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The major portion of the research effort was devoted to the development of unified framework for the description of approximation reasoning methods that facilitates the study of their fundamental characteristics. This objective was attained by consideration of structures, defined in spaces of possible worlds that measure either the relative size of certain subsets (for probabilistic methods) or the similarity between possible states (for possibilistic methods). Possible worlds are formalizations of the notion of possible state or behavior of a system. Using this concept, an approximate reasoning problem may be described as one where available evidence is insufficient to determine if the actual state of the world lied among those conceivable possibilities, conceivable where a statement about the system is true.

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ADVANCED CONCEPTS AND METHODS OF APPROXIMATE REASONING

SRI Project 6488 Final Report

December 1, 1989

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ADVANCED CONCEPTS AND METHODS OF APPROXIMATE REASONING

Final Technical Report
December 1, 1989

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1 Background

The research program on advanced concepts and methods of approximate seeks to establish clear formal foundations that advance the understanding of approximate reasoning methodologies. The approaches that are being studied are fundamental techniques for the analysis of imprecise, uncertain, and unreliable data that are applicable in a wide variety of important contexts.

In particular, we want to identify and study frameworks that facilitate the comparison of the features of each approach allowing the determination of its utility in the solution of specific problems. Our research also seeks to broaden the scope of applicability of existing methods by consideration of approximate reasoning mechanisms that, going beyond the mere extension of classical deductive techniques, seek to develop intelligent systems capable of performing inductive (i.e., learning), abductive (i.e., discovery or explanation), and analogical (i.e., similarity-based) functions. Furthermore, we are interested in expanding the scope of our knowledge sources beyond behavioral knowledge (e.g., expert-generated rules) and current observations, to include historical databases of relevant experience.

2 U.S. Air Force Relevance

The questions addressed by this program of research are related to basic issues of knowledge and information and, as such, applicable results will have a wide impact accross a variety of important applications of USAF interest.

Practically every important real-life problem is characterized by the presence of information that is not totally precise, certain, or credible. These undesirable knowledge features are often found in the military domain where the size and complexity of systems, coupled with the presence of agents actively seeking to deny and falsify information, renders their precise observation difficult or impossible.

The need to process imprecise and uncertain knowledge is obvious in military intelligence problems, where the objectives are situation assessment and decision-support on the basis of the information provided by multiple items of evidence that, typically, are imprecise, incomplete, and of limited reliability. In many other problems of Air Force interest, however, availability of tools for approximate reasoning (including methods to determine applicability and usefulness of specific techniques) is of paramount importance.

Probabilistic reasoning, for example, is a key element of the command and control process beyond situation assessment, due to its direct relevance to issues such as the determination of the viability of missions and the reliability of information sources and control chains. In a broader context, probabilistic analysis is an essential tool in the failure-diagnosis and reliability-analysis problems that are commonly found any organization that utilizes large-scale systems.

Possibilistic reasoning methods, because of their relations with analogical reasoning (which were elaborated and clarified in the task being described), are also of direct relevance to a myriad of problems of interest. Situation analysis, plan construction (e.g., mission planning), and system design are just a few of the potential applications of methods that exploit databases of historical experience to determine solutions to new problems. For example, in a command and control application, lessons learned in previous situations may be directly retrieved and analyzed to determine courses of action that are applicable in the current context. Similarly, system design (e.g., an aircraft subsystem) might be considerably simplified by use of similarity-based tools that suggest *plausible* design choices on the basis of existing knowledge.

Beyond these applications of "case-based reasoning," recent experience with the development of large-scale controllers based on possibilistic logic indicates that this type of reasoning leads to the development of autonomous, robust controllers for unstable systems. Among these, the control of active flexible wings using a fuzzy-logic approach (being currently considered by Rockwell International) deserves special mention due to its USAF relevance. Similar controllers might be also conceivably used to stabilize autonomous walking robots and to plan their activities.

3 Accomplishments

The major portion of our investigative effort was devoted to the development of a unified framework for the description of approximate reasoning methods that facilitates the study of their fundamental characteristics. This objective was attained by consideration of structures, defined in spaces of *possible worlds* that measure either the relative size of certain subsets (for probabilistic methods) or the similarity between possible states (for possibilistic methods).

