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SATELLITE MOTION IN AN AXI-SYMMETRIC GRAVITATIONAL FIELD PART 2: PERTURBATIONS DUE TO AN ARBITRARY J_ℓ

by

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SATELLITE MOTION IN AN AXI-SYMMETRIC GRAVITATIONAL FIELD
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SUMMARY

This Report continues the presentation of the untruncated orbital theory begun in Technical Report 88061. The effects of the general zonal harmonic, J_ℓ, are now covered, the main results being a trio of formulae for perturbations in the spherical-polar coordinates introduced in the previous paper. The formulae are only first-order in J_ℓ, but, in conjunction with the second-order results for J_2 published in Part 1, the complete set of formulae may be regarded as constituting a second-order theory, the Earth's J_2 being much larger than J_ℓ for $\ell > 2$.

The mean elements of the theory are defined in such a way that, for each J_ℓ, the coordinate-perturbation formulae have their simplest possible form, with no occurrence of zero denominators. The general formulae are used in a rederivation of the results for J_3, given in Part 1, and in a derivation of results for J_4.

Numerical comparisons with reference orbits are held over to a later report (Part 3).

Departmental Reference: Space 675

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1 INTRODUCTION

This Report is the second of an intended trilogy devoted to satellite motion about an axi-symmetric primary, i.e. about a gravitating solid of revolution. Thus it continues the exposition of Ref 1, which will henceforth be referred to as 'Part 1'. Part 1 brought together the principles of an approach to orbit modelling in which lengthy expressions for short-period perturbations in the usual osculating elements are compressed into concise expressions for perturbations in a particular set of spherical-polar coordinates; it then proceeded into the presentation of a complete second-order theory for perturbations due to the zonal harmonic J_2, and a complete first-order theory for J_3. When the primary body is the Earth, J_3 (and every subsequent J_ℓ) is of order J_2^2, so Part 1 may be regarded as describing (for J_2 and J_3 only) a complete second-order theory for Earth satellites, where 'first order' refers to effects of relative magnitude 10^{-3}. Though Part 1 has only recently been published, a résumé[2] of the theory had been given much earlier.

Two other papers are relevant to the maturation of the trilogy: a recent one[3] on mean elements (as used in Part 1), with particular reference to the relation between mean semi-major axis and mean mean motion; and a much earlier (and more important) paper[4], of similar title to the trilogy's, that established formulae for secular and long-period perturbations due to the general J_ℓ (so general, in fact, that ℓ could be negative, the formulae then being applicable to lunisolar perturbations). The present Report effectively combines the new approach of Part 1 with the general principles and notation of Ref 4, the result being a complete theory for the zonal harmonics; secular and long-period perturbations are applied to mean orbital elements, and short-period perturbations to coordinates.

Part 3 of the trilogy will be largely devoted to the way in which the mean elements evolve over periods of time longer than just a small number of orbital revolutions. This topic, which was given limited attention in Part 1, is entirely neglected in Part 2. It is intended that Part 3 will also give details of the Fortran program(s) written to evaluate the accuracy of the overall approach, using harmonics up to J_4. (The variations in the mean elements are computed by a technique that involves a numerical component of an otherwise analytical model, aspects of this technique were described in the paper[5] that originally outlined the author's philosophy of coupling a hybrid computational procedure to the coordinate-perturbation approach.)

Other authors have published first-order formulae for satellite perturbations due to the geopotential; they usually address the subject more generally than here, by covering the tesseral harmonics as well as the zonal harmonics. The first entirely general results were derived by Groves[6], in an analysis of formidable complexity, whilst the classic reference is the text-book of Kaula[7]. The very generality of the formulae in Refs 6 and 7 makes it difficult to write down expressions for individual effects, however, and it is not even easy to show that the two sets of formulae are formally equivalent (the full first-order expression for the perturbation in mean anomaly is omitted in Ref 6, and the supplementary terms are only added as an afterthought in Ref 7).

Much of the difficulty in the general analysis arises from the need, when covering the tesseral harmonics as well as the zonal harmonics, to allow for the rotation of the primary. The uniformity of this rotation with time makes it natural to work with M (mean anomaly), rather than v (true anomaly), as integration variable, but this inevitably leads to infinite summations. When the analysis is restricted to the zonal harmonics, however, use of v (rather than M) leads to expressions that are free of infinite summation, and Zafiropoulos[8] has recently published untruncated formulae for the first-order perturbations in the orbital elements due to the general J_ℓ. The formulae of Ref 8 are much more explicit than those in Refs 6 and 7, but this is unfortunately at the expense of some very long expressions - it takes more than five pages to express the basic formulae, and even then the supplementary terms of the perturbation in M are again absent. Now it will emerge from the present Report that the formulae of Zafiropoulos can be expressed much more concisely. The real breakthrough comes, however, when the short-period perturbations in elements are replaced by perturbations in coordinates. If it were not for the rotation of the primary, this procedure could be immediately extended to the tesseral harmonics*; for orbits of sufficiently low eccentricity there is no difficulty, and very simple general formulae were given in Refs 5 and 9, having originally been derived during a study[10] of Navstar/GPS.

As with Part 1, a List of Symbols is appended to the Report; it is almost entirely consistent with the List of Part 1, the few exceptions being noted. The meaning of every new symbol is fully specified in the text, but only minimal

* Appendix A, which is in the nature of a postscript, outlines what is involved in the extension for a non-rotating primary, and a separate paper is planned for later publication.

explanation is given for those carried over from Part 1. This is true, in particular, for standard symbolism: thus we note, straight away, that the orbital elements used are a, e, i, Ω, ω and M, an arbitrary one of which is denoted (generically) by ζ; also $M = \sigma + \int$, where $\int$ is shorthand for $\int n\,dt$, the integral being taken from epoch to current time. We continue to make use of the quasi-elements ψ, ρ and L, really only defined at the differential level; thus, $d\psi = d\omega + c\,d\Omega$ (where $c = \cos i$), $d\rho = d\sigma + q\,d\psi$ (where $q^2 = 1 - e^2$) and $dL = dM + q\,d\psi$.

As explained in Part 1, each osculating element, ζ, may be regarded as the sum of a mean element, $\bar{\zeta}$, and a short-period perturbation, $\delta\zeta$, so that

$$\zeta = \bar{\zeta} + \delta\zeta \,. \tag{1}$$

A 'semi-mean' element, $\tilde{\zeta}$, is also needed (see section 3.2 of Part 1), but in Part 2 we will usually ignore the distinction between $\bar{\zeta}$ and $\tilde{\zeta}$. The effect of this neglect is that we do not distinguish between the quantities denoted by $\delta\zeta$, $\delta_p\zeta$ and $\delta\zeta$ in Part 1, normally using $\delta\zeta$ here in the sense of $\delta_p\zeta$ of Part 1; towards the end of the Report (in deriving the perturbations due to J_4, in section 8.5), we remind the reader of the additional terms (split between $\tilde{\zeta}$ and $\delta\zeta$, as explained in Part 1) that are needed to express (first-order) perturbations in full. Not even the distinction between osculating elements and mean elements is of significance in evaluating the right-hand sides of equations in general, since second-order perturbations are not taken into account in Part 2, but the following important distinction (on left-hand sides) between $\dot{\zeta}$ and $\dot{\bar{\zeta}}$ is worth noting: Lagrange's planetary equation for $\dot{\zeta}$ constitutes the <u>starting point</u> of analysis for the element ζ, whereas a formula for $\dot{\bar{\zeta}}$ is part of the <u>goal</u> of that analysis.

The analysis is greatly facilitated by using, instead of ω and u (argument of latitude), the modified quantities ω' and u', where

$$\omega' = \omega - \tfrac{1}{2}\pi \quad \text{and} \quad u' = u - \tfrac{1}{2}\pi \,. \tag{2}$$

To avoid any confusion, it is remarked that the use of the accent (prime sign) here has a connotation entirely different from that which distinguishes a (osculating semi-major axis) from a'; the latter quantity is an absolute constant of the motion (under zonal harmonics only), which (as shown in Part 1) constitutes the best choice for mean semi-major axis ($\bar{a}$), to whatever order the analysis is conducted. With u' now introduced, it is convenient to define here the much-used quantities C_j^k and S_j^k ; thus*

$$C_j^k = \cos(jv + ku') \quad \text{and} \quad S_j^k = \sin(jv + ku') . \tag{3}$$

When there is no ambiguity in regard to k, the superfix (but never the suffix) will often be omitted. (Warning: C_j and S_j, as used in Part 1, identify with $-C_{j-2}^2$ and $-S_{j-2}^2$ here.)

As the primary is assumed axi-symmetric, we start from the potential function $\mu/r + \sum_\ell U_\ell$, where the individual terms of the disturbing function are given (in the usual notation) by

$$U_\ell = -\frac{\mu}{r} J_\ell (R/r)^\ell P_\ell(\sin\beta) . \tag{4}$$

The value of ℓ in the summation is normally taken to run from 2 upwards, since the cases $\ell = 0$ and $\ell = 1$ are essentially trivial, but the general formulae to be developed cover the case $\ell = 1$ without difficulty; both 'trivial' cases are instructive and are interpreted in section 8, following Ref 4. If the concept of an axi-symmetric primary is generalized to allow for mass outside the orbital region, as well as inside, then (4) can be extended to cover negative ℓ, as in Ref 4; the only change needed in the expression for U_ℓ is that P_ℓ is replaced by $P_{-\ell-1}$. Our overall requirement is to integrate the planetary equations for the general U_ℓ, thereby obtaining the first-order contributions to each $\dot{\zeta}$ and $\delta\zeta$, and then to combine the $\delta\zeta$ (for the six elements) into δr, δb and δw, the corresponding perturbations in the spherical-polar coordinate system attached to the mean orbital plane; the latter is specified by $\bar{I}$ and $\bar{\Omega}$, and the transformation from the (r,b,w)-coordinate system to the usual rectangular equatorial system is described in detail in Part 1.

In section 2 we decompose U_ℓ as

$$U_\ell = \sum_k U_\ell^k , \tag{5}$$

---

* This notation leads to more concise expressions than if the trigonometric argument was $jv + k\omega'$, as was originally planned; the disadvantage is that the Kepler-constant quantities are now C_{-k} and S_{-k}, rather than C_0 and S_0.

where $0 \leq k \leq \ell$ and U_ℓ^k is only non-zero when k has the same parity as ℓ (or as $-\ell - 1$, in the extension to $\ell < 0$). The decomposition arises as β, in U_ℓ, is effectively replaced by i, and this involves the introduction of certain families of inclination functions. The functions $A_\ell^k(i)$ were originally introduced in Ref 4 and are used again, but quantities $A_{\ell k}$, proportional to the $A_\ell^k(i)$ values, are actually more convenient. Related functions, and associated quantities, are also introduced, and recurrence relations are given. These relations (and corresponding relations for the eccentricity functions, referred to in the next paragraph) are required here in the development of the theory, but they are also important as computational aids in the implementation of the theory. The U_ℓ^k can be treated separately in all the analysis up to the derivation of the δr and the δw, but there is a complication in the derivation of δb; this will be handled by the introduction of another index, κ, which is always of the opposite parity to ℓ (and hence k).

Following the elimination of β, we must also eliminate r, using the basic formula of the ellipse

$$\frac{p}{r} = 1 + e \cos v , \tag{6}$$

before the planetary equations can be integrated. This involves families of eccentricity functions, which are introduced in section 3. The functions $B_\ell^j(e)$ were originally introduced in Ref 4, but the quantities $B_{\ell j}$, proportional to them, are actually more useful. Recurrence relations are given, and these are even more important than the relations for the inclination functions. It is implicit in the use of the $B_{\ell j}$ that every U_ℓ^k could be further decomposed, into $\sum_j U_\ell^{kj}$ say, but we prefer not to do this, postponing the introduction of the $B_{\ell j}$ until U_ℓ^k, in each planetary equation, has been eliminated in favour of an explicit expression, thus the notation U_ℓ^{kj} is not needed.

The development of each planetary equation, via first the $A_{\ell k}$ and then the $B_{\ell j}$, is the topic of section 4. As already remarked, we avoid infinite summations by retaining v as an argument of the equation (as opposed to eliminating it in favour of M), and indeed we make it the integration variable by applying the relation

$$\frac{dv}{dt} = n\, q^{-3} (p/r)^2 \tag{7}$$

Each equation now expresses d /dv , rather than $\dot{\zeta}$, as a (finite) sum of terms in v . Each such term is just a multiple of C_j^k or S_j^k , as it turns out, so the integration of the equation is immediate. Terms with $k + j \neq 0$ lead to the $\delta\zeta$ by definition. Terms with $k + j = 0$, on the other hand, are effectively constant over the short term, and contribute directly to $\dot{\bar{\zeta}}$; when $k = 0$ (and so also $j = 0$), the integrated contribution is a secular perturbation, whilst the terms with $k \neq 0$ contribute to the long-period perturbation. The distinction is important for Earth satellites, because the secular variation of ω (due to J_2) must be allowed for in integrating the perturbation, but the subject was dealt with in Part 1 and will be picked up again in Part 3; there will be no further reference in Part 2 to this coupling between J_2 and the other J_ℓ .

Formulae for the $\dot{\bar{\zeta}}_{\ell k}$ ($\dot{\bar{\zeta}}$ due to U_ℓ^k) are collected in section 5, whilst the reduction of the appropriate $\delta\zeta$ to formulae for δr , δb and δw is the subject of the next two (and much longer) sections. As described in previous papers there is an important distinction between the integrations required for the two types of term: for the $\dot{\bar{\zeta}}_{\ell k}$, the process leads to definite integrals (see Parts 1 and 3), necessarily zero if taken over zero time from epoch; in ℓk-components of the $\delta\zeta$, on the other hand, the process leads to epoch-independent indefinite integrals that (apart from the complication of semi-mean elements) satisfy (1). But indefinite integrals contain arbitrary constants, where a 'constant' in the present context is any quantity that is independent of the fast-varying v , i.e. would be a true constant for motion in a fixed ellipse. It is only when these constants are all assigned that (at the first-order level) the mean elements, $\bar{\zeta}$, are fully defined.

It has been noted that enormous advantage accrues from taking $\bar{a}$ to be the exact quantity a' (defined by the energy integral, as explained in Part 1), but there are no immediately compelling reasons for associating particular constants with any of the other five elements. We therefore base our choice on the philosophy of making the expressions for δr , δb and δw as simple as possible. These expressions, which constitute the most important results of the Report, are presented in their general form in section 6; each of the three expressions involves a summation over the index j, with the integration constants for the elements (other than a) not yet taken into account.

In the general formulae just referred to, certain values of j in the summations would involve terms of zero denominator, and it is by the elimination of all these terms that the integration constants (other than for a) are chosen. This is the subject matter for section 7, which completes the entire analysis. An outline of the material in this section is as follows. First, the formula for the mandatory constant in δa (for each U_ℓ^k) is recorded, essentially as a matter of completeness. Second, the constants are derived for δe and δM that validate the omission of the terms with particular j that would otherwise arise in δr. Third, the constants are derived for δi and $\delta\Omega$ that do the same thing for δb. Fourth, special terms (with particular j) in δw, that could not be included in section 6 because they are induced by the constants in δe and δM, are obtained. Finally, the constant in $\delta\omega$ (for each U_ℓ^k) is derived.

Examples of the general formulae of section 6, together with the special terms in δw derived in section 7, are given in section 8: first, for the trivial cases $\ell = 0$ and $\ell = 1$, the interest in which has been remarked; then for $\ell = 2$ and $\ell = 3$, leading (as a useful overall check) to results already known from Part 1; finally, for $\ell = 4$, leading to formulae not hitherto published.

2 FUNCTIONS OF INCLINATION REQUIRED IN EXPANDING THE POTENTIAL

Following Ref 4, we expand $P_\ell(\sin\beta)$, required in (4), via the addition theorem for zonal harmonics (or Legendre polynomials); thus

$$P_\ell(\sin\beta) = \sum_{k=0}^{\ell} u_k \frac{(\ell-k)!}{(\ell+k)!} P_\ell^k(0)\, P_\ell^k(c) \cos ku' \,. \tag{8}$$

Here $u_0 = 1$, $u_k = 2$ if $k > 0$, and the Legendre function P_ℓ^k is defined by

$$P_\ell^k(c) = s^k \frac{d^k P_\ell(c)}{dc^k} \,. \tag{9}$$

The second factor (the k'th derivative) in (9) is a polynomial in c, which (with $k \le \ell$) does not vanish when $c = 1$, its value then being $(\ell + k)!/\{2^k\, k!\, (\ell - k)!\}$. Hence this factor may be normalized*, in a certain useful sense, and we write

$$\frac{d^k P_\ell(c)}{dc^k} = \frac{(\ell + k)!}{2^k\, k!\, (\ell - k)!} A_\ell^k(i) , \tag{10}$$

where $A_\ell^k(i)$ is a pure polynomial in s (= sin i) if k has the same parity as ℓ, but has an additional factor c if k and ℓ are of opposite parity; in each case the constant term in the polynomial is unity, by the normalization. Explicit expressions for the $A_\ell^k(i)$ are given in Table 1, for values of ℓ and k up to 6.

We can now rewrite (8) as

$$P_\ell(\sin \beta) = \sum_{k=0}^{\ell} \alpha_{\ell k}\, s^k A_\ell^k(i) \cos ku' , \tag{11}$$

where the constant, $\alpha_{\ell k}$, is given by

$$\alpha_{\ell k} = u_k\, P_\ell^k(0) / (2^k\, k!) . \tag{12}$$

A different constant, C_ℓ^k, was used in Ref 4, incorporating a factor associated with the eccentricity functions of section 3; it is given by $-2^{-k}\binom{\ell - 1}{k} \alpha_{\ell k}$, where $\binom{m}{k}$ is the usual binomial coefficient, m here (and throughout the paper) being used to denote a general integer, with negative values allowed; when $\ell < 0$, P_ℓ^k in (12) must be replaced by $P_{-\ell-1}^k$, so that $\alpha_{\ell k} = \alpha_{-\ell-1,k}$, but the relation of $\alpha_{\ell k}$ to the C_ℓ^k of Ref 4 is unchanged. It is clear, from the last paragraph, that $P_\ell^k(0)$ (or $P_{-\ell-1}^k(0)$) vanishes when k and ℓ (or $-\ell - 1$) are of opposite parity, and it may be shown that when the parity is the same,

* This 'normalization, which has nothing to do with the standard normalization of the spherical harmonics and their J-coefficients, leads directly to $A_k^k(i) = 1$, one of the pair of starting values for the recurrence relation (21). For some purposes a different normalization is preferable such that $A_\ell^k(i)$ is defined for all $\ell \ge 0$, and $A_0^0(i) = 1$; the family of normalized functions can then be extended in a unified manner when the orbital theory is to cover the tesseral harmonics (Appendi A).

$$P_\ell^k(0) = (-1)^{\frac{1}{2}(\ell-k)} \frac{(\ell + k)!}{2^\ell \{\frac{1}{2}(\ell + k)\}! \{\frac{1}{2}(\ell - k)\}!} . \tag{13}$$

In substituting (11) into (4) it is of great benefit to introduce a new quantity, $A_{\ell k}$, defined by

$$A_{\ell k} = J_\ell (R/p)^\ell \alpha_{\ell k} s^k A_\ell^k(i) , \tag{14}$$

where p ($= a(1 - e^2)$) is the semi-latus rectum (or parameter) of the orbit; the equation applies when $\ell < 0$, so long as the suffix of A is replaced by $-\ell - 1$. The use of $A_{\ell k}$ permits us to write the general term of (5), using also the notation of (3), as

$$U_\ell^k = - \frac{\mu}{p} (p/r)^{\ell+1} A_{\ell k} C_0^k . \tag{15}$$

It will be noted that, whereas $A_\ell^k(i)$ is defined and useful regardless of parity, $A_{\ell k}$ (and hence U_ℓ^k) is only non-zero when k and ℓ are of the same parity. The zeroes come from $\alpha_{\ell k}$, for which the non-zero values, up to $k = \ell = 6$, are given by the like-parity entries of Table 2. (The Table has been extended back to $\ell = -7$, to illustrate the identity of $\alpha_{\ell k}$ with $\alpha_{-\ell-1,k}$ when $\ell < 0$.) However, a use will be found for quantities that behave in the opposite way from $\alpha_{\ell k}$ and $A_{\ell k}$, anticipating which we define (with bold letters to make the distinction)

$$\boldsymbol{\alpha}_{\ell\kappa} = u_\kappa (\ell - \kappa + 1) P_{\ell+1}^\kappa(0) / (2^\kappa \kappa! \ \ell) \tag{16}$$

and

$$\mathbf{A}_{\ell\kappa} = J_\ell (R/p)^\ell \boldsymbol{\alpha}_{\ell\kappa} s^\kappa A_\ell^\kappa(i) , \tag{17}$$

where k has been replaced by κ to signify that we now have quantities that are non-zero only when κ and ℓ are of <u>opposite</u> parity. Half of Table 2 (for all ℓ) is devoted to $\boldsymbol{\alpha}_{\ell\kappa}$, since these quantities can be included with $\alpha_{\ell k}$ on a chequer-board basis.

We will require derivatives of the inclination functions. It is evident from (10) that

$$\frac{d}{di}\{A_\ell^k(i)\} = -\frac{(\ell - k)(\ell + k + 1)}{2(k + 1)} s\, A_\ell^{k+1}(i) ; \tag{18}$$

from this and (14) it follows that the (partial) derivative of $A_{\ell k}$ with respect to i is given by

$$A'_{\ell k} = J_\ell\, (R/p)^\ell\, \alpha_{\ell k}\, s^{k-1} \left\{ kc\, A_\ell^k(i) - \frac{(\ell - k)(\ell + k + 1)}{2(k + 1)} f\, A_\ell^{k+1}(i) \right\} , \tag{19}$$

where $f = s^2$. The quantity in (curly) brackets is the $D_\ell^k(i)$ of Ref 4. We will also require, finally, the particular combinations of $A_{\ell k}$ and $A'_{\ell k}$ denoted by $A^+_{\ell k}$ and $A^-_{\ell k}$, and given by

$$A^\pm_{\ell k} = ks^{-1}\, A_{\ell k} \pm c^{-1}\, A'_{\ell k} ; \tag{20}$$

the s^{-1} and c^{-1} factors do not imply singularities, as they must always cancel via $k\, A_{\ell k}$ and $A'_{\ell k}$ respectively.

