeton, NJ, 1952.
- [18] R. Seeley, Spherical harmonics, *Amer. Math. Monthly* 73 (1966), 115-121.

- [19] G. Watson, A Treatise on the theory of Bessel functions, Cambridge Univ. Press, 1966.
- [20] L. Zalcman, "Offbeat integral geometry," Amer. Math. Monthly 87 (1980), 161-175.
- [21] L. Zalcman, Analyticity and the Pompeiu problem, Arch Rat. Mech. Anal. 47 (1972), 237-254.

CONCLUSIONS

We have demonstrated the feasibility of building super-resolution systems using multiple detectors. The deconvolution method provides a real time linear implementation of the reconstruction problem which is robust with respect to noise and perturbations of the overall system.

The next task that is starting to be carried out by doctoral students of Dr. Berenstein's in Maryland, Drs. Gay and Yger in Bordeaux, and Dr. Taylor in Michigan, is to incorporate postprocessing features adequate for particular tasks. For instance, in many problems we need not only to reconstruct the pixel by pixel values of signal to a desired degree of resolution but also we need to automatically identify it, e.g. using segmentations, Voronov diagrams, etc. How to incorporate this image (post) processing into the deconvolutions is the challenge.

Another thing that still remains to be implemented, since it requires considerable manpower, is to write a user friendly menu to design and simulate multidetector systems. It is for this task we feel the newer completely numerical algorithm of finding deconvolvers might be most useful.

The local Pompeiu problem that "peeled-off" the deconvolution problem presents new challenges. The nature of the inversion formulas for Chapter 4, indicates that the deconvolution and local deconvolution formulas must be well

adapted to implementation in terms of wavelets. If this were the case, it will also help to produce a significant data reduction and reduction of processing time in ATR and similar problems.

Finally, the purely algebraic problems of the size and degree of solutions to the Bezout equation have applications to robotics, control of distributed parameter systems, computational geometry, etc., and they show the unexpected payoffs of the use of the powerful methods of several complex variables in applied mathematics.