Possible worlds are formalizations of the notion of possible state or behavior of a system (e.g., the possible, but typically unknown, situation in a battlefield, possibly encompassing its potential modifications in time). Using this concept, an approximate reasoning problem may be described as one where available evidence (e.g., battlefield intelligence) is insufficient to determine if the actual state of the world lies among those conceivable possibilities (i.e., possible worlds), where a statement ('hypothesis") about the system is true (e.g., whether a SAM battery is currently at a specific location).

The major contribution of the research performed during the reporting period has been the interpretation of possibilistic methods in terms of similarity functions between possible worlds. The formal results derived in this research, which are summarized in the paper "The Semantics of Vague Knowledge," which is enclosed as an integral part of this report, show that possibilistic methods are substantially different in nature from their probabilistic counterparts. Furthermore, as discussed in detail in that work, these results have shown that all major technologies proposed for the analysis of imprecise information, including nonmonotonic logic and "qualitative reasoning" approaches, may be easily described and understood in terms of models based on possible worlds.

For example, probabilistic methods may be characterized as being concerned with the estimation of measures of the sets of possible worlds that are both compatible with the evidence and are such that the hypothesis is true. Since any proposition is equivalent to a set of possible worlds, these set measures are usually estimated by the past frequency of truth of the hypothesis under similar circumstances. Probabilistic assessments describe therefore the "tendency" or "propensity" of a system to behave in certain ways (for example, to break down after so many hours of operation). Except in extreme cases, these assessments do not assert that the hypothesis is true or false but rather that there is a likelihood (expressed numerically) or chance that the hypothesis will be true.

Possibilistic methods, on the other hand, are concerned with the identification of statements that are true and that resemble, in some respect, the hypothesis. Their bases are certain measures (metrics) that describe how "similar" or "close" are pairs of possible worlds rather than to measures that characterize the "size" of subsets of possible worlds. These metrics formally capture the notion that two possible states of affairs are similar in that certain propositions that are true in one resemble those that are true in the other (e.g., "the pressure is greater than 100 lb/sq.in." and "the pressure is greater than 110 lb./sq.in."). While a probabilistic statement describes tendency towards truth (e.g., "the probability of runway destruction is 80%"), the possibilistic answer asserts the truth of a related proposition (e.g., "the runway will be definitely inoperative for all aircraft of type A or type B").

Contrary to the opinions held by some, the results of our research show that possibilistic methods are not easily interpreted or explained by probabilistic structures. Possibilistic structures, on the other hand, have been shown to be close in character to the discretizations used in "qualitative reasoning," where scalar variables are substituted by coarser frameworks that replace all numbers by three possible values: zero, negative, and positive. The possibilistic schemes generalize this idea in that significant groups of variable values (or "granules") may be arbitrarily defined and in that these granules are "fuzzy," in the sense that whenever the value of the variable is "close" to some *typical* value in the granule, results applicable to the typical value may be "extrapolated" to the actual value.

Furthermore, our research indicates that it is also improper to regard probabilistic and possibilistic methods as competitive technologies. Since their aims and output are fundamentally different, the proper attitude is to regard these methodologies as complementary tools that help, in different ways, in assessing the state of the world.

The formal model leading to our results is a Kripke-type semantic model with the customary relation of *accessibility* replaced by multiple relations indexed by a parameter α. Although it is easier to think of this parameter in numerical terms, our model is very general allowing the use of symbolic, nonnumeric, scales to assess resemblance. Furthermore, our formulation justifies certain formal requirements that any similarity measure must obey. The major highlights of the model are described in the technical note "On the Semantics of Fuzzy Logic," which is enclosed as part of this report. These developments may be summarized as follows:

- Definition of multiple accessibility relations by a similarity function that defines a metric in a space of possible worlds (thus allowing use of "continuity" arguments to "extrapolate" results from one world to those that are close to it)
- Generalization of the modal notion of possibility to a graded notion of possibility that is related to the so-called "de re" interpretation of conditional statements in modal logic.
- Characterization of similarities as being defined either from the joint viewpoint of several variables or descriptors (joint similarities), or being limited to considerations from some limited respect (marginal similarities).
- Identification of relationships of marginal similarities with topological and metric concepts (mainly, the so-called "Hausdorff" distance).
- Definition of unconditioned and conditional possibility functions from similarity functions.
- Formal justification of the *generalized modus ponens* of Zadeh as an extension of the corresponding classical inferential rule. This central result generalizes the transitivity of set inclusion that makes the modus ponens valid (i.e., if A is a subset of B and if B is a subset of C, then A is a subset of C) into a relationship between the sizes of the "neighborhoods" of sets that include each other (e.g., if A is in a neighborhood of size α of B, and if B is in a neighborhood of size β of C, then A is in a neighborhood of size $\gamma = f(\alpha, \beta)$ of C). The generalized modus ponens, therefore, combines logical principles with the properties of a metric relation to provide a sound, correct, form of logical "extrapolation."
- Characterization of the problem (important in practice) of derivation of similarity functions from possibility functions.