The $A_\ell^k(i)$ and $\alpha_{\ell k}$ (and hence the $A_{\ell k}$) may be computed with the aid of recurrence relations. A fixed k was stipulated in Ref 4 for the formula

$$(\ell + k)\, A_\ell^k(i) = (2\ell - 1)\, c\, A_{\ell-1}^k(i) - (\ell - k - 1)\, A_{\ell-2}^k(i) , \tag{21}$$

valid for $\ell \geq k + 2$ with the starting values $A_k^k(i) = 1$ and $A_{k+1}^k(i) = c$; (21) is even valid for $\ell = k + 1$, if an arbitrary (but finite) $A_{k-1}^k(i)$ is assumed. However, it is usually more useful to stipulate a fixed ℓ ; the required formula was given by Merson[11], being

$$A_\ell^k(i) = c\, A_\ell^{k+1}(i) - \frac{(\ell - k - 1)(\ell + k + 2)}{4(k + 1)(k + 2)} f\, A_\ell^{k+2}(i) , \tag{22}$$

valid for $\ell - 2 \geq k \geq 0$ with the starters $A_\ell^\ell(i) = 1$ and $A_\ell^{\ell-1}(i) = c$; (22) is also valid for $k = \ell - 1$, with an arbitrary (finite) $A_\ell^{\ell+1}(i)$. Either of the two preceding 'pure' three-term recurrence relations, (21) or (22), can be used with just one 'mixed' such relation to generate all the relations connecting the $A_\ell^k(i)$; perhaps the simplest mixed relation (with neither ℓ nor k fixed) is

$$(\ell + k)\, c\, A_{\ell}^{k}(i) = (\ell - k)\, A_{\ell-1}^{k}(i) + 2k\, A_{\ell}^{k-1}(i)\ . \tag{23}$$

For the $\alpha_{\ell k}$ we have the relation, for proceeding along a 'fixed diagonal' of Table 2 (with $\ell > 0$ and a constant value of $\ell - k$),

$$\alpha_{\ell k} = \frac{\ell + k - 1}{u_{k-1}\, k}\, \alpha_{\ell-1,k-1}\ , \tag{24}$$

whilst to proceed to a lower diagonal we have

$$\alpha_{\ell k} = -\frac{u_k\,(k + 1)}{\ell - k}\, \alpha_{\ell-1,k+1}\ . \tag{25}$$

These relations suffice to generate all the $\alpha_{\ell k}$ from $\alpha_{0,0} = 1$. Similar relations permit the generation of all the $\alpha_{\ell\kappa}$ from $\alpha_{1,0} = -1$, they can be dispensed with, however, since it follows from (12), (13) and (16) that

$$\alpha_{\ell\kappa} = \frac{\ell - \kappa + 1}{\ell}\, \alpha_{\ell+1,\kappa} = -\frac{\ell + \kappa}{\ell}\, \alpha_{\ell-1,\kappa}\ . \tag{26}$$

Though it is the $A_{\ell k}$ (and $A_{\ell\kappa}$) that we actually require to carry through the paper, recurrence relations are not offered for these. To preserve parity if one suffix is fixed, it would be necessary to use alternate values of the other; there seems little point in doing this, though a valid relation could easily be obtained, for example by applying (21) three times. There are simple relations <u>between</u> the $A_{\ell k}$ and $A_{\ell\kappa}$, however. We will need two of these in section 7.3, namely,

$$\ell\left(\frac{A_{\ell,k+1}}{u_{k+1}} - \frac{A_{\ell,k-1}}{u_{k-1}}\right) = 2kcs^{-1}\,\frac{A_{\ell k}}{u_k} \tag{27}$$

and

$$\ell\left(\frac{A_{\ell,k+1}}{u_{k+1}} + \frac{A_{\ell,k-1}}{u_{k-1}}\right) = -\frac{2A'_{\ell k}}{u_k}\ , \tag{28}$$

Postscript. In regard to equations (27) - (31), it should have been noted that $A^{+}_{\ell k}/u_k$ and $A^{-}_{\ell k}/u_k$ actually reduce to $-\ell c^{-1} A_{\ell,k-1}/u_{k-1}$ and $\ell c^{-1} A_{\ell,k+1}/u_{k+1}$ respectively, results that are implicit in the analysis of section 6.2; the u_k factors could be avoided by allowing negative k and κ (see the footnote of page 40).

these being true for $1 \leq k \leq \ell$ (with ℓ and k of the same parity). For $k = 0$ we only have one relation, given by direct addition of (27) and (28) (and really only one, involving direct subtraction, when $k = \ell$); thus,

$$\ell A_{\ell,1} = -2A'_{\ell,0} . \tag{29}$$

We could use (27) and (28) to obtain expressions for $A^{\pm}_{\ell k}$, defined by (20), but instead derive them directly. On substituting for $A_{\ell k}$ from (14) and for $A'_{\ell k}$ from (19), then in forming $A^{-}_{\ell k}$ we find that the term in $A^{k}_{\ell}(i)$ cancels out and we get (for $0 \leq k \leq \ell$, and ℓ and k of like parity)

$$A^{-}_{\ell k} = J_{\ell}\,(R/p)^{\ell}\,\alpha_{\ell k}\,\frac{(\ell - k)(\ell + k + 1)}{2(k + 1)}\,c^{-1}\,s^{k+1}\,A^{k+1}_{\ell}(i) . \tag{30}$$

For $A^{+}_{\ell k}$, on the other hand, the combination of $A^{k}_{\ell}(i)$ and $A^{k+1}_{\ell}(i)$ is such that (22), with k replaced by $k - 1$, is immediately applicable, leading to (but only for $1 \leq k \leq \ell$ now)

$$A^{+}_{\ell k} = 2k\,J_{\ell}\,(R/p)^{\ell}\,\alpha_{\ell k}\,c^{-1}\,s^{k-1}\,A^{k-1}_{\ell}(i) ; \tag{31}$$

for $k = 0$, this would give a false value of zero, the correct result being the same as is given by (30), since $A^{+}_{\ell,0} = -A^{-}_{\ell,0}$. The formulae (30) and (31) are used in the analysis of δb in section 6.2. (See also the footnote to page 15.)

3 FUNCTIONS OF ECCENTRICITY USED IN THE SUBSEQUENT ANALYSIS

The term U_{ℓ} of the potential, specified by (4), has now been decomposed into the U^{k}_{ℓ} defined by (15), the latitude (β) having been eliminated. The longitude was absent from U_{ℓ} from the beginning, because of axial symmetry, so it remains to eliminate the radius vector (r). This can be done by appeal to (6), but (as noted in section 1) we will in practice postpone the use of (6) until the setting up of each planetary equation, so the present section is preparatory in nature. Further, it is not $(p/r)^{\ell+1}$, in (15), that must be eliminated, but $(p/r)^{\ell-1}$, as a factor $(p/r)^{2}$ is retained to effect the change of integration variable defined by (7).

It is evident from (6) that an expansion of the form

$$(p/r)^{\ell-1} = \sum_{j=0}^{\ell-1} u_j\,B_{\ell j}\,\cos jv \tag{32}$$

is possible, for $\ell \geq 1$, and we regard $B_{\ell j}$ as defined by this expansion; clearly, $B_{\ell j}$ is a polynomial in e. We shall find it useful, and entirely natural, to extend the definition of $B_{\ell j}$ to negative j, by defining $B_{\ell j} = B_{\ell,|j|}$, and to take $B_{\ell j} = 0$ when $|j| \geq \ell$. On this basis, and using the notation of (3), we can replace (32) by

$$(p/r)^{\ell-1} = \sum B_{\ell j} C_j^0 , \tag{33}$$

where the summation effectively runs from $j = -\infty$ to $j = +\infty$, so that there is no need for explicit summation limits.

To make use of some results from Ref 4, we first demonstrate that the $B_{\ell j}$ are directly related to the Hansen X functions of classical celestial mechanics[12], such that

$$B_{\ell j} = q^{2\ell-1} X_0^{-\ell-1,j} . \tag{34}$$

Hansen's functions (of eccentricity) are defined (uniquely) by the existence of the expansion, for all integral ℓ and j, regardless of sign,

$$(r/a)^{\ell} \exp(\imath jv) = \sum_m X_m^{\ell j} \exp(\imath mM) , \tag{35}$$

where $\imath^2 = -1$ and the summation runs from $-\infty$ to $+\infty$. Only when $m = 0$, which is the case with which we are concerned, is $X_m^{\ell j}$ a simple (finitely expressible in elementary functions) function of e (and it is precisely because of this that we change the variable from t to v in the planetary equations, thus avoiding infinite expansions in M).

To demonstrate (34), we first replace the index ℓ by $-(\ell + 1)$ in (35), and then integrate over a revolution of M. We get (from the real part of the result)

$$\int_0^{2\pi} (p/r)^{\ell+1} \cos jv \, dM = 2\pi \, q^{2(\ell+1)} X_0^{-\ell-1,j} . \tag{36}$$

But from (33),

$$\int_0^{2\pi} (p/r)^{\ell+1} \cos jv \, dM = \int_0^{2\pi} \sum_m B_{\ell m} (p/r)^2 \cos mv \cos jv \, dM . \qquad (37)$$

We apply (7) to change the integration variable to v on the right-hand side of (37); only if $m = \pm j$ do we retain a non-zero term, and in fact

$$\int_0^{2\pi} (p/r)^{\ell+1} \cos jv \, dM = 2\pi q^3 B_{\ell j} . \qquad (38)$$

Then (34) is immediate from (36) and (38).

Some comments related to the notation are worth making before we proceed further. In principle we are reserving the suffix k for the A functions and j for the B functions, but in section 5, where only the value $-k$ arises for j, we will naturally encounter $B_{\ell k}$. We would also rather naturally change the notation from j to k in (35) if we were following the traditional path[7] in which the integration variable is M and the expansion of (15) is by (35) directly. After replacement of ℓ by $-(\ell + 1)$, the Hansen function would then appear as $X_m^{-\ell-1,k}$, which is nowadays (following Kaula[7]) usually expressed (when $\ell \geq 0$) as $G_{\ell pq}(e)$; here $p = \frac{1}{2}(\ell - k)$, which must be integral (assuming ℓ and k to be of the same parity), and $q = m - k$. Introducing also the notation $G_{\ell q}^{k}$, which the present author[9] has recommended as preferable to $G_{\ell pq}$, we may extend (34) by writing

$$q^{-(2\ell-1)} B_{\ell j} = X_0^{-\ell-1,j} = G_{\ell,\frac{1}{2}(\ell-j),-j}(e) = G_{\ell,-j}^{j}(e) . \qquad (39)$$

Gooding and King-Hele[13] have recently reported on the G functions that are relevant to resonant satellite orbits. Ref 13 includes the listing of a Fortran program (by Alfred Odell) that computes the functions for arbitrary values of ℓ, k and Kaula's q, by quadrature.

We can now use the identity (34) to tie into the analysis of Ref 4. Thus, we may express the e-polynomial $B_{\ell j}$, when $\ell \geq 1$ and $0 \leq j < \ell$, in terms of a normalized such polynomial, the connecting relation being

$$B_{\ell j} = \binom{\ell - 1}{j} (e/2)^j B_\ell^j(e) , \qquad (40)$$

$B_\ell^j(e)$ in (40) is a polynomial in e^2, with constant term unity by the normalization. Explicit expressions for the $B_\ell^j(e)$ are given in Table 3, for values of ℓ and j up to 7 and 6 respectively. There is an evident resemblance between the $B_\ell^j(e)$ and the $A_\ell^k(i)$, a significant difference being that the new functions run from $\ell = 1$ and not $\ell = 0$; the resemblance is not fortuitous, since it can be shown that[4]

$$B_\ell^j(e) = \frac{j!\,(\ell-1-j)!}{(\ell-1+j)!}(ie/2)^{-j}\,q^{\ell-1}\,P_{\ell-1}^j(q^{-1}) , \tag{41}$$

from which it follows that

$$B_\ell^j(e) = q^{\ell-j-1}\,A_{\ell-1}^j(\tan^{-1} ie) . \tag{42}$$

In contradistinction with the $A_{\ell k}$, however, it is usually much better to work with the $B_{\ell j}$ directly (in recurrence relations, for example), rather than through $B_\ell^j(e)$ and (40). One reason for this is that only alternate values of the $A_{\ell k}$ are non-zero, whereas (for $|j| < \ell$ and $e \neq 0$) all the $B_{\ell j}$ are non-zero. Further, no difficulty arises with the $B_{\ell j}$ when $j < 0$ (as already noted, and see also Appendix B), whereas $B_\ell^j(e)$ would then be infinite (if $|j| < \ell$). We can even allow ℓ to be negative (or zero) as well as j. The validity of this follows from the universality of (34) - the universality is brought out by Table 4, which lists $B_{\ell j}$ for ℓ running from -3 to +4 and j from -1 to +3.

The entries in Table 4 form triangular blocks of four types. First, for $\ell > 0$ and $|j| < \ell$, we have the quantities that can be expressed by (40) when $j \geq 0$. Secondly, for $\ell > 0$ and $|j| \geq \ell$, we have (two blocks of) zeroes. Thirdly, for $\ell \leq 0$ and $|j| \leq -\ell$, we have quantities that, when $j \geq 0$, can be expressed by a formula complementary to (40), viz

$$B_{\ell j} = \binom{\ell}{j}\,(e/2)^j\,q^{2\ell-1}\,B_{-\ell+1}^j(e) ; \tag{43}$$

a formula equivalent to this was given in Ref 4, the application being (as noted in section 1) to secular and long-period perturbations associated with exterior (rather than interior) mass. Finally, for $\ell \leq 0$ and $|j| > \ell$, we have (two blocks of) quantities that are most conveniently expressed in terms of β (not now denoting latitude, as previously) and q, rather than e and q, where

$$\beta = \frac{e}{1+q} , \tag{44}$$

no attention was paid to these quantities (or their equivalents) in Ref 4, there being no application for them, but the formula for $B_{0,1}$ is derived here, for completeness, in Appendix B. (Other entries in this last pair of blocks are then derivable from recurrence relations.) Before leaving Table 4, we note that the formula for the X or G function corresponding to $B_{\ell j}$ is immediate from the Table, in view of (34); thus it is only necessary to apply the factor $q^{1-2\ell}$, which introduces a negative power of q when there is not one already present and cancels it out when there is.

In regard to derivatives of the eccentricity functions, it can be shown (by working from (41)), and easily verified from Table 3 that, for $1 \le j < \ell$,

$$\frac{d}{de}\{B_\ell^j(e)\} = 2je^{-1}\{B_{\ell-1}^{j-1}(e) - B_\ell^j(e)\} . \tag{45}$$

The universal formula for the derivative of $B_{\ell j}$ is

$$B'_{\ell j} = (\ell - 1)\, B_{\ell-1,j-1} - j\, e^{-1}\, B_{\ell j} . \tag{46}$$

For $1 \le j < \ell$, this follows from (45); for general entries in Table 3, it can be verified with the aid of q' and β', which may be expressed as $-e/q$ and β/eq respectively. However, because we only introduce the $B_{\ell j}$ after each planetary equation has been set up, we effectively only use (46) in expressing the rates of change of the mean elements. Since this involves

$$\frac{\partial}{\partial e}(q\, A_{\ell k}\, B_{\ell k}) = q^{-1}\ {}_{\ell k}\{q^2\, B'_{\ell k} + (2\ell - 1)\, e\, B_{\ell k}\} , \tag{47}$$

we define

$$E_{\ell k} = q^2\, B'_{\ell k} + (2\ell - 1)\, e\, B_{\ell k} ; \tag{48}$$

then (46), re-expressed via the recurrence relation (56), leads to

$$E_{\ell k} = e^{-1}\,(\ell e^2 - k)\, B_{\ell k} + (\ell - k)\, B_{\ell,k-1} . \tag{49}$$

(By symmetry, there is a parallel expression for $E_{\ell k}$ that involves $B_{\ell k}$ and $B_{\ell,k+1}$.) Table 5 gives explicit expressions for the $E_{\ell k}$, with ℓ running from -3 to +4 as in Table 4; only the entries in which k has the same parity as ℓ (or $-\ell - 1$ if $\ell < 0$) are useful in practice, and entries for $k \geq \ell$ (or $-\ell + 1$ if $\ell \leq 0$) are omitted entirely (for $k \geq \ell > 0$ they would all be zero). The $E_{\ell k}$ are related to the $E_\ell^k(e)$ of Ref 4 by

$$E_{\ell k} = e^{-1} (e/2)^k \binom{\ell - 1}{k} E_\ell^k(e) , \tag{50}$$

when $\ell > 0$; for $\ell \leq 0$, the extra factor $q^{2\ell+1}$ is required (cf (43), where the additional factor, in relation to (40), is $q^{2\ell-1}$).

The $B_{\ell j}$ may be computed from recurrence relations for the $B_\ell^j(e)$, which will now be given, but in developing the theory it is more useful to have such relations for the $B_{\ell j}$ themselves, so these will also be given. For fixed j (≥ 0), the recurrence formula (from Ref 4) is

$$(\ell + j - 1) B_\ell^j(e) = (2\ell - 3) B_{\ell-1}^j - (\ell - j - 2) q^2 B_{\ell-2}^j(e) , \tag{51}$$

valid for $\ell \geq j + 3$ with the starters $B_{j+1}^j(e)$ and $B_{j+2}^j(e)$ both unity. For fixed ℓ (≥ 3), on the other hand, the formula is[11]

$$B_\ell^j(e) = B_\ell^{j+1}(e) + \frac{(\ell - j - 2)(\ell + j + 1)}{4(j + 1)(j + 2)} e^2 B_\ell^{j+2}(e) , \tag{52}$$

valid for $\ell - 3 \geq j \geq 0$ with the starters $B_\ell^{\ell-1}(e)$ and $B_\ell^{\ell-2}(e)$ again both unity. The resemblance of (51) and (52) to (21) and (22) respectively follows from the remark leading up to (42).

The recurrence relations for the $B_{\ell j}$, that correspond to (51) and (52), respectively, and are valid for all ℓ and j, are (when symmetrically expressed)

$$\ell(\ell - 1) q^2 B_{\ell-1,j} - \ell(2\ell - 1) B_{\ell j} + (\ell^2 - j^2) B_{\ell+1,j} = 0 \tag{53}$$

and

$$(j - \ell) e B_{\ell,j-1} + 2j B_{\ell j} + (j + \ell) e B_{\ell,j+1} = 0 ; \tag{54}$$

the mandatory symmetry in (54), to satisfy the unchanging value of $B_{\ell j}$ under the operation $j \to -j$, is evident.

As remarked for the inclination functions, either of the 'pure' relations, (53) and (54), can be used with a 'mixed' relation* to generate all the recurrence relations connecting the $B_{\ell j}$. Here the mixed relations are, in particular, those that connect three out of four of the $B_{\ell j}$ lying 'around a square' of index duplets; if the square consists of the duplets (ℓ, j), $(\ell + 1, j)$, $(\ell, j + 1)$ and $(\ell + 1, j + 1)$, then the four mixed relations connecting them (all of which we shall require in the sequel) are

$$\ell B_{\ell j} - (\ell - j) B_{\ell+1,j} + \ell e B_{\ell,j+1} = 0 , \qquad (55)$$

$$\ell q^2 B_{\ell j} - (\ell - j) B_{\ell+1,j} + (\ell + j + 1) e B_{\ell+1,j+1} = 0 , \qquad (56)$$

$$\ell e B_{\ell j} + \ell B_{\ell,j+1} - (\ell + j + 1) B_{\ell+1,j+1} = 0 \qquad (57)$$

and

$$(\ell - j) e B_{\ell+1,j} + \ell q^2 B_{\ell,j+1} - (\ell + j + 1) B_{\ell+1,j+1} = 0 . \qquad (58)$$

If we re-order the terms in the last two relations and replace j by $j - 1$, we get relations which are symmetric pairings of (55) and (56), viz

$$\ell B_{\ell j} - (\ell + j) B_{\ell+1,j} + \ell e B_{\ell,j-1} = 0 \qquad (59)$$

and

$$\ell q^2 B_{\ell j} - (\ell + j) B_{\ell+1,j} + (\ell - j + 1) e B_{\ell+1,j-1} = 0 . \qquad (60)$$

Of this set of relations, (55) and (59) can be obtained at once from (54) and the relation equivalent to one given (for the Hansen functions) by Zafiropoulos[8], viz

$$\ell e (B_{\ell,j-1} - B_{\ell,j+1}) = 2j B_{\ell+1,j} ; \qquad (61)$$

this is of a different 'shape' from our triangles-around-the-square relations, but is perhaps the simplest recurrence relation of all.

* Note added in proof: Ref 19 indicates that, for inclination functions, pure relations are computationally preferable to mixed relations (see also Ref 20).

4 RATES OF CHANGE OF OSCULATING ELEMENTS

In this section we use Lagrange's planetary equations to develop the rate of change of each of the orbital elements (a, e, i, Ω, ω and M) due to U_ℓ^k , the term of the disturbing function specified by (15). Each rate of change is to be with respect to v , rather than t , expressed as a finite trigonometric series (assuming $\ell > 0$, as we now always do, except in section 8.1), with v as the variable. The v-independent terms of each $d\zeta/dv$ are then isolated; they effectively contribute to the time rate of change, $\dot{\bar{\zeta}}$, of the mean element, $\bar{\zeta}$, expressions for the $\dot{\bar{\zeta}}$ being held over to section 5. The remaining terms of $d\zeta/dv$ can at once be integrated to provide contributions to the short-period perturbation, $\delta\zeta$. The result of the integration is, in fact, so 'immediate' (apart from the question of the integration 'constants' already referred to in section 1) that we will not bother to write down formal expressions for the five $\delta\zeta$ other than δa ; this is to emphasize the fact that it is the combinations of the $\delta\zeta$ into δr, δb and δw that are of interest (being the topic of section 6), not the $\delta\zeta$ themselves.

The perturbation δa is a special case because it can be obtained without integration. As in Part 1, however, we also derive da/dv from the appropriate planetary equation, as a prototype for the derivation of the other $d\zeta/dv$. By bringing in the quasi-elements, ψ and ρ , it is possible to develop each equation in terms of the partial derivative of U_ℓ^k with respect to a single quantity.

4.1 Semi-major axis

As in Part 1, there is an absolute constant of the motion, which we denote by a' , such that

$$a = a'(1 + 2aU/\mu) ; \tag{62}$$

this is an exact relationship for any time-independent disturbing function, U , and in particular for the axi-symmetric U_ℓ^k . It follows that there is no long-term variation in a , to whatever order of magnitude the perturbation analysis is conducted. Further, the short-period perturbation, δa , is given exactly, on substituting for U_ℓ^k from (15), thus

$$\delta a = -2a'q^{-2} A_{\ell k} (p/r)^{\ell+1} C_0^k . \tag{63}$$

This does not mean that an exact perturbation can be written down for semi-major axis, however, as the right-hand side of (63) is expressed in terms of osculating elements; as soon as mean elements are introduced, the result is no more than a first-order perturbation expression, as with any other ζ .

To present δa in the form appropriate for use in section 6.1, we combine C_0^k with one of the factors p/r . Thus,

$$\delta a = -aq^{-2} A_{\ell k} (p/r)^{\ell} (eC_{-1}^k + 2C_0^k + eC_1^k) . \tag{64}$$

We retain another p/r factor explicitly, and expand the remaining $(p/r)^{\ell-1}$ by (33). By this means the term $2C_0$, for example, in (64) is effectively transformed, for each j , into $C_j + C_{-j}$. But each pair of terms (such as this) for positive j , in the infinite summation of (33), is matched by the same pair (in reverse order) for negative j , so we can express the result of the expansion as

$$\delta a = -aq^{-2} A_{\ell k} (p/r) \sum B_{\ell j} (eC_{j-1} + 2C_j + eC_{j+1}) . \tag{65}$$

We now develop an expression for da/dv ab initio, using the general procedure that involves the planetary equation for $\dot{a}$. This equation is

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial U}{\partial M} , \tag{66}$$

and on substituting for U_ℓ^k we get

$$\frac{da}{dt} = -2naq^{-2} A_{\ell k} \frac{\partial}{\partial M}\{(p/r)^{\ell+1} C_0\} . \tag{67}$$

The M-differentiation is immediate, since p/r is given by (6) and $\partial v/\partial M$ is $q^{-3}(p/r)^2$ (cf (7)). We transform from dt to dv (again using (7)), and all this leads to

$$\frac{da}{dv} = aq^{-2} A_{\ell k} (p/r)^{\ell} \{(k - \ell - 1) e S_{-1} + 2k S_0 + (k + \ell + 1) e S_1\} . \tag{68}$$

Finally, we split $(p/r)^{\ell}$ into $(p/r)^{\ell-1}$ and p/r ; then, applying (33) and taking advantage of the matching of terms for positive and negative j, we get

$$\frac{da}{dv} = \tfrac{1}{4}aq^{-2} A_{\ell k} \sum B_{\ell j} \{(k - \ell - 1) e^2 S_{j-2} + 2(2k - \ell - 1) e S_{j-1} + 2k(2 + e^2) S_j + 2(2k + \ell + 1) e S_{j+1} + (k + \ell + 1) e^2 S_{j+2}\} . \tag{69}$$

This may be regarded as a prototype for all the $d\zeta/dv$; in addition, (69) is used in the derivation of de/dv in section 4.2. The equivalence of this result with the v-derivative of δa obtained by the special procedure may be verified, most easily by the v-differentiation of (64).

In dealing with the subsequent ζ, we will be isolating the component of $d\zeta/dv$ that leads to secular and long-period perturbations. We know that for da/dv this component must be zero, both from the special procedure and from the form of (66) (since a term of U that is free of short-period variation must tautologously have zero M-derivative), but it is instructive (as part of the prototype for the other ζ) to obtain this result from (69). The terms independent of v in the overall j-sum are the terms in S_{-k} ($= S^k_{-k}$) . Since $B_{\ell,-j} = B_{\ell j}$, the combination of all such terms involves the factor

$$(k - \ell - 1) e^2 B_{\ell,k-2} + 2(2k - \ell - 1) e B_{\ell,k-1} + 2k(2 + e^2) B_{\ell k} + 2(2k + \ell + 1) e B_{\ell,k+1} + (k + \ell + 1) e^2 B_{\ell,k+2},$$

and it follows from three applications of (54) that this is zero.

4.2 Eccentricity

We develope the perturbation in e by first obtaining the perturbation in p, since

$$\dot{p} = q^2 \dot{a} - 2ae \dot{e} \tag{70}$$

and the planetary equation for p is just

$$\frac{dp}{dt} = \frac{2q}{na} \frac{\partial U}{\partial \omega} . \tag{71}$$

On substituting for U_ℓ^k we get

$$\dot{p} = -2naq^{-1} A_{\ell k} (p/r)^{\ell+1} \frac{\partial}{\partial\omega}(C_0) . \tag{72}$$

Transformation of the integration variable to v, by (7), yields

$$\frac{dp}{dv} = 2kp\, A_{\ell k} (p/r)^{\ell-1} S_0 ; \tag{73}$$

hence we get, from (33) and the usual argument concerning positive and negative j,

$$\frac{dp}{dv} = 2kp\, A_{\ell k} \sum B_{\ell j} S_j . \tag{74}$$

We get the long-term variation by setting $j = -k$; thus

$$\dot{\bar{p}} = 2knp\, A_{\ell k} B_{\ell k} S_{-k} . \tag{75}$$

The expression for $\dot{\bar{e}}$ is now immediate from (70), since $\dot{\bar{a}} = 0$, but it is not given here, as the complete list of the $\dot{\bar{\zeta}}$ is given in section 5.

