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Toward a General Theory of C3 Processes

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This report continues the past effort by the author in establishing an approach to a general theory of C^3 processes and an ensuing decision game. Previous work is corrected, modified, and extended here. An outline of a general C^3 decision game is first presented, utilizing as an integral part, a formal theory describing the evolution of a typical C^3 node state vector. The determination of the C^3 decision game also requires an algebraic logic description pair assigned to the formal theory, such as probability logic or fuzzy logic, depending on the type of information considered. In the probability logic case, a general result concerning uniform approximation by linear-Gaussian mixtures of distributions is presented. This is shown to have potentially good applications for carrying out reduced calculations of evolving node state distributions, as functionals of pertinent distributions of various C^3 variables and relations.

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SUMMARY

OBJECTIVE

This report is part of an ongoing effort to establish a rigorous, comprehensive tactical theory of command, control, and communications (C^3) processes.

RESULTS

A formal theory of C^3 processes is presented in conjunction with the evaluation through algebraic logic considerations. These take into account linguistic and other nonstochastic information as well as ordinary probabilistic ones. The theory involves a number of C^3 variables and their relations such as node descriptions, equations of motions, input-output signals, weapon firings, etc. As a result of the formal theory, recursive forms are developed for the evolution of typical node states as explicit functions of C^3variables and their relations. In turn, a game theoretic structure of C^3 processes is also established where moves of the game represent the allowable designs of C^3relations.

In the probability logic case, a general result concerning uniform approximation by linear-Gaussian mixtures of distribution is presented for improving implementations.

CONCLUSION

A coherent theory of tactical C^3 processes appears possible through this work. Future efforts will address problems of implementation.

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1. INTRODUCTION

For the past 10 years, command, control, and communications (C^3), as an organized discipline, has evolved greatly from the fledging concepts of J. Lawson, M. Athans, and others as presented in the Proceedings of the First MIT/ESL-ONR Workshop on Distributed Communication and Decision Problems, (Athans, 1978) to the more sophisticated views presented in the succeeding Proceedings of the Command and Control Research Symposium sponsored by the Joint Directorate of Laboratories (1988).

In addition to the seminal work presented in the proceedings during the 8 years of the MIT/ONR workshop on C^3 systems and in the proceedings of the past 2 years of the Command and Control Research Symposium, other sources for unclassified, unlimited analysis of general C^3 systems include:

1. JDL, TPC^3, BRG, C^3 Handbook (Draft), San Diego, CA: NOSC and NPRDC, 1988.
2. Institute of Electrical Engineers (IEE), Computing, Control, and Electronics Division, Proceedings of the First and Second International Conferences on Command, Control, Communications and Management Information Systems, Great Britain, 1987, 1988.
3. Institute of Electrical and Electronics Engineers (IEEE), Special Issue on Information Technology for Command and Control, IEEE Trans.Sys.Man, Cybern., Vol SMC-16(6), Nov - Dec 1986.
4. Armed Forces Communications and Electronics Association (AFCEA), Science of Command and Control: Coping with Uncertainty, Fairfax, VA.: AFCEA International Press, 1988.
5. R. Sweet, M. Metersky, and M. Sovereign, eds., Command and Control Evaluation Workshop (Proceedings), Based on Military Operations Research Society (MORS) MOE Workshop, Jan 1985, Naval Postgraduate School, MORS and MITRE Corp., Nov 1985.
6. C. Strack, Elements of C^2 Theory, McLean, VA.: Defense Systems Inc. (DSI), draft version, 15 Oct 1985, abridged version, 30 Jan 1985.
7. Office of the Secretary of Defense and MITRE Corp., C^3 Division, Proceedings for Quantitative Assessment of Utility of Command and Control Systems, proceedings of conference held at National Defense University (NDU), Ft. McNair, Washington, DC, Oct 1979, McLean, VA.: MITRE Corp., Jan 1980.

For an overview of the more important papers from the MIT/ONR workshop on C^3 up to 1985, see Goodman (1985). More recently, Van Trees (1988) has presented an excellent critical analysis of the state-of-the-art in C^3 work. Based on these studies of C^3 systems, we can conclude that past and current work in this area can be divided roughly into the following taxonomy:

1. C^3 systems on long-range strategies, global and political ramifications, and high-level planning.
2. Tactical or midlevel C^3 systems, emphasizing the actual dynamics of C^3 events during typical engagements. Note: throughout this report, little or no distinction

is made between the use of the terms "systems" and "processes," nor between "measures of effectiveness" (MOEs) and "measures of performance" (MOPs). However, for a different viewpoint, see Metersky (1985).

a. Qualitative studies, emphasizing the use of graphs, flowcharts, and verbal descriptions.

b. Quantitative studies, emphasizing numerical measures or algebraic relations, usually centered about the stochastic state space approach.

(1) Aspect-oriented quantitative work, where one, or possibly more, particular facets of tactical C^3 processes are considered. Typical examples of this are surveillance and tracking/correlation, tactics for a particular class of scenarios, deployment for air-to-ground conflicts, determination of MOEs for C^3 systems, resource allocation of troops or weapons studies, communications analysis, C^3 decision aids, aspects of distributive decision-making, and game theory applied to restricted situations.

(2) General-structured C^3 analysis, where an attempt is made toward developing a general view of C^3 processes as a whole entity.

3. Technological hardware/software considerations, database management, computer design, field exercise studies, teaching and training of C^3 concepts.

Suffice it to say, the vast majority of endeavor in C^3 analysis has been directed to subparagraphs 2a, 2b(1), and 3, with little attention paid to 1 (understandable relative to the open literature format) and 2b(2). Certainly, qualitative studies are still useful due not only to the inherently great complexity of C^3 systems, but also since heuristic analysis must always precede any quantitative analysis. Furthermore, quantitative aspect-oriented work is also of prime importance, since the various parts of the whole C^3 picture are in themselves challenging and complex problems. Understanding the bits and pieces of C^3 can only contribute to the entire view. Nevertheless, we contend that more effort must be made to discern the general pattern of C^3 dynamics if C^3 work is to gain the stature of other scientific disciplines.

For some attempts at considering C^3 tactical processes as a whole entity, see e.g., the independent works of Ingber (1986) and Rubin and Mayk (1985), as well as followup papers by these authors in the proceedings of the MIT/ONR workshop on C^3 systems and later proceedings. Ingber's approach to the problem is a mesoscopic-macroscopic one, considering C^3 systems analogous to systems of interacting molecules or neurons, suitably modified and analyzable from a statistical mechanics viewpoint. On the other hand, Rubin and Mayk's approach is more microscopic in nature, also utilizing a purely stochastic technique, originally centering about numerical supply and attrition levels, generalizing the well-known Lanchester equations for mutual growth and decay of interacting populations. Although the thrust of their work has greatly expanded, modeling of the internal behavior of C^3 node complexes of decision-makers is not explicitly taken into account, as Levis (1983) and Tomovic and Levis (1984) have done. In the latter, the behavior of individual decision-makers and their constraints and interactions with others are modeled from an organizational viewpoint, using in one key part Conant's decomposition of entropy, as well as other criteria. (See also the subsequent papers by Levis and his students at the Laboratory for Information and Decision Systems (LIDS), MIT in these proceedings and those of the MIT/ONR workshop on C^3 systems.)

With modest changes, we can attempt to model the entire tactical C^3 process, using individual node structures and states as the basic building blocks, without necessarily making stochastic assumptions. This author has proposed such an approach, (Goodman, 1986a) with a followup paper (Goodman, 1987) elaborating upon the use of nonprobabilistic models in C^3 analysis. Much of this was based on previous work concerning the modeling and combining of uncertainties arriving from possibly disparate sources, stochastic or linguistic (Goodman and Nguyen, 1985). [See also Goodman, (1986b) for a practical implementation of such an approach for tracking and correlation of data.]

2. SCOPE OF THE PRESENT WORK

The current task is an improvement and enlargement of the previous efforts in establishing an overall model for C^3 dynamics (Goodman, 1986a, 1987). The basic goal of this report is twofold:

1. To show that tactical C^3 processes can be modeled within a game theory setting, using a formal system of axioms, capturing a minimally required number of relations among C^3 variables and operators.
2. To provide an outline for a feasible implementation of this program as an aid in designing and predicting global behavior of C^3 systems.

In modeling the C^3 processes, we must tradeoff the fidelity of a theory with the complexity of resulting computations required for implementation. The thesis here is that we should first attempt to model C^3 processes as a whole, despite its complexities, then seek *reasonable reductions of computations.*

In section 3, the basic C^3 model is developed as a decision game. This requires five steps:

1. Select a set of axioms involving relevant C^3 variables and operators so that a formal description of dynamically evolving node states can be obtained.
2. Select an algebraic logic description pair to numerically evaluate the formal theory developed in step 1.
3. Specify an averaging procedure that can apply to all C^3 node state evaluations in step 2.
4. Use outputs of step 3 to determine the overall "state of health" (i.e., tendency for winning) of each C^3 system.
5. Use figures of merit (FOM) in step 4 to define the overall loss function, and hence, C^3 decision game.

As in previous work, C^3 processes here are viewed as interacting networks of node complexes of decision-makers. Relevant C^3 variables are first identified, including nodal ones such as equations of motion, attrition level, detection state, algorithm selection, and hypotheses evaluations. Other C^3 variables treated are reception of signals or incoming exploding weapons, and the responses following data processing. In addition, logical operators, such as conjunction, disjunction, negation, and conditioning operators among C^3 variables are also taken into account.

To implement step 1, a collection of axioms is presented in section 4, formally reproducing the essential relations among C^3 variables and operators. These axioms modify and extend analogous axioms presented by Goodman (1986a). These axioms reflect the typical time cycle of a node, somewhat simplified. They begin with the reception of arriving "signals" (i.e., information, weapons, any other incoming entities which can provoke change in the node state) followed by information processing and related activity. These axioms end with the time of output response by the node to other nodes, friendly or adversary. With the formal language established through the axioms and using the standard rules of deduction, a basic theorem is derived. Theorem 4.1 gives a formal description of the dynamic evolution of a typical node state vector at the end of the input-output cycle as a functional of the same node state at the beginning of the cycle just prior to the reception of the "signal" and of all other pertinent C^3 variables. This immediately leads to a recursive form for each node state's evolution, beginning with the original intitialization of the C^3 system.

In step 2, an algebraic logic description pair (ALDP) is chosen, compatible with the above formal descriptions, in order to obtain the full quantitative evaluation. Typically, probability logic can play this role, but in order to utilize linguistic-based and other types of information, other logics can be used just as well, such as Zadeh's fuzzy logic or Dempster-Shafer's belief logic. A scheme is presented for using these logics in a suboptimal or marginal sense as inputs to "two-person" (actually friendly versus adversary C^3 processes) decision game. For more details on steps 3 to 5, see the concluding part of section 3.

By choosing probability logic for implementation, a fundamental result (corollary 5.2) can be invoked that is of potentially good use in evaluating the overall dynamic evolution of C^3 node states. Section 5 presents the details of this logic where essentially, a uniform close approximation by finite linear-Gaussian mixtures of distributions can be used to represent distributions of C^3 variables and in turn, the evolving node states.

Finally, section 6 presents a brief discussion for application of the model, in general, and to a simplified inner-outer air battle scenario, in particular.

3. THE C^3 DECISION GAME

In all that follows, for simplicity, we will consider only two C^3 processes, one friendly and one adversary.