In addition to our basic research in the semantics of possibilistic approaches, we have continued our research into the definition and utilization of conditional belief measures in the Dempster-Shafer calculus of evidence. Applicable formulas are currently being evaluated on the basis of their applicability to general cases (in general, the combination of conditioned and unconditioned evidence does not lead to functions that are compatible with the axioms of the evidential calculus) and in terms of the computational complexities of the algorithms required for their evaluation.

4 Status and Plan

Our basic semantic model of fuzzy logic is complete. Our immediate concern is the evaluation of alternative formulations that rely on classes of similarity functions that satisfy certain important properties (mainly assuring that the value of the similarity between two objects from some respect, like *color*, be always higher than the value of the similarity between those objects from multiple respects, e.g., *color* and *shape*).

Our long term plans, however, focus on the important problem of deriving similarity values from possibility distributions. In our model, possibility distributions may be thought of as similarities from some respect (e.g., "pressure") that measure how close is a particular situation (e.g., "pressure greater than 50 lb./sq.in.) to a set of "typical examples" (e.g., "pressure greater than 100 lb./sq.in."). This measure of object-to-set resemblance defines a "linguistic value" (e.g., "very high pressure") that may be used as the basis to extrapolate from statements that are true in any prototype to statements that are true in the particular case under consideration.

The role of similarities in our formulation, however, is primarily conceptual; intended to explain a complex notion (i.e., possibility) in terms of a more primitive concept (i.e., similarity). Although our formulas permit the computation of possibility values from similarity values, similarities (representing proximity from the joint viewpoint of several respects) will be derived, in practical applications such as similarity-driven case-based reasoning, from possibility distributions (characterizing proximity between sets of objects from a limited perspective). For this reason, it is our intent to focus future attention on the problems associated with the derivation of similarity functions from possibility distributions. Our point of departure is existing work linking similarity relations with certain classes of subsets of possible worlds. The derivation of specific formulas must await, however, the evaluation of models based on restricted classes of similarity functions characterized both by desirable theoretical properties (such as mentioned above) and by their utility in practical applications (primarily, case-based reasoning).

In addition, we plan to utilize the formulas and relations derived in our semantic model to further extend possibilistic calculi by identification of relationships between distributions that may be used to compute some of them as a function of others (e.g., conditional possibility distributions from joint and marginal unconditional distributions). In order to assess the applicability and efficiency of algorithms based on such relations, we plan to develop (in collaboration with Dr. Leonard Wesley of the Artificial Intelligence Center, SRI International) a computational environment (**ANALOG**) for the testing of similarity-based analogical reasoning procedures. As part of these activities, Dr. Wesley is currently engaged in the collection of suitable databases that may be used in our computational experiments.

5 Conference Participation. Publications

1. E.H. Ruspini. *The Semantics of Vague Knowledge.* Presented at the Second International Conference on the Processing and Management of Uncertainty by Expert Systems, Urbino, Italy, 1988.

2. E.H. Ruspini. *Generalized Similarity Relations and the Semantics of Fuzzy Logic.* Presented at the Workshop on Approximate Reasoning in Expert Systems, Blanes, Spain, 1989.

3. E.H. Ruspini. *The Semantics of Fuzzy Logic.* Presented at the Third International Fuzzy Systems Associations Conference, Seattle, Washington, 1989.

4. E.H. Ruspini participated as an invited discussant in the Workshop on Nonstandard Logics, Rocamadour, France, 1988. His discussion of papers presented by panelists presenting position papers in approximate reasoning will appear in a volume to be published by Academic Press in 1989.