For the v-derivative of the short-period perturbation, δe, we have all the terms with $j + k \neq 0$ in the expression given by the combination of (69) and (73), according to (70). This combination leads to

$$\begin{aligned}\frac{de}{dv} = \tfrac{1}{4} A_{\ell k} \sum B_{\ell j} \{ &(k - \ell - 1)e\, S_{j-2} + 2(2k - \ell - 1)S_{j-1} \\ &+ 6ke\, S_j + 2(2k + \ell + 1)S_{j+1} + (k + \ell + 1)e\, S_{j+2} \} .\end{aligned} \tag{76}$$

As already indicated, we will not write down the expression for δe, involving C_{j-2} etc, given immediately by integration of (76); immediate, that is, apart from the 'constant term', in C_{-k}, that we are not yet in a position to assign. This term effectively replaces the infinite term (in C_{-k}) that would arise if we had not removed the S_{-k} term from (76) in advance.

4.3 Inclination

We can develop the perturbation in i from the perturbations in p and pc^2, since

$$d(pc^2)/dt = c^2\dot{p} - 2pcs\,\dot{i} \tag{77}$$

and the planetary equation for pc^2 is just

$$\frac{d(pc^2)}{dt} = \frac{2qc}{na}\frac{\partial U}{\partial \Omega}. \tag{78}$$

But U_ℓ^k is independent of longitude, and hence of Ω, so pc^2 is an invariant. Using (77), therefore, we have $\dagger$ at once from (75), whilst δi will be based on the expression for di/dv derived from (74), viz

$$\frac{di}{dv} = kcs^{-1} A_{\ell k} \sum B_{\ell j} S_j . \tag{79}$$

(Since $A_{\ell k}$ contains s^k as a factor, there will never be a non-zero multiple of an uncancelled s^{-1}.)

4.4 Right ascension of the node

The perturbation in Ω comes from the planetary equation

$$\frac{d\Omega}{dt} = \frac{1}{na^2qs}\frac{\partial U}{\partial i}. \tag{80}$$

On substituting for U_ℓ^k we get

$$\dot{\Omega} = -nq^{-3} s^{-1} (p/r)^{\ell+1} A'_{\ell k} C_0 . \tag{81}$$

We now apply (7) and (33) as usual, getting an isolated contribution to $\dagger\Omega$, together with the expression for $d\Omega/dv$, viz

$$\frac{d\Omega}{dv} = -s^{-1} A'_{\ell k} \sum B_{\ell j} C_j . \tag{82}$$

4.5 Argument of perigee

Introduction of ψ gives us a one-term planetary equation, since

$$\frac{d\psi}{dt} = \frac{q}{na^2e}\frac{\partial U}{\partial e}. \tag{83}$$

The e-derivative is much the most complicated of the partial derivatives of U_ℓ^k, since e is an argument of each of the four factors on the right-hand side of (15). Thus we get

$$\dot{\psi} = -ne^{-1}q\frac{\partial}{\partial e}\left\{A_{\ell k}\, q^{-2}\,(p/r)^{\ell+1}\, C_0\right\}. \tag{84}$$

But

$$\frac{\partial}{\partial e}(q^{-2}A_{\ell k}) = 2(\ell+1)\, eq^{-4}A_{\ell k} \tag{85}$$

and, using the expressions for $\partial r/\partial e$ and $\partial v/\partial e$ (equations (41) and (42) of Part 1),

$$\begin{aligned}\frac{\partial}{\partial e}\left\{(p/r)^{\ell+1}C_0\right\} = q^{-2}(p/r)^{\ell+1}\{&(\ell+1)(\cos v - e - e\sin^2 v)\, C_0\\ &- k\sin v\,(2+e\cos v)\, S_0\},\end{aligned} \tag{86}$$

so that (84) reduces to

$$\begin{aligned}\dot{\psi} = ne^{-1}q^{-3}A_{\ell k}\,(p/r)^{\ell+1}\{&k\sin v\,(2+e\cos v)\, S_0 -\\ &(\ell+1)\cos v\,(1+e\cos v)\, C_0\}.\end{aligned} \tag{87}$$

We make the standard expansions of the trigonometric products in (87), and then apply (7) and (33) as usual. This leads to

$$\begin{aligned}\frac{d\psi}{dv} = -\tfrac{1}{4}e^{-1}A_{\ell k}\sum B_{\ell j}\{&(\ell+1-k)e\, C_{j-2} + 2(\ell+1-2k)C_{j-1}\\ &+ 2(\ell+1)e\, C_j + 2(\ell+1+2k)C_{j+1} + (\ell+1+k)e\, C_{j+2}\}.\end{aligned} \tag{88}$$

To get $\overset{+}{\psi}$, we pick out the coefficient of C_{-k}. Thus

$$\dot{\psi} = -\tfrac{1}{4} n e^{-1} A_{\ell k} \{(\ell + 1 - k)e\, B_{\ell,k-2} + 2(\ell + 1 - 2k)B_{\ell,k-1} + 2(\ell + 1)e\, B_{\ell k} + 2(\ell + 1 + 2k)B_{\ell,k+1} + (\ell + 1 + k)e\, B_{\ell,k+2}\} C_{-k} . \tag{89}$$

But this can be simplified by three applications of (54), which lead to

$$\overset{+}{\psi} = -n e^{-2} A_{\ell k} \{(\ell e^2 - k)B_{\ell k} + (\ell - k)e\, B_{\ell,k-1}\} C_{-k} . \tag{90}$$

We can now introduce the quantity $E_{\ell k}$, to get a concise expression, since by (49) we have

$$\overset{+}{\psi} = -n e^{-1} A_{\ell k} E_{\ell k} C_{-k} . \tag{91}$$

To get $\dot{\varpi}$ and the appropriate terms of $d\omega/dv$, we combine (91) and (the residual terms of) (88) with $\overset{+}{\Omega}$ and (82), respectively, using

$$\dot{\omega} = \dot{\psi} - c\dot{\Omega} . \tag{92}$$

4.6 Mean anomaly

We start by studying ρ, since our final one-term planetary equation is

$$\frac{d\rho}{dt} = -\frac{2}{na}\frac{\partial U}{\partial a} . \tag{93}$$

Remembering that $A_{\ell k}$, in U_ℓ^k, is itself a function of semi-major axis, we obtain

$$\dot{\rho} = -2(\ell + 1)nq^{-2} (p/r)^{\ell+1} A_{\ell k} C_0 \tag{94}$$

and hence

$$\frac{d\rho}{dv} = -2(\ell + 1)q\, A_{\ell k} \sum B_{\ell j} C_j . \tag{95}$$

In particular,

$$\overset{+}{\rho} = -2(\ell + 1)nq\, A_{\ell k} B_{\ell k} C_{-k} . \tag{96}$$

and from (91) we therefore also have

$$\dot{\sigma} = -nq A_{\ell k} \{2(\ell + 1)B_{\ell k} - e^{-1} E_{\ell k}\} C_{-k} , \tag{97}$$

since

$$\dot{\sigma} = \dot{\rho} - q\dot{\psi} . \tag{98}$$

From (88), similarly, the v-derivative of $\delta\sigma$ is given by the v-dependent terms of

$$\begin{aligned} \frac{d\sigma}{dv} = \tfrac{1}{4} e^{-1} q A_{\ell k} \sum B_{\ell j} \{&(\ell + 1 - k)e C_{j-2} + 2(\ell + 1 - 2k)C_{j-1} \\ &- 6(\ell + 1)e C_j + 2(\ell + 1 + 2k)C_{j+1} + (\ell + 1 + k)e C_{j+2}\} . \end{aligned} \tag{99}$$

But

$$\dot{M} = \dot{\sigma} + \dot{\int} , \tag{100}$$

where (with τ standing for time)

$$\int = \int_0^t n \, d\tau \tag{101}$$

and (assuming only $U_{\ell k}$ to be operating)

$$n - n' = 3nq^{-2} A_{\ell k} (p/r)^{\ell+1} C_0 \tag{102}$$

by (63) and Kepler's third law.

From (101) and (102) it follows that

$$\frac{d\int}{dv} = \frac{n'}{\dot{v}} + 3q A_{\ell k} \sum B_{\ell j} C_j , \tag{103}$$

by the usual procedure. We may then write

$$\dot{\int} = n' + 3nq A_{\ell k} B_{\ell k} C_{-k} , \tag{104}$$

from which $\dot{M}$ is available on combining with (97). Finally, on combining the residual terms (those with $k + j = 0$) of (103) with (99), we find that the v-derivative of δM is given by the v-dependent terms of

$$\frac{dM}{dv} = \frac{1}{4} e^{-1} q\, A_{\ell k} \sum B_{\ell j} \{(\ell + 1 - k)e\, C_{j-2} + 2(\ell + 1 - 2k)C_{j-1} - 6(\ell - 1)e\, C_j + 2(\ell + 1 + 2k)C_{j+1} + (\ell + 1 + k)e\, C_{j+2}\} . \quad (105)$$

To conclude, we note that a very much simpler result than (105) is available for the non-singular δL . Thus from (88) and (105) we get

$$\frac{dL}{dv} = -(2\ell - 1)q\, A_{\ell k} \sum B_{\ell j}\, C_j . \quad (106)$$

5 SECULAR AND LONG-PERIOD ELEMENT RATES

In this section we collect the expressions for the rates of change of the mean elements, i.e. the $\dot{\zeta}$ associated with U_ℓ^k . As we have seen in section 4, this simply amounts to listing the components of the $d\zeta/dv$ that are multiples of either S_{-k}^k or C_{-k}^k . When $k = 0$, the rate of change is secular; for $k > 0$, it is long-period. We shall not be concerned with the build-up of actual perturbations from the $\dot{\zeta}$, since this is fully dealt with in Parts 1 and 3; suffice it to say that there is no difficulty in the secular perturbations, but that (even in a first-order analysis) difficulties arise with the long-period perturbations, in particular due to the singularities associated with zero e and zero s .

Another point must be mentioned before we list the $\dot{\zeta}$. As the expressions arise from terms in $d\zeta/dv$, but were treated (in section 4) as if from terms in $d\zeta/dM$, each $\dot{\zeta}$ produces a short-period component of the perturbation in ζ ; i.e. a contribution to $\delta\zeta$ is induced. These contributions may be amalgamated into components of δr , δb and δw , as done for J_3 in Part 1 (section 7). The issue relates to the definition of semi-mean elements (section 3.2 ibid), which is outside the scope of Part 2; it should be clear, from equations (120) - (122) in section 6, however, that no difficulty arises in the amalgamating.

In the list of the $\dot{\zeta}$ that follows, we note that the maximum value of k is $\ell - 2$, since $B_{\ell\ell} = E_{\ell\ell} = 0$. We attach an explicit subscript (ℓk) to each $\dot{\zeta}$; then our first result is

$$\dot{a}_{\ell k} = 0 . \tag{107}$$

For e, it follows from (70) and (75) that

$$\dot{e}_{\ell k} = - kne^{-1}q^2 A_{\ell k} B_{\ell k} S^k_{-k} . \tag{108}$$

There will always be a positive power of e to cancel the factor e^{-1}, it will be noted, coming from $k B_{\ell k}$.

For i, similarly, it follows from (75) and (77) that

$$\dot{i}_{\ell k} = kncs^{-1} A_{\ell k} B_{\ell k} S^k_{-k} . \tag{109}$$

Here there will always be a positive power of s, coming from $k A_{\ell k}$, to cancel the factor s^{-1}.

For Ω, our analysis of (81) gives

$$s \dot{\Omega}_{\ell k} = - n A'_{\ell k} B_{\ell k} C^k_{-k} . \tag{110}$$

The formula is expressed in this way, with a factor s on the left-hand side, to avoid the possibility of an uncancellable s^{-1} on the right-hand side. For long-period perturbations, there is a singularity difficulty here, which can be dealt with as indicated in Part 1 (section 3.5). For secular perturbations (and here is our first non-zero $\dot{\zeta}$ when $k = 0$) there is no problem, since in the expression for $A'_{\ell k}$, given by (19), $A^k_\ell(i)$ appears with the multiplying factor k, and $A^{k+1}_\ell(i)$ with the factor f.

For ω, we use the final result, (91), for $\dot{\psi}$. Then from (92) and (110) it follows that

$$es \dot{\varpi}_{\ell k} = n(ec A'_{\ell k} B_{\ell k} - s A_{\ell k} E_{\ell k}) C^k_{-k} . \tag{111}$$

Again the formula is expressed like this to make the right-hand side non-singular; and again (because $E_{\ell k}$ contains a factor e when $k = 0$, as seen from Table 5) there is no difficulty with secular perturbations.

For M, we combine the results for $\dot{\gamma}$ and $\dot{J} - n'$, given by (97) and (104) respectively; thus

$$e \dot{M}_{\ell k} = nq A_{\ell k} \{E_{\ell k} - (2\ell - 1)e B_{\ell k}\} C^{k}_{-k} . \quad (112)$$

From (48), this may also be written as

$$e \dot{M}_{\ell k} = nq^3 A_{\ell k} B'_{\ell k} C^{k}_{-k} . \quad (113)$$

From the definition of L, we may also combine (112) with (91); this gives the non-singular result

$$\dot{L}_{\ell k} = -(2\ell - 1) nq A_{\ell k} B_{\ell k} C^{k}_{-k} . \quad (114)$$

As usual, a factor e can be cancelled from both sides of (113) when $k = 0$. However, there is a simpler way of dealing with secular perturbations in M, as indicated in Part 1; Ref 3 was largely devoted to this topic, and the rest of this section conforms with the account therein.

The basic idea is that we represent the secular perturbations in mean anomaly by modifying the value of the mean mean motion. In view of (113), in fact, we write

$$\bar{n} = n'(1 + e^{-1}q^3 \sum A_{\ell,0} B'_{\ell,0}) , \quad (115)$$

where the summation is now on ℓ, and we have set C^0_0 to unity. This agrees with equation (11) of Ref 3, since $A_{\ell,0}$ here may be identified with $-J_\ell C_\ell (R/p)^\ell A_\ell(i)$ from that paper.

The logic for using a' as mean semi-major axis ($\bar{a}$) is, as we have seen, compelling, so if $\bar{n} \neq n'$, we do not retain Kepler's third law in its simplest form. This is of no consequence, however, and we simply write, from (115),

$$\bar{n}^2 \bar{a}^3 = \mu(1 + 2e^{-1}q^3 \sum A_{\ell,0} B'_{\ell,0}) . \quad (116)$$

Ref 3 gives, as equation (15), an explicit version of (116) with (even) values of ℓ up to 8; the two versions can easily be verified as equivalent if we note that Q_ℓ (in Ref 3) is just a normalized form of $B'_{\ell,0}$ such that

$$B'_{\ell,0} = \tfrac{1}{2}(\ell - 1)(\ell - 2)eQ_\ell \ . \tag{117}$$

The following recurrence relation was given in Ref 3:

$$(\ell - 1)Q_\ell = (2\ell - 5)Q_{\ell-1} - (\ell - 4)\, q^2\, Q_{\ell-2} \ ; \tag{118}$$

this is valid for $\ell \geq 4$, with $Q_2 = 0$ and $Q_3 = 1$. The relation may be obtained from (117) and (46), together with

$$\begin{aligned}(\ell - 1)(\ell - 3)\, B_{\ell-1,1} = {} & (\ell - 2)(2\ell - 5)\, B_{\ell-2,1} \\ & - (\ell - 2)(\ell - 3)\, q^2\, B_{\ell-3,1}\end{aligned} \tag{119}$$

which derives from (53) on replacing (ℓ, j) by $(\ell - 2, 1)$.

It is very convenient that $Q_2 = 0$. It means that for first-order analysis associated with J_2 (the dominant harmonic of the geopotential) π is the same as n'.

6 PERTURBATIONS (SHORT-PERIOD) IN COORDINATES - GENERAL CASE

In this section we develop general expressions for the δr, δb and δw that can be derived from the first-order $\delta\zeta$ via the formulae (taken from section 3.3 of Part 1)

$$\delta r = (r/a)\, \delta a - (a \cos v)\, \delta e + (aeq^{-1} \sin v)\, \delta M \ , \tag{120}$$

$$\delta b = (\cos u')\, \delta i + (s \sin u')\, \delta\Omega \tag{121}$$

and

$$\delta w = \delta\psi + \{q^{-2} \sin v\, (1 + p/r)\}\, \delta e + q^{-3}(p/r)^2\, \delta M \ . \tag{122}$$

Special cases (derived from the choice of integration constants in the $\delta\zeta$) are reserved to section 7, but in counting the number of terms associated with the general U_ℓ we have regard to the basis on which these constants are chosen.

The $\delta\zeta$ are available at once as the v-integrals of the expressions for $d\zeta/dv$ in section 4. Generation of the expressions for δr and δw is essentially straightforward in that the analysis starts with the $\delta\zeta$ due to U_ℓ^k (which $\delta\zeta$ we can denote by $\delta\zeta_{\ell k}$) and finishes with $\delta r_{\ell k}$ and $\delta w_{\ell k}$. With δb , however, there is a complication, due to the appearance of u' in (121), as opposed to v in (120) and (122); as already noted in section 1, we deal with the difficulty by deriving $\delta b_{\ell\kappa}$, rather than $\delta b_{\ell k}$, where κ has values of opposite parity to those of k .

We do not give expressions for $\delta\dot{r}$, $\delta\dot{b}$ and $\delta\dot{w}$, but (as is clear from Part 1) these are immediately available from the expressions for δr , δb and δw , just by replacing S_j and C_j by (respectively) $(k + j)\,\bar{n}\,C_j$ and $-(k + j)\,\bar{n}\,S_j$. We can do better than this if we allow for the (overall) rate of change of $\bar{\omega}$, replacing $(k + j)\,\bar{n}$ by $(k + j)\,\bar{n} + k\,\dot{\bar{\omega}}$, assuming C_j and S_j still to be shorthand for C_j^k and S_j^k .

6.1 The perturbation δr

We have to apply (120) with δa , δe and δM given by (65) and (the integrals of) (76) and (105). We find that the integrals combine in a very natural way, as a result of which we can write (with δr short for $\delta r_{\ell k}$)

$$\delta r - (r/a)\,\delta a = \tfrac{1}{4}\,a\,A_{\ell k}\sum B_{\ell j}\left\{e\left(\frac{k-\ell-1}{k+j-2} + 3\,\frac{k-\ell+1}{k+j}\right)C_{j-1} + 2\left(\frac{2k-\ell-1}{k+j-1} + \frac{2k+\ell+1}{k+j+1}\right)C_j + e\left(3\,\frac{k+\ell-1}{k+j} + \frac{k+\ell+1}{k+j+2}\right)C_{j+1}\right\} . \tag{123}$$

The simplest way to incorporate (65) is to note that this can be decomposed into

$$\frac{r}{a}\,\delta a = -\tfrac{1}{4}\,a\,A_{\ell k}\sum B_{\ell j}\left\{e\left(\frac{k+j-2}{k+j-2} + 3\,\frac{k+j}{k+j}\right)C_{j-1} + 2\left(2\,\frac{k+j-1}{k+j-1} + 2\,\frac{k+j+1}{k+j+1}\right)C_j + e\left(3\,\frac{k+j}{k+j} + \frac{k+j+2}{k+j+2}\right)C_{j+1}\right\} . \tag{124}$$

By this trick, we can combine (123) and (124) at once, to get, say,

$$\delta r = -\tfrac{1}{4}\,a\,A_{\ell k}\sum B_{\ell j}\,R_j , \tag{125}$$

where R_j (or $R_{\ell kj}$ to display all the index parameters) is given by

$$\begin{aligned} R_j = \; & e(j + \ell - 1)\left(\frac{1}{k + j - 2} + \frac{3}{k + j}\right)C_{j-1} \\ & + 2\left(\frac{2j + \ell - 1}{k + j - 1} + \frac{2j - \ell + 1}{k + j + 1}\right)C_j \\ & + e(j - \ell + 1)\left(\frac{3}{k + j} + \frac{1}{k + j + 2}\right)C_{j+1} \; . \end{aligned} \tag{126}$$

It can be seen that (125) is a summation in which R_j, as given by (126), has three components; each component is expressed as the sum of two multiples of the same 'C quantity'. Let us separate the first multiple from the second (in each component of R_j), feeding them back separately into the summation of (125), so that we have two distinct summations that we can denote by Σ_- and Σ_+. Thus Σ_- involves $\Sigma\, B_{\ell j}\, R_{j-}$, where

$$R_{j-} = \frac{j + \ell - 1}{k + j - 2}\, eC_{j-1} + 2\,\frac{2j + \ell - 1}{k + j - 1}\, C_j + 3\,\frac{j - \ell + 1}{k + j}\, eC_{j+1} \; . \tag{127}$$

Now we have seen (in section 3) that all sums over $B_{\ell j}$ can be regarded as running from $-\infty$ to $+\infty$. It follows that we can rearrange the three sets of terms in $\Sigma B_{\ell j}\, R_{j-}$ such that (with j now used in a different way)

$$\begin{aligned} \sum B_{\ell j}\, R_{j-} = \; & \sum (k + j - 1)^{-1}\{(j + \ell)e\, B_{\ell,j+1} + 2(2j + \ell - 1)B_{\ell j} \\ & + 3(j - \ell)e\, B_{\ell,j-1}\}\, C_j \; . \end{aligned} \tag{128}$$

We now invoke the recurrence relation (54) to eliminate $B_{\ell,j+1}$, so that the quantity in curly brackets in (128) becomes

$$2(j + \ell - 1)B_{\ell j} + 2(j - \ell)e\, B_{\ell,j-1} \, ,$$

and then simplify further, using (58) with both ℓ and j reduced by 1, to reduce this to $2(\ell - 1)q^2\, B_{\ell-1,j}$. (We get the same result by using (54) to eliminate $B_{\ell,j-1}$ first, and then simplifying further via (56).) Thus

$$\sum B_{\ell j}\, R_{j-} = 2(\ell - 1)q^2 \sum (k + j - 1)^{-1}\, B_{\ell-1,j}\, C_j \; . \tag{129}$$

Similarly,

$$\sum B_{\ell j} R_{j+} = -2(\ell - 1)q^2 \sum (k + j + 1)^{-1} B_{\ell-1,j} C_j . \qquad (130)$$

The final result we require now follows from (125), (129) and (130). Because of its importance, we write C_j in full. Thus

$$\delta r_{\ell k} = -(\ell - 1) p A_{\ell k} \sum_j \frac{1}{(k + j + 1)(k + j - 1)} B_{\ell-1,j} \cos (ku' + jv) . \qquad (131)$$

Equation (131) provides a general formula for δr due to U_ℓ^k, valid for $\ell \geq 1$. (This restriction on ℓ has been operative from the beginning of section 4.) In view of the fact that k only takes non-negative values of the same parity as ℓ, it should be noted that j takes all values, but with $B_{\ell-1,j}$ only non-zero if $|j| \leq \ell - 2$. (This applies if $\ell \geq 2$; but the case $\ell = 1$ is trivial because there is an overall factor $\ell - 1$.)

If $j = -k \pm 1$, there is a zero denominator in (131), and terms with these values of j must be excluded from the formula; they are associated with the terms in $\delta\zeta/dv$ that were hived off in the generation of the $\overset{+}{\zeta}$. In section 7 we shall determine constants for $\delta e_{\ell k}$ and $\delta M_{\ell k}$ such that the terms with these two values of j are forced to zero. It will be noted that all the cosine terms occurring in (131), for a given J_ℓ and all possible k, are distinct, except that if $k = 0$ (ℓ even) then equal and opposite values of j lead to identical terms in $\cos jv$.

We use the remarks in the last paragraph to provide a pair of formulae for $N_{\ell r}$, the total number of terms required to express δr for a given value of ℓ. One formula applies when ℓ is odd, the other when ℓ is even. In both cases the number of j values for each k (regardless of the excluded values, if any) is $2\ell - 3$, if $\ell \geq 2$.

If ℓ is odd, there are $\frac{1}{2}(\ell + 1)$ possible values of k, so a priori the value of $N_{\ell r}$ is $\frac{1}{2}(\ell + 1)(2\ell - 3)$, if $\ell \geq 3$. But this must be reduced by the number of excluded values of the duplet (k, j). If $k = \ell$, j cannot be $-k \pm 1$, so there is no value to exclude. If $k = \ell - 2$, j can (a priori) be $-k + 1$ and this value must be excluded. If $k \leq \ell - 4$, it will always be necessary to exclude both $-k + 1$ and $-k - 1$. Thus the total number of exclusions is $\ell - 2$. Subtracting this from the a-priori value, we get

$$N_{\ell r} = \ell^2 - \tfrac{1}{2}(3\ell - 1) . \qquad (132)$$

Values for ℓ up to 15 (including $N_{1,r} = 0$, which is the correct value, even though the above analysis only applies for $\ell \geq 3$) are given in Table 6.