There are three types of variables describing a C^3 process:

M, *node complex variables*, representing the decision-makers, human or automated, and their immediate environment,

S, *node input "signal" variables*, where, as explained before, "signal" need not refer to just an ordinary array of incoming information, but may, as well, denote incoming activated weapons or other disturbances to the initial node complex,

R, *node output responses* immediately following the complete "signal" processing.

Symbolically, one can represent the temporal relation among M,S,R, as

$$\ldots \rightarrow M \rightarrow R \rightarrow S \rightarrow M \rightarrow R \rightarrow S \rightarrow M \rightarrow R \rightarrow S \rightarrow \ldots, \tag{3.1}$$

regardless of the particular nodes interacting with other nodes and the multiplicity of "signals" and responses. In general, an arriving "signal" originates, possibly from several, nodes M as initial responses R, but due to intermediate media distortion and change, becomes S. See figure 1 for an example of a C^3 process, showing how one can scope out the roles of "signals", responses, and nodes.

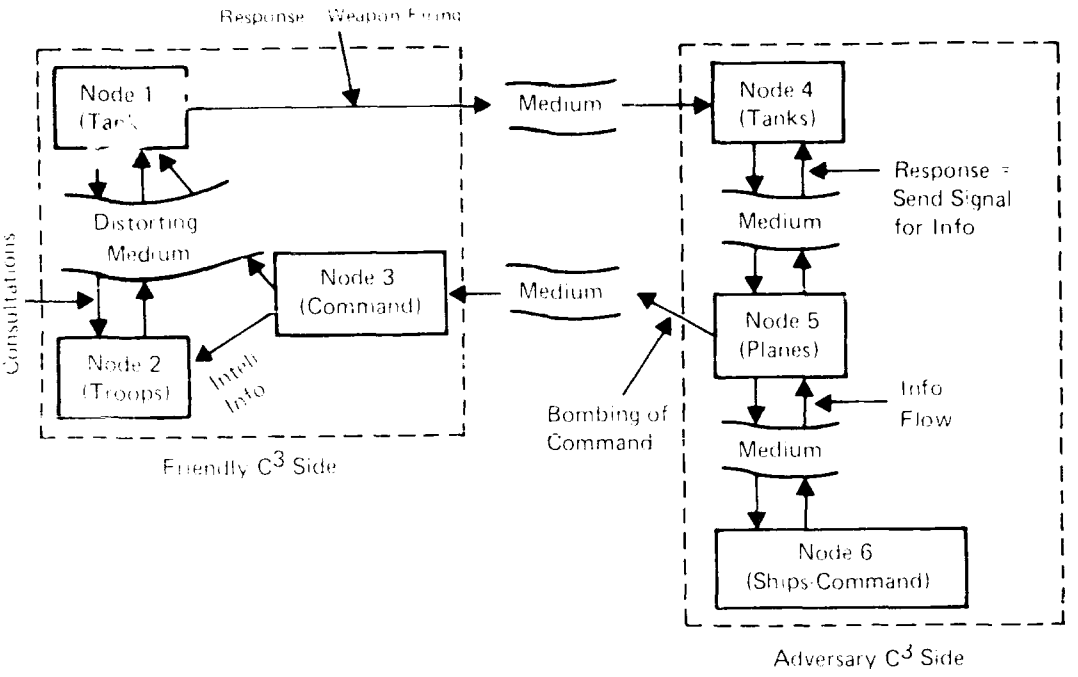


Figure 1. Simplified external dynamics of C^3 processes.

Variables, M,S,R can be indexed, sub or super, to indicate the particular C^3 process, friendly or adversary, the specific node involved, and the time. Node variables can be decomposed into two parts, with similar remarks holding for indexing of these and all subsequent decompositions of the main variables into subvariables:

$$M = (N, I) \tag{3.2}$$

where N represents the *node* state, while I represents the *internal node structure*. See figure 2 for an illustration of a typical node structure (relative to the processing of incoming "signals"). See figure 3 for the basic evolution cycle due to the processing of a "signal" as in figure 2. Note the superscripts $(\,)^-$, $(\,)^+$ to indicate relative times and $(S|R)$ to indicate *conditioning*. For simplicity, changes are only noted at times of response and times of reception, following medium distortion of responses.

Typically, node states involve: threat levels; numbers of troops; numbers of weapons of different types; supply levels; equations of motion of the node if not stationary, such as a tank or a formation of airplanes, or if stationary but its locations is in doubt; as well as possible knowledge or estimates of other node states, friendly or adversary. See table I for a simple example. Symbolically, one can write N and I in subvariable form such as

$$N = (\#WP_1, \#WP_2, \#WP_3, \#TROOP, EQMO, INFO), \tag{3.3}$$

$$I = (DET, ALG, HYP, FUS, CONS, DEC), \tag{3.4}$$

where the above symbols are essentially self-explanatory, such as: $\#WP_2$ = number of type 2 weapons onboard, EQMO = equations of motion (of entire node); INFO = knowledge or estimates of other node states; DET = detection state (usually 0 or 1, for probabilistic interpretations), ALG = algorithms chosen for possible response; FUS = data fusion variables, further

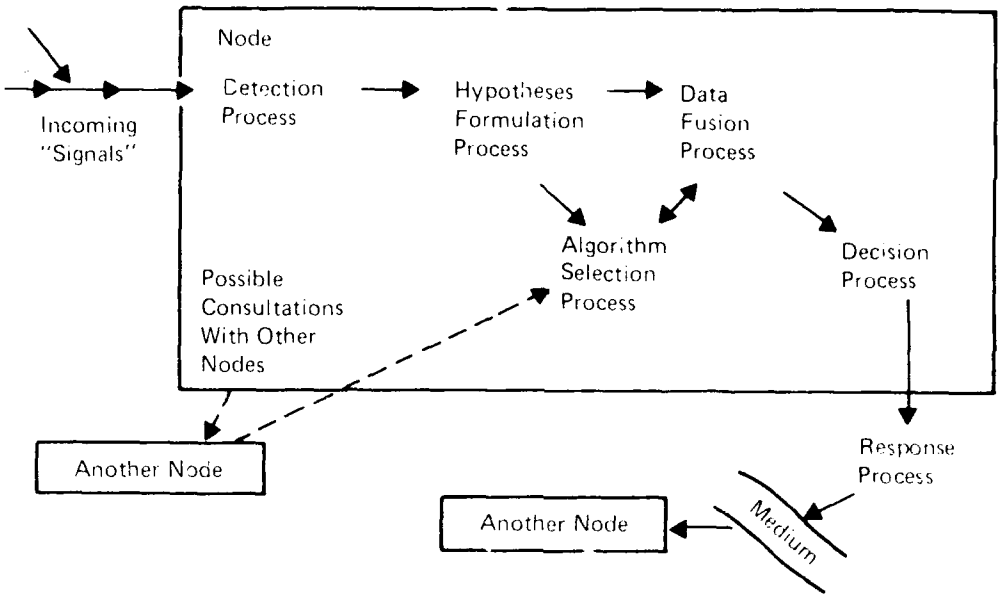


Figure 2. Simplified internal dynamics of C^3 processes.

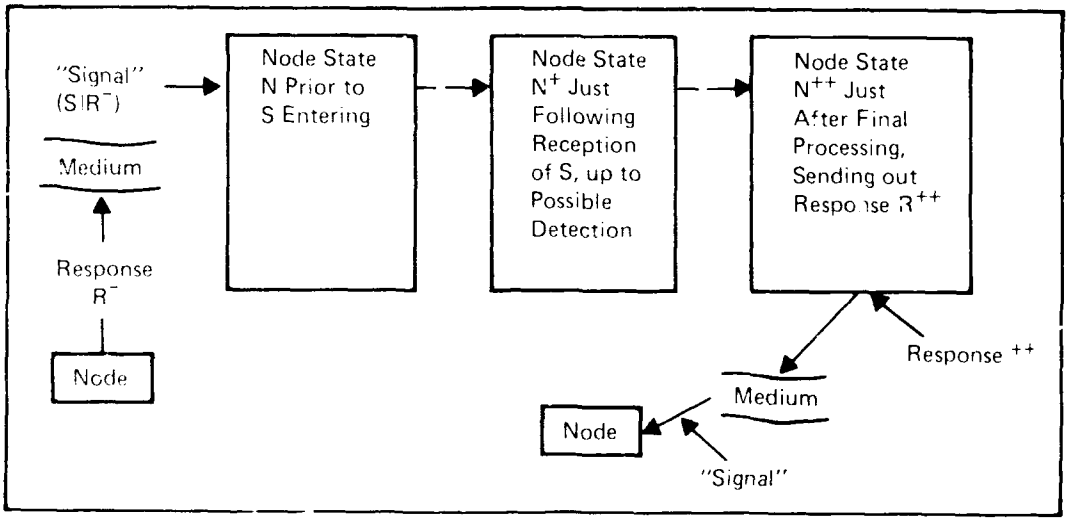
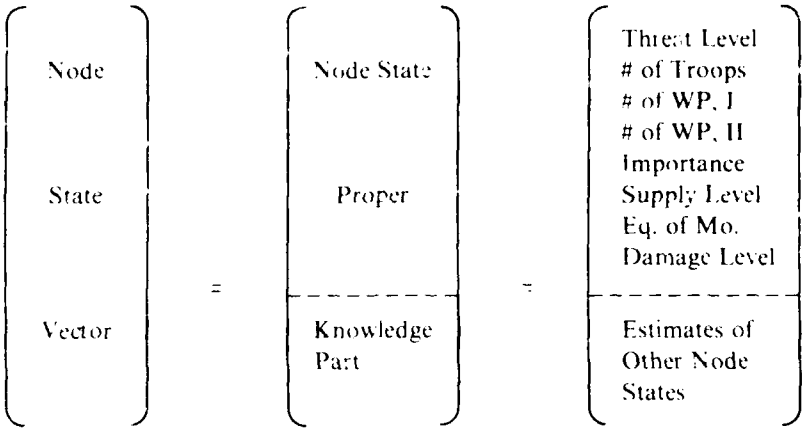


Figure 3. Basic evolution cycle of a node due to "signal" processing and response.

Table 1. Components of C^3 node states.



NODE = (NODE STATE, NODE STRUCTURE)

decomposable into appropriate subvariables, when required, such as in correlation of data in multitarget tracking; and DEC = decision variables representing actual decisions to be made based upon arriving "signals".

Consider now the basic C^3 structure indicated in equations (3.1) through (3.4), as well as time-shift markers $(\ )^-$, $(\ )^+$, $(\ )_0$, and conditioning operator $(|)$. Also, note logical connectors & (conjunction, and, intersection), v (disjunction, or, union), $(\ )'$ (negation, not, complement) and set event relations DOM (domain of possible values of the associated variable), ϵ (set membership relation), Ω (universal set of discourse) and ϕ (null or empty set). Clearly, any qualitative or quantitative description of a C^3 process as a whole entity must entail descriptions of the nodes constituting the process using, in some sense, the above concepts.

With all of the above C^3 variables and operations noted and interpreted, the fundamental steps in establishing an overall model of tactical C^3 processes can now be attempted:

(1) Obtain a formal theory for the dynamic evolution of a typical node state N using an appropriate set of axioms AX involving C^3 variables and operators. In this case, N can be expressed as some functional of C^3 variables and operators symbolically described as

$$N = \mathcal{F}(AX) . \tag{3.5}$$

Details of these relations are given in table 2, in section 4.