5. E.H. Ruspini participated as a reviewer in the DRUMS/RP3 program sponsored by the European Economic Community.

6. E.H. Ruspini. The Semantics of Vague Knowledge. *Revue Internationale de Systémique,* to appear, 1990.

7. E.H. Ruspini. *On the Semantics of Fuzzy Logic.* Technical Note No. 475, SRI International, Menlo Park, California, November 1989.

In addition the principal investigator was the recipient of a Fulbright Fellowship to conduct a course in Approximate Reasoning in Spain in the Spring 1989.

RESEARCH PUBLICATIONS

The Semantics of Vague Knowledge

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Abstract

This paper is devoted to the discussion of basic issues related to the meaning of imprecise, uncertain, and vague knowledge, its manipulation, and its utilization. The informational deficiencies that characterize this type of knowledge are described in terms of the impossibility to determine, without ambiguity, the truth value of certain hypotheses — i.e., statements of interest to those seeking to understand the state and behavior of a real-world system.

Using a "possible worlds" perspective, this inability may also be characterized by the presence of conceivable (i.e., consistent with evidence) circumstances where the proposition is true, and of equally admissible circumstances where it is false. From such a viewpoint, approximate reasoning techniques are presented as producers of correct descriptions of properties of the class of possible worlds that are consistent with observed evidence, rather than as the results of some relaxation of the notion of "truth-value."

Two major classes of approximate reasoning systems are identified — probabilistic and possibilistic — and their major conceptual differences are described. The theoretical underpinnings of each methodological approach are described, and the current level of understanding of their major functional structures and concepts is discussed.

The discussion of probabilistic approaches encompasses both subjectivist and objectivist perspectives, and also includes nonclassical approaches (such as the Dempster/Shafer calculus of evidence) that are related to the notion of interval probabilities. The discussion of possibilistic approaches, on the other hand, stresses the relations between the concepts of possibility and similarity that have been recently studied by the author.

Finally, nonmonotonic logic and qualitative process theory concepts are briefly examined from the perspective of possible-world semantics.

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1 Introduction

This paper is devoted to the discussion of basic issues relevant to the purpose of approximate reasoning methodologies with emphasis on the meaning of their basic structures and concepts. Approximate reasoning systems may be briefly characterized as automated agents (e.g., computer programs and systems) that seek to identify the state of a real-world system on the basis of knowledge that it is *imprecise* — i.e., available information does not possess the desired degree of detail — and *uncertain* — i.e., we are not absolutely certain about the correctness of such information.

Under these conditions it is possible, usually easily so, to conceive of situations where, given available information, some statement about the real world is true. Under other conceivable circumstances — equally admissible given the available knowledge — that statement is false. In a majority of weather-forecasting applications, for example, the information collected by a variety of sensors is often insufficient to determine if rain will fall at a given location at a given future time. Depending on the evolution and interaction of the different components and subsystems of the atmosphere, rain may actually fall or may not fall.

The importance and ubiquity of problems characterized by information that is imprecise and uncertain make the development of so-called "approximate reasoning" systems one of the most important technological requirements to be met by artificial intelligence procedures that, going beyond the foundations of classical deductive techniques, must cope with the undesirable features of the underlying knowledge. The current lack of understanding of the principles that underlie these methodologies combined with their present state of technological development — often exemplified by the use of questionable "ad hoc" methods — has led to considerable controversy among practitioners who have, in recent years, debated their relative advantages and disadvantages.

The absence of a formal unified framework for the description of the underlying concepts and structures of various applicable technologies has complicated their understanding and comparison, making it nearly impossible to develop even a partial consensus about the relative applicability of each methodology. Lacking formal structures to guide, in a rigorous fashion, the use of terms such as "probability" and "possibility," each capable of being interpreted in a variety of ways, it is nearly impossible to evaluate arguments advanced for or against particular positions. Furthermore, problems such as the determination of the validity of the output of approximate reasoning systems, or of their usefulness in specific circumstances (or even establishing the meaning of such notions), have remained largely unaddressed.

This paper reports on the results of research toward the development of firm foundations for the unified description of approximate reasoning methods, with emphasis on the interpretation of their underlying concepts and structures. The formal framework derived in this research is based on the notion of "possible worlds" as introduced in modal logics [15]. In this paper, our attention will be mainly focused on various types of probabilistic, discussed in Section 2, and possibilistic reasoning methods, presented in Section 3. Included is a discussion of relations with qualitative and nonmonotonic reasoning methods, which are also concerned with problems associated with imprecise and uncertain information. Before presenting such issues, it is important to consider the general nature of the approximate reasoning problem.