If ℓ is even, there are $\frac{1}{2}\ell + 1$ possible values of k, so a priori the value of $N_{\ell r}$ is $\frac{1}{2}(\ell + 2)(2\ell - 3)$. The exclusions are as recorded before, amounting to $\ell - 1$ now if $\ell \geq 4$, but it is also natural* for $N_{\ell r}$ not to count the 'duplications' that arise when $k = 0$; there are $\ell - 3$ of these duplications if $\ell \geq 4$, viz for $2 \leq |j| \leq \ell - 2$ (we cannot 'discount' for $|j| = 1$, since both values have already been 'excluded'). Thus the total number of exclusions is effectively increased to $2\ell - 4$, and this is the right number even when $\ell = 2$ (not covered by the argument that applies for $\ell \geq 4$ only). Subtracting this value from the a-priori value, we get

$$N_{\ell r} = \ell^2 - \tfrac{1}{2}(3\ell - 2) . \qquad (133)$$

Table 6 gives values for ℓ up to 16. It is remarkable that, as a result of the discounting of the duplications, we have a formula that is so close to what the improper use of (132) would give, the value by (133) being larger by just $\frac{1}{2}$. Further, if we did not discount, the formula for $\ell \geq 4$, viz $\ell^2 - \frac{1}{2}(\ell + 4)$, would give 1, instead of the correct 2, when $\ell = 2$.

It is noted, in conclusion, that, due to the multiplier (r/a) of δa in (120), it would not be a simple matter to null the 'constant' terms of δr with a choice of $\bar{a} \neq a'$, but that in any case we would prefer not to make such a choice. Further, the constants in δa and δr are not the same, partly due to the multiplier of δa referred to, but mainly to the way in which the terms in δe and δM combine. For even ℓ, we will have, in particular, a coefficient of $C_0^0 (= 1)$ equal to $(\ell - 1)pA_{\ell,0}B_{\ell-1,0}$. (See (159) for the constant in δa .)

6.2 The perturbation δb

We get δb from (121), where δi and $\delta\Omega$ are given by the integrals of (79) and (82). This is on the assumption that $\delta b\ (= \delta b_{\ell k})$ is associated with U_ℓ^k, following the decomposition of U_ℓ by (5). We shall shortly find, however, that it is much more convenient to decompose the total

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* In software in particular, we would rather double a computed quantity than have to compute it again; in the general analytical formula, (131), however, there is no easy way to indicate a special situation when $k = 0$ and $j \neq 0$.

δb (associated with U_ℓ) as $\sum \delta b_{\ell\kappa}$, where the summation is for values of κ that are of opposite parity to ℓ and we no longer associate the individual δb $(= \delta b_{\ell\kappa})$ with specific components of U_ℓ.

In relation to U_ℓ^k, we get

$$\delta b_{\ell k} = -\sum B_{\ell j} \left\{ \frac{kcs^{-1}}{k+j} A_{\ell k} C_j^k \cos u' + \frac{1}{k+j} A'_{\ell k} S_j^k \sin u' \right\} . \tag{134}$$

The trigonometrical products can be replaced by sums, in the usual way, and we can then invoke the notation of (20) to write

$$\delta b_{\ell k} = -\tfrac{1}{2}c \sum B_{\ell j} (k+j)^{-1} (A^+_{\ell k} C_j^{k-1} + A^-_{\ell k} C_j^{k+1}) . \tag{135}$$

This expression may be contrasted with (125) and (126) for δr. In view of the difference in superfix, as well as suffix (which alone varied in the terms of R_j), in the two C terms of (135), we would now like to combine a pair of terms with different k indices, before the summation over the j index operates. We note that $A^+_{\ell k}$ and $A^-_{\ell k}$, though under the summation sign in (135), are actually independent of j.

With the philosophy just referred to, we make the new decomposition

$$\delta b_\ell = \sum \delta b_{\ell\kappa} , \tag{136}$$

where each δb $(= \delta b_{\ell\kappa})$ is of the form

$$\delta b = \sum T_j B_{\ell j} C_j^\kappa \tag{137}$$

and we require an expression for T_j (or $T_{\ell\kappa j}$ to display all the index parameters). We note first that since (for non-trivial results) k runs from 0 or 1 to ℓ (taking alternate values), it follows that, in principle, κ runs from -1 or 0 to $\ell + 1$ (again alternate values, but of opposite parity to k): for the minimum value of κ, only the term in $A^+_{\ell k}$, in (135), contributes to T_j, whilst for the maximum value of κ, only the term in $A^-_{\ell k}$ contributes; for intermediate values (if any), both terms contribute. But we can straight away dismiss the 'maximum value' $(\kappa = \ell + 1)$, because $A_{\ell\ell}$ is just a multiple of s^ℓ; from this it follows that $A^-_{\ell\ell}$, defined by (20), is zero. (Also $B_{\ell\ell} = 0$ anyway!) We shall find that we do not require the 'minimum value' $(\kappa = -1)$ either.

To evaluate T_j, in general, we use (30) and (31) for $A^-_{\ell k}$ and $A^+_{\ell k}$, respectively. It is fortunate that we require the first with $k = \kappa - 1$ and the second with $k = \kappa + 1$, since this means that we pick up the same inclination function, $A^\kappa_\ell(i)$, for both; moreover, it is aesthetically satisfying to have a direct application for the inclination functions for which subscript and superscript are of opposite parity, as opposed to merely an application in the propagation of like-parity functions*. We have

$$T_j = -J_\ell \left\{\frac{\kappa + 1}{\kappa + j + 1}\alpha_{\ell,\kappa+1} + \frac{(\ell - \kappa + 1)(\ell + \kappa)}{4\kappa(\kappa + j - 1)}\alpha_{\ell,\kappa-1}\right\}(R/p)^\ell s^\kappa A^\kappa_\ell(i) \ . \quad (138)$$

The quantity in curly brackets in (138) is a pure constant, in which the $\alpha_{\ell k}$ are given by (12): thus the first α involves $P^{\kappa+1}_\ell(0)$ and the second involves $P^{\kappa-1}_\ell(0)$, these being given by (13). By relating these to $P^\kappa_{\ell+1}(0)$, we may express the aforesaid quantity (after some algebraic reduction) as

$$-\frac{(\ell - \kappa + 1)}{2^{\kappa+1}\ \kappa!} P^\kappa_{\ell+1}(0) \left\{\frac{u_{\kappa+1}}{\kappa + j + 1} - \frac{u_{\kappa-1}}{\kappa + j - 1}\right\} .$$

But $P^\kappa_{\ell+1}(0)$ is related to $\alpha_{\ell\kappa}$ by (16), and thus to $A_{\ell\kappa}$ by (17). Hence (138) gives

$$T_j = \frac{\ell A_{\ell\kappa}}{2u_\kappa}\left\{\frac{u_{\kappa+1}}{\kappa + j + 1} - \frac{u_{\kappa-1}}{\kappa + j - 1}\right\} . \quad (139)$$

The preceding is 'general' in that it applies for $1 \le \kappa \le \ell - 1$ (or, more precisely, with 2 as lower bound when ℓ is odd); further, if $\kappa \ge 2$ we can obviously cancel the three appearances of u. We still have to cover the cases $\kappa = 0$ (ℓ odd) and $\kappa = -1$ (ℓ even), in which (in principle) only the first term in curly brackets is to be taken. To count only non-zero terms when T_j is substituted in (137), the restriction on j is that $|j| \le \ell - 1$ (cf an upper bound of $\ell - 2$ in the analysis for r); we shall be excluding the values $j = \kappa \pm 1$, of course.

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* This strengthens the view (noted in other papers, and in section 8 of Ref 13 in particular) that κ (of either parity) is a much better index parameter than Kaula's p (where $2p = \ell - k$, referred to in section 3 here). When the analysis includes the tesseral harmonics (Ref 9, and see Appendix A here), k takes negative values (with $|k| \le \ell$) as well as positive, but the factor u_k in (12) is not required.

When $\kappa = 0$, we require just $2/(j+1)$ from the curly brackets in (139); but for each $j > 0$, half this quantity may be combined with half the corresponding quantity for $j < 0$, to give $1/(j+1) + 1/(-j+1)$, which can be rewritten as $1/(j+1) - 1/(j-1)$. It follows that (139), with the three occurrences of u deleted, again gives the right results (counted separately for $j > 0$ and $j < 0$). The modified formula may be seen to apply, finally, for $j = 0$.

When $\kappa = -1$, the position is more complicated. First, the expression by (139) is not even legitimate now, since $A_{\ell\kappa}$ is not defined for $\kappa < 0$; the illegitimacy arose in the substitution for $A^{+}_{\ell k}$, since (31) does not apply when $k = 0$. But (20) indicates that $A^{+}_{\ell,0} = -A^{-}_{\ell,0}$, and this suggests that we can relate the required term, involving $u_{\kappa+1}$ with $\kappa = -1$, to the term in $u_{\kappa-1}$ when $\kappa = 1$. Since $C^{-1}_{j} = C^{1}_{-j}$, the relating will involve the transposition of positive and negative values of j, and this is also necessary to identify $u_{\kappa+1}/(\kappa+j+1)$ for $\kappa = -1$ with $u_{\kappa-1}/(\kappa+j-1)$ for $\kappa = 1$. In short, we can deal with $\kappa = -1$ just by doubling the second term in curly brackets in (139) that is associated with $\kappa = 1$. This means that, yet again*, we get the right result from (139) if we cancel the three appearances of u.

We can now write down the final result we require, on substituting (139) into (137) and expressing C^{κ}_{ℓ} in full. Thus

$$\delta b_{\ell\kappa} = -\ell A_{\ell\kappa} \sum_{j} \frac{1}{(\kappa + j + 1)(\kappa + j - 1)} B_{\ell j} \cos(\kappa u' + jv) . \qquad (140)$$

As already indicated, this formula is unlike (131), the corresponding one for δr, in that it cannot be taken in isolation as relating to a sub-component of U_{ℓ}. It is like (131) in one respect, however, in that terms of $\delta b_{\ell\kappa}$ with $j = -\kappa \pm 1$ are excluded. In section 7 we shall determine constants for $\delta i_{\ell k}$ and $\delta\Omega_{\ell k}$ (k, not κ, now being the appropriate symbol) such that these terms are forced to zero.

* The universality of this procedure (cancelling the u) stems from the original introduction of u_{κ} into the definition of $\alpha_{\ell\kappa}$. If we dispensed with this factor, but used positive values of κ as well as negative ones (see also Appendix A), then we would find nothing special about the values ± 1 and 0 in the first place.

We proceed to obtain a pair of formulae for $N_{\ell b}$, the total number of terms (without duplication of C_j) required to express δb for a given value of ℓ. Whether ℓ is odd or even, the number of j values for each κ (not discounting excluded values) is $2\ell - 1$, correct for all $\ell\ (\geq 1)$ this time.

If ℓ is even (which we have seen to be the simpler case), there are $\frac{1}{2}\ell$ possible values of κ, so a priori the value of $N_{\ell b}$ is $\frac{1}{2}\ell(2\ell - 1)$. When $\kappa = \ell - 1$, j can be $-\kappa + 1$ but not $-\kappa - 1$, so there is just one value to exclude. For all other κ, values of $-\kappa \pm 1$ are both possible, so the total number of exclusions is $\ell - 1$. Subtracting this from the a-priori value, we get

$$N_{\ell b} = \ell^2 - \tfrac{1}{2}(3\ell - 2) . \tag{141}$$

Interestingly, this is the same as $N_{\ell r}$ given by (133). Values for ℓ up to 16 are given in Table 6.

If ℓ is odd, there are $\frac{1}{2}(\ell + 1)$ possible values of κ, so a priori the value of $N_{\ell b}$ is $\frac{1}{2}(\ell + 1)(2\ell - 1)$. There is again a single exclusion if $\kappa = \ell - 1$, and two otherwise, so there are ℓ basic exclusions (assuming $\ell \geq 3$). In addition, however, there are $\ell - 2$ duplications when $\kappa = 0$, and these can be discounted for $N_{\ell b}$ (though not for δb itself - see also the footnote in section 6.1), so the effective number of exclusions is $2\ell - 2$; this value applies even when $\ell = 1$. Subtracting this total number of exclusions from the a-priori values we get

$$N_{\ell b} = \ell^2 - \tfrac{1}{2}(\ell - 1) , \tag{142}$$

which is one more than for the corresponding $N_{\ell r}$. Values for ℓ up to 15 are given in Table 6.

In conclusion, it is worth remarking that if the planetary equations are used in Gauss's form, as opposed to Lagrange's, (and this is done in Ref 8), then the resulting form of the expressions for δi and $\delta \Omega$ is such as to provide an easier route to our δb (with κ, rather that k, effectively involved from the outset). For both δr and δw, however, the approach via Lagrange's form of the equations is much simpler.

6.3 The perturbation δw

The analysis for δw is much more like the δr analysis than the δb analysis, because each U_ℓ^k can again be treated separately throughout. There are two complications, however. First, (122) effectively involves $\cos 2v$ and $\sin 2v$, not just $\cos v$ and $\sin v$ (we see this at equation (143), following), and this means that the values $j = -k \pm 2$ are special as well as $j = -k \pm 1$. Second, we cannot take δw to be zero for any of these special cases, since the constants in δe and δM must now be assumed to have been already assigned, formula for the four special δw will be obtained in section 7.4. Actually, a fifth special case emerges, corresponding to $j = -k$ and a zero denominator $k + j$; δw for this case <u>can</u> be set to zero, since we still have (for each k) the constant in $\delta\omega$, as yet unassigned, available for the purpose - the constants for $\delta\omega$ are determined in section 7.5.

We start by rewriting (122) as

$$\delta w = 2q^{-2}(\delta e \sin v + eq^{-1}\,\delta M \cos v)$$
$$+ \tfrac{1}{2}eq^{-2}(\delta e \sin 2v + eq^{-1}\,\delta M \cos 2v) + \tfrac{1}{2}e^2q^{-3}\,\delta M + q^{-1}\,\delta L\,, \qquad (143)$$

where δe, δM and δL are available from the integrals of (76), (105) and (106). The integrals for δe and δM combine in a very natural way and we get

$$2q^{-2}(\delta e \sin v + eq^{-1}\,\delta M \cos v) = \tfrac{1}{2}q^{-2} A_{\ell k} \sum B_{\ell j} \left\{ e\left(\frac{1+\ell-k}{k+j-2}\right.\right.$$
$$\left. + 3\,\frac{1-\ell+k}{k+j}\right) S_{j-1} + 2\left(\frac{1+\ell-2k}{k+j-1} + \frac{1+\ell+2k}{k+j+1}\right) S_j$$
$$\left. + e\left(3\,\frac{1-\ell-k}{k+j} + \frac{1+\ell+k}{k+j+2}\right) S_{j+1} \right\}, \qquad (144)$$

$$\tfrac{1}{2}eq^{-2}(\delta e \sin 2v + eq^{-1}\,\delta M \cos 2v) = \tfrac{1}{2}eq^{-2} A_{\ell k} \sum B_{\ell j} \times$$
$$\left\{ 3e\,\frac{1-\ell+k}{k+j}\,S_{j-2} + 2\,\frac{1+\ell+2k}{k+j+1}\,S_{j-1} + e\left(\frac{1+\ell+k}{k+j+2} + \frac{1+\ell-k}{k+j-2}\right) S_j \right.$$
$$\left. + 2\,\frac{1+\ell-2k}{k+j-1}\,S_{j+1} + 3e\,\frac{1-\ell-k}{k+j}\,S_{j+2} \right\}, \qquad (145)$$

$$\tfrac{3}{2} e^2 q^{-3}\ \delta M = \tfrac{3}{8} e q^{-2} A_{\ell k} \sum B_{\ell j} \Big\{ e \frac{1 + \ell - k}{k + j - 2} S_{j-2} + 2 \frac{1 + \ell - 2k}{k + j - 1} S_{j-1} + 6e \frac{1 - \ell}{k + j} S_j + 2 \frac{1 + \ell + 2k}{k + j + 1} S_{j+1} + e \frac{1 + \ell + k}{k + j + 2} S_{j+2} \Big\} \tag{146}$$

and

$$q^{-1}\ \delta L = (1 - 2\ell) A_{\ell k} \sum B_{\ell j} \frac{1}{k + j} S_j . \tag{147}$$

We substitute the last four results in (143) and at the same time (as in the analysis for δr) change the interpretation of j so that we can use the same S_j in each term. This leads to

$$\begin{aligned} \delta w = \tfrac{1}{8} q^{-2} A_{\ell k} \sum \Big\{ & 3e^2 \left(\frac{1 - \ell + k}{k + j + 2} + \frac{1 + \ell - k}{k + j} \right) B_{\ell, j+2} \\ & + 2e \left(\frac{1 + \ell + 2k}{k + j + 2} + 6 \frac{1 - \ell + k}{k + j + 1} + 3 \frac{1 + \ell - 2k}{k + j} + 2 \frac{1 + \ell - k}{k + j - 1} \right) B_{\ell, j+1} + \left(e^2 \frac{1 + \ell + k}{k + j + 2} + 8 \frac{1 + \ell + 2k}{k + j + 1} \right. \\ & \left. + 2 \frac{4(1 - 2\ell) + e^2 (5 - \ell)}{k + j} + 8 \frac{1 + \ell - 2k}{k + j - 1} + e^2 \frac{1 + \ell - k}{k + j - 2} \right) B_{\ell j} \\ & + 2e \left(2 \frac{1 + \ell + k}{k + j + 1} + 3 \frac{1 + \ell + 2k}{k + j} + 6 \frac{1 - \ell - k}{k + j - 1} + \frac{1 + \ell - 2k}{k + j - 2} \right) B_{\ell, j-1} \\ & + 3e^2 \left(\frac{1 + \ell + k}{k + j} + \frac{1 - \ell - k}{k + j - 2} \right) B_{\ell, j-2} \Big\} S_j . \end{aligned} \tag{148}$$

Though the algebra is tedious, we can now eliminate $B_{\ell, j+2}$ and $B_{\ell, j-2}$, by the appropriate versions of (54). If we express the result as

$$\delta w = \tfrac{1}{8} q^{-2} A_{\ell k} \sum (V_{j,1} B_{\ell, j+1} + V_{j,0} B_{\ell j} + V_{j,-1} B_{\ell, j-1}) S_j , \tag{149}$$

the formulae for $V_{j,1}$, $V_{j,0}$ and $V_{j,-1}$ are initially very complicated. For $V_{j,1}$, in particular, we start with

$$2e \Big\{ \frac{3\ell (k - \ell + 1)}{(j + \ell + 1)(k + j + 2)} - \frac{k - 4\ell + 2}{k + j + 2} + \frac{6(k - \ell + 1)}{k + j + 1} - \frac{3\ell (k - \ell - 1)}{(j + \ell + 1)(k + j)} - \frac{3k}{k + j} - \frac{2(k - \ell - 1)}{k + j - 1} \Big\} ;$$

$V_{j,-1}$ is symmetrically related to this, but $V_{j,0}$ is a great deal more complicated. All three formulae can be greatly simplified, however; for $V_{j,0}$ this was done by a technique akin to partial fractions. The resulting expressions are

$$V_{j,1} = 2e(\ell + j)\left(\frac{1}{k + j + 2} - \frac{6}{k + j + 1} + \frac{3}{k + j} + \frac{2}{k + j - 1}\right), \quad (150)$$

$$V_{j,0} = 8\left(\frac{\ell + 2k + 1}{k + j + 1} - \frac{2\ell - 1}{k + j} + \frac{\ell - 2k + 1}{k + j - 1}\right)$$
$$- 2e^2\left(\frac{\ell + k + 1}{k + j + 2} - \frac{2(\ell + 1)}{k + j} + \frac{\ell - k + 1}{k + j - 2}\right) \quad (151)$$

and

$$V_{j,-1} = 2e(\ell - j)\left(\frac{2}{k + j + 1} + \frac{3}{k + j} - \frac{6}{k + j - 1} + \frac{1}{k + j - 2}\right). \quad (152)$$

As a result of this remarkable simplification, it will be observed that $V_{j,1}\, B_{\ell,j+1}$ and $V_{j,-1}\, B_{\ell,j-1}$, in (149), have been expressed in a very suitable form for the application of (56) and (60), with ℓ replaced by $\ell - 1$ in both relations, to eliminate $B_{\ell,j+1}$ and $B_{\ell,j-1}$, respectively, in favour of $B_{\ell j}$ (already present in (149)) and $B_{\ell-1,j}$. Thus, if we now write

$$\delta w = \tfrac{1}{4}A_{\ell k} \sum (W_{\ell,0}\, B_{\ell j} + W_{\ell,1}\, B_{\ell-1,j})S_j , \quad (153)$$

we get

$$W_{\ell,0} = 2\left(\frac{\ell + k + 1}{k + j + 2} - 2\,\frac{\ell + 1}{k + j} + \frac{\ell - k + 1}{k + j - 2}\right) \quad (154)$$

and

$$W_{\ell,-1} = -2(\ell - 1)\left(\frac{1}{k + j + 2} - \frac{4}{k + j + 1} + \frac{6}{k + j}\right.$$
$$\left.- \frac{4}{k + j - 1} + \frac{1}{k + j - 2}\right). \quad (155)$$

The final result we require follows from the substitution of (154) and (155) into (153). Writing S_j in full, we get

$$\delta w_{\ell k} = A_{\ell k} \sum_j \frac{1}{(k+j+2)(k+j)(k+j-2)} \Bigg\{ [2(\ell+1) - k(k+j)] B_{\ell j} - \frac{6(\ell-1)}{(k+j+1)(k+j-1)} B_{\ell-1,j} \Bigg\} \sin(ku' + jv) . \qquad (156)$$

Equation (156) is the general formula for δw due to U_ℓ^k. As with (131) and (140), for δr and δb respectively, it applies for all $\ell \geq 1$; like (140) but unlike (131), on the other hand, values of $|j|$ up to $\ell - 1$ are required to cover all the non-zero terms. For each k, zero denominators exist for five different values of j: for four of these values ($j = -k \pm 1$ and $j = -k \pm 2$), special formulae are required, in place of (156), as already noted; only for the fifth value ($j = -k$) can a term (for each k) be actually excluded.

Before proceeding to a pair of formulae for $N_{\ell w}$, the total number of terms required to express δw for a given value of ℓ, we note (and make allowance for) one specific null term that arises for each even value of ℓ. For $k = 2$ and $j = \ell - 1$, we see from (156) that the coefficient of $B_{\ell j}$ is identically zero (i.e. independently of ℓ). But $B_{\ell-1,j}$ is itself zero when $j = \ell - 1$, so this specific term of $\delta w_{\ell,2}$ always vanishes. Proceeding to $N_{\ell w}$, we first note that the number of j values for each k (regardless of any exclusion) is $2\ell - 1$ (for all $\ell \geq 1$).

If ℓ is odd, then a priori the value of $N_{\ell w}$ is $\frac{1}{2}(\ell + 1)(2\ell - 1)$. There is one excluded value of j for each $k \neq \ell$, so there are $\frac{1}{2}(\ell - 1)$ exclusions altogether. It follows that

$$N_{\ell w} = \ell^2 , \qquad (157)$$

and values for ℓ up to 15 are given in Table 6.

If ℓ is even, the a-priori value of $N_{\ell w}$ is $\frac{1}{2}(\ell + 2)(2\ell - 1)$. There is again an excluded j for each $k \neq \ell$, amounting to $\frac{1}{2}\ell$ basic exclusions, but there are now two other sources of discounted terms. We have just remarked on the particular zero term that arises for $k = 2$; we might prefer to allow zero actually to be computed in a general computer program, but here we regard this term as an exclusion. The other source of discounted terms consists of the $\ell - 1$ duplications that occur when $k = 0$ (see the footnote of section 6.1). Thus the number of effective exclusions is $\frac{1}{2}\ell$, and from subtraction we obtain

$$N_{\ell w} = \ell^2 - 1 . \tag{158}$$

Values for ℓ up to 16 are give in Table 6.

6.4 Universality of results (non-elliptic orbits)

Equations (131), (1'0) and (156) give, on summing over k or κ as appropriate, general formulae for the perturbations δr, δb and δw, respectively, due to U_ℓ. It is being tacitly assumed, in the rest of the Report, that we are only considering elliptic orbits. It is worth remarking here, therefore, that (as follows by a continuity argument) the formulae are also valid for parabolic and hyperbolic orbits. The formulae are effectively universal[14], in other words, though they inevitably fail for rectilinear orbits (with infinities arising from zero p).