(2) Evaluate/quantify typical C^3 node states as given in eq. (3.5) by choice of some *algebraic logic description pair* (ALDP). Such a pair consists of a syntax space with algebraic structure and a compatible semantic evaluation/logic function with range in the positive real line. Examples include:

$$CL(\text{classical logic}) = (\text{Boolean algebra, 0-1 truth function}) . \tag{3.6}$$

$$PL(\text{probability logic}) = (\text{Boolean algebra, probability measure}) . \tag{3.7}$$

$$FL(\text{Zadeh's fuzzy logic}) = (\text{Browerian lattice, possibility function}) . \tag{3.8}$$

$$DSBL(\text{Dempster-Shafer belief logic}) = (\text{Boolean algebra, belief function}) . \tag{3.9}$$

[See Goodman (1987) and Goodman & Nguyen (1985) for further details.]

Indeed, one can view the ALDP evaluation as the final part of knowledge flow in describing not only C^3 processes, but also of situations in general, beginning with cognition. (See figure 4 for a brief outline of this.)

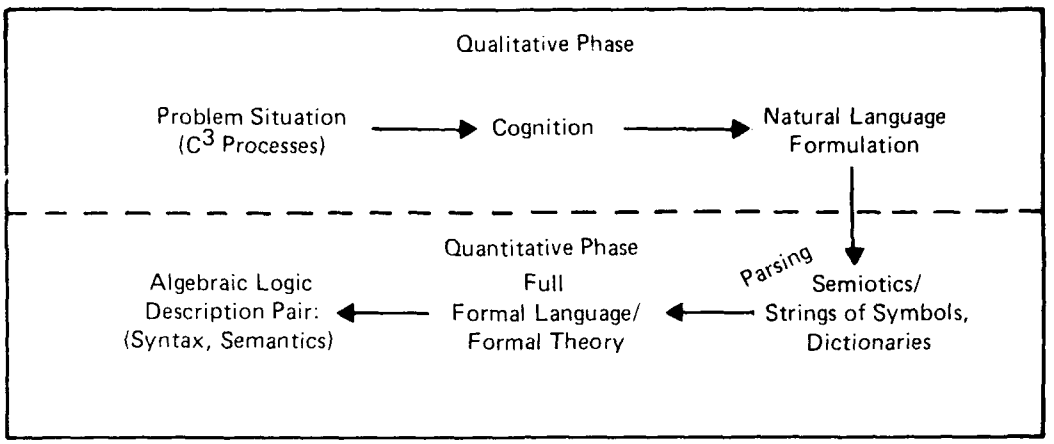


Figure 4. Knowledge flow in describing situations.

Denote the numerical evaluation of each formal node state description symbolically as

$$p_{ALDP}(N,C^3) = ||(N,C^3)||_{ALDP}, \tag{3.10}$$

where p_{ALDP} or $||\ ||_{ALDP}$ indicates the logic function associated with the chosen ALDP. Thus, for pl, p_{ALDP} means a probability measure (or function, etc.), or if one chooses ALDP = FL, p_{FL} = poss, a possibility function, etc. For an ALDP, in general, call p_{ALDP} a *dispersion*, unless otherwise specified.

Using eq. (3.5) in eq. (3.10) and applying the compatibility property of ALDPs yields the relation

$$p_{ALDP}(N,C^3) = G_{ALDP,AX}[p_{ALDP}(AX,C^3)], \tag{3.11}$$

where N is a given node state of a C^3 process, denoted simply as C^3, $G_{ALDP,AX}$ is a functional and $p_{ALDP}(AX,C^3)$, extending somewhat the notation in eq. (3.10), is the collection of all dispersions in evaluating the RHS (right-hand side) of eq. (3.10) for a given C^3 process (friendly or adversary) and choice of ALDP. Note that in a real-world situation, for given AX and ALDP, the set of dispersions $p_{ALDP}(AX,C^3)$ is highly dependent in form upon just what C^3 is, or equivalently, any allowable collection of dispersions represented by $p_{ALDP}(AX,C^3)$ completely determines C^3 and thus, may be identified with any of its primitive set of dispersions determining all of C^3.

In the approach in steps (1) and (2), each C^3 node is considered separately or marginally taking into account, however, all of the nodes that interact with it as well as all relevant C^3 variables, friendly or adversary. This is assumed at the outset in order to obtain a simpler model than if all possible joint interactions at a given time were taken into account, analogous to the suboptimal determination of individual marginal distributions of a more (exponentially) complex joint distribution from which the marginals arise.

(3) In turn, combine the results of step (2) -- individual updated marginal node state descriptions back into a simple global description of the C^3 process at hand. As mentioned above, this is done considering the very real tradeoff of model fidelity versus complexity of

calculations, in place of attempting to carry out the optimal full joint descripton of C^3 behavior. A reasonable combination operator AV here can be simple arithmetic, or more generally, a weighted sum, averaging the corresponding node state entry dispersions, representing thus an updated overall C^3 process behavior. If arithmetic averaging is not desired, other suitable measures of central tendency can be used, such as modes or medians, among others. The basic output at this stage can be symbolized as

$$\bar{p}_{ALDP}(C^3) = AV(p_{ALDP}(N,C^3), \text{ all } N \in C^3) , \tag{3.12}$$

where C^3 represents the particular C^3 process of interest.

(4) Next, determine a functional $\mathcal{H}_{ALDP}$ which directly relates the updated averaged dispersions of a given C^3 process with a single or multiple attribute vector describing the overall "health" state of C^3:

$$HLTH_{ALDP}(C^3) = \mathcal{H}_{ALDP}[(\bar{p}_{ALDP}(C^3)] . \tag{3.13}$$

For example, $\mathcal{H}_{ALDP}$ could be a certain weighted mean, or perhaps involve set thresholds, of a number of MOEs and or MOPs of the C^3 process as a whole, each of which, in turn, depends upon $\bar{p}_{ALDP}(C^3)$. Examples of such MOEs and MOPs can include overall supply attrition level, overall entropy level, various system performance criteria, including timeliness, damage levels, kills, destruction of enemy level, etc. For a good systematic exposition on MOEs and MOPs and related measures, see Sweet (1987) or Rubin and Mayk's discussion (1987 pp. 15-16).

Using eqs. (3.11) and (3.12) in (3.13) yields

$$HLTH_{ALDP}(C^3) = \mathcal{G}_{ALDP}[p_{ALDP}(AX,C^3)] , \tag{3.14}$$

for some functional $\mathcal{G}$ depending in part on $\mathcal{G}$, $\mathcal{H}$.

(5) Finally, one can establish the desired two-person zero sum decision game Γ as shown schematically in figure 5:

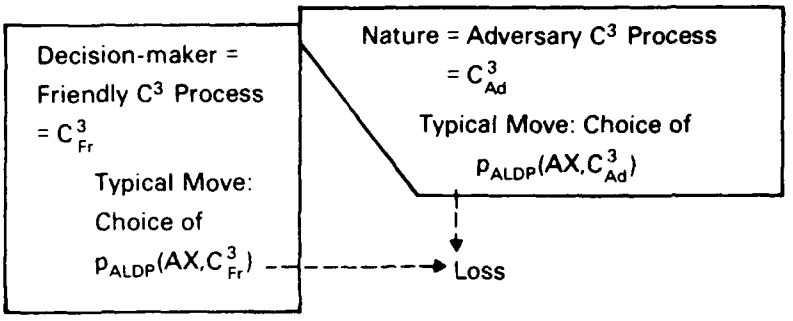
The loss function in figure 5 resulting from the moves of each player can be expressed as

$$\begin{aligned} \text{Loss} &= \mathcal{J}_{ALDP}[HLTH_{ALDP}(C^3_{Fr}), HLTH_{ALDP}(C^3_{Ad})] \\ &= \mathcal{K}_{ALDP}[p_{ALDP}(AX,C^3_{Fr}), p_{ALDP}(AX,C^3_{Ad})] , \end{aligned} \tag{3.15}$$

where $\mathcal{K}$ is determined by equations (3.11) through (3.14).

Thus, one can inquire whether Γ has a game value, what the Bayes strategies are for each side, what the least favorable move or strategy is for the adversary versus the friendly side, what the friendly side's minimax strategy is, and what the class of admissible strategies is for each side, etc. All answers must depend upon the allowable classes of choices for the primitive relations $p_{ALDP}(AX,C^3)$.

Such a game can be of value in the design and study of sensitivity of outcomes for C^3 processes, provided these questions can be addressed using feasible computations. This of course, depends on whether eq. (3.5) and hence eq. (3.11) can be effectively limited in complexity.



Loss is determined through a functional $\mathcal{J}_{ALDP}$. See equation (3.15) below.

Figure 5. Schematic for C^3 decision game Γ.

A summary of steps (1) through (5) is given below in figure 6:

In the next section, a key part of the development of Γ is given as the choice of axioms AX and the resulting formal structure for node state evaluation as in step (1).

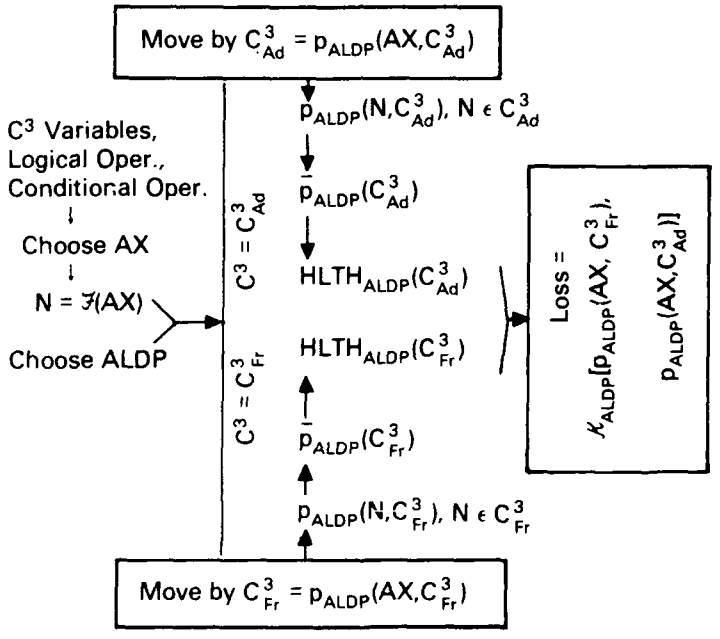


Figure 6. Functional development of C^3 decision game Γ.

4. FORMAL THEORY AND EVALUATION FOR THE EVOLUTION OF A TYPICAL C^3 NODE STATE

In the past, a number of attempts have been made to develop formal theories for various scientific and social disciplines which traditionally had not been analyzed from such viewpoints. See, e.g., Woodger (1987) for biological systems and Carnap (1958), for other applications including legal and social considerations. In this report, a simplified list of axioms in non-quantifier propositional form is proposed, characterizing the essential features required to evaluate dynamically evolving C^3 node states. In turn, as oulined in section 3, these axioms can be of use in developing a C^3 decision game. These axioms are of three kinds: logical connections, properties of conditioning, and reduction of relations among C^3 variables, or sufficiency conditions. The axiom collection AX is summarized in table 2 as follows:

Table 2. Formal language description of a C^3 node evolution.