1.1 The Nature of Approximate Reasoning

The goal of any system that relies on inference techniques is to assign a *truth value*, which may be either **true** or **false**, to statements — called *hypotheses* — about the state or behavior of a real world system. Due to its very nature, however, the approximate reasoning problem is unsolvable, because of either fundamental or practical limitations.

Available information is often insufficient to determine, by means of conventional inference procedures, if a hypothesis is true or false. In some problems, the impossibility is of a more practical nature: there are not enough resources (e.g., memory, computer time) to determine if the hypothesis is true or not.

Whether the impossibility is fundamental or practical, the important fact is that, as posed, an approximate reasoning problem is not solvable. Information *constrains* the possible truth values of hypotheses but rarely restricts them to unique values. In general, those constraints determine a set of possible solutions. Each such solution is an assignment of truth values that is logically consistent with observed facts and system knowledge (typically expressing laws of system behavior). For example, an observation, made several days earlier about the location of an automobile on a highway, augmented by knowledge about the capability of such a vehicle to proceed at certain speeds through some roads, may be sufficient to determine a set of its possible current locations, but it will usually be unable to pinpoint any one of them as the only possible place where the vehicle could be at the present time.

The solution of an approximate reasoning problem is therefore a *set of possibilities*[1] that are logically consistent with available information. In this document we use the term *possible worlds*, which is borrowed from logic (specifically modal logic), to denote each such possibility [4].

In most approximate reasoning problems it is not practically possible to describe a set of possible worlds to an acceptable level of detail. Different methodologies have been developed, however, to describe some properties of the set of possible solutions or, more generally, certain constraints on values that measure such properties. For example, probabilistic methods seek to identify the probability distribution of some of the variables that are used to characterize each possible world. As we will see, often even this level of detail may not be attained, and the best we can do is to indicate that certain probability distribution values are possible while others are not (e.g., the probability of rain will be between 60% and 80%).

[1] Note that this use of the term *possibility* is different from that used below in connection with possibilistic reasoning.

1.2 Possible Worlds

Possible worlds, as informally described above, are the solutions of an approximate reasoning problem that are consistent with existing information and knowledge. In many problems, each of these solutions corresponds to the state of a real-world system at a given instant in time. In other examples, each possible world may also include descriptions of past, present, and future (predicted) states of the real world. In some planning and control problems (e.g., autonomous robot path and activity planning), each possible world may correspond to a description of the characteristics of a plan formulated by *rational agents* seeking to control certain aspects of system behavior together with its resulting effects on the planned system and its environment.

The characteristics and complexity of each possible solution are, therefore, highly dependent on the particular real-world system being studied and the analytical requirements of the users of the approximate reasoning system. Although, as we have just seen, this diversity of needs leads to widely different types of possible worlds, there exists a high-level, logical characterization of the concept of possible world in terms of the possible truth of statements (propositions) about the real-world system being studied. This characterization was derived by Carnap [5], who also proposed a conceptual procedure for the generation of descriptions of all possible states of affairs.

While Carnap considered first-order-logic systems in his characterization of the concept, we shall confine ourselves to a simpler, proposition-based description that captures the essence of his construction procedure. Before proceeding to its discussion it is very important to remark, however, that the Carnap procedure is a *conceptual process* intended primarily to formalize the notion of possible world while providing clear foundations for the discussion of other concepts (e.g., *possible truth*). The combinatorial explosion associated with Carnap's process makes unfeasible the actual enumeration and representation of possible-world spaces in real-life problems.

The procedure of Carnap starts with consideration of a finite number of *ground* propositions

$$p_1, p_2, \ldots, p_m$$

that describe characteristics of a real-world system. For example, in a weather-forecasting application, these propositions may include declarative knowledge statements such as: "The total rainfall will be less than 1 cm." These statements are intended to capture those aspects of the behavior of the world that are important to analysts and to identify that behavior to the necessary degree of precision.