7 THE SPECIAL CASES, AND INTEGRATION CONSTANTS

The main results in this section, obtained in section 7.4, are the formulae required to supplement (156), the general formula for δw. These formulae, covering the cases $j = -k \pm 1$ and $-k \pm 2$, are forced by the 'constants' for δe and δM, which are determined so that certain terms (those for $j = -k \pm 1$) can be excluded from δr. Though we have omitted (in section 4) the full expressions for the short-period perturbations, $\delta\zeta$, in the elements, we give here the adopted 'constants' for all the ζ. Five of the elements have constants chosen to suit δr, δb and δw; for completeness, we start with the semi-major axis, for which the constants are mandated by the use of a' as $\bar{a}$.

7.1 Mandatory constants for δa

We go back to the original expression for δa due to U_ℓ^k, viz (63). We can expand the complete factor $(p/r)^{\ell+1}$ in terms of the $B_{\ell+2,j}$ (cf the expansion via the $B_{\ell,j}$ in (65)). On taking just the term of the expansion with $j = -k$, we isolate the constant term that (for each k, and a given J_ℓ) is mandated by taking $\bar{a} = a'$.

The result can be written in the form (for the 'constant' component of $\delta a_{\ell k}$)

$$\delta a_{\ell k(c)} = -2aq^{-2} A_{\ell k} B_{\ell+2,k} \cos k\omega' . \tag{159}$$

7.2 Constants for δe and δM

The task in this section is to derive the formulae for $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$ that will legitimize our taking the terms in $\delta r_{\ell k}$ for $j = -k + 1$ and $-k - 1$ to be zero. These 'constants' will complete the formulae, for δe and δM, given by the integrals of (76) and (105) respectively.

We start by observing that (131), the general formula for $\delta r_{\ell k}$, was obtained by combining the two different denominators from (129) and (130). If we do not combine the denominators, we can rewrite the formula as

$$\delta r_{\ell k} = -\tfrac{1}{4}(\ell - 1)p\, A_{\ell k} \sum \left(\frac{1}{k + j - 1} - \frac{1}{k + j + 1}\right) B_{\ell-1,j}\, C_j . \tag{160}$$

The first denominator here is associated with the Σ_- summation of section 6.1. If this summation still applied for $j = -k + 1$, then the result would be an infinite coefficient of $B_{\ell-1,-k+1}\, C_{-k+1}$. We actually want this coefficient to be $-\tfrac{1}{4}(\ell - 1)p\, A_{\ell k}$, since it will then neutralize the coefficient, $\tfrac{1}{4}(\ell - 1)p\, A_{\ell k}$, that arises without difficulty from the second denominator in (160). The situation is similar when $j = -k - 1$ and we want the coefficient of $B_{\ell-1,-k-1}\, C_{-k-1}$, from the second term of (160), to be $-\tfrac{1}{4}(\ell - 1)p\, A_{\ell k}$ (and not infinity) to neutralize the first term. (It is recalled that infinite coefficients are avoided, simply because we deal separately, in section 5, with the relevant terms of de/dv and dM/dv.) What we do, therefore, is to obtain the coefficients of C_{-k+1} and C_{-k-1} that would apply in the absence of the constants $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$; we can then derive the appropriate values of these constants to cancel these putative coefficients.

So what would the first-denominator coefficient of C_j be, with $j = -k + 1$, in the absence of the constants? There would then be no contribution from equation (123), but still a cont ibution from the complementary (124), given by δa. Its value may be obtained from the first term of each pair in (124) - the second term does not apply because it feeds separately into the second-denominator coefficient of C_{-k+1} which behaves normally as we have seen.

But equation (124) was written down before the 'rearrangement', from (127) to (128), in which the use of j changed. This change affects the B subscripts, and it may be seen that the required first-denominator coefficient of C_{-k+1} is

$$-\tfrac{1}{4}\, a\, A_{\ell k}\, (eB_{\ell,-k+2} + 4B_{\ell,-k+1} + 3eB_{\ell,-k}) \;.$$

The (normally behaved) second-denominator coefficient, on the other hand, may be written

$$\tfrac{1}{4}\, a\, A_{\ell k}\, (\ell - 1)\, q^2\, B_{\ell-1,-k+1} \;.$$

To cancel the combined coefficient by use of $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$, let us suppose that

$$\delta e_{\ell k(c)} = A_{\ell k}\, x\, C_{-k} \tag{161}$$

and

$$\delta M_{\ell k(c)} = A_{\ell k}\, e^{-1}\, q\, y\, S_{-k} \;, \tag{162}$$

where x and y are quantities to be determined. On combining these for a contribution to δr (cf (120)), we get a coefficient of C_{-k+1} given by

$$-\tfrac{1}{2}\, a\, A_{\ell k}\, (x + y) \;,$$

so that one equation to be satisfied by x and y is

$$2(x + y) + eB_{\ell,-k+2} + 4B_{\ell,-k+1} + 3eB_{\ell,-k} - (\ell - 1)\, q^2\, B_{\ell-1,-k+1} = 0 \;. \tag{163}$$

The complementary contribution to δr from (161) and (162) leads to a term in C_{-k-1}, of coefficient

$$-\tfrac{1}{2}\, a\, A_{\ell k}\, (x - y) \;,$$

and this combines with two other coefficients of C_{-k-1}, obtained as in the last paragraph; the result is another equation in x and y,

$$2(x - y) + eB_{\ell,-k-2} + 4B_{\ell,-k-1} + 3eB_{\ell,-k} - (\ell - 1)\, q^2\, B_{\ell-1,-k-1} = 0 \;. \tag{164}$$

Solution of (163) and (164) gives

$$x = \tfrac{1}{4}\{(\ell - 1)q^2 (B_{\ell-1,-k+1} + B_{\ell-1,-k-1}) - 6eB_{\ell,-k} - 4(B_{\ell,-k+1} + B_{\ell,-k-1}) - e(B_{\ell,-k+2} + B_{\ell,-k-2})\} \tag{165}$$

and

$$y = \tfrac{1}{4}\{(\ell - 1)q^2 (B_{\ell-1,-k+1} - B_{\ell-1,-k-1}) - 4(B_{\ell,-k+1} - B_{\ell,-k-1}) - e(B_{\ell,-k+2} - B_{\ell,-k-2})\} . \tag{166}$$

To get the formulae for $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$ that we require, it remains to substitute (165) and (166) into (161) and (162). In doing this, we make two simplifications: we eliminate $B_{\ell-1,-k+1}$ and $B_{\ell-1,-k-1}$ by use of (60) and (56), respectively (with ℓ replaced by $\ell - 1$ in each case); and we write $B_{\ell,k}$ etc rather than $B_{\ell,-k}$.

Finally, then, we have

$$\delta e_{\ell k(c)} = -\tfrac{1}{4} A_{\ell k} \{eB_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} + 2(\ell + 2)eB_{\ell k} - (\ell - k - 4)B_{\ell,k-1} + eB_{\ell,k-2}\} \cos k\omega' \tag{167}$$

and

$$\delta M_{\ell k(c)} = \tfrac{1}{4} e^{-1} q A_{\ell k} \{eB_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} - 2keB_{\ell k} + (\ell - k - 4)B_{\ell,k-1} - eB_{\ell,k-2}\} \sin k\omega' . \tag{168}$$

The formulae could, of course, be reduced to a smaller number of terms, by use of the fixed-ℓ recurrence relation, (54), but the coefficients would then be much more awkward; no genuinely simpler versions of (167) and (168) have been found.

7.3 Constants for δi and $\delta\Omega$

In this section we derive formulae for $\delta i_{\ell k(c)}$ and $\delta\Omega_{\ell k(c)}$ to legitimize our taking the terms for $j = -\kappa + 1$ and $-\kappa - 1$ in (140), the general expression for $\delta b_{\ell\kappa}$, to be zero. The analysis is somewhat simpler than that in the preceding section, in spite of the complexity entailed by the need to work with both k and κ .

As with $\delta r_{\ell k}$, we start by observing that (140) was obtained by combining two denominators, which appear separately in the preceding (139). When $j = -\kappa + 1$, the second denominator becomes zero and no longer operates; from the first alone we get, as the effective term in (140), $\frac{1}{4}\ell\, A_{\ell\kappa}\, B_{\ell,-\kappa+1}\, C^{\kappa}_{-\kappa+1}$. When $j = -\kappa - 1$, similarly, the first denominator in (139) does not operate, and (140) effectively reduces to $\frac{1}{4}\ell\, A_{\ell\kappa}\, B_{\ell,-\kappa-1}\, C^{\kappa}_{-\kappa-1}$. These terms have to be cancelled by the use of $\delta i_{\ell k(c)}$ and $\delta\Omega_{\ell k(c)}$, with appropriate k, so we suppose that

$$\delta i_{\ell k(c)} = x\, C^{k}_{-k} \tag{169}$$

and

$$\delta\Omega_{\ell k(c)} = s^{-1}\, y\, S^{k}_{-k} . \tag{170}$$

(In the last section we were able to include $A_{\ell k}$ in the corresponding expressions, (161) and (162), but there is no common factor available now.) On combining (169) and (170) for a contribution to δb (cf (121)), we get

$$\tfrac{1}{2}(x - y)\, C^{k+1}_{-k} + \tfrac{1}{2}(x + y)\, C^{k-1}_{-k} .$$

If we postpone consideration of any difficulties associated with the extreme values of k that arise, then we get a pair of equations for x and y, on identifying κ with $k + 1$ and $k - 1$, respectively, in the coefficients of $C^{\kappa}_{-\kappa+1}$ and $C^{\kappa}_{-\kappa-1}$ that have been recorded. Thus the equations are

$$2(x - y) + \ell\, A_{\ell,k+1}\, B_{\ell,-k} = 0 \tag{171}$$

and

$$2(x + y) + \ell\, A_{\ell,k-1}\, B_{\ell,-k} = 0 . \tag{172}$$

Solution gives

$$x = -\tfrac{1}{4}\ell(A_{\ell,k+1} + A_{\ell,k-1})B_{\ell,-k} \tag{173}$$

and

$$y = \tfrac{1}{4}\ell(A_{\ell,k+1} - A_{\ell,k-1})B_{\ell,-k} . \tag{174}$$

The expressions in brackets can be replaced by simpler expressions, following (27) and (28), and we also replace $B_{\ell,-k}$ by $B_{\ell k}$. Then substitution in (169) and (170) gives us the formulae we require. Thus

$$\delta i_{\ell k(c)} = \tfrac{1}{2}A'_{\ell k} B_{\ell k} \cos k\omega' \tag{175}$$

and

$$\delta\Omega_{\ell k(c)} = \tfrac{1}{2}kcs^{-2} A_{\ell k} B_{\ell k} \sin k\omega' . \tag{176}$$

We now have to show that (175) and (176) are valid for the extreme values of k as well as 'general values'. For $k = \ell$, the maximum value of k, there is no difficulty: (171) involves a value of $\ell + 1$ for κ in δb, but this is all right as $B_{\ell\ell} = 0$ (cf section 6.2); thus $x = y = 0$ for $k = \ell$. For the minimum value of k (0 or 1, according to whether ℓ is even or odd), it is a little more complicated: we consider the two cases separately.

If ℓ is even, with $k = 0$, C_0^1 and C_0^{-1} are the same and we cannot separate $x + y$ from $x - y$ in the combination of (169) and (170). Thus there is only one equation to be satisfied (instead of both (171) and (172)), and it involves only the identification of κ with $k + 1$ (not also with $k - 1$). The equation reduces to

$$x = -\tfrac{1}{2}\ell A_{\ell,1} B_{\ell,0} , \tag{177}$$

which is consistent with (175), in view of (29). The value of y is indeterminate, but this is appropriate for the coefficient of $\sin k\omega'$, which is itself zero when $k = 0$. If ℓ is odd (with $k = 1$), on the other hand, the validity of (175) and (176) follows in essence from the argument just prior to the establishment of equation (140), and the details are omitted.

A last point in this section is noted as no more than a curiosity. Whereas the rest of $\delta i_{\ell k}$ and $\delta\Omega_{\ell k}$ contain $A_{\ell k}$ and $A'_{\ell k}$, respectively, as a factor of every term, as indicated by (79) and (82), these factors are reversed in the 'constant' terms, as indicated by (175) and (176). The point was noted, for $\ell = 3$, in Part 1.

7.4 Forced terms in δw

We now have, for each U_ℓ^k , only one 'constant' at our disposal; denoted by $\delta\omega_{\ell k(c)}$, it will be determined in section 7.5 so as to validate the nulling of the term for $j = -k$ in the formula, (156), for $\delta w_{\ell k}$. For $j = -k \pm 1$ and $-k \pm 2$, on the other hand, we are forced to accept non-null terms that arise, via (122), from (167) and (168), the formulae for $\delta e_{\ell k(c)}$ and $\delta M_{\ell k(c)}$. We derive the formulae for these terms in the present section, which must therefore be regarded (from the reference viewpoint) as a completion of section 6.

For each of the four special values of j , in principle we embark on a procedure that is similar to that employed in section 7.2. The basis of this procedure is that we re-determine the expression for δw with the appropriate terms in δe and δM (which in their general form would lead to infinities) replaced by 'constants' proportional to x and y (as defined by (161) and (162)); the only difference (in principle) from section 7.2 is that we do not have unknowns to solve for, so that the procedure is 'direct'. In practice, however, because the basic formula for δw , (122), is so much more complicated than the corresponding formula for δr , (120), it is better to proceed a little differently from this: instead of developing our four special formulae more or less ab initio, we start four times from the (final) general formula for δw , (156), and modify it each time in the appropriate manner.

We start with $j = -k + 1$; it will be useful to have a shorthand for the denominator that becomes zero, so we define

$$d = k + j - 1 \tag{178}$$

for general j and will eventually set $d = 0$. In terms of d , we can rewrite (154) and (155), which on substitution into (153) give the final (156), as

$$W_{\ell,0} = 2\left(\frac{\ell + k + 1}{d + 3} - 2\,\frac{\ell + 1}{d + 1} + \frac{\ell - k + 1}{d - 1}\right) \tag{179}$$

and

$$W_{\ell,-1} = -2(\ell - 1)\left(\frac{1}{d + 3} - \frac{4}{d + 2} + \frac{6}{d + 1} - \frac{4}{d} + \frac{1}{d - 1}\right). \tag{180}$$

There is no difficulty with (179), but in (180) a zero denominator appears when d is set to zero. We will find that this denominator disappears when the 'replacement procedure' is complete.

The terms we have to replace come from the first term of (143), for which the general expression is given by (144). In the latter equation the 'offending terms' consist of the first of each pair; on changing the way j is used, as usual, the combination of the three terms in question is

$$\tfrac{1}{2}q^{-2} A_{\ell k} \left\{ e\,\frac{1 + \ell - k}{k + j - 1} B_{\ell,j+1} + 2\,\frac{1 + \ell - 2k}{k + j - 1} B_{\ell j} + 3e\,\frac{1 - \ell - k}{k + j - 1} B_{\ell,j-1} \right\} S_j ,$$

where the zero denominators are evident as soon as we set $j = -k + 1$. The replacement term, also based on the first term of (143), emerges when δe and δM are set, following (161) and (162), to $A_{\ell k}\, x\, C_{-k}$ and $A_{\ell k}\, e^{-1} q\, y\, S_{-k}$ respectively. The resulting term in S_{-k+1} is $A_{\ell k}\, q^{-2} (x + y)\, S_{-k+1}$ (in the analysis for $j = -k - 1$ it will be the term in S_{-k-1} that we need), so the required change in δw is

$$q^{-2} A_{\ell k} \{x + y + \tfrac{1}{2}d^{-1}[e(k - \ell - 1)\, B_{\ell,-k+2} + 2(2k - \ell - 1)\, B_{\ell,-k+1} + 3e(k + \ell - 1)\, B_{\ell,-k}]\}\, S_{-k+1} .$$

But $x + y$ is given by (163), so this can be expressed as

$$\tfrac{1}{2}q^{-2} A_{\ell k} \{(\ell - 1)\, q^2 B_{\ell-1,-k+1} - d^{-1}[e(j + \ell)\, B_{\ell,-k+2} + 2(2j + \ell - 1)\, B_{\ell,-k+1} + 3e(j - \ell)\, B_{\ell,-k}]\}\, S_{-k+1} .$$

We set $j = -k + 1$ in this expression and then invoke versions of (56) and (58); as a result it simplifies to

$$\tfrac{1}{2}(\ell - 1)\, A_{\ell k}\, (1 - 2d^{-1})\, B_{\ell-1,-k+1}\, S_{-k+1} .$$

Since only $B_{\ell-1,-k+1}$, i.e. $B_{\ell-1,j}$, is present in this expression, the change in δw can be represented as a change in $W_{\ell,-1}$ (cf (153) and (155)) of amount $4(\ell - 1)(1 - 2d^{-1})$; thus (180) is to be replaced by

$$W_{\ell,-1} = -2(\ell - 1)\left[\frac{1}{d + 3} - \frac{4}{d + 2} + \frac{6}{d + 1} - 2 + \frac{1}{d - 1}\right]. \qquad (181)$$

It is now legitimate to set $d = 0$ in (179) and (181), the result being

$$W_{\ell,0} = -\tfrac{1}{4}(2\ell - k + 2)$$

and

$$W_{\ell,-1} = -\tfrac{1}{4}(\ell - 1) .$$

On substituting in (153), we get the first of our four special formulae; it can be written (with a change of sign in the second suffix of each B)

$$\delta w_{\ell k,-k+1} = -\tfrac{1}{4} A_{\ell k}\{(2\ell - k + 2) B_{\ell,k-1} + (\ell - 1) B_{\ell-1,k-1}\} \sin (k\omega' + v) . \tag{182}$$

The second special case is with $j = -k - 1$. This is the twin of the first case, so it involves

$$d = k + j + 1 \tag{183}$$

and the expression for $x - y$ given by (164). There is no point in a virtual repetition of the analysis in detail, so we proceed direct to our second special formula; it can be written

$$\delta w_{\ell k,-k-1} = \tfrac{1}{4} A_{\ell k}\{(2\ell + k + 2) B_{\ell,k+1} + (\ell - 1) B_{\ell-1,k+1}\} \sin (k\omega' - v) . \tag{184}$$

The symmetry between (182) and (184) is obvious. For $k = 0$, of course, the equations reduce to the same formula, and the footnote of section 6.1 is again relevant.

For our third special case, we require to set $j = -k + 2$, so we start by defining

$$d = k + j - 2 . \tag{185}$$

Then (154) and (155) take the form

$$W_{\ell,0} = 2\left(\frac{\ell + k + 1}{d + 4} - 2\,\frac{\ell + 1}{d + 2} + \frac{\ell - k + 1}{d}\right) \tag{186}$$

and

$$W_{\ell,-1} = -2(\ell - 1)\left(\frac{1}{d+4} - \frac{4}{d+3} + \frac{6}{d+2} - \frac{4}{d+1} + \frac{1}{d}\right), \qquad (187)$$

with the potentially zero denominator in both expressions.

The terms of the general (156) that we have to replace now come from the second term of (143), for which the general expression is given by (145). The 'offending terms' are the last three of the six that occur in the latter equation; with the usual re-interpretation of j, these three terms in combination give

$$\tfrac{1}{4}\, eq^{-2} A_{\ell k}\left\{e\,\frac{j+\ell-k}{k+j-2} B_{\ell,j} + 2\,\frac{1+\ell-2k}{k+j-2} B_{\ell,j-1} + 3e\,\frac{1-\ell-k}{k+j-2} B_{\ell,j-2}\right\} S_j .$$

The replacement term is now $\frac{1}{4} A_{\ell k}\, eq^{-2}(x+y) S_{-k+2}$, so the required change in δw is

$$\tfrac{1}{4}\, eq^{-2} A_{\ell k}\,\{x + y + \tfrac{1}{4}d^{-1}[e(k-\ell-1)B_{\ell,-k+2} + 2(2k-\ell-1)\, B_{\ell,-k+1} + 3e(k+\ell-1)\, B_{\ell,-k}]\}\, S_{-k+2} .$$

With $x + y$ given by (163), this can be expressed as

$$\tfrac{1}{4}\, eq^{-2} A_{\ell k}\,\{(\ell-1)q^2 B_{\ell-1,-k+1} - d^{-1}[e(j+\ell-1)B_{\ell,-k+2} + 2(2j+\ell-3)B_{\ell,-k+1} + 3e(j-\ell-1)B_{\ell,-k}]\}\, S_{-k+2} .$$

We set $j = -k + 2$ in this expression and then invoke versions of (54), (57) and (58) to eliminate $B_{\ell-1,-k+1}$, $B_{\ell,-k+1}$ and $B_{\ell,-k}$ in favour of $B_{\ell-1,-k+2}$. This simplifies the result (for the change) to

$$-\tfrac{1}{4} A_{\ell k}(2d^{-1} - 1)\{(\ell - k + 1 + d)B_{\ell,-k+2} - (\ell-1)B_{\ell-1,-k+2}\}\, S_{-k+2} .$$

It now follows, from (186) and (187), that the altered values of $W_{\ell,0}$ and $W_{\ell,-1}$, to be substituted in (153), are given by

$$W_{\ell,0} = 2\left(\frac{\ell+k+1}{d+4} - 2\,\frac{\ell+1}{d+2}\right) + (\ell - k - 1 + d) \qquad (188)$$

and

$$W_{\ell,-1} = -(\ell - 1)\left(\frac{2}{d+4} - \frac{8}{d+3} + \frac{12}{d+2} - \frac{8}{d+1} + 1\right) . \qquad (189)$$

We can now set $d = 0$, getting

$$W_{\ell,0} = -\tfrac{1}{2}(\ell + k + 5) \qquad (190)$$

and

$$W_{\ell,-1} = \tfrac{19}{6}(\ell - 1) . \qquad (191)$$

Thus the substitution gives, for our third formula,

$$\delta w_{\ell k,-k+2} = -\tfrac{1}{48} A_{\ell k} \{3(\ell + k + 5)B_{\ell,k-2} - 19(\ell - 1)B_{\ell-1,k-2}\} \sin(k\omega' + 2v) . \qquad \ldots(192)$$

Our final special case, with $j = -k - 2$, is the twin of the preceding (third) case and involves

$$d = k + j + 2 . \qquad (193)$$

We will not go through the analysis in detail, in view of the symmetry, but proceed directly to the final formula; it is

$$\delta w_{\ell k,-k-2} = \tfrac{1}{48} A_{\ell k} \{3(\ell - k + 5)B_{\ell,k+2} - 19(\ell - 1)B_{\ell-1,k+2}\} \sin(k\omega' - 2v) . \qquad \ldots(194)$$

7.5 Constants for $\delta\omega$

To complete section 7, it remains to determine the constant, $\delta\omega_{\ell k(c)}$, that legitimizes our taking the term for $j = -k$ in (156), the general expression for $\delta w_{\ell k}$, to be zero. We already have $\delta M_{\ell k(c)}$, given by (168), so we only need to determine $\delta L_{\ell k(c)}$, the constant in $\delta L_{\ell k}$, for $\delta\omega_{\ell k(c)}$ to be known at once.

We start by now defining

$$d = k + j \tag{195}$$

so that (154) and (155), in their general form with potentially zero denominators, are rewritten as

$$W_{\ell,0} = 2\left(\frac{\ell + k + 1}{d + 2} - 2\,\frac{\ell + 1}{d} + \frac{\ell - k + 1}{d - 2}\right) \tag{196}$$

and

$$W_{\ell,-1} = -2(\ell - 1)\left(\frac{1}{d + 2} - \frac{4}{d + 1} + \frac{6}{d} - \frac{4}{d - 1} + \frac{1}{d - 2}\right). \tag{197}$$

The zero denominators will disappear when, for the coefficient of S_{-k} in (153), we replace a quantity occurring in the general analysis with a quantity based on $\delta M_{\ell k(c)}$ ($\delta e_{\ell k(c)}$ not being involved); then $\delta L_{\ell k(c)}$ is defined to null this resulting coefficient.