Equality Symbol: =

Constants: Ω, ϕ

Dummy Variables: α, β, γ

Specific Variables: N, R, S, T

N and T can be partitioned into subvectors, e.g.,

$$N = (\#WP_1, \#WP_2, \#WP_3, \#TROOP, EQMO, INFO)$$
$$T = (DET, ALG, HYP, FUS, CONS, DEC)$$

Operators: $(\,)^+$, $(\,)^-$, $(\,)_0$, $(|)$, &, v, DOM, ϵ

General Axioms: For all α, β, γ, and for $\star$ = &,v:

Ring structure for &,v

$$\alpha\star\beta = \beta\star\alpha, \quad \alpha\star(\beta\star\gamma) = (\alpha\star\beta)\star\gamma,$$
$$\alpha\&\phi = \phi, \quad \alpha\&\Omega = \alpha = \alpha v\phi, \quad \alpha v\Omega = \Omega,$$
$$\alpha\&(\beta v\gamma) = (\alpha\&\beta)v(\alpha\&\gamma).$$

General Axioms: For all α, β, γ, and for $\star$ = &,v:

Implicative Conditional structure for &,v:

$$(\alpha|\Omega) = \alpha, \quad (\alpha|\beta) = (\alpha\&\beta|\beta),$$
$$(\alpha\&\beta|\gamma) = (\alpha|\beta\&\gamma)\&(\beta|\gamma),$$
$$(\alpha\star\beta|\gamma) = (\alpha|\gamma)\star(\beta|\gamma),$$
$$v\alpha = \Omega.$$
$$\alpha\epsilon DOM(\alpha)$$

Sufficiency Axioms: For $\tilde{N} \stackrel{d}{=} (N\&N \quad \&N^{==}\&..\&N_0)$:

$$(N^{++}|R^{++}\&T^{+} \;\&\; N^{+} \;\&\; S \;\&\; R \;\&\; \tilde{N}) = (N^{++}|R^{++} \;\&\; N^{+}),$$
$$(R^{++}|T^{+} \;\&\; N^{+} \;\&\; S \;\&\; R \;\&\; \tilde{N}) = (R^{++}|DEC^{+} \;\&\; N^{+}).$$
$$(T^{+}|N^{+} \;\&\; S \;\&\; R \;\&\; \tilde{N}) = (T^{+}|N^{+})$$
$$(N^{+}|S \;\&\; R \;\&\; \tilde{N}) = (N^{+}|S \;\&\; N),$$
$$(S|R \;\&\; \tilde{N}) = (S|R),$$
$$(R \;|\; \tilde{N}) = (R \;|N)$$

The ring structure axioms are the minimum needed to characterize conjunction and disjunction. (Note that no inverses are postulated.) The axioms for conditioning, while reflecting well-known properties, such as for probability interpretations, are new in that they refer to *measure-free* conditioning (see discussion later in this section). The sufficiency axioms show the relationships among the C^3 variables of interest, based on sequence of events as outlined in figures 2 and 3.

The axiom set in table 2 leads to a particular form of equation (3.5) describing recursively node state evolution:

Theorem 4.1.

Under the assumptions in table 2:

(i)

$$N^{++} = \bigvee [(N^{++} \mid R^{++} \,\&\, T^{+} \,\&\, N^{+} \,\&\, S \,\&\, R \;\&\, \widetilde{N}) \,\&\, (R^{++} \,\&\, T^{+} \,\&\, N^{+} \,\&\, S \,\&\, R \;\&\, \widetilde{N})]$$

$$\left[\begin{array}{l} R^{++} \epsilon DOM(R^{++}), \\ T^{+} \epsilon DOM(T^{+}),\ S \epsilon DOM(S), \\ N^{+} \epsilon DOM(N^{+}),\ R \;\epsilon DOM(R\;), \\ \widetilde{N} \epsilon DOM(\widetilde{N}) \end{array}\right] \tag{4.1}$$

$$= \bigvee [(N^{++} \mid R^{++} \,\&\, N^{+}) \,\&\, (R^{++} \mid DEC^{+} \,\&\, N^{+}) \,\&\, (T^{+} \mid N^{+}) \,\&\, (N^{+} \mid S \,\&\, N) \,\&\, (S \mid R\;) \,\&\, (R\; \mid N) \,\&\, N]$$

$$\left[\begin{array}{l} R^{++} \epsilon DOM(R^{++}),\ T^{+} \epsilon DOM(T^{+}), \\ N^{+} \epsilon DOM(N^{+}),\ S \epsilon DOM(S),\ R \;\epsilon DOM(R\;),\ N \epsilon DOM(N) \end{array}\right].$$

(ii) Regrouping terms above, compatible with table 2:

$$N^{++} = \bigvee_{N \epsilon DOM(N)} [(N^{++} \mid N) \,\&\, N], \tag{4.2}$$

$$(N^{++} \mid N) = \bigvee_{\left[\begin{array}{l} R^{++} \epsilon\ DOM(R^{++}) \\ N^{+}\ \epsilon\ DOM(N^{+}) \end{array}\right]} [(N^{++} \mid R^{++} \,\&\, N^{+}) \,\&\, (R^{++} \mid N^{+}) \,\&\, (N^{+} \mid N)] \tag{4.3}$$

where

$$(N^{+} \mid N) = \bigvee_{R\; \epsilon DOM(R\;)} [(N^{+} \mid R\; \,\&\, N) \,\&\, (R\; \mid N)], \tag{4.4}$$

$$(R^{++}|N^{+}) = \bigvee_{T^{+}\epsilon\, DOM(T^{+})} [(R^{++}|DEC^{+}\&\ N^{+})\&(T^{+}|N^{+})]\ , \tag{4.5}$$

$$(N^{+}|R\ \&N) = \bigvee_{S\epsilon\, DOM(S)} [(N^{+}|S\&N)\&(S|R\ )]\ , \tag{4.6}$$

$$\begin{aligned}(T^{+}|N^{+}) = {} & (DEC^{+}|CONS^{+}\&FUS^{+}\&HYP^{+}\&ALG^{+}\&DET^{+}\&N^{+}) \\ & \&(CONS^{+}|FUS^{+}\&HYP^{+}\&ALG^{+}\&DET^{+}\&N^{+}) \\ & \&(FUS^{+}|HYP^{+}\&ALG^{+}\&DET^{+}\&N^{+}) \\ & \&(HYP^{+}|ALG^{+}\&DET^{+}\&N^{+}) \\ & \&(ALG^{+}|DET^{+}\&N^{+})\&(DET^{+}|N^{+})\ .\end{aligned} \tag{4.7}$$

Proof: Details omitted, but straightforward application of the usual rules of substitution and the axioms in table 2.

■

Table 3 presents interpretations of the formal language used in table 2 for presenting the axioms and consequent structure of node state evolution in theorem 4.1.

Table 3. Interpretations of the formal language for C^3 node evolution.

N	= Node state vector, T = Node structure
R	= Response vector, S = "Signal" vector
$(\ )^{+}$	= Positive time shift to new phase
$(\ )$	= Negative time shift to old phase
$(\ )_0$	= Initialization of state (time-wise)
(|)	= Implication or conditioning
&	= AND, v = OR, $(\ )'$ = NOT (explained earlier)
DOM	= Domain of possible values
ϵ	= Set membership relation as used before
ϕ	= Null set
Ω	= Universal set

Thus, using the interpretation in table 2, compatible with figure 3,

$(R^{++}|T^{+}\&N^{+})$ = response following processing, (4.8)

$(N^{++}|R^{++}\&N^{+})$ = new node state due to its sending out response , (4.9)

$(T^{+}|N^{+})$ = processing data, (4.10)

$(N^{++}|N)$ = full cycle of node change due to "signals" received, over all possible processing, and responses, etc. (4.11)

Following the five-step procedure presented in section 3 for developing the overall decision game, the next stage requires the numerical evaluation of N^{++} given in theorem 4.1 through choice of a suitable ALDP.

Remark 4.1

Note that one can readily verify that all Boolean algebras — and more generally, all Browerian lattices -- to satisfy the axioms in table 2. Hence, all four examples of ALDPs given in equations (3.6) through (3.9) can be used to evaluate theorem 4.1. For background and discussions in considering the most appropriate ALDP for a given situation, see Goodman and Nguyen (1985, 1988). Thus, if PL were chosen as the ALDP by assuming only stochastic relations should be used, then eq. (4.3) becomes

$$p(N^{++}|N) = \int_{\left[\begin{array}{l}\text{over all}\\ R^{++}\epsilon DOM(R^{++}),\\ N^{+}\epsilon DOM(N^{+})\end{array}\right]} p(N^{++}|R^{++},N^{+})\cdot p(R^{++}|N^{+})p(N^{+}|N)dR^{++}dN^{+} \tag{4.12}$$

yielding, in turn, the counterpart of (4.2)

$$p(N^{++}) = \int_{N\epsilon DOM(N)} p(N^{++}|N)\cdot p(N)dN\ . \tag{4.13}$$

Or, if FL were chosen as the ALDP, by assuming only fuzzy relations should be used, then eq. (4.3) becomes

$$poss(N^{++}|N) = \max_{\left[\begin{array}{l}\text{over all}\\ R^{++}\epsilon DOM(R^{++}),\\ N^{+}\epsilon DOM(N^{+})\end{array}\right]} \{\min[poss(N^{++}|R^{++},N^{+}).poss(R^{++}|N+)poss(N^{+}|N)]\} \tag{4.14}$$

yielding, in turn, the counterpart of eq. (4.2)

$$poss(N^{++}) = \max_{N\epsilon DOM(N)} \{\min[poss(N^{++}|N),poss(N)]\}\ . \tag{4.15}$$

One could also choose combinations of PL and FL or other ALDPs. Again, see Goodman (1987).

Similarly, by applying, e.g., PL, one can evaluate in turn

$$p(N^{+}|N) = \int_{R\ \epsilon DOM(R\ )} p(N^{+}|R\ ,N)\cdot p(R\ |N)dR \tag{4.16}$$

$$p(R^{++}|N^{+}) = \int_{T^{+}\epsilon DOM(T^{+})} p(R^{++}|DEC^{+},N^{+})\cdot P(T^{+}|N^{+})dT^{+} \tag{4.17}$$

$$p(N^+|R^-,N) = \int_{S\epsilon DOM(S)} p(N^+|S,N)\cdot p(S|R^-)dS \tag{4.18}$$

$$p(S|R^-) = p[W = S - f(R^-)], \tag{4.19}$$

when the nonlinear additive regression relation holds

$$S = f(R^-) + W, \tag{4.20}$$

where W is a random vector representing additive medium error between responses and "signals" and f is a known function representing medium distortion.

Thus, one can identify the probabilities in equations (4.12), (4.13), and (4.16) through (4.19) with $p_{PL}(AX,C^3)$ as given in step (2), following equation (3.11). Hence, by specifying $p_{PL}(AX,C^3)$ for each process, friendly or adversary, one can then proceed, at least in theory, with the construction of the C^3 decision game, as outlined in steps (3) through (5), section 3. Similar remarks hold true for the use of FL or any other of the ALDPs.