After these propositions have been identified, the process proceeds to consider all the conjunctions of the type[2]

$$p_1 \wedge p_2 \wedge \neg p_3 \wedge \ldots \wedge p_m ,$$

[2]Throughout this paper we use the conjunction symbol $\wedge$ to mean "and," the disjunction symbol $\vee$ to mean "or," and the negation symbol $\neg$ to mean "not."

where each of the ground propositions appears once either as given or negated. If m ground propositions had been identified, this process leads to 2^m conjunctions. We eliminate from this set conjunctions that represent logical impossibilities like, for example: "the total rainfall will be less than 1 cm **and** the total rainfall will be more than 3 cm," and those that are logically inconsistent with several prespecified propositions — *axioms* about the behavior of the system being studied — $a_1, a_2, \ldots, a_l$, that are always assumed to be true.

The remaining members of this propositional set, or *Carnapian Universe*, are called *possible worlds*. Each possible world is a description (to the maximum level of detail allowed by our original set of ground propositions) of a possible, although typically unknown, state of the system under study. Each such description is consistent both with the laws of logic and with the axioms that constrain system behavior and may be thought of as a function (called a *valuation*) that assigns to each relevant proposition a truth-value that is either "true" or "false." Similarly, possible worlds may be thought of as sets of propositions that contain all propositions that are true and the negation of those that are false, as illustrated in Figure 1 where each possible world is revealed, through the help of a hypothetical "logical"

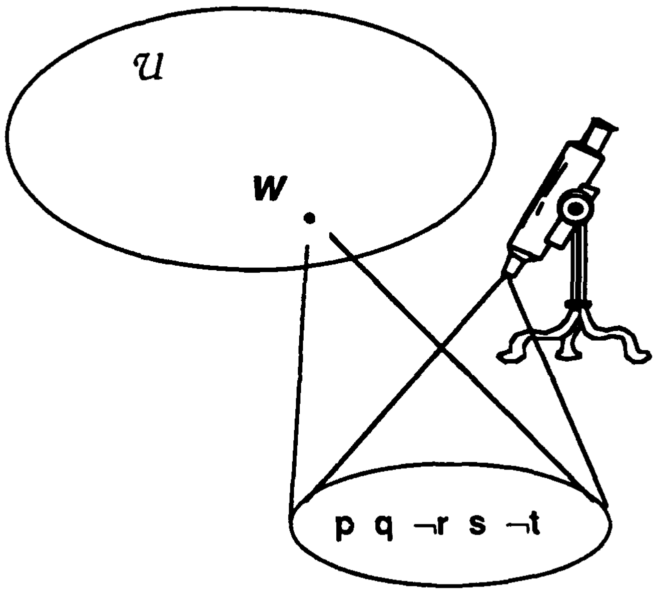

Figure 1: The Carnapian Universe.

microscope as a collection of true propositions. Furthermore, each possible world differs from any other in that at least one proposition that is true in one world is false in the other.

From this logical perspective, which is particularly useful in artificial intelligence applications, the observations in a body of evidence, which correspond to the truth of certain propositions, may be thought of as constraints on the subsets of possible worlds where the state of the real-world system actually lies. Possible worlds that are logically consistent with those propositions (said to be *compatible* with the evidence) are, generally, a proper subset of the Carnapian universe of possibilities.

It is generally agreed that "stronger" or "better" evidence results in subsets of possible worlds that are smaller, in some sense, than "weak" evidence. The quality of evidence, however, should be judged from a variety of standards. Among those, domain-dependent criteria are usually the most important in assessing the quality of informational bodies. In general, it is desirable that the evidence be such as to allow unambiguous answers to certain questions of importance (i.e., hypotheses). To rephrase this statement with the help of the Carnapian characterization, it is desirable that the evidence be such that propositions of importance be true (or false) for every possible world compatible with the evidence, rather than true for some and false for others.