The quantity to be replaced derives from the combination of (146) and (147). With our usual re-interpretation of j, we can write the resulting coefficient of S_{-k} as the sum of

$$\tfrac{1}{4}\, eq^{-2} A_{\ell k}\, d^{-1} \{e(1 + \ell - k)B_{\ell,j+2} + 2(1 + \ell - 2k)B_{\ell,j+1}$$
$$+6e(1 - \ell)B_{\ell j} + 2(1 + \ell + 2k)B_{\ell,j-1} + e(1 + \ell + k)B_{\ell,j-2}\}$$

and

$$(1 - 2\ell)\, A_{\ell k}\, d^{-1}\, B_{\ell j} ,$$

associated with $\frac{1}{2} e^2 q^{-3}\, \delta M$ and $q^{-1}\, \delta L$ respectively. The quantity that has to replace this coefficient is available immediately from (168), but it is more convenient to back-track a little and take it instead as $\frac{1}{4} eq^{-2} A_{\ell k}\, y$, with y given by (166). Thus the replacement coefficient may be written, with j rather than $-k$ in the B subscripts, as

$$-\tfrac{3}{4}\, eq^{-2} A_{\ell k} \{e\, B_{\ell,j+2} + 4\, B_{\ell,j+1} - 4\, B_{\ell,j-1} - e\, B_{\ell,j-2}$$
$$- (\ell - 1)q^2\, (B_{\ell-1,j+1} - B_{\ell-1,j-1})\} .$$

On subtracting from this the coefficient being replaced, we get

$$- \tfrac{1}{4} A_{\ell k}\, q^{-2}\, d^{-1} \{3e^2(\ell + j + 1)B_{\ell,j+2} + 6e(\ell + 2j + 1)B_{\ell,j+1}$$
$$- 2[4(2\ell - 1) + e^2(\ell - 5)]B_{\ell j} + 6e(\ell - 2j + 1)B_{\ell,j-1}$$
$$+ 3e^2(\ell - j + 1)B_{\ell,j-2} - 3eq^2\, d(\ell - 1)(B_{\ell-1,j+1} - B_{\ell-1,j-1})\} .$$

By application of versions of the five relations (54) to (58), we can eliminate $B_{\ell,j+2}$ and $B_{\ell,j-2}$, then $B_{\ell,j+1}$ and $B_{\ell,j-1}$, and finally $B_{\ell-1,j+1}$ and $B_{\ell-1,j-1}$. This reduces the foregoing expression to

$$\tfrac{1}{4} A_{\ell k}\, d^{-1}\{[2(\ell + 1) - 3jd]\, B_{\ell j} + 6(\ell - 1)B_{\ell-1,j}\} ,$$

which represents (when $j = -k$) the adjustment required to the coefficient of S_{-k} in δw. On adding the appropriate contributions to (196) and (197) we get

$$W_{\ell,0} = 2\left(\frac{\ell + k + 1}{d + 2} + \frac{\ell - k + 1}{d - 2} - 3j\right) \tag{198}$$

and

$$W_{\ell,-1} = -2(\ell - 1)\left(\frac{1}{d + 2} - \frac{4}{d + 1} - \frac{4}{d - 1} + \frac{1}{d - 2}\right) . \tag{199}$$

We can now set $d = 0$ (i.e. $j = -k$), getting

$$W_{\ell,0} = 8k \tag{200}$$

and

$$W_{\ell,-1} = 0 . \tag{201}$$

These results mean that, in the absence of $\delta L_{\ell k(c)}$, we would have $k A_{\ell k} B_{\ell k}$ as the coefficient of S_{-k} in $\delta w_{\ell k}$; so, to null this, we take

$$\delta L_{\ell k(c)} = -kq\, A_{\ell k}\, B_{\ell k} \sin k\omega' . \tag{202}$$

From (168) and the definition of ψ , we now get

$$\delta\psi_{\ell k(c)} = -\tfrac{1}{4} e^{-1} A_{\ell k} \{e B_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} + 2keB_{\ell k} + (\ell - k - 4)B_{\ell,k-1} - e B_{\ell,k-2}\} \sin k\omega' . \qquad (203)$$

Finally, $\delta\Omega_{\ell k(c)}$ is given by (176), so our desired formula for $\delta\omega_{\ell k(c)}$ is

$$\delta\omega_{\ell k(c)} = -\tfrac{1}{4} e^{-1} A_{\ell k} \{e B_{\ell,k+2} - (\ell + k - 4)B_{\ell,k+1} + 2kes^{-2} B_{\ell k} + (\ell - k - 4)B_{\ell,k-1} - e B_{\ell,k-2}\} \sin k\omega' . \qquad (204)$$

It is remarked that $es\ \delta\omega_{\ell k(c)}$ is free of singularity (as would be expected), since $A_{\ell k}$ contains s^k as a factor, so that $ks^{-1} A_{\ell k}$ is non-singular. We also have the non-singularity of $s\ \delta\Omega_{\ell k(c)}$, given by (176), for the same reason.

8 RESULTS EXEMPLIFIED FOR ℓ FROM 0 THROUGH 4

To illustrate the main results of this Report, derived for general ℓ (> 0) , we use them to derive results for the particular cases $\ell = 1$, 2, 3 and 4 . We start with an analysis for $\ell = 0$, a case not covered by the general formulae - their failure for $\ell \leq 0$ stems from the fact that the expansion (33) is then inherently infinite, and not just 'effectively' so (cf Table 4). Both the cases $\ell = 0$ and $\ell = 1$ (analysed next) are actually trivial, since the 'perturbed motion' can (in each case) be looked at from a viewpoint which makes it pure Keplerian (unperturbed). The interest in these cases then lies in the interpretation of the perturbation formulae, which relate to the nominal mean elements $\bar{\zeta}$, in terms of the 'true' (fixed) elements of the effective Keplerian orbit - the elements of the latter will be denoted by ζ_T .

For $\ell = 2$ and 3 we write down, from the general results, the specific formulae for δr , δb and δw that were given before in Refs 1 ('Part 1') and 2. Both these papers gave also the specific $\dot{\bar{\zeta}}$ that complement δr , δb and δw , and Part 1 gave the $\delta\zeta$ that underlie them - the $\delta\zeta$ for $\ell = 2$ are well known, having been given by many authors. For $\ell = 4$ we summarize a complete (first-order) solution, giving the $\dot{\bar{\zeta}}$ as well as δr , δb and δw , the coordinate perturbations (δr, δb, δw), like the general formulae from which they are derived, have not been published before.

8.1 The trivial (but exceptional) case $\ell = 0$

From (4), we have

$$U_0 = -\mu J_0/r \ . \tag{205}$$

This is confirmed by (15), in which k is restricted to zero so that $U_0 = U_0^0$; $\alpha_{0,0}$ and $A_0^0(i)$ are both unity, so $A_{0,0} = J_0$ by (14). Thus the effect of J_0 is to reduce the power of the central force as indicated in Ref 4, the value of the overall 'true' power being given by

$$\mu_T = \mu(1 - J_0) \ . \tag{206}$$

The orbit can be fully represented by μ_T and the 'true elements' ζ_T , but it is instructive to exhibit the behaviour of the osculating elements (as well as the perturbations δr , δb and δw) relative to the μ originally assumed. As the general results of the paper do not apply when $\ell = 0$, it is simplest to derive formulae from the original planetary equations directly. There are no out-of-plane effects, even as a 'trivial' phenomenon, since we at once get

$$\delta i = \delta\Omega = \delta b = 0 \tag{207}$$

so that

$$\bar{i} = i_T \text{ and } \bar{\Omega} = \Omega_T \ . \tag{208}$$

From (62) and (205) it follows that

$$\delta a = a - a' = -2J_0 a^2/r \ . \tag{209}$$

This is a first-order relation, as usual, with a on the right-hand side interpreted as a' $(= \bar{a})$; it becomes exact if a^2 is interpreted as aa' . There should be no surprise that (osculating) a varies around the orbit (unless it is circular): this results from the use of the 'wrong' μ ; with the 'right' μ (μ_T), the osculating a would have the fixed value a_T . Since (with $r_T = r$)

$$\mu_T\left(\frac{2}{r_T} - \frac{1}{a_T}\right) = \mu\left(\frac{2}{r} - \frac{1}{a}\right) = \mu\left(\frac{2}{r} - \frac{1}{a'} - \frac{2J_0}{r}\right), \tag{210}$$

where all three expressions identify with V^2, V being the orbital speed, it follows that

$$a_T = a'(1 - J_0) ; \tag{211}$$

this is an exact relation.

The planetary equation for p, (71), gives

$$\dot{p} = 0 , \tag{212}$$

and it is found best to take the integral of this to be

$$\delta p = p - \overline{p} = -2J_0 a . \tag{213}$$

Then (209) and (213) lead to the non-singular perturbation for e given by

$$\delta e = e - \overline{e} = -J_0 \cos v ; \tag{214}$$

it turns out that we cannot get a simpler expression for our eventual δr by altering the implicit constant in (214), based on the explicit constant in (213). We introduce p_T, and hence e_T, by noting that

$$\mu p = \mu_T p_T , \tag{215}$$

since both quantities identify with h^2, where h is the angular momentum. It follows from this, using (206) and (213), that

$$p_T = \overline{p} - J_0 a(1 + e^2) . \tag{216}$$

Thence, using (211), we have

$$e_T = \overline{e}(1 + J_0) . \tag{217}$$

The planetary equation for ψ, (83), leads to

$$\frac{d\psi}{dv} = -J_0\, e^{-1} \cos v \, , \tag{218}$$

and we take the integral of this to be

$$\delta\psi = \psi - \overline{\psi} = -J_0\, e^{-1} \sin v \, , \tag{219}$$

since (it turns out) we cannot get a simpler expression for δw by changing the (implicit) constant. Hence also

$$\delta\omega = \omega - \overline{\omega} = -J_0\, e^{-1} \sin v \, . \tag{220}$$

We introduce ω_T via v_T, noting that (6), taken with and without T-suffixes (and with $r_T = r$), yields

$$(p - p_T)/r = (e - e_T) \cos v - e(v - v_T) \sin v \, . \tag{221}$$

This leads to

$$v - v_T = J_0\, e^{-1} \sin v \tag{222}$$

and hence (taking $u = u_T$)

$$\omega - \omega_T = -J_0\, e^{-1} \sin v \, . \tag{223}$$

From (220), we now see that

$$\omega_T = \overline{\omega} \, . \tag{224}$$

The planetary equation for ρ, (93), leads to

$$\frac{d\rho}{dv} = -2J_0\, q\, r/p \, , \tag{225}$$

and this introduces a difficulty in the analysis for M, since (for the first time) we effectively have a negative power of $1 + e \cos v$. The simplest way to deal with the situation is to change the integration variable* from true anomaly, v, to eccentric anomaly, E. Instead of (225) we have

$$\frac{d\rho}{dE} = -2J_0 ; \tag{226}$$

the integral of this is evidently secular, rather than short-periodic, but for convenience we use the notation appropriate to short-period perturbations and write (with the most useful integration constant)

$$\delta\rho = -2J_0 E . \tag{227}$$

The secular perturbation in ρ, that has just emerged, is dealt with in the usual way by choice of a suitable value for $\bar{n}$, not compelled to be equal to n'. With

$$n_T^2 a_T^3 = \mu_T = \mu(1 - J_0) , \tag{228}$$

we naturally take

$$\bar{n} = n_T = n'/(1 - J_0) \tag{229}$$

exactly, the formula being compatible with (211). (Equation (229) is in the spirit of (115), though not just a particular case of the earlier equation, which is only valid for $\ell > 0$.) Then (209) and (229) give

$$\frac{d\mathit{f}}{dt} = n = \bar{n} + J_0 n(3a/r - 1) , \tag{230}$$

and hence

$$\mathit{f} = \bar{n}t + J_0(3E - M) . \tag{231}$$

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* The use of E as integration variable leads to a (finite) solution of the general problem when $\ell < 0$. The analysis is more complicated now, however, as v has to be replaced by E in occurrences of $\sin u$, induced by the factor $P_{-\ell-1}(\sin\beta)$ of U_ℓ, as well as in the basic (negative) power of $1 + e \cos v$ that arises.

In combination with (227) we may now write

$$\delta L = J_0(E - M) = J_0 e \sin E \tag{232}$$

(using Kepler's equation). Using (219), we finally get

$$\delta M = M - \bar{M} = J_0 e^{-1}(q \sin v + e^2 \sin E) . \tag{233}$$

We introduce M_T via E_T, since from the equation

$$r = a(1 - e \cos E) \tag{234}$$

we get (with $r = r_T$, and both $a = a_T$ and $e = e_T$ known)

$$E - E_T = J_0 e^{-1} q^{-1} \sin v . \tag{235}$$

Then subtraction of the versions of Kepler's equation for M and M_T, with the aid of $e = e_T$ again, leads to

$$M - M_T = J_0 e^{-1} (q \sin v + e^2 \sin E) , \tag{236}$$

so that (233) yields

$$M_T = \bar{M} . \tag{237}$$

Thus four of the ζ_T are the same as the corresponding $\bar{\zeta}$, the only differences being for the elements a and e. Further, we can apply (120) and (122) to the $\delta\zeta$, getting

$$\delta r = J_0 a(e^2 q^{-1} \sin v \sin E - 1) \tag{238}$$

and

$$\delta w = J_0 e q^{-2} \sin v (2 + e \cos v) . \tag{239}$$

In view of (207), this completes the analysis for $\ell = 0$.

8.2 The trivial case $\ell = 1$

From (4),

$$U_1 = -\mu J_1 \frac{R}{r^2} \sin \beta \ . \tag{240}$$

This transforms to

$$U_1 = U_1^1 = -\mu A_{1,1} \frac{p}{r^2} \cos u' \tag{241}$$

by (15), in the general analysis, where

$$A_{1,1} = J_1 \frac{R}{p} s \ , \tag{242}$$

by (14), and $\sin \beta = s \sin u = s \cos u'$. As noted in Ref 4, (240) implies that

$$\frac{\mu}{r} + U_1 = \mu[r^2 + z^2 - 2rz \cos(\tfrac{1}{2}\pi - \beta)]^{-\frac{1}{2}} + O(J_1{}^2) \tag{243}$$

where $z = -J_1 R$, so the overall potential is the same (to first order) as for a central force towards the point at distance z, and axially 'north', from the nominal centre of 'unperturbed' attraction. (The precise representation of this configuration requires that for each $\ell > 0$, J_ℓ has a specific value, given by $-(-J_1)^\ell$.)

We have three essentially equivalent parameters (J_1, $A_{1,1}$ and z), and our formulae can be expressed in terms of any one of these, but it is more convenient to use a fourth parameter, λ, defined by

$$\lambda = z/p = -s^{-1} A_{1,1} = -J_1 R/p \ . \tag{244}$$

Then the general formulae, taken with $\ell = 1$, lead to the following: (131) gives

$$\delta r = 0 \ ; \tag{245}$$

(140), with $\kappa = 0$ and $j = 0$, gives

$$\delta b = \lambda c \ ; \tag{246}$$

and (182), with $k = 1$, gives

$$\delta w = -\lambda s \cos u . \tag{247}$$

Though (247) is taken from a special-case formula (because $j = -k + 1$), the general formula, (156), actually gives the same result because the second term of this formula, which is responsible for the zero denominator, does not arise. Our formulae are consistent with (132), (142) and (157), which give

$$(N_{1,r}, N_{1,b}, N_{1,w}) = (0, 1, 1) , \tag{248}$$

as seen also from Table 6.

Expressions for the $\delta\zeta$ can be written down easily enough, following the general analysis, but it is of more interest to obtain formulae relating the $\bar{\zeta}$ (mean elements relative to the nominal attraction centre) to the ζ_T (unchanging osculating elements relative to the 'true' centre). This can be done via the ζ (varying osculating elements) and derived quantities, since no conventional definitions are involved in relating the ζ to the ζ_T.

We start with (243), which may be taken to express μ/r_T. It leads to

$$r - r_T = \lambda p \sin \beta , \tag{249}$$

so that, in view of (245),

$$\bar{r} = r_T + \lambda p s \sin u . \tag{250}$$

Now this is true for all u; but $\bar{a} - a_T$, $\bar{e} - e_T$ and $\bar{M} - M_T$ must all be independent of u, whilst defining $\bar{r} - r_T$ via (120). This is only possible if

$$\bar{a} = a_T , \tag{251}$$

$$\bar{e} = e_T - \lambda q^2 s \sin \omega \tag{252}$$

and

$$\bar{M} = M_T + \lambda e^{-1} q^3 s \cos \omega . \tag{253}$$

Thus we have established three of the desired relationships; (251) could also be obtained, more directly, by identifying two different expressions for V^2 (cf (210)).

We can proceed in a similar way to get the relationships for i and Ω. Some geometrical visualization is needed, and we may regard the difference between b and b_T as validly defined, independently of the precise location of the 'mean orbital plane' which is involved in defining the coordinate b. This difference is given by a projection of the displacement z perpendicular to the (mean) orbital plane, such that

$$b - b_T = cz/r = \lambda cp/r . \tag{254}$$

Then from (246),

$$\bar{b} = b_T + \lambda ce \cos v \tag{255}$$

(where $\bar{b}$ is actually zero, by definition, but this is not relevant to the argument). As with (250), this is true for all u, whilst $\bar{b} - b_T$ may be expresed in terms of $\bar{i} - i_T$ and $\bar{\Omega} - \Omega_T$ via (121). It follows that

$$\bar{i} = i_T + \lambda ce \sin \omega \tag{256}$$

and

$$\bar{\Omega} = \Omega_T - \lambda cs^{-1}e \cos \omega . \tag{257}$$

This only leaves the relationship between $\bar{\omega}$ and ω_T to be established. It was not obvious how to proceed, analogously to (249) and (254), via a formula for $w - w_T$, so the procedure adopted was based on formulae for $\bar{v} - v_T$ and $\bar{u} - u_T$. The first of these comes easily from (252) and (253); thus

$$\bar{v} = v_T + \tfrac{1}{4}\lambda e^{-1}s \{e^2 \cos (u + v) + 4e \cos u + (2 + e^2) \cos \omega\} . \tag{258}$$

For the other relationship, we need the special formula (that can be derived for the given geometry)

$$c(u - u_T) = \sin u \cos u (i - i_T) - (1 - s^2 \sin^2 u)(\Omega - \Omega_T) . \tag{259}$$

From this, using (256) and (257) together with the expressions (omitted here) for δi and $\delta\Omega$, we get

$$u - u_T = \tfrac{1}{2}\lambda s^{-1}\{es^2\cos(u+v) + 2\cos u + e(1+c^2)\cos\omega\}. \qquad (260)$$

If we introduce also the (omitted) expression for δu, (260) gives

$$\overline{u} = u_T + \tfrac{1}{2}\lambda s^{-1}\{es^2\cos(u+v) + 4s^2\cos u + e(1+c^2)\cos\omega\}. \qquad (261)$$

From (258 and (261) we have, finally,

$$\varpi = \omega_T - \lambda e^{-1}s^{-1}(s^2 - e^2c^2)\cos\omega. \qquad (262)$$

It is worth remarking, in conclusion, that (262) can be used to infer the formula for $w - w_T$ that seemed less obvious, intuitively, than the formula for $b - b_T$. We find that

$$w - w_T = (\lambda p/rs)\cos u. \qquad (263)$$

Now that the missing formula is available, it is much easier to visualize its geometrical interpretation, especially for polar orbits $(s = 1)$.

8.3 The case $\ell = 2$

This time we start by noting, from Table 6 or the underlying formulae, that

$$(N_{2,r}, N_{2,b}, N_{2,w}) = (2, 2, 3), \qquad (264)$$

so that there are altogether seven terms in the coordinate perturbations. As in previous papers, we simplify the coordinate expressions by using the notation K and h, where

$$K = \tfrac{1}{2}J_2(R/p)^2 \qquad (265)$$

and

$$h = 1 - \tfrac{3}{2}f. \qquad (266)$$

Then Tables 1 and 2 give

$$A_{2,0} = -\tfrac{1}{4}Kh, \quad A_{2,1} = -Kcs , \quad A_{2,2} = \tfrac{1}{4}Kf , \tag{267}$$

and Table 4 gives

$$B_{1,0} = B_{2,0} = 1 , \quad B_{2,1} = B_{2,-1} = \tfrac{1}{2}e . \tag{268}$$

The two terms of δr are given by (131) with $(k, j) = (2, 0)$ and $(0, 0)$. We get, immediately,

$$\delta r = \tfrac{1}{4}Kp(f \cos 2u - 2h) , \tag{269}$$

confirming equation (188) of Part 1. (The single-term variable part of this formula has been given by other authors, of course; the best-known derivation was probably that of Kozai[15], but King-Hele and Gilmore established the result somewhat earlier, in equation (A-59) of Ref 16.)

The two terms of δb are given by (140) with $(\kappa, j) = (1, 1)$ and $(1, -1)$, since $(1, 0)$ corresponds to an 'excluded term'. We get

$$\delta b = \tfrac{1}{4}Kecs\{\sin(u + v) - 3 \sin \omega\} , \tag{270}$$

confirming equation (189) of Part 1.

For δw , we might have expected five terms after the exclusion of $(k, j) = (0, 0)$. But $(2, 1)$ is an example of the specific null term given in general by $(2, \ell - 1)$, whilst the terms for $(0, 1)$ and $(0, -1)$, being identical, are combined. None of the terms is given by the general formula, (156): the term associated with $(2, 0)$ is given by (192); the term associated with $(2, -1)$ is given by (182), and the pair of terms associated with $(0, \pm 1)$ are each given by either* (182) or (184). Overall, we get

$$\delta w = \tfrac{1}{16}K\{f \sin 2u + 4ef \sin(u + \omega) + 8eh \sin v\} , \tag{271}$$

confirming equation (190) of Part 1.

* In writing down specific formulae for δr and δw when ℓ is even and k zero, we must always remember to double the coefficient of each term with $j = 0$, when our intention is to cover the corresponding term with $-j$ (cf the footnote of section 6.1).

The long-term motion, for $\ell = 2$, comes entirely from the secular rates of change, $\dot{\Omega}$ and $\dot{\omega}$, given (from section 5) by $-Knc$ and $\frac{1}{4}Kn(4 - 5f)$ respectively. They are quoted here, only to remind the reader of the additional 'carry-over' terms in δr, δb and δw that they induce. In Part 1, these terms are included in equations (194) - (196), as opposed to equations (188) - (190); the intervening equations, (191) - (193), refer to the velocity-coordinate perturbations, namely, $\delta\dot{r}$, $\delta\dot{b}$ and $\delta\dot{w}$.

Finally, of course, since J_2 for the Earth is of order $\sqrt{J_\ell}$ for $\ell > 2$, the perturbations of order $J_2{}^2$ have to be taken into account for Earth satellites. Part 1 gives a detailed analysis of these perturbations, and the resulting formulae constitute the principal results of that Report: equations (320), (343), and (359) of Ref 1 give the contributions to δr, δb and δw, respectively, whilst the long-term effects are covered by equations (297) to (309).

8.4 The case $\ell = 3$

From Table 6,

$$(N_{3,r}, N_{3,b}, N_{3,w}) = (5, 6, 9) , \qquad (272)$$

so that there are 20 terms, in total, in the coordinate perturbations. As in Part 1 we write

$$H = \tfrac{1}{2} J_3 (R/p)^3 . \qquad (273)$$

Tables 1 and 2 give

$$A_{3,0} = \tfrac{1}{4} Hc(2 - 5f), \quad A_{3,1} = -\tfrac{1}{4} Hs(4 - 5f) , \qquad (274)$$

$$A_{3,2} = -\tfrac{5}{4} Hfc, \qquad A_{3,3} = \tfrac{5}{12} Hsf ; \qquad (275)$$

also Table 4 gives, in addition to quantities we already have from (268),

$$B_{3,0} = 1 + \tfrac{1}{4}e^2, \quad B_{3,1} = e, \quad B_{3,2} = \tfrac{1}{4}e^2 . \qquad (276)$$

We start with δr, separating (for convenience) the effects for $k = 3$ and $k = 1$. For $k = 3$, all the a-priori values of j must be included, namely, 1, 0 and -1; for $k = 1$, on the other hand, we exclude $j = 0$. Then (131) gives, corresponding to the two values of k,

$$\delta r = \tfrac{1}{144} Hpsf \{4e \sin (3u + v) + 15 \sin 3u + 20e \sin (2u + \omega)\} \tag{277}$$

and

$$\delta r = \tfrac{1}{12} Hpes (4 - 5f) \{\sin (u + v) - 3 \sin \omega\} ; \tag{278}$$

these conform with equation (408) of Part 1. (The total δr is, of course, given by adding the two contributions.)

For δb, the effects are for $\kappa = 2$ and $\kappa = 0$, and the a-priori values of j are the five with $|j| \leq 2$. For $\kappa = 2$ we exclude $j = -1$, and for $\kappa = 0$ we exclude $j = \pm 1$; for $\kappa = 0$ we also lose a term on combining* the terms with $j = \pm 2$. Then (140) gives, corresponding to the two values of κ,

$$\begin{aligned} \delta b = - \tfrac{1}{48} Hcf \{&2e^2 \cos 2(u + v) + 15e \cos (2u + v) \\ &+ 20(2 + e^2) \cos 2u - 30e^2 \cos 2\omega\} \end{aligned} \tag{279}$$

and

$$\delta b = - \tfrac{1}{12} Hc(2 - 5f)\{e^2 \cos 2v - 3(2 + e^2)\} ; \tag{280}$$

these conform with equation (411) of Part 1.