■

In particular, for PL, it follows from theorem 4.1 that for each C^3 process, $p_{PL}(AX,C^3)$ is determined by first specifying 12 relatively primitive relations. Thus, one can write:

$$\left.\begin{array}{r} p(N^{++}|R^{++},N^+), \\ p(R^+|DEC^+,N^+), \\ p(DEC^+|CONS^+,FUS^+,HYP^+,ALG^+,DET^+,N^+), \\ p(CONS^+|FUS^+,HYP^+,ALG^+,DET^+,N^+), \\ p(FUS^+|HYP^+,ALG^+,DET^+,N^+), \\ p(HYP^+|ALG^+,DET^+,N^+), \\ p(ALG^+|DET^+,N^+), \\ p(DET^+|N^+), \\ p(N^+|S,N), \\ p(S|R^-), \\ p(R^-|N), \\ p(N_o) \end{array}\right\} = p_{PL})AX,C^3). \tag{4.21}$$

Of course, it is understood in eq. (4.21) that the relations (except for the last) must be specified for each time cycle of node-processing input "signal" to output response.

In another direction, noting that many of the relations in table 2 and in subsequent equations are in conditional form, it is of some interest to inquire whether this theory can treat operations on conditional forms when the antecedents need not be identical, generalizing the axiom in table 2

$$(\alpha\star\beta|\gamma) = (\alpha|\gamma)\star(\beta|\gamma) \tag{4.22}$$

to the case

$$(\alpha\star\beta|\rho) = (\alpha|\gamma)\star(\beta|\eta), \tag{4.23}$$

where now η and γ need not be the same, and ρ is some computable function (another event or set) of α,β,γ,η, for $\star = \&,\vee$. For example, one may wish to determine Q, and by choosing PL, p(Q), ..., some appropriate joint probability measure p, where

$$Q = [(a|b) \,\&\, (c|d)] \vee (e|f) , \tag{4.24}$$

where

a = a(x) = enemy will move up about x troops tomorrow; x = 0, 50, 100, 150.

b = b(y) = it will y tomorrow; y = be clear, snow, rain.

c = c(z) = enemy will use pass z to approach us; z = I, II, III, IV.

d = d(r,s) = morale of enemy node 17 is at level r and number of their troops left is s ; r = very low, low, medium, high, very high, s = 0, 100, 200, 300.

e = e(w) = enemy will w tomorrow; w = surrender, not surrender.

f = f(q) = enemy overall damage level is q ; q = 0, 1, 2, . . . , 10.

Of course, if the antecedents in eq. (4.24) were all the same, then no real problem would arise, since it is readily justified that for any choice of ALDP — certainly for PL ! — that

$$\begin{aligned} Q &= [(a|d) \,\&\, (c|d)] \vee (e|d) \\ &= \{[(a\&c)\vee e]|d\} , \end{aligned} \tag{4.25}$$

even though normally one does not talk about such *measure-free* entities (up to now). Indeed, since the goal is the evaluation of Q, for PL, choosing a probability measure p over all the relevant events, one would usually evaluate Q as simply

$$\begin{aligned} p(Q) &= p\{[(a|d) \,\&\, (c|d)] \vee (e|d)\} \\ &= p\{[(a\&c)\vee e]|d\} \\ &= p\{[(a\&c)\vee e] \,\&d\}/p(d) , \end{aligned} \tag{4.26}$$

etc., assuming $p(d) > 0$.

But the point of the above example given in eq. (4.24) is that the antecedents in the conditional forms are *not* identical! What to do?

Contrary to popular belief (author's note: this author and his colleague Prof. H.T. Nguyen, Mathematics Department New Mexico State University, Las Cruces, have undertaken and extensive informal survey of the probability community — both applied and theoretical — resulting in the following conclusions — see Goodman (1988) and Goodman and Nguyen (1989): there is *no* systematic and mathematically sound procedure for computing p(Q) (or Q, for that matter) in eq. (4.24) or, in fact, for any similar problem!

Indeed, there are "folk" remedies to this situation which reduce to either identifying conditioning with material implication, forcing conditioning to be a closed operation over the Boolean algebra of events, or identifying conditional events as marginal to a universal joint

event having a fixed antecedent common to these marginal ones. In either case, serious difficulties arise, either mathematically or computationally. For a satisfactory solution to this problem, see Goodman (1988) and Goodman & Nguyen (1989), where a sound and complete ALDP (among other properties) is developed, compatible with and extending, classical PL, called CPL (conditional probability logic).

One consequence of the calculus of operations in CPL is that the evaluation of Q in eq. (4.24) becomes

$$Q = (\alpha|\beta) , \tag{4.27}$$

resulting in the value

$$p(Q) = p(\alpha) \cdot p(\beta) , \tag{4.28}$$

where

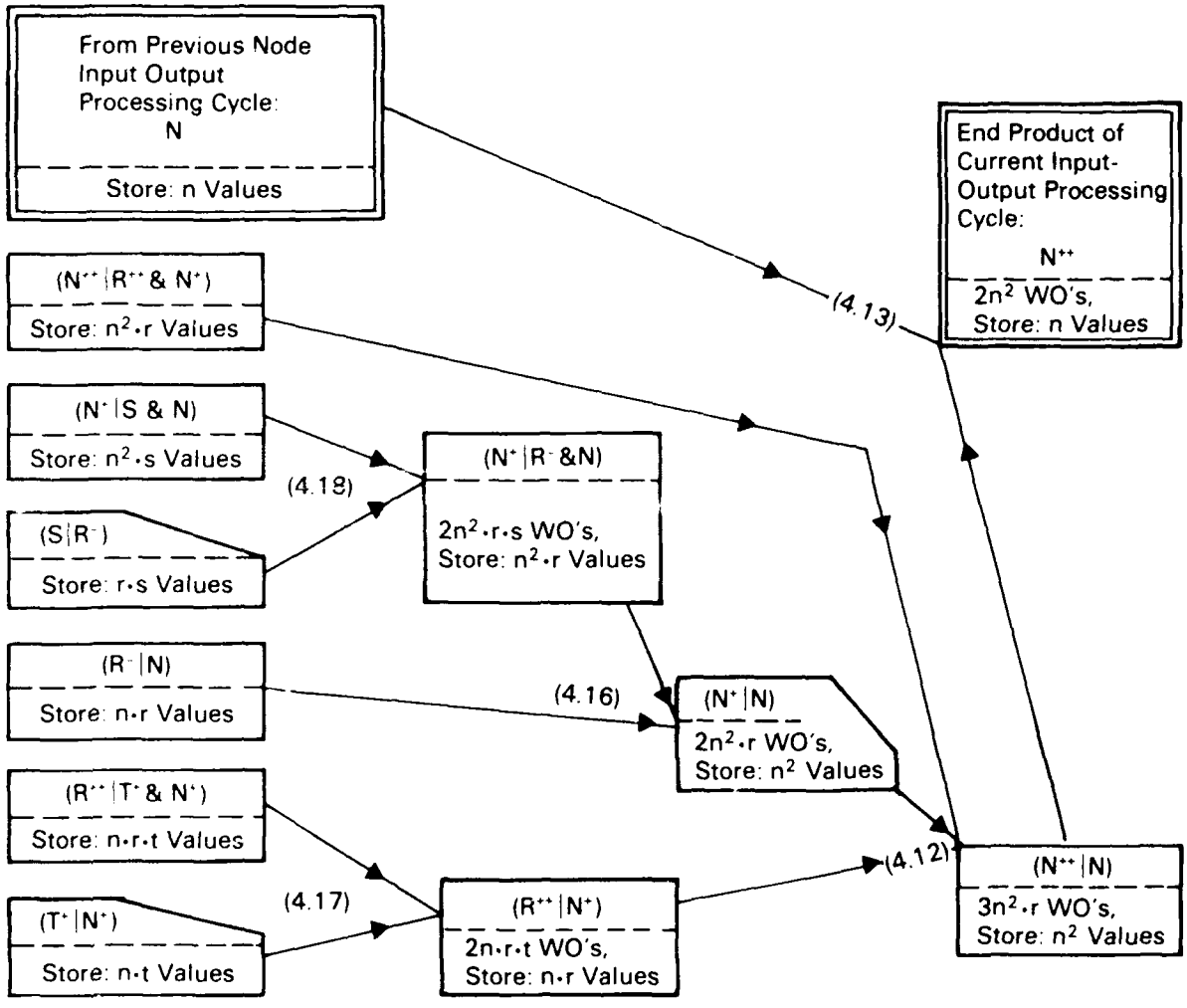
$$\alpha \triangleq (a\&b\&c\&d)v(e\&f) , \tag{4.29}$$

$$\beta \triangleq \alpha \vee \{[(a'\&b)v(c'\&d)v(b\&d)]\&f\} , \tag{4.30}$$

differing considerably from the "folk" approaches.

Returning to the construction of the C^3 decision game, again note that by utilizing the evaluations as in equations (4.12), (4.13), and (4.16) through (4.19) for PL, or equations (4.14), and (4.15), and analogous calculations for FL, etc., one can then recursively evaluate the dispersion of N^+ in eq. (4.1). However, as gleaned from figure 7 below, regardless of the ALDP chosen, even the basic scheme for evaluating a typical single node-processing cycle from input to output without further decomposition of the C^3 variables and subvariables, especially that of T^+ and N^+, still requires on the order of q^4 matrix/vector addition or multiplication operations, when, for simplicity, the domains of possible values for N,R,S,T have the same cardinality q. Under the same assumptions, one must also store about $2 \cdot q3$ domain values in order to accomplish all the computations.

The evaluation procedure can become extremely tedious due to the multiplicative forms of the terms used, compounded by lengthy iterative disjunction operations over the domains of the variables. In the next section, a possible solution to this problem is offered for the case of PL.



Totals: #(Initial Stored Values) = $n \cdot (1 + r) \cdot (1 + t) + n^2 \cdot (r + s) + r \cdot s$

#(WO's) = $n^2 \cdot (2 + 5r + 2r \cdot s) + 2n \cdot r \cdot t$

( ) Counts as 1 Weighted Operation (WO)

( )★( ) Counts as 2 Weighted Operations (WOs), etc.

$n \stackrel{d}{=} \text{Card}[\text{DOM}(N)]$, $r \stackrel{d}{=} \text{Card}[\text{DOM}(R)]$, $s \stackrel{d}{=} \text{Card}[\text{DOM}(S)]$, $t \stackrel{d}{=} \text{Card}[\text{DOM}(T)]$

Figure 7. Computation requirements for high-level processing of a typical node input-output processing cycle.

5. UTILIZATION OF LINEAR-GAUSSIAN MIXTURE APPROXIMATIONS

As stated before, all modeling of C^3 processes, in general, must take into account the very real problem of accuracy of model versus complexity of calculations. One possible reasonable solution to this difficulty, at least for the choice of ALDP = PL, involves the use of some fundamental approximation and representation theorems, which are useful in their own right.

First, before considering a theorem based in part on material in Goodman and Boyer (1975), the following definitions and notational conventions will be introduced:

Letting r be any positive integer and $\mathcal{R}^r$ the real r-dimensional Euclidean space, with [0,1] being the real unit interval, call any cumulative probability distribution function (cdf) $F: \mathcal{R}^r \to [0,1]$ *well-behaved* iff in its Jordan-Lebesque decomposition (see, e.g., Feller, 1966, pp. 135–140) no singular component appears, the discrete component distribution, if present is at most finite, and the absolutely continuous component admits a probability density function (pdf) which is bounded and uniformly continuous over $\mathcal{R}^r$. Thus,

$$F = \lambda \cdot F^{(1)} + (1 - \lambda) \cdot F^{(2)}, \tag{5.1}$$

where $F^{(j)}: \mathcal{R}^r \to [0,1]$ corresponds to a finitely discrete probability measure for j = 1 and an absolutely continuous probability measure for j = 2, with pdf $f^{(2)}$ being bounded and uniformly continuous over $\mathcal{R}^r$.