As we have stressed before, however, an approximate reasoning problem is such that the evidence is incapable of determining whether a hypothesis is true or false, as illustrated in Figure 2. Approximate reasoning systems are concerned with the description of certain

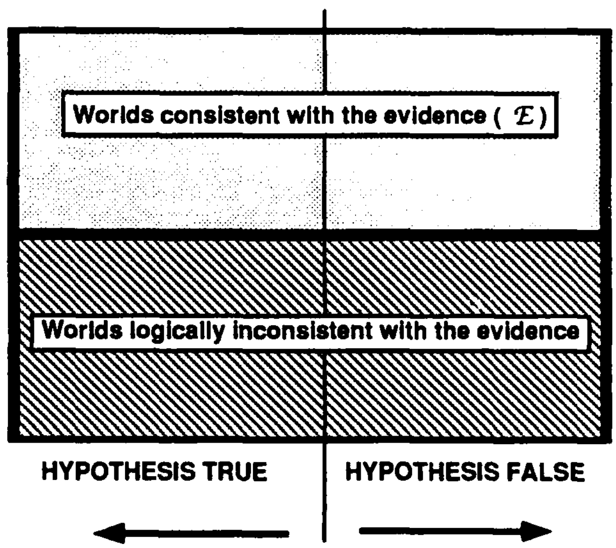

Figure 2: The Approximate Reasoning Problem.

properties of the set $\mathcal{E}$ of possible worlds that are consistent with the evidence, seeking primarily to characterize the subsets $\mathcal{H} \cap \mathcal{E}$ and $\overline{\mathcal{H}} \cap \mathcal{E}$ of worlds compatible with the evidence where a hypothesis is either true or false, respectively. The descriptions that they provide, however, are of a substantially different nature for different approaches — not being all based or explained, as often erroneously claimed, by probabilistic notions.

1.3 Probabilistic and Possibilistic Reasoning

In this paper we will be concerned primarily with the two major types of approximate reasoning methodologies that are being actively used to treat practical situation-assessment and planning/decision problems. These methodologies are commonly said to be *probabilistic* or *possibilistic*, respectively.

Probabilistic methods seek to describe the structure of a set of possible worlds by means of certain conditional probability distributions (the condition being the actual evidence at hand). If these distributions are considered to represent the *tendency* or *propensity* of the world to act in a repetitive fashion that may be described by a *frequency of occurrence*, they are said to have an *objectivist* interpretation; if they represent, on the other hand, the degrees of belief (or of commitment to certain courses of action) of certain rational agents, then they are said to have a *subjectivist* interpretation.

Irrespectively of the particular interpretation used, probabilistic reasoning methods are concerned with the *likelihood* (either measured by previous experience or believed by an agent) that a particular hypothesis will be true in a given situation. Save for exceptional cases (i.e., probabilities equal to 0 or 1), no firm assurances are given to the user of any probabilistic methodology about the actual state of the world or its behavior. The probabilistic assessment is one of *tendency* and is primarily useful in the "long run," that is, when evaluated by criteria that take into account the aggregate performance of the approximate reasoner over many situation-assessment and decision-aid examples.

Probabilistic results are particularly useful in organizations such as insurance companies or gambling houses, where success is evaluated in terms of a population of examples (i.e., all insurance policies or all gambling customers). By this statement we do not mean that probabilistic information is useless for single cases or "short runs."[3] Our point is that, for all we know, the hypothesis may be true or may be false (that is the nature of the approximate reasoning problem). Under such circumstances, decisions that could possibly lead to an undesirable state of affairs may deserve to be analyzed from other viewpoints.

Possibilistic reasoning, on the other hand, seeks to describe possible worlds in terms of their similarity to other sets of possible worlds by placing emphasis on assessments that may be assured to be valid in each particular case and situation. Rather than describing relative proportions (of occurrence) of possible worlds where a hypothesis of interest is true or false, as done by probabilistic methods, possibilistic reasoning seeks to describe all possible worlds that are compatible with evidence, in terms of their resemblance to members of certain sets of "exemplary" or "typical" worlds.

For example, a probabilistic method may determine that a corporation has a probability of 80% of exceeding its profit goal for the year. This assessment is not an assurance that such a goal will be attained. It does provide, however, some basis for subsequent management policy. While there is a chance that profits will fall short of the goal, if management policy be consistently applied in every fiscal period, then, in the long run, proper rational decisions would have been made and the company could be expected to prosper (despite possible occasional setbacks). A possibilistic method, on the other hand, may assert that profits will amount to *at least* 70% of the goal figure. On some previously agreed similarity

[3]Our view, that decisions that are best in the "long run" may not be the same as those that are best in single instances, does not agree with current subjectivist orthodoxy.

scale such a statement may be translated into the possibilistic statement: "the possibility of achieving the profit-goal is 0.7." Note that the emphasis is on certainty and *comparison* between statements rather than on *likelihood* and *chance*.