For δw, the effects are for $k = 3$ and $k = 1$, with the same a-priori j values as for δb. For $k = 3$, all five values yield terms, but only three of them come from the general (156); for $j = -1$ we use (192) and for $j = -2$ we use (182). For $k = 1$, the term with $j = -1$ is excluded, and the only general term is for $j = 2$; the terms for $j = 1$, 0 and -2 come from (192), (182) and (184) respectively. Corresponding to the two values for k, we get

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* In contradistinction to the previous footnote, and as noted in general after (141) in section 6, the two terms do not have the same numerical coefficient.

$$\delta w = -\tfrac{1}{256}Hsf\{2e^2\cos(3u+2v) + 11e\cos(3u+v) + 4(5+e^2)\cos 3u + 25e\cos(2u+\omega) + 50e^2\cos(u+2\omega)\} \qquad (281)$$

and

$$\delta w = \tfrac{1}{64}Hs(4-5f)\{e^2\cos(u+2v) - 2e\cos(u+v) - 2(18+7e^2)\cos u + 9e^2\cos(v-\omega)\}\ ; \qquad (282)$$

these conform with equation (413) of Part 1.

Expressions for the long-period rates of change of the mean elements can be written down from the formulas of section 5. The results agree with (from Ref 1) (373), (376), (384), (389) and (399), for $\bar{e}$, $\bar{I}$, $\bar{\Omega}$, $\bar{\varpi}$ and $\bar{M}$, respectively.

8.5 The (new) case $\ell = 4$

From Table 6,

$$(N_{4,r}, N_{4,b}, N_{4,w}) = (11, 11, 15)\ , \qquad (283)$$

so that there are altogether 37 terms in the coordinate perturbations, which we obtain first. To simplify our expressions, we define

$$G = \tfrac{1}{1024}J_4(R/p)^4\ . \qquad (284)$$

Then Tables 1 and 2 give

$$A_{4,0} = 48G(8 - 40f + 35f^2),\ A_{4,1} = 480Gcs(4 - 7f)\ , \qquad (285)$$

$$A_{4,2} = -320Gf(6 - 7f),\quad A_{4,3} = -1120Gcsf,\quad A_{4,4} = 560Gf^2\ ; \qquad (286)$$

also Table 4 gives, in addition to quantities we already have from (276),

$$B_{4,0} = 1 + \tfrac{3}{2}e^2,\ B_{4,1} = \tfrac{3}{2}e(1 + \tfrac{1}{4}e^2),\ B_{4,2} = \tfrac{3}{4}e^2,\ B_{4,3} = \tfrac{1}{4}e^3\ . \qquad (287)$$

We start with δr, as usual, separating the effects for $k = 4$, $k = 2$ and $k = 0$. The a-priori values of j are the five values for which $|j| \le 2$. All values apply when $k = 4$; for $k = 2$ we exclude $j = -1$; and for $k = 0$ we exclude $j = \pm 1$, whilst the terms for $j = \pm 2$ are identical. Then (131) gives, corresponding to the three values of k,

$$\delta r = -2Gpf^2 \{6e^2 \cos 2(2u + v) + 35e \cos (4u + v) + 28(2 + e^2) \cos 4u + 105e \cos (3u + \omega) + 70e^2 \cos 2(u + \omega)\} \tag{288}$$

$$\delta r = -8Gpf(6 - 7f)\{2e^2 \cos 2(u + v) + 15e \cos (2u + v) + 20(2 + e^2) \cos 2u - 30e^2 \cos 2\omega\} \tag{289}$$

and

$$\delta r = -24Gp(8 - 40f + 35f^2)\{e^2 \cos 2v - 3(2 + e^2)\} . \tag{290}$$

(The dominant (e-free) terms of (288) and (289) were originally given in equation (A108) of Ref 16.)

For δb, the effects are for $\kappa = 3$ and $\kappa = 1$, the a-priori values of j being the seven with $|j| \le 3$. For $\kappa = 3$ we exclude $j = -2$, and for $\kappa = 1$ we exclude $j = 0$ and $j = -2$. Then (140) gives, corresponding to the two values of κ,

$$\delta b = -4Gcsf \{4e^3 \sin 3(u + v) + 35e^2 \sin (3u + 2v) + 28e(4 + e^2) \sin (3u + v) + 70(2 + 3e^2) \sin 3u + 140e(4 + e^2) \sin (2u + \omega) - 140e^3 \sin 3\omega\} \tag{291}$$

and

$$\delta b = -4Gcs(4 - 7f)\{4e^3 \sin (u + 3v) + 45e^2 \sin (u + 2v) + 60e(4 + e^2) \sin (u + v) - 180e(4 + e^2) \sin \omega - 20e^3 \sin (v - \omega)\} . \tag{292}$$

For δw, the effects are again for $k = 4$, 2 and 0, with the same a-priori j values as for δb. For $k = 4$, all seven j values yield terms, of which five come from the general (156), for $j = -2$ we use (192) and for $j = -3$ we use (182). For $k = 2$, the term with $j = -2$ is excluded, whilst for $j = 3$, (156) gives another example of a 'specifically null' term (as in section 8.3); there are non-null general terms for $j = 2$ and $j = 1$; and the terms for $j = 0$, -1 and -3 come from (192), (182) and (184) respectively. Finally, for $k = 0$ the term with $j = 0$ is excluded; the other terms come in pairs, being 'general' for $j = \pm 3$, from (192) and (194) for $j = \pm 2$, and from (182) and (184) for $j = \pm 1$. Corresponding to the three values of k, we get

$$\begin{aligned}\delta w = -Gf^2\{&4e^3 \sin(4u + 3v) + 31e^2 \sin 2(2u + v) + 4e(21 + 5e^2) \times \\ &\times \sin(4u + v) + 28(3 \quad 4e^2) \sin 4u + 28e(7 + e^2) \sin(3u + \omega) \\ &+ 175e^2 \sin 2(u + \omega) + 140e^3 \sin(u + 3\omega)\}\,, \end{aligned} \tag{293}$$

$$\begin{aligned}\delta w = 4Gf(6 - 7f)\{&2e^2 \sin 2(u + v) + 4e(5 + 2e^2) \sin(2u + v) \\ &+ 5(8 - 7e^2) \sin 2u - 80e(5 + e^2) \sin(u + \omega) \\ &- 40e^3 \sin(v - 2\omega)\} \end{aligned} \tag{294}$$

and

$$\begin{aligned}\delta w = 4Ge(8 - 40f + 35f^2)\{&2e^2 \sin 3v - 3e \sin 2v \\ &- 6(24 + 5e^2) \sin v\}\,. \end{aligned} \tag{295}$$

It only remains to give the expressions for the $\dot{\zeta}$ (secular and long-period) from section 5. They may also be derived (as a check) from the author's early Ref 4; also, the version of Kepler's third law, given here as (303), checks with equation (15) of Ref 3.

There are, of course, no secular rates of change in ϖ or Υ. Their long-period rates are given by (108) and (109), with just $k = 2$. Thus

$$\dot{\varpi} = -480\, Gneq^2f(6 - 7f) \sin 2\omega \tag{296}$$

and

$$\dot{\Upsilon} = 480\, Gne^2cs(6 - 7f) \sin 2\omega\,. \tag{297}$$

The secular rate of change of Π, given by (110) with $k = 0$, is

$$\dot{\Pi} = 480\, Gnc(4 - 7f)(2 + 3e^2)\,, \tag{298}$$

and the long-period rate (given just with $k = 2$) is

$$\dot{\Omega} = -960\, Gne^2c(3 - 7f) \cos 2\omega \,. \tag{299}$$

For the variation of ϖ, we work with $\dot{\Psi}$, given by the second term of (111). Thus the secular rate is given by

$$\dot{\Psi} = -120\, Gn(8 - 40f + 35f^2)(4 + 3e^2) \,, \tag{300}$$

so that from (298) and (300) the secular rate for ϖ is

$$\dot{\varpi} = -120\, Gn\{4(16 - 62f + 49f^2) + 9e^2(8 - 28f + 21f^2)\} \,. \tag{301}$$

Similarly, the long-period rate for Ψ is given by

$$\dot{\Psi} = -240\, Gnf(6 - 7f)(2 + 5e^2) \cos 2\omega \,, \tag{302}$$

from which the rate for ϖ is at once available.

Finally, for the variation of $\bar{M}$, we deal with the secular perturbation by the modification of Kepler's third law given by (116). This gives

$$\bar{n}^2 \bar{a}^3 = \mu\{1 + 288\, Gq^3(8 - 40f + 35f^2)\} \,, \tag{303}$$

based on the perturbation rate (residual to the mean motion)

$$\dot{M} = 144\, Gnq^3(8 - 40f + 35f^2) \tag{304}$$

given by (113). Again, the long-period rate, from (113), is given by

$$\dot{M} = 480\, Gnq^3f(6 - 7f) \cos 2\omega \,; \tag{305}$$

this checks with (302) and the long-period rate for Σ, which from (114) is

$$\dot{\Sigma} = -1680\, Gne^2qf(6 - 7f) \cos 2\omega \,. \tag{306}$$

For completeness in implementing the (first-order) effects of J_4 (and any other J_ℓ), it is necessary to incorporate terms relating to what Part 1 describes as the difference between mean and semi-mean elements. The effects induced by the secular variation are included by adding $(\dot{\bar{\Omega}}/n)m$, $(\dot{\bar{\omega}}/n)m$ and $(\dot{\bar{M}}/n)m$ to $\bar{\Omega}$, $\bar{\omega}$ and $\bar{M}$, respectively, where $\dot{\bar{\Omega}}$, $\dot{\bar{\omega}}$ and $\dot{\bar{M}}$ are given by (298), (301) and (304), and where $m = v - M$ as in Part 1. The effects induced by the long-period variation, on the other hand, are allowed for via additional terms in the expressions for δr , δb and δw . Using (120) - (122), we find that these additional terms are given by

$$\delta r = 480\, Gpemf(6 - 7f) \sin (u + \omega) , \tag{307}$$

$$\delta b = 240\, Ge^2mcs \{3(4 - 7f) \cos (v - \omega) - 7f \cos (u + 2\omega)\} \tag{308}$$

and

$$\delta w = 240\, Gemf(6 - 7f)\{e \cos 2u + 4 \cos (u + \omega) - 4e \cos 2\omega\} . \tag{309}$$

9 CONCLUSIONS

The main function of Part 1 of the present trilogy of Reports was to provide details of a new theory of satellite motion, largely based on the use of a particular system of spherical-polar coordinates in the representation of the short-period components of the orbital perturbations. The emphasis was on the derivation of the second-order perturbations due to the zonal harmonic J_2 , but the first-order perturbations due to J_3 were derived as well. The latter derivation has now been extended to an arbitary zonal harmonic, J_ℓ (where ℓ is positive), with the development of general formulae of which those for J_3 were just a particular case.

The main formulae, which (in their generality) are believed to be entirely novel, are those for the perturbations in coordinates. The general terms of these formulae are given by the summations in (131), (140) and (156), for the perturbations in r , b and w , respectively. Terms that would have a zero denominator are excluded from these summations, as a consequence of the optimal definition of mean elements, except that replacement terms are needed for the perturbations in w ; the formulae for the replacement terms are (182), (184), (192) and (194).

The formulae for coordinate perturbations are complemented by the formulae, given in section 5, for the rates of change of the mean elements. In principle, the integration of the rate-of-change expressions is immediate, leading to the secular and long-period perturbations in the elements. In practice, however, there are complications, as was indicated in Part 1. One of these complications results from the fact that the expressions really arise as rates of change with respect to true anomaly, rather than time, and this leads to additional effects that are short-periodic in nature. However, the difficulty can easily be dealt with via the concept of semi-mean elements; the matter was fully discussed in Part 1, and has been touched on here in the context of the derivation of the appropriate perturbation terms for $\ell = 4$ (section 8.5). The other complications arise in the long-term evolution of the mean elements, the chief source of difficulty being the well-known singularities in the standard set of elements. A preliminary consideration of these difficulties was included in Part 1, but a full analysis is held over to Part 3, which will also give some numerical results.

The main limitation of the theory presented by the trilogy is apparent from its overall title - the gravitational field is assumed to be axi-symmetric, i.e. represented by zonal harmonics alone. For a complete field, with the tesseral harmonics included, the author has already published some general formulae (in terms of cylindrical coordinates rather than spherical coordinates, though that is a minor detail), but they apply only to near-circular orbits. The formulae were originally given in Ref 10, then in Ref 5, and finally as equations (92) - (94) of Ref 9.

In the formulae referred to, the inclination functions involve an additional suffix, m, to cover the longitude-dependent harmonics. For $m = 0$, the functions reduce to the $A_\ell^k(i)$ and $A_{\ell k}$ of the present paper, whilst the formulae themselves are then equivalent to truncated versions of the present equations (131), (140) and (156). Since we now have one set of formulae that relate to all inclination functions, though the formulae are truncated in regard to eccentricity, and another set of formulae that are valid for any eccentricity, though only relating to inclination functions for which $m = 0$, an obvious goal is the derivation of formulae that are 'general' in both respects. There is a fundamental difficulty, however, arising from the rotation of the gravitating primary, which we are able to neglect in the trilogy because it is assumed to take place about the axis of symmetry.

The root of the trouble is that the disturbing function (U) is no longer time-independent when the rotation of an arbitrary primary is allowed for. This nullifies our key constant, a', and leads ipso facto to the important phenomenon of resonance[13]. It will not be easy to develop a unified theory that covers resonant effects by the same formulae as non-resonant ones. However, a starting point is obviously the generation of the formulae referred to (in the preceding paragraph) as being 'general in both respects'; Appendix A gives an outline of what is involved.

Appendix A

EXTENSION TO THE GENERAL GRAVITATIONAL FIELD

Extending the theory of this Report to the tesseral harmonics (sectorial included) is an easier matter than might have been expected, so long as the rotation of the gravitating body is neglected; i.e., we suppose the sidereal angle, ν , to be fixed. We assume the potential to be described by the usual harmonic coefficients, $C_{\ell m}$ and $S_{\ell m}$, where $-C_{\ell,0}$ can be identified with the zonal coefficient J_ℓ and $S_{\ell,0}$ is taken as zero. For convenience, we introduce the polar equivalents, $J_{\ell m}$ and $\lambda_{\ell m}$, where λ is longitude and

$$(C_{\ell m}, S_{\ell m}) = J_{\ell m}(\cos m\lambda_{\ell m}, \sin m\lambda_{\ell m}) . \qquad \text{(A-1)}$$

(Note: $\lambda_{\ell m}$ is not uniquely defined if $m > 1$, and if $m = 0$ we set $J_{\ell,0} = -J_\ell$, so $J_{\ell,0}$ must be allowed to be negative.) It is usual, in practice, to work with normalized versions of $C_{\ell m}$ and $S_{\ell m}$ (and hence $J_{\ell m}$), but this is an irrelevant complication here. The potential due to $J_{\ell m}$, generalizing equation (4) of the main text, is given by

$$U_{\ell m} = \frac{\mu}{r} J_{\ell m} (R/r)^\ell P_\ell^m(\sin\beta) \cos m(\lambda - \lambda_{\ell m}) . \qquad \text{(A-2)}$$

The expansion of $U_{\ell m}$, in terms of the orbital elements, is customarily based on the family of inclination functions, $F_{\ell mp}(i)$, such that, generalizing equation (8),

$$P_\ell^m(\sin\beta)\exp(\imath m\lambda) = \sum_{p=0}^{\ell} F_{\ell mp}(i) \exp \imath\{(\ell - 2p)u + m(\Omega - \nu)\} . \qquad \text{(A-3)}$$

As already indicated in the main text of the Report, however, we prefer to use the index k $(= \ell - 2p)$, rather than p . This index only takes values that are of the same parity as ℓ , but in the extension to $m > 0$ we have to allow negative values of k , so that its range is now from $-\ell$ to $+\ell$; as compensation, we no longer require the factor u_k introduced at equation (8). Further, we prefer the inclination functions to be real for all values of the indices, so we define, as an unnormalized equivalent of the $\overline{F}_{\ell m}^k(i)$ in Ref 9,

$$F_{\ell m}^k(i) = \imath^{m+k} F_{\ell mp}(i) . \qquad \text{(A-4)}$$

The $U_{\ell m}$ expansion will involve Ω , as well as the other elements, to reflect the abandonment of axial symmetry, but for convenience we work with Ω' , defined by

$$\Omega' = \Omega - \nu - \tfrac{1}{2}\pi . \tag{A-5}$$

Then $U_{\ell m}$ can be decomposed into $\sum_k U^k_{\ell m}$, where

$$U^k_{\ell m} = \frac{\mu}{r} (R/r)^\ell J_{\ell m} F^k_{\ell m}(i) \cos \{ku' + m(\Omega' - \lambda_{\ell m})\} ; \tag{A-6}$$

this is compatible with equation (71) of Ref 9, in which Υ and X were the negatives of the present u' and Ω' . (Compatibility with two other papers can be obtained by noting that $F^k_{\ell m}$ here is $\imath^{\ell+k}$ times the unnormalized equivalent of the $\bar{F}_{\ell mp}$ used in Ref 13, whilst $F_{\ell mp}$ in Ref 17 is identical with $F_{\ell mp}$ introduced at (A-3) here.)

Next we introduce quantities $A_{\ell mk}$ that directly generalize the $A_{\ell m}$ of the main text, defining

$$A_{\ell mk} = -J_{\ell m} (R/p)^\ell F^k_{\ell m}(i) . \tag{A-7}$$

We also generalize C^k_j and S^k_j , by defining

$$\left.\begin{aligned} C^{\ell mk}_j &= \cos [jv + ku' + m(\Omega' - \lambda_{\ell m})] \\ S^{\ell mk}_j &= \sin [jv + ku' + m(\Omega' - \lambda_{\ell m})] \end{aligned}\right\} , \tag{A-8}$$

and henceforth we will omit the superfixes.

Then (A-6) to (A-8) give

$$U^k_{\ell m} = -\frac{\mu}{p} (p/r)^{\ell+1} A_{\ell mk} C_0 , \tag{A-9}$$

which is a straight generalization of equation (15) of the main text. When $m \le k$, we can also generalize the preceding equation (14) by writing

$$A_{\ell mk} = -J_{\ell m} (R/p)^\ell \alpha_{\ell mk} s^{k-m} (1 + c)^m A^k_{\ell m}(i) , \tag{A-10}$$

where $A^k_{\ell m}(i)$ is from the family of 'normalized' inclination functions introduced in Ref 17 to generalize the $A^k_\ell(i)$, and $\alpha_{\ell mk}$ generalizes $\alpha_{\ell m}$; the formula for $\alpha_{\ell mk}$ can be inferred from equation (11) ibid. It then follows from (A-7) and (A-10) that

$$F^k_{\ell m}(i) = \alpha_{\ell mk}\, s^{k-m}\, (1+c)^m\, A^k_{\ell m}(i) . \tag{A-11}$$

It was assumed, in (A-10), that $m \le k$. When $0 \le k \le m$, a different generalization became necessary in Ref 17, leading to

$$A_{\ell mk} = -J_{\ell m}\, (R/p)^\ell\, \alpha_{\ell mk}\, s^{m-k}\, (1+c)^k\, A^k_{\ell m}(i) , \tag{A-12}$$

where now the formula for $\alpha_{\ell mk}$ can be inferred from equation (13) ibid; also (A-7) and (A-12) lead to

$$F^k_{\ell m}(i) = \alpha_{\ell mk}\, s^{m-k}\, (1+c)^k\, A^k_{\ell m}(i) . \tag{A-13}$$

For $k = m$, (A-12) and (A-13) are consistent with (A-10) and (A-11) respectively, but otherwise the dual definitions of $A^k_{\ell m}(i)$ and $\alpha_{\ell mk}$ are distinct. A further complication is that an extension of (A-10) and (A-11) to negative k is not generally available; (A-12) and (A-13) still operate for $k < 0$, with $|k| \le m$, but there is a marked lack of symmetry between the forms of $A^k_{\ell m}(i)$ and $\alpha_{\ell mk}$ for $k < 0$ in relation to $k > 0$. The difficulty for negative k is not too serious, however, as $F^k_{\ell m}(i)$ can then be derived from

$$F^k_{\ell m}(i) = (-)^m\, F^{-k}_{\ell m}(\pi - i) . \tag{A-14}$$

By appeal to (A-14) as required, the $A_{\ell mk}$ can always be obtained. There are advantages in the adoption of a different 'normalization' for the $A^k_{\ell m}(i)$, however, such that they constitute a fully unified family of functions, defined for all k and symmetric in regard to the sign of k. The constants $\alpha_{\ell mk}$ must then also be redefined, to preserve the $A_{\ell mk}$ unchanged, and the connecting formulae is based on (A-12) rather than (A-10). We introduce a variable sign into (A-12), to make the $\alpha_{\ell mk}$ always positive; expressed symmetrically in regard to the sign of k, the connecting formula is then

$$A_{\ell mk} = (-)^{\frac{1}{2}(\ell+k-2)}\, J_{\ell m}\, (R/p)^\ell\, \alpha_{\ell mk}\, s^m \left(\frac{1+c}{1-c}\right)^{\frac{1}{2}k} A^k_{\ell m}(i) . \tag{A-15}$$

The new $A^k_{\ell m}(i)$ are defined for $\ell \geq m$, as before, but for all k now, not just for $|k| \leq \ell$ (quite apart from the problem with $k < 0$). The recurrence relation for fixed m and k is

$$(\ell - 1)(\ell^2 - m^2)A^k_{\ell m} - (2\ell - 1)\{\ell(\ell - 1)c - mk\}\, A^k_{\ell-1,m} + \ell\{(\ell - 1)^2 - k^2\}\, A^k_{\ell-2,m} = 0 , \qquad \text{(A-16)}$$

which is slightly simpler than the relation in Ref 17; the starting values for this are

$$A^k_{mm}(i) = 1 \quad \text{and} \quad A^k_{m+1,m}(i) = (m + 1)c - k , \qquad \text{(A-17)}$$

though the second of (A-17) can be dispensed with if we define $A^k_{m-1,m}(i)$ to be zero. A recurrence relation for fixed ℓ and m is also available, viz

$$(\ell - k)(1 + c)A^{k+1}_{\ell m} - 2(m - kc)A^k_{\ell m} + (\ell + k)(1 - c)A^{k-1}_{\ell m} = 0 , \qquad \text{(A-18)}$$

in which the symmetry (in regard to the sign of k) is obvious; expressions for $A^k_{\ell m}(i)$, with $|k| < \ell$, can be generated 'from either end' by use of just one starting value from the pair

$$A^{\pm\ell}_{\ell m}(i) = [\tfrac{1}{2}(c \mp 1)]^{\ell-m} \binom{2\ell}{\ell + m} . \qquad \text{(A-19)}$$

There is also a recurrence relation for fixed ℓ and k, but instead of giving it we note that the set of 15 relations, each involving $A^k_{\ell m}(i)$ and two 'adjacent' functions from a three-dimensional table, can all be generated from various subsets of just three relations; one of the simplest such subsets consists of (A-18) and the following pair of relations:

$$(\ell + k)A^k_{\ell-1,m} - (m + \ell c)A^k_{\ell m} + \ell(1 + c)A^{k+1}_{\ell m} = 0 \qquad \text{(A-20)}$$

and

$$2(\ell - m + 1)A^k_{\ell,m-1} + (m + k)(1 - c)A^k_{\ell m} - (\ell - k)(1 + c)A^{k+1}_{\ell m} = 0 . \qquad \text{(A-21)}$$

(Note in proof: see Refs 19 and 20 for computational aspects of these relations.)

We will not enlarge on the advantages of the redefinition of $A^k_{\ell m}(I)$ and $\alpha_{\ell mk}$, but two disadvantages must be mentioned. First, since the factors in s and c in (A-15) can be combined as $(1 + c)^{\frac{1}{2}(m+k)}(1 - c)^{\frac{1}{2}(m-k)}$, we see that a negative power of either $1 + c$ or $1 - c$ appears whenever $m < |k|$, and this has to be cancelled by a corresponding positive power that is present in the new $A^k_{\ell m}(I)$. Second, the use of (A-16) and (A-17) to compute $A^k_{\ell m}(I)$ under these circumstances $(m < |k|)$ is inefficient, since the recurrence process has to work through the unwanted functions with $m < \ell < |k|$.

The $A^k_{\ell m}(I)$ are, like the $A^k_\ell(I)$ of the main text, defined regardless of parity. The constants $\alpha_{\ell mk}$ (and hence the quantities $A_{\ell mk}$) are only defined for ℓ and k of the same parity, however, and (as redefined) their only property to be stated here is that of complete symmetry, so that

$$\alpha^k_{\ell m} = \alpha^{-k}_{\ell m} \quad (> 0) . \tag{A-22}$$

But, just as in the main text, we require another set of constants, $\alpha_{\ell m\kappa}$, and quantities, $A_{\ell m\kappa}$, defined when ℓ and k are of opposite parity, to allow the formula for δb to be expressed. The connecting formula corresponding to (A-15) is

$$A_{\ell m\kappa} = (-)^{\frac{1}{2}(\ell+\kappa-1)} J_{\ell m} (R/p)^\ell \alpha_{\ell mk} s^m \left(\frac{1 + c}{1 - c}\right)^{\frac{1}{2}\kappa} A^\kappa_{\ell m}(I) . \tag{A-23}$$

The $\alpha_{\ell m\kappa}$ are available at once from the $\alpha_{\ell m\kappa}$, since (cf (26) of the main text, which, because of the redefinition, is not being directly generalized)

$$\alpha_{\ell m\kappa} = \alpha_{\ell-1,m,\kappa} . \tag{A-24}$$

Tables and further properties of the redefined inclination functions, and the associated constants, will be given in a separate paper.