Clearly, large classes of common probability measures have cdf's satisfying these criteria.

For notational purposes, also let

$$F^{(1)} = \sum_{j \in J} p_j \cdot \delta(\cdot - \mu_j), \tag{5.2}$$

where $\mu_j \in \mathcal{R}^r$ is a mass point for $F^{(1)}$, δ is the dirac delta function, p_j represents the probability of occurrence relative to $F^{(1)}$ at μ_j, $0 \le p_j \le 1$, $j \in J$, $\sum_{j \in J} p_j = 1$,

J being finite or even vacuous. By convention, if $J = \phi$, $F^{(1)}$ is vacuous, and F reduces to the purely absolutely continuous cdf $F^{(2)}$.

Depending on the context used, denote $g_{\underset{\sim}{\Sigma}}$ to mean either the pdf, cdf, or probability distribution, corresponding to an r-dimensional Gaussian distribution with mean 0_r and positive definite covariance matrix Σ. Next, for any sequences

$$\underset{\sim}{\tau} \stackrel{d}{=} (\tau_1, \ldots, \tau_m), \quad \underset{\sim}{\mu} \stackrel{d}{=} (\mu_1, \ldots, \mu_m), \quad \underset{\sim}{\Sigma} \stackrel{d}{=} (\Sigma_1, \ldots, \Sigma_m), \tag{5.3}$$

$0 \le \tau_j \le 1$, $\tau_1 + \cdots + \tau_m = 1$, noting the pdf for $g_{\underset{\sim}{\Sigma}}$ is

$$g_{\underset{\sim}{\Sigma}}(x) = [(2\pi)^r \cdot \text{determ}(\Sigma)]^{-1/2} \cdot e^{(-1/2)x^T \Sigma^{-1} x^T}, \tag{5.4}$$

for all $x \in \mathcal{R}^r$, so that $g_{\underset{\sim}{\Sigma}}(\cdot - \mu)$ corresponds to an r-dimensional Gaussian distribution with mean μ and covariance matrix Σ, let the finite Gaussian mixture

$$g_{\underset{\sim}{\tau}, \underset{\sim}{\mu}, \underset{\sim}{\Sigma}} \overset{d}{=} \sum_{i=1}^{m} \tau_i \cdot g_{\Sigma_i}(\cdot \; \mu_i) , \tag{5.5}$$

which can represent a pdf, cdf, or probability distribution, depending on the context.

With all of the above established, a uniform approximation theorem can now be stated.

Theorem 5.1.

Let $F: \mathcal{R}^r \to [0,1]$ be any given well-behaved cdf with possibly vacuous J containing mass points μ_j, $j \in J$, in accordance with the above notation.

Then, F can be arbitarily uniformly closely approximated over $\mathcal{R}^r$, except at all of F's mass points μ_j, $j \in J$, by a sequence of cdf's which are finite Gaussian mixtures. Denote this relation as

$$g_{\underset{\sim}{\tau}, \underset{\sim}{\mu}, \underset{\sim}{\Sigma}} \approx\approx F . \tag{5.6}$$

Proof: First, consider separately the truth of the theorem for the purely finitely discrete case $F = F^{(1)}$. Clearly in this case, the sequence

$$\underset{\sim}{g}_1 \overset{d}{=} (g_{\underset{\sim}{\tau}_1, \underset{\sim}{\mu}_1, \underset{\sim}{\Sigma}_{1,n}})_{n = 1,2,\ldots} \tag{5.7}$$

where

$$\underset{\sim}{\tau}_1 \overset{d}{=} (p_j)_{j \in J} \; , \quad \underset{\sim}{\mu}_1 \overset{d}{=} (\mu_j)_{j \in J} \; , \quad \underset{\sim}{\Sigma}_{1,n} \overset{d}{=} [(1/n) \cdot I_r]_{j \in J} , \tag{5.8}$$

obviously, uniformly approaches $F^{(1)}$ as a cdf, except over $\underset{\sim}{\mu}_1$.

Next, consider the validity of the theorem for the purely absolutely continuous case $F = F^{(2)}$. Now, from (19), theorem 1, there exists a sequence of finite Gaussian mixtures

$$\underset{\sim}{g}_2 \overset{d}{=} (g_{\underset{\sim}{\tau}_{2,n}, \underset{\sim}{\mu}_{2,n}, \underset{\sim}{\Sigma}_{2,n}})_{n = 1,2,\ldots} \; , \tag{5.9}$$

which approaches $f^{(2)}$ in L^1-norm. But using the basic absolute inequality relation for all $A \in \mathcal{R}^r$ (measurable),

$$\left| \int_{x \in A} g_{\underset{\sim}{\tau}_{2n}, \underset{\sim}{\mu}_{2n}, \underset{\sim}{\Sigma}_{2n}}(x) dx - \int_{x \in A} f^{(2)}(x) dx \right| \leq$$

$$\int_{x \in \mathcal{R}^r} \left| g_{\underset{\sim}{\tau}_{2n}, \underset{\sim}{\mu}_{2n}, \underset{\sim}{\Sigma}_{2n}}(x) - f^{(2)}(x) \right| dx , \tag{5.10}$$

and letting $A = A_s \overset{d}{=} \{x : x \in \mathcal{R}^r \text{ and } x \leq s\}$, for any $s \in \mathcal{R}^r$, it follows that as cdf's, $\underset{\sim}{g}_2$ approaches $F^{(2)}$ uniformly over $\mathcal{R}^r$.

Finally, since F is a linear combination of $F^{(1)}$ and $F^{(2)}$, it follows that one can let $\approx\approx$ in eq. (5.6)

$$g_{\tau,\underset{\sim}{\mu},\underset{\approx}{\Sigma}} \stackrel{d}{=} \lambda \cdot g_1 + (1 - \lambda) \cdot g_2 \,, \tag{5.11}$$

which is again a legitimate finite Gaussian mixture.

■

Remark 5.1

It should be noted that in the construction of g_2 above, the $\underset{\approx}{\Sigma}_{2n}$ may approach zero, a not necessarily desirable property due to the resulting fluctuations of form of the pdf's. However, in Goodman and Boyer (1975) and in Medgyessy (1961) alternative forms can be utilized. Of course, in the case of g_1, this cannot be avoided.

■

The next theorem establishes a unique linear regression relation among *any* given pair of random vectors, provided sufficient joint second moments exist. Although this is a basic result, appearing in many places in one form or another [see, e.g., Deutsch (1965) sections 3.3 and 3.4], it is surprisingly not often in the full form to be given here with direct application to reducing nonlinear relations to "exact" linear ones, without employing approximating expansions.

Theorem 5.2

Let $(\Omega,\mathcal{A},p)$ be a probability space and $X:\Omega\to\mathcal{R}^k$ and $Y:\Omega\to\mathcal{R}^m$ be random vectors such that $\mathrm{Cov}\begin{pmatrix} Y \\ X \end{pmatrix}$ exists and Cov(X) is positive definite.

Then, there exists constant m by k (real) matrix B, called the *regression transition matrix* from Y to X, and random vector $W:\Omega\to\mathcal{R}^m$ such that the linear regression relation holds

$$Y = B \cdot X + W \,, \tag{5.12}$$

such that X and W are uncorrelated. W and B are uniquely determined with

$$W = Y - B \cdot X \,, \quad B = \mathrm{Cov}(Y,X) \cdot \mathrm{Cov}^{-1}(X) \,, \tag{5.13}$$

$$\mathrm{Cov}(W) = \mathrm{Cov}(Y) - \mathrm{Cov}(Y,X) \cdot \mathrm{Cov}^{-1}(X) \cdot \mathrm{Cov}^{T}(Y,X) \,, \tag{5.14}$$

and a constant $A = E(W) = E(Y) - B \cdot X$, etc. can be added to the RHS of (5.12), provided W is replaced by the zero mean random vector $W - E(W)$.

Proof: The result is self-evident, once equation (5.13) is noted, using the calculus of multivariate moments and matrices.

■

Remark 5.2.

Note that theorem 5.2 can be applied to produce an "exact" linearization of nonlinear relations such as given below where X and V are uncorrelated random vectors with $E(V) = 0_r$ so that

$$Y = f(X) + V \,, \tag{5.15}$$

where $f:\mathcal{R}^k\to\mathcal{R}^m$ is some fixed (measurable) function. The result immediately follows that eq. (5.12) becomes here

$$Y = B \cdot X + W ,$$

but where

$$B = Cov[f(X),X] \cdot Cov^{-1}(X) . \tag{5.16}$$

Now, if one were to apply a standard linearization approach to the model in eq. (5.15), by linearizing f(X),

$$f(X) \approx A_o + B_o \cdot (X - X_o) , \tag{5.17}$$

where

$$A_o \stackrel{d}{=} f(x_o) , \quad B_o \stackrel{d}{=} [df(X)/dX]_{X=x_o} . \tag{5.18}$$

for some constant $x_o \in \mathcal{R}^k$ such as E(X). Substituting eq. (5.17) into eq. (5.15) yields the approximation

$$Y \approx A_o + B_o \cdot (X - x_o) + V . \tag{5.19}$$

Next, if one defines the true error in eq. (5.19) as

$$\begin{aligned} Z &\stackrel{d}{=} Y - A_o - B_o \cdot (X - x_o) \\ &= W + (B - B_o) \cdot X + C , \end{aligned} \tag{5.20}$$

where

$$C \stackrel{d}{=} B_o \cdot x_o - A_o , \tag{5.21}$$

then by the above equations and standard matrix manipulations of covariances and means

$$\begin{aligned} E(Z \cdot Z^T) &= Cov(W) + (B - B_o) \cdot Cov(X) \cdot (B - B_o)^T + D \cdot D^T \\ &\gtrsim Cov(W) \\ &= Cov(V) + H \\ &\gtrsim Cov(V) , \end{aligned} \tag{5.22}$$

where

$$D \stackrel{d}{=} E(W) + (B - B_o) \cdot E(X) + C , \tag{5.23}$$

$$H \stackrel{d}{=} Cov[f(X)] - Cov[f(X),X] \cdot Cov^{-1}(X) \cdot Cov^T[f(X),X] . \tag{5.24}$$

$\gtrsim$ indicates positive semidefinite partial ordering among all m by m matrices, where one has $M_1 \gtrsim M_2$ iff $M_1 - M_2$ is a positive semidefinite matrix.

Thus, eq. (5.22) quantitatively shows that, although one pays a penalty (H) for using the exactly linearized form in eqs. (5.12), and (5.16), in place of the original nonlinear additive form in eq. (5.15), the form is still a more accurate approach than use of the standard linearization (without reiteration) in eq. (5.19).

Finally, note that the above remarks are valid for the exact linearization of the conditional "signal" relation in eqs. (4.19) and (4.20).

Corollary 5.1.

Suppose $(\Omega,\mathcal{A},p)$ is a probability space and $X:\Omega\rightarrow\mathcal{R}^k$, $Y:\Omega\rightarrow\mathcal{R}^m$ are random vectors such that $\begin{pmatrix} Y \\ X \end{pmatrix}$ has a well-behaved cdf over $\mathcal{R}^{m+k}$ and $\mathrm{Cov}\begin{pmatrix} Y \\ X \end{pmatrix}$ exists with Cov(X) positive definite.