By making use of the quantities $A_{\ell mk}$ and $A_{\ell m\kappa}$, we find no difficulty on extending the theory, largely because the treatment of $(p/r)^{\ell+1}$, in (A-9), via the $B_{\ell j}$, goes through unchanged from the main text. Further, the energy-based exact quantity, a', is still available, following the assumption that the attracting body does not rotate. Thus, equations (65), (76), (88) and (105), for δa, de/dv, $d\omega/dv$ and dM/dv, respectively, are unchanged apart from the appearance of $A_{\ell mk}$ in place of $A_{\ell m}$. Equation (82), for $d\Omega/dv$, requires a correspondıng change, such that the derivative $A'_{\ell mk}$ replaces $A'_{\ell m}$. This just leaves (79), for di/dv, for which a slightly more complicated expression is now required, to reflect the fact that pc^2 is no longer an invariant. We have, in fact,

$$\frac{d(pc^2)}{dv} = 2mpc\, A_{\ell mk} \sum B_{\ell j} S_j \;. \tag{A-25}$$

From this, using the version of dp/dv corresponding to equation (74), we get

$$\frac{di}{dv} = s^{-1}(kc - m)\, A_{\ell mk} \sum B_{\ell j} S_j \;; \tag{A-26}$$

in comparison with equation (79), we see that the only additional change is the replacement of kc by $kc - m$.

Six of the seven formulae that define δr, δb and δw completely, for the zonal harmonics, are immediately applicable to the zonal harmonics, so long as $A_{\ell mk}$ replaces $A_{\ell m}$ and the trigonometric argument includes the term $m(\Omega' - \lambda_{\ell m})$. These six are (131) and (156), for the general δr and δw, and (182), (184), (192) and (194), the four special formulae for δw. In the seventh formula, (140) for δb, $\ell A_{\ell K}$ must be replaced by $(\ell + m)A_{\ell mK}$, in addition to the inclusion of the extra term in the trigonometric argument. (It is, perhaps, suprising that the change to (140) is as slight as this, but it would have been even less if $A_{\ell K}$ and $A_{\ell mK}$ had been defined to include the factors ℓ and $\ell + m$ respectively; the reason for excluding these factors was, essentially, to give a degree of homogeneity to (26) and (A-24.)

Finally, of course, the numbers of terms in δr, δb and δw, for a given $J_{\ell m}$, are greater for $m > 0$ than for $m = 0$, to reflect the distinction between positive and negative k. These numbers are otherwise independent of m, however, in consequence of which we write the formulae as follows:

$$N'_{\ell r} = 2\ell^2 - 3\ell + 1 \;, \tag{A-27}$$

$$N'_{\ell b} = 2\ell^2 - 3\ell + 2 \tag{A-28}$$

and

$$N'_{\ell w} = \begin{Bmatrix} 2\ell^2 & \text{for odd } \ell \\ 2(\ell^2 - 1) & \text{for even } \ell \end{Bmatrix}. \tag{A-29}$$

Appendix B

THE QUANTITIES $B_{\ell j}$, AND $B_{0,1}$ IN PARTICULAR

When $j \geq 0$, $B_{\ell j}$ may be expressed in terms of the hypergeometric function as:

$$B_{\ell j} = \binom{\ell - 1}{j} (e/2)^j \, F\left(\frac{j - \ell + 1}{2} , \frac{j - \ell + 2}{2} ; j + 1 ; e^2\right) , \qquad \text{(B-1)}$$

where (B-1) applies for all ℓ . This result is proved, in terms of the function $B_\ell^j(e)$, in Appendix E of Ref 4; it is also quoted, in terms of the equivalent Hansen function, at equation (32) of Ref 8. For $(0 \leq) \; j < \ell$, (B-1) gives a polynomial in e^2 ; for $j \geq \ell > 0$ it gives zero; and for $\ell \leq 0$ it gives a power series in e^2 , which can be transformed into a closed expression involving $q \; (= \sqrt{(1 - e^2)})$ and perhaps $\beta \; (= e/(1 + q))$. All this is consistent with Table 4.

Equation (B-1) breaks down when $j < 0$, owing to the eventual occurrence of a zero denominator when the hypergeometric function is expanded. Since

$$B_{\ell j} = B_{\ell,-j} , \qquad \text{(B-2)}$$

however, this break-down is of no account. The justification of (B-2) comes from the Hansen-function equivalence and the relation

$$X_m^{\ell j} = X_{-m}^{\ell,-j} , \qquad \text{(B-3)}$$

which follows immediately from the definition of Hansen's functions by equation (35) of the main text.

The quantity $B_{0,1}$ is of particular interest, being the simplest of the $B_{\ell j}$ that involve β . Once it is known, the other such $B_{\ell j}$ can be progressively derived using the recurrence relations of the main text.

From (B-1) we have

$$B_{0,1} = -\tfrac{1}{2}eF(1, 1\tfrac{1}{2}; \; 2; \; e^2) . \qquad \text{(B-4)}$$

But as a particular case of the hypergeometric relation proved as equation (9.2) of Ref 18, we have

$$F(1,\ 1\tfrac{1}{2};\ 2;\ e^2) = 2q^{-1}F(\tfrac{1}{2},\ 1;\ \tfrac{1}{2};\ q^2) - 2F(1,\ 1\tfrac{1}{2};\ 1\tfrac{1}{2};\ q^2)\ . \tag{B-5}$$

Also, it is immediate (from the expansion) that

$$F(a,\ b;\ a;\ q^2) = (1 - q^2)^{-b}\ , \tag{B-6}$$

independently of a, so that (B-5) gives

$$F(1,\ 1\tfrac{1}{2};\ 2;\ e^2) = 2/q(1 + q)\ . \tag{B-7}$$

Finally, (B-4) and (B-7) give

$$B_{0,1} = -\beta q^{-1}\ , \tag{B-8}$$

in conformity with the entry in Table 4.

Table 1

THE FUNCTIONS $A_\ell^k(1)$

k / ℓ	0	1	2	3	4	5	6
0	1						
1	c	1					
2	$1 - \frac{3}{2}f$	c	1				
3	$c(1 - \frac{5}{2}f)$	$1 - \frac{5}{4}f$	c	1			
4	$1 - 5f + \frac{35}{8}f^2$	$c(1 - \frac{7}{4}f)$	$1 - \frac{7}{6}f$	c	1		
5	$c(1 - 7f + \frac{21}{8}f^2)$	$1 - \frac{7}{2}f + \frac{21}{8}f^2$	$c(1 - \frac{3}{2}f)$	$1 - \frac{9}{8}f$	c	1	
6	$1 - \frac{21}{2}f + \frac{189}{8}f^2 - \frac{231}{16}f^3$	$c(1 - \frac{9}{2}f + \frac{33}{8}f^2)$	$1 - 3f + \frac{33}{16}f^2$	$c(1 - \frac{11}{8}f)$	$1 - \frac{11}{10}f$	c	1

Table 2

THE CONSTANTS $\alpha_{\ell k}$ AND $\alpha_{\ell \kappa}$

k/κ	0	1	2	3	4	5	6
ℓ							
-7	$-\frac{5}{16}$	$-\frac{35}{16}$	$\frac{105}{32}$	$\frac{105}{32}$	$-\frac{315}{128}$	$-\frac{231}{256}$	$\frac{231}{512}$
-6	$-\frac{3}{8}$	$\frac{15}{8}$	$\frac{21}{8}$	$-\frac{35}{16}$	$-\frac{63}{64}$	$\frac{63}{128}$	
-5	$\frac{3}{8}$	$\frac{15}{8}$	$-\frac{15}{8}$	$-\frac{35}{32}$	$\frac{35}{64}$		
-4	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{4}$	$\frac{5}{8}$			
-3	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{4}$				
-2	-1	1					
-1	1						
0	1						
1	-1	1					
2	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{3}{4}$				
3	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{5}{4}$	$\frac{5}{8}$			
4	$\frac{3}{8}$	$\frac{15}{8}$	$-\frac{15}{8}$	$-\frac{35}{32}$	$\frac{35}{64}$		
5	$-\frac{3}{8}$	$\frac{15}{8}$	$\frac{21}{8}$	$-\frac{35}{16}$	$-\frac{63}{64}$	$\frac{63}{128}$	
6	$-\frac{5}{16}$	$-\frac{35}{16}$	$\frac{105}{32}$	$\frac{105}{32}$	$-\frac{315}{128}$	$-\frac{231}{256}$	$\frac{231}{512}$

Table 3

THE FUNCTIONS $B_{\ell}^{j}(e)$

j / ℓ	0	1	2	3	4	5	6
1	1						
2	1	1					
3	$1 + \frac{1}{2}e^2$	1	1				
4	$1 + \frac{3}{2}e^2$	$1 + \frac{1}{4}e^2$	1	1			
5	$1 + 3e^2 + \frac{3}{8}e^4$	$1 + \frac{3}{4}e^2$	$1 + \frac{1}{6}e^2$	1	1		
6	$1 + 5e^2 + \frac{15}{8}e^4$	$1 + \frac{3}{2}e^2 + \frac{1}{8}e^4$	$1 + \frac{1}{2}e^2$	$1 + \frac{1}{8}e^2$	1	1	
7	$1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6$	$1 + \frac{3}{2}e^2 + \frac{3}{8}e^4$	$1 + e^2 + \frac{1}{16}e^4$	$1 + \frac{3}{8}e^2$	$1 + \frac{1}{10}e^2$	1	1

Table 4

THE QUANTITIES $B_{\ell j}$

j / ℓ	-1	0	1	2	3
-3	$-\frac{1}{2}eq^{-7}(4+e^2)$	$\frac{1}{2}q^{-7}(2+3e^2)$	$-\frac{1}{2}eq^{-7}(4+e^2)$	$\frac{3}{2}e^2q^{-7}$	$-\frac{5}{2}e^3q^{-7}$
-2	$-\frac{3}{2}eq^{-5}$	$\frac{1}{2}q^{-5}(2+e^2)$	$-\frac{3}{2}eq^{-5}$	$\frac{3}{2}e^2q^{-5}$	$-\frac{1}{2}\beta^3q^{-5}(3+9q+8q^2)$
-1	$-eq^{-3}$	q^{-3}	$-eq^{-3}$	$\beta^2q^{-3}(1+2q)$	$-\beta^3q^{-3}(1+3q)$
0	$-\beta q^{-1}$	q^{-1}	$-\beta q^{-1}$	β^2q^{-1}	$-\beta^3q^{-1}$
1	0	1	0	0	0
2	$\frac{1}{2}e$	1	$\frac{1}{2}e$	0	0
3	e	$\frac{1}{2}(2+e^2)$	e	$\frac{1}{4}e^2$	0
4	$\frac{3}{8}e(4+e^2)$	$\frac{1}{2}(2+3e^2)$	$\frac{3}{8}e(4+e^2)$	$\frac{3}{4}e^2$	$\frac{1}{8}e^3$

Table 5

THE QUANTITIES $E_{\ell k}$

k / ℓ	0	1	2	3
-3	$3eq^{-5}$	$-\frac{1}{2}q^{-5}(4 + 3e^2)$	$5eq^{-5}$	$-\frac{13}{2}e^2q^{-5}$
-2	eq^{-3}	$-\frac{1}{2}q^{-3}$	$3eq^{-3}$	
-1	0	$-q^{-1}$		
0	0			
1	e			
2	$3e$	$\frac{1}{2}(1 + 2e^2)$		
3	$\frac{3}{2}e(4 + e^2)$	$1 + 4e^2$	$\frac{1}{4}e(2 + 3e^2)$	
4	$\frac{3}{2}e(4 + 3e^2)$	$\frac{3}{8}(4 + 27e^2 + 4e^4)$	$\frac{1}{6}e(2 + 5e^2)$	$\frac{1}{8}e^2(3 + 4e^2)$

Table 6

THE NUMBER OF TERMS IN δr, δb AND δw

ℓ	$N_{\ell,r}$	$N_{\ell,b}$	$N_{\ell,w}$
1	0	1	1
2	2	2	3
3	5	6	9
4	11	11	15
5	18	19	25
6	28	28	35
7	39	40	49
8	53	53	63
9	68	69	81
10	86	86	99
11	105	106	121
12	127	127	143
13	150	151	169
14	176	176	195
15	203	204	225
16	233	233	255

LIST OF SYMBOLS

(Usage for the main text only)

a	semi-major axis
a'	energy-based fixed value of a
$A_\ell^k(i)$	'normalized' function of inclination
$A_{\ell k}$	quantity, based on $A_\ell^k(i)$, defined by (14)
$A_{\ell k}$	similar to $A_{\ell k}$, but defined by (17)
$A_{\ell k}$	derivative of $A_{\ell k}$ with respect to i
$A_{\ell k}^{\pm}$	$ks^{-1} A_{\ell k} \pm c^{-1} A'_{\ell k}$
b	latitude-like spherical coordinate of (r, b, w)
$B_\ell^j(e)$	normalized function of eccentricity
$B_{\ell j}$	quantity related to $B_\ell^j(e)$, defined by (32)
$B'_{\ell j}$	derivative of $B_{\ell j}$ with respect to e
c	cos i
C_j^k (or C_j)	cos $(jv + ku')$ (different meaning in Part 1)
d	shorthand for $k + j - 1$ etc in sections 7.4 and 7.5
$D_\ell^k(i)$	inclination function, quoted from Ref 4
e	eccentricity
E	eccentric anomaly (only required in section 8)

LIST OF SYMBOLS (continued)

$E_\ell^j(e)$	eccentricity function, quoted from Ref 4
$E_{\ell j}$	quantity related to $E_\ell^j(e)$, defined by (48)
f	$\sin^2 i$
G	$\cancel{\frac{1}{1024}} J_4 \, (R/p)^4$ (in section 8.5)
h	angular momentum (but $1 - \frac{5}{4} f$ in section 8.3)
H	$\frac{1}{2} J_3 \, (R/p)^3$ (in section 8.4)
i	inclination
j	index associated with multiples of v
J_ℓ	zonal harmonic coefficient for the Earth
k	index associated with multiples of u
K	$\frac{3}{4} J_2 \, (R/p)^2$ (in section 8.3)
ℓ	index of J_ℓ
L	quantity such that $\dot{L} = \dot{M} + q\dot{\psi} = n + \dot{\rho}$
m	$v - M$ in section 8.5 (and Part 1); otherwise an arbitrary integer
M	mean anomaly
n	mean motion

We will require derivatives of the inclination functions. It is evident from (10) that

$$\frac{d}{di}\{A_\ell^k(i)\} = -\frac{(\ell - k)(\ell + k + 1)}{2(k + 1)}\, s\, A_\ell^{k+1}(i)\ , \qquad (18)$$

from this and (14) it follows that the (partial) derivative of $A_{\ell k}$ with respect to i is given by

$$A'_{\ell k} = J_\ell\, (R/p)^\ell\, \alpha_{\ell k}\, s^{k-1} \left\{ kc\, A_\ell^k(i) - \frac{(\ell - k)(\ell + k + 1)}{2(k + 1)}\, f\, A_\ell^{k+1}(i) \right\}, \qquad (19)$$

where $f = s^2$. The quantity in (curly) brackets is the $D_\ell^k(i)$ of Ref 4. We will also require, finally, the particular combinations of $A_{\ell k}$ and $A'_{\ell k}$ denoted by $A^+_{\ell k}$ and $A^-_{\ell k}$, and given by

$$A^\pm_{\ell k} = ks^{-1}\, A_{\ell k} \pm c^{-1}\, A'_{\ell k}\ , \qquad (20)$$

the s^{-1} and c^{-1} factors do not imply singularities, as they must always cancel via $k\,A_{\ell k}$ and $A'_{\ell k}$ respectively.

The $A_\ell^k(i)$ and $\alpha_{\ell k}$ (and hence the $A_{\ell k}$) may be computed with the aid of recurrence relations. A fixed k was stipulated in Ref 4 for the formula

$$(\ell + k)\, A_\ell^k(i) = (2\ell - 1)\, c\, A_{\ell-1}^k(i) - (\ell - k - 1)\, A_{\ell-2}^k(i)\ , \qquad (21)$$

valid for $\ell \geq k + 2$ with the starting values $A_k^k(i) = 1$ and $A_{k+1}^k(i) = c$; (21) is even valid for $\ell = k + 1$, if an arbitrary (but finite) $A_{k-1}^k(i)$ is assumed. However, it is usually more useful to stipulate a fixed ℓ, the required formula was given by Merson[11], being

$$A_\ell^k(i) = c\, A_\ell^{k+1}(i) - \frac{(\ell - k - 1)(\ell + k + 2)}{4(k + 1)(k + 2)}\, f\, A_\ell^{k+2}(i)\ , \qquad (22)$$

valid for $\ell - 2 \geq k \geq 0$ with the starters $A_\ell^\ell(i) = 1$ and $A_\ell^{\ell-1}(i) = c$, (22) is also valid for $k = \ell - 1$, with an arbitrary (finite) $A_\ell^{\ell+1}(i)$. Either of the two preceding 'pure' three-term recurrence relations, (21) or (22), can be used with just one 'mixed' such relation to generate all the relations connecting the $A_\ell^k(i)$, perhaps the simplest mixed relation (with neither ℓ nor k fixed)[13]

LIST OF SYMBOLS (continued)

$N_{\ell,r}$, $N_{\ell,b}$, $N_{\ell,w}$	number of terms (for given ℓ) in δr, δb, δw
p	parameter (semi-latus rectum) of orbit
$P_\ell(\ )$	Legendre polynomial (of argument supplied)
$P_\ell^k(\ )$	Legendre associated function
q	$\sqrt{(1 - e^2)}$
$Q_\ell(e)$	normalized eccentricity function quoted from Ref 3
r	radius-vector coordinate of (r, b, w)
R	Earth's equatorial radius
R_j (for $R_{\ell kj}$)	quantity defined by (125) (different in Part 1)
s	$\sin i$
S_j^k (or S_j)	$\sin (jv + ku')$ (different meaning in Part 1)
t	time
T_j (for $T_{\ell kj}$)	quantity defined by (137)
u	argument of latitude, $v + \omega$
u'	modified u $(= v + \omega' = u - \frac{1}{2}\pi)$
U_ℓ	potential due to J_ℓ
U_ℓ^k	component of U_ℓ

LIST OF SYMBOLS (continued)

v	true anomaly
V	orbital speed (used only in section 8)
$V_{j,1}$, $V_{j,0}$, $V_{j,-1}$	quantities introduced at (149)
w	longitude-like spherical coordinate of (r, b, w)
$W_{\ell,0}$, $W_{\ell,-1}$	quantities introduced at (153)
x, y	general unknown quantities (different in Part 1)
$X_m^{\ell j}$	generic Hansen function (of eccentricity)
z	$-J_1 R$ (section 8.2)
$\alpha_{\ell k}$	fixed constant, defined by (12)
$\alpha_{\ell \kappa}$	fixed constant, defined by (16)
β	geocentric latitude (declination); $e/(1 + q)$ in section 3
δ	symbol for pure short-period perturbation (δ_p in Part 1)
ζ	generic orbital element (osculating)
$\overline{\zeta}$	mean element corresponding to ζ
$\tilde{\zeta}$	semi-mean element
$\dot{\zeta}$	rate of change of $\overline{\zeta}_{\ell k}$ due to $U_{\ell k}$
ι	$(-1)^{\frac{1}{2}}$

LIST OF SYMBOLS (concluded)

κ	index related to k, but of opposite parity
λ	$-J_1 R/p$ (section 8.2)
μ	Earth's gravitational constant
ρ	quantity such that $\dot{\rho} = \dot{\sigma} + q\dot{\psi}$
σ	modified mean anomaly at epoch
$\sum$	summation (different use of Σ in Part 1)
u_k	1 if $k = 0$, 2 if $k > 0$
ψ	quantity such that $\dot{\phi} = \dot{\omega} + c\dot{\Omega}$
ω	argument of perigee
ω'	$\omega - \frac{1}{2}\pi$
Ω	right ascension of the node
$\int$	$\int n \, dt$

REFERENCES

No.	Author	Title, etc
1	R.H. Gooding	Satellite motion in an axi-symmetric gravitational field Part 1: Perturbations due to J_2 (second order) and J_3. RAE Technical Report 88068 (1988)
2	R.H. Gooding	Complete second-order satellite perturbations due to J_2 and J_3, compactly expressed in spherical-polar coordinates. <u>Acta Astronautica</u>, <u>10</u>, 309-317 (1983) RAE Technical Report 83033 (1983)
3	R.H. Gooding	On mean elements for satellite orbits perturbed by the zonal harmonics of the geopolential. <u>Adv. Space Res.</u>, <u>10</u>, 279-283 (1990) RAE Technical Memo. Space 371 (1989)
4	R.H. Gooding	Satellite motion in an axi-symmetric field, with an application to luni-solar perturbations. RAE Technical Report 66018 (1966)
5	R.H. Gooding	On the generation of satellite position (and velocity) by a mixed analytical-numerical procedure. <u>Adv. Space Res.</u>, <u>1</u>, 83-93 (1981) RAE Technical Memo. Space 311 (1982)
6	G.V. Groves	Motion of a satellite in the earth's gravitational field. <u>Proc. Roy. Soc.</u>, A <u>254</u>, 48-65 (1960)
7	W.M. Kaula	<u>Theory of satellite geodesy</u> Waltham, Mass. etc., Blaisdell Publ. Co. (1966)
8	B. Zafiropoulos	The motion of a satellite in an axi-symmetric gravitational field. <u>Astrophys. Space Sci.</u>, <u>139</u>, 111-128 (1987)
9	R.H. Gooding	A second-order satellite theory, with compact results in cylindrical coordinates. <u>Phil. Trans. Roy. Soc.</u>, A <u>299</u>, 425-474 (1981)

REFERENCES (continued)

No.	Author	Title, etc
10	R.H. Gooding	Satellite location by GPS (Navstar) RAE Technical Report 78154 (1978)
11	R.H. Merson	The dynamic model of PROP, a computer program for the refinement of the orbital parameters of an earth satellite. RAE Technical Report 66255 (1966)
12	H.C. Plummer	An introductory treatise on dynamical astronomy New York, Dover Publications (1960) [from CUP (1918)]
13	R.H. Gooding D.G. King-Hele	Explicit forms of some functions arising in the analysis of resonant satellite orbits. Proc. Roy. Soc., A, 422, 241-259 (1989) RAE Technical Report 88035 (1988)
14	R.H. Gooding	Universal procedures for conversion of orbital elements to and from position and velocity (unperturbed orbits). RAE Technical Report 87043 (1987) [Shortened version published as Celestial Mech., 44, 283-298 (1988)]
15	Y. Kozai	The motion of a close earth satellite. Astron. J., 64, 367-377 (1959)
16	D.G. King-Hele Doreen M.C. Gilmore	The effect of the Earth's oblateness on the orbit of a near satellite. Proc. R. Soc., A, 247, 49-72 (1958) RAE Technical Note GW 475 (1957)
17	R.H. Gooding	A recurrence relation for inclination functions. Celestial Mech., 4, 91-98 (1971) RAE Technical Report 70074 (1970)
18	I.N. Sneddon	Special functions of mathematical physics and chemistry. Second edition (reprinted), Edinburgh and London, Oliver and Boyd (1966)

REFERENCES (concluded)

No.	Author	Title, etc
19	J.B. Lundberg B.E. Schutz	Recursion formulas of Legendre functions for use with nonsingular geopotential models. J. Guidance, 11, 31-38 (1988)
20	W. Gautschi	Computational aspects of three-term recurrence relations. SIAM review, 9, 24-82 (1967)

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17. Abstract

This Report continues the presentation of the untruncated orbital theory begun in Technical Report 88068. The effects of the general zonal harmonic, J_ℓ, are now covered, the main results being a trio of formulae for perturbations in the spherical-polar coordinates introduced in the previous paper. The formulae are only first-order in J_ℓ, but, in conjunction with the second-order results for J_2 published in Part 1, the complete set of formulae may be regarded as constituting a second-order theory, the Earth's J_2 being much larger than J_ℓ for $\ell > 2$.

The mean elements of the theory are defined in such a way that, for each J_ℓ, the coordinate-perturbation formulae have their simplest possible form, with no occurrence of zero denominators. The general formulae are used in a rederivation of the results for J_2, given in Part 1, and in a derivation of results for J_4.

Numerical comparisons with reference orbits are held over to a later report (Part 3).

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