Then:

(i) An exact linear regression relation holds as in eq. (5.12) between X and Y.

(ii) For conditional random vector (Y|X), for all outcomes y of Y and x of X, for some $\underset{\sim}{\tau}_1$, $\underset{\sim}{\mu}_1$, $\underset{\approx}{\Sigma}_1$,

$$g_{\underset{\sim}{\tau}_1,\,\underset{\sim}{\mu}_1,\,\underset{\approx}{\Sigma}_1}(y - B\cdot x) \;\approxeq\; F(Y = y|X = x)\,. \tag{5.25}$$

(iii) For unconditional random vector X at any possible outcome x, for some $\underset{\sim}{\tau}_2$, $\underset{\sim}{\mu}_2$, $\underset{\approx}{\Sigma}_2$,

$$\underset{\underset{\sim}{\tau}_2,\,\underset{\sim}{\mu}_2,\,\underset{\approx}{\Sigma}_2}{g(x)} \;\approxeq\; F(X = x)\,. \tag{5.26}$$

(iv) For the marginal integrated-out cdf of Y, at any outcome y

$$\underset{\underset{\sim}{\tau},\,\underset{\sim}{\mu},\,\underset{\approx}{\Sigma}}{g\,(y)} \;\approxeq\; F(y) = \int_{x\in\mathcal{R}^k} F(Y = y|X = x)\,dF(X = x)\,, \tag{5.27}$$

where letting

$$\underset{\sim}{\tau}_i \underset{=}{\overset{d}{=}} (\tau_{i,j})_{j\in J_i}\,,\quad \underset{\sim}{\mu}_i \underset{=}{\overset{d}{=}} (\mu_{i,j})_{j\in J_i}\,,\quad \underset{\approx}{\Sigma}_i \underset{=}{\overset{d}{=}} (\Sigma_{i,j})_{j\in J_i}\,, \tag{5.28}$$

$i = 1,2,$

$$\underset{\sim}{\tau} \underset{=}{\overset{d}{=}} \underbrace{(\tau_{1,j_1}\cdot\tau_{2,j_2})_{j_i\in J_i}}_{i=1,2},\quad \underset{\sim}{\mu} \underset{=}{\overset{d}{=}} \underbrace{(B\cdot\mu_{2,j_2} + \mu_{1,j_1})_{j_i\in J_i}}_{i=1,2},$$

$$\underset{\underset{\sim}{\tau}^{(j_1,j_2)}}{\underbrace{\qquad\qquad}},\qquad \underset{\underset{\sim}{\mu}^{(j_1,j_2)}}{\underbrace{\qquad\qquad}}$$

$$\underset{\approx}{\Sigma} \underset{=}{\overset{d}{=}} \underbrace{(B\cdot\Sigma_{2,j_2}\cdot B^T + \Sigma_{1,j_1})_{j_i\in J_i}}_{i=1,2}\,. \tag{5.29}$$

$$\underset{\underset{\approx}{\Sigma}^{(j_1,j_2)}}{\underbrace{\qquad\qquad}}$$

Proof: Note that all linear transforms on $\begin{pmatrix} Y \\ X \end{pmatrix}$ preserve the well-behaved property. Then, applying theorems 5.1 and 5.2 yields (i)-(iii), with (iv) obtained by multiplying out all mixture terms and using the well-known convolution of Gaussian pdf's which here takes the form, for any typical term

$$g_{\Sigma(j_1,j_2)}[y - \mu^{(j_1,j_2)}] = \int_{x\epsilon\mathcal{R}^k} g_{\Sigma_{1,j_1}}(y - Bx - \mu_{1,j_1}) g_{\Sigma_{2,j_2}}(x - \mu_{2,j_2})dx \tag{5.30}$$

■

Before establishing the main result, actually a corollary of the above two theorems, the following additional notation will be useful:

As before, suppose Y and X are random vectors with $Y:\Omega\rightarrow\mathcal{R}^m$ and $X:\Omega\rightarrow\mathcal{R}^k$, but now where X is partitioned into

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_s \end{pmatrix} \quad , \quad X_i:\Omega\rightarrow\mathcal{R}^{k_i} \quad , \quad i=1,\ldots,s, \tag{5.31}$$

where necessarily $k_1 + \cdots + k_s = k$. Partition, similarly, any $x\epsilon\mathcal{R}^k$ into

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_s \end{pmatrix}, \tag{5.32}$$

and let for any i , i = 1 ,...,s,

$$X^{(i)} \stackrel{d}{=} \begin{pmatrix} X_i \\ \vdots \\ X_s \end{pmatrix}, \tag{5.33}$$

so that one notes

$$X^{(1)} = X \quad , \quad X^{(s)} = X_s \; . \tag{5.34}$$

Also, extend the notation, where

$$X_0 \stackrel{d}{=} Y \quad , \quad X_{s+1} \stackrel{d}{=} 0 \quad , \quad X^{(s+1)} \stackrel{d}{=} \emptyset \; , \tag{5.35}$$

etc., with similar remarks holding for x.

Corollary 5.2.

Suppose the hypotheses of corollary 5.1 hold, with the notation introduced in equations (5.31) through (5.35) valid.

Then:

(i) For each i , i = 0,1,...,s, the exact linear regression relation holds

$$X_i = B_i \cdot X^{(i+1)} + W_i \; , \tag{5.36}$$

where $X^{(i+1)}$ and W_i are uncorrelated random vectors and each B_i is a constant k_i by $k_{i+1} + \cdots + k_s$ regression transition matrix, obtained analogous to B in (5.13). Denote the partitioning of each B_i

$$B_i = (B_{i,i+1}, B_{i,i+2}, \ldots, B_{i,s}) = B^{(i,i+1)} \ , \tag{5.37}$$

$$B^{(i,i+t)} \triangleq (B_{i,i+t}, \ldots, B_{i,s}) \ , \quad t = 1, \ldots, s - i \ . \tag{5.38}$$

(ii) In turn, there exists finite Gaussian mixtures, where for all possible outcomes x_i of X_i

$$g_{\underset{\sim}{\tau}_i, \underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i}[x_i - B_i \cdot x^{(i+1)}] \approx F[X_i = x_i | X^{(i+1)} = x^{(i+1)}] \ , \tag{5.39}$$

where $\underset{\sim}{\tau}_i, \underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_i$ are all formally the same as the corresponding values in eq. (5.28), except here, $i = 0, 1, 2, \ldots, s$, noting for $i = s$, RHS of eq. (5.36) is identical to $X_s = W_s$ and, in effect, B_s is 0.

(iii) Finally, the marginal, integrated-out cdf of Y, compatible with (i) and (ii), is uniformly approximated as, for any outcome y of Y,

$$g_{\underset{\sim}{\tau}, \underset{\sim}{\mu}, \underset{\sim}{\Sigma}}(y) \approx F(y) = \int_{x \in \mathcal{R}^k} F(Y = y | X = x) \cdot \prod_{i=1}^{S} F[X_i = x_i | X^{(i+1)} = x^{(i+1)}] dx \ , \tag{5.40}$$

where

$$\underset{\sim}{\tau} \triangleq \underbrace{(\tau_{0,j_0} \cdots \tau_{s,j_s})}_{\triangleq \tau(\underset{\sim}{j})}{}_{i=0,\ldots,s} \quad {}_{j_i \in J_i,} \qquad \underset{\sim}{\mu} \triangleq \underbrace{\left[\sum_{i=0}^{s} B^{(i)} \cdot \mu_{i,j_i} \right]}_{\triangleq \mu(\underset{\sim}{j})}{}_{i=0,\ldots s} \quad {}_{j_i \in J_i,}$$

$$\underset{\sim}{\Sigma} \triangleq \underbrace{\left[\sum_{i=0}^{s} B^{(i)} \cdot \Sigma_{i,j_i} \cdot B^{(i)\prime} \right]}_{\triangleq \Sigma(\underset{\sim}{j})}{}_{i=0,\ldots,s} \quad {}_{j_i \in J_i,} \ , \ \underset{\sim}{j} \triangleq (j_0, \ldots, j_s) \in J_0 \times \cdots \times J_s \ , \tag{5.41}$$

and where

$$B^{(0)} \triangleq I_m \ , \ B^{(1)} \triangleq B_{0,1} \ , \ B^{(2)} \triangleq B_{0,2} + B_{0,1} \cdot B_{1,2} \ ,$$

$$B^{(3)} \triangleq B_{0,3} + B_{0,2} \cdot B_{2,3} + B_{0,1} \cdot B_{1,3} + B_{0,1} \cdot B_{1,2} \cdot B_{2,3} \ ,$$

and more generally,

$$B^{(i)} \triangleq \sum_{\substack{\text{over all possible} \\ \text{paths } 0 < i_1 < \cdots < i_t < i, \\ t = 1,2,\ldots}} B_{0,i_1} \cdot B_{i_1 i_2} \cdots B_{i_t,i} = \sum_{\ell=1}^{i} B^{(i-\ell)} \cdot B_{\ell,i} \ . \tag{5.42}$$

Proof: (i) and (ii) follow from corollary 5.1. For (iii), consider

$$F(y) = \int_{x \epsilon \mathcal{R}^k} \left\{ \prod_{i=0}^{s} F[X_i = x_i | X^{(i+1)} = x^{(i+1)}] \right\} dx$$

$$\approx \int_{x \epsilon \mathcal{R}^k} \left\{ \prod_{i=0}^{s} g_{\Sigma_i, \mu_i, \nu_i}[x_i - B_i \cdot x^{(i+1)}] \right\} dx$$

$$= \sum_{\underset{\sim}{j} \epsilon J_0 \times \cdots \times J_s} h(x_0; \underset{\sim}{j}) . \tag{5.43}$$

where

$$h(x_0; \underset{\sim}{j}) \stackrel{d}{=} \int_{x \epsilon \mathcal{R}^k} \sum_{i=0}^{s} g_{\Sigma_{i,j}}[x_i - B_i \cdot x^{(i+1)} - \mu_{i,j_i}] \; dx$$

$$= h_s(x_0; \underset{\sim}{j}; x^{(s+1)}) , \tag{5.44}$$

where recursively h_i is defined as follows:

$$h_0[x_0; \underset{\sim}{j}; x^{(1)}] \stackrel{d}{=} g_{\Sigma_{0,j_0}}[x_0 - B_0 \cdot x^{(1)} - \mu_{0,j_0}] , \tag{5.45}$$

and for i = 1,...,s,

$$h_i[x_0; \underset{\sim}{j}; x^{(i+1)}] \stackrel{d}{=} \int_{x_i \epsilon \mathcal{R}^{k_i}} \left\{ h_{i-1}[x_0; \underset{\sim}{j}; x^{(i)}] \cdot g_{\Sigma_{i,j_i}}[x_i - B_i \cdot x^{(i+1)} - \mu_{i,j_i}] \right\} dx_i . \tag{5.46}$$

Then, beginning with i = 1, one applies the identity in eq. (5.30) to obtain h_1, followed in turn by a similar procedure for i = 2, using eq. (5.46), . . . , until step i = s is reached.

■

Remark 5.3.

Note that the results in corollary 5.2 (iii) are equivalent to specifying a finite Gaussian mixture, where for each mixing index $\underset{\sim}{j} \epsilon J_0 \times \cdots \times J_s$, the corresponding component distribution is Gaussian determining a random vector, say $U_{\underset{\sim}{j}}$, which has $E(U_{\underset{\sim}{j}}) = \mu_{\underset{\sim}{j}}$, $Cov(U_{\underset{\sim}{j}}) = \Sigma(\underset{\sim}{j})$, values given in eq. (5.41). In turn, from the forms in eq. (5.41), each $U_{\underset{\sim}{j}}$ is seen to be a collection of fixed linear combination of statistically independent Gaussian random vectors, say $U_{\underset{\sim}{j},\ell}$, where symbolically

$$U_{\underset{\sim}{j}} = B^{(0)} \cdot U_{\underset{\sim}{j},0} + \cdots + B^{(s)} \cdot U_{\underset{\sim}{j},s} . \tag{5.47}$$

and each $\underset{\sim}{U}_{j_\ell,\ell}$ corresponds to $[g_{\Sigma_{\ell,j_\ell}}(\cdot - \mu_{\ell,j_\ell})]_{j_\ell \in J_\ell}$. Each coefficient matrix $B^{(i)}$ in the linear combination via eq. (5.42) represents the overall regression transition matrix from Y multiplicatively up to X_i via all possible distinct path combinations of individual regression transition matrices among pairs of random vectors from $(Y,X_1,\ldots,X_i)$ connecting Y to X_i. Possible connections may exist also with Ingber's mesoscopic-macroscopic approach to C^3 modeling, where the path-integral representation relative to nonlinear, nonequilibrium Gaussian-Markovian statistical mechanics is used. This avenue remains to be explored. [Again, see Ingber (1986).]

■

Remark 5.4.

In applying corollary 5.2, the philosophy of approach is as follows: One does *not* know *a priori* the distribution of Y, the desired goal, but one does know — or has control in assigning — all of the intermediate or auxiliary conditional distributions, conditioning Y on X, X_1 on $X^{(2)}$, X_2 on $X^{(3)}$, ..., and finally X_s [$X^{(s+1)}$ being trivialized to make X_s an unconditional random vector]. Further help in reducing calculations in eqs. (5.41) and (5.42) will occur if sufficiency or Markovian-like assumptions can be made, thereby causing in effect a number of individual transition matrices B_{i_1,i_2} to be zero.

In particular, consider now applying corollary 5.2 to the evolution of node states as in steps (i) and (ii), sections 3 and 4, where ALDP=PL:

$$Y = X_0 \stackrel{d}{=} N^{++} = (\#WP_1{}^+, \#WP_2{}^+, \#WP_3{}^+, \#TROOP^+, EQMO^+, INFO^+) , \tag{5.48}$$

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{21} \end{pmatrix} , \quad s = 21 \tag{5.49}$$

where, replacing directly N, and hence N^+, as well as T^+, by their component subvariables, as given in eqs. (3.3) and (3.4),

$$X_1 \stackrel{d}{=} R^{++} , \tag{5.50}$$

$$X_2 \stackrel{d}{=} DEC^+, X_3 \stackrel{d}{=} CONS^+, X_4 \stackrel{d}{=} FUS^+, X_5 \stackrel{d}{=} HYP^+, X_6 \stackrel{d}{=} ALG^+, X_7 \stackrel{d}{=} DET^+ , \tag{5.51}$$

$$X_8 \stackrel{d}{=} \#WP_1{}^+, X_9 \stackrel{d}{=} \#WP_2{}^+ \; X_{10} \stackrel{d}{=} \#WP_3{}^+ ,$$

$$X_{11} \stackrel{d}{=} \#TROOP^+, X_{12} \stackrel{d}{=} EQMO^+, X_{13} \stackrel{d}{=} INFO^+ , \tag{5.52}$$

$$X_{14} \stackrel{d}{=} S , \tag{5.53}$$

$$X_{15} \stackrel{d}{=} R \tag{5.54}$$

$$X_{16} \stackrel{d}{=} \#WP_1, X_{17} \stackrel{d}{=} \#WP_2, X_{18} \stackrel{d}{=} \#WP_3,$$

$$X_{19} \stackrel{d}{=} \#TROOP, X_{20} \stackrel{d}{=} EQMO, X_{21} \stackrel{d}{=} INFO . \tag{5.55}$$

In turn, due to the assumptions in table 2, a number of transition matrices can be set equal to zero, as mentioned earlier:

$$\left.\begin{array}{l} B_{0,i}\ ,\ i=2\ ,\ldots,7\ ,14\ ,\ldots,21\ , \\ B_{1,i}\ ,\ i=3\ ,\ldots,7\ ,14\ ,\ldots,21\ , \\ B_{i_1,i_2},\ i_1=2\ ,\ldots,7\ ,\ i_2=14\ ,\ldots,21\ , \\ B_{i,15}\ ,\ i=8\ ,\ldots,13\ , \\ B_{14,i}\ ,\ i=16\ ,\ldots,\ 21 \end{array}\right\} = 0\ . \tag{5.56}$$

Thus, the key calculations in equations (4.12), (4.13), and (4.16) through (4.19) can all be replaced, using eqs. (5.48) through (5.56) in corollary 5.2 (iii).

If one considers the number of operations required in eqs. (4.12), (4.13) and (4.16) through (4.19) directly to compute $p(N^{++})$, assuming equally sized domains of values, say q (as noted at the end of section 4), one obtains here on the order of q^{21}, 21 being the number of variables being integrated out in eq. (4.1). On the other hand, the linear-Gaussian mixture approximation of corollary 5.2 (iii) requires about $\text{card}(J_0)^{21}$ mixing coefficients, when $\text{card}(J_i) = \text{card}(J_0)$, for all i. In addition, for each mixture distribution corresponding to a mixing term, there are $2 \times 22 + 3 \times 22$ entire matrix multiplication and addition operations for obtaining the characterizing Gaussian parameters: the mean and covariance matrix. There is also an upper bound of 2^{21} — reduced by used of eq. (5.56) — number of matrix additions of matrix multiplication required to obtain $B^{(0)},\ldots,B^{(21)}$, recursively, as in eq. (5.42). Hence, for the linear-Gaussian approach, a multiplicative value of $[2\text{card}(J_0)]^{21}$ could be required for implementation.

Thus, if the average number of mixing coefficients can be reduced so that

$$\text{card}(J_0) \ll q/2\ , \tag{5.57}$$

then the linear-Gaussian mixture approach can be of real use. This will occur especially when the number of C^3 variables describable by single-Gaussian distributions or by some absolutely continuous distributions which are bimodal or at least relatively minimum in number of modes. This is opposite in kind to the situation where most C^3 variables are discrete with the only good Gaussian mixture approximations being essentially the same as the original discrete distributions, but with each corresponding dirac delta function replaced by a Gaussian distribution of sufficiently small covariance matrix, as in the proof of theorem 5.1. In the latter case, eq. (5.57) will be violated and the better approach is to stay with the original integrals, possibly discretizing them.

■

One desirable property of Gaussian mixtures is the ease in computing means and covariance matrices, once the mixing parameters are all determined. Thus, it follows easily that for Y in corollary 5.2

$$E(Y) \approx \sum_{\underset{\sim}{j} \in J_0 \times \cdots \times J_s} \tau(\underset{\sim}{j}) \cdot \mu(\underset{\sim}{j})\ , \tag{5.58}$$

$$\mathrm{Cov}(Y) \approx \sum_{\underset{\sim}{j} \epsilon J_0 \times \cdots \times J_s} \left(\pi^{(\underset{\sim}{j})} \cdot \Sigma^{(\underset{\sim}{j})} + [\mu^{(\underset{\sim}{j})} - E(Y)] \cdot [\mu^{(\underset{\sim}{j})} - E(Y)]^T \right). \tag{5.59}$$

Finally, mention must be made of another possible source of difficulty in implementing the linear-Gaussian mixture method. This involves the actual construction of the Gaussian mixtures, mentioned previously. [Again, see Goodman and Boyer (1975) and Medgyessy (1961) for techniques.]

6. CONCLUSIONS

To implement the proposed general C^3 decision game, as outlined in the previous sections, for a given simulated scenario, one must be careful in defining the boundaries for what constitutes the relevant C^3 variables. This is a relative concept. In a given situation, a node may represent simply a single person or machine, such as a tank crew, or it may represent an entire group of tanks, depending on the desired aggregation or hierarchical level considered. In addition, before implementing the model proposed here, one must scope out what constitutes a "signal" input-output node cycle and over what time periods the "signals" occur.

Consider, for example, the following possible simplified inner-outer air battle:

(1) Enemy bombers (I) arrive in formation towards grouping of friendly ships of two types (IV,V).

(2) Friendly scout airplanes (II) detect/surveil I and pass information to friendly fighter airplanes (III) as well as to IV,V.

(3) III meet I and attack, I being passive.

(4) Remaining I continue toward IV,V, with III now ceasing attacks.

(5) IV,V send missiles against remaining I, before themselves are bombed by I.

(6) Remaining I, following now above missile attack, bomb IV,V.

(7) End of scenario as I turns away.

For an outline of approach, the aggregation level here is to make each individual combatant a distinct node. Thus, there are five types of nodes here: enemy bombers (I), friendly surveillance airplanes (II), friendly fighter planes (III), friendly ships of types (IV) and (V). Within each type, one can designate individual nodes by suitable indexing. In addition, one *must* determine for each side, friendly or adversary, all of the relative primitive relations such as given in eq. (4.21) in probability form. All of these will also depend on what constitute node cycles. Based on this scenario, table 4 presents a tentative collection of epochs of node interaction which can be identified as input-output node-processing cycles.

Future efforts will be directed toward further implementation of the C^3 decision game relative to particular scenarios such as the inner-outer air battle.

Table 4. Categorization of events occurring during inner-outer air battle.

Node Type	Input-output node cycles occur at phase changes which are by doctrine and game design								
I: Enemy Bombers	Initial	BFA by III BFA by IV BFA⁻ by V	BFA by III BFA by IV BFA by V	BFA⁺ by III BFA⁺ by IV BFA⁺ by V	BOMB IV BOMB⁻ V	BOMB IV BOMB V	BOMB⁺ IV BOMB⁺ V	End	
II: Friend Surv.	Initial	SPOT I	SPOT I	SPOT⁺ I	SINF III SINF IV SINF V	End			
III: Friend Fight	Initial	RINF from II	RINF from II	RINF⁺ from II	FIR⁻ at I	FIR at I	FIR⁺ at I	End	
IV: Friend Ship	Initial	RINF from II	RINF from II	FIR at I	FIR at I	FIR⁺ at I	BBOM⁻ by I	BBOM by I	BBOM⁺ by I	End
V: Friend Ship	Initial	RINF⁻ from II	RINF from II	FIR at I	FIR at I	FIR⁺ at I	BBOM by I	BBOM by I	BBOM⁺ by I	End

Symbols: BFA = Being Fired At , SPOT = Spotting , SINF = Sending Information
RINF = Receiving Information , FIR = firing , BBOM = Being Bombed ,
BOM = Bombing , ( ) = Just Before , ( )⁺ = Just After

EACH ROW LISTS EVENTS IN SEQUENCE OF OCCURRENCE. COLUMNS ARE NOT RELATED TO SUCH OCCURRENCES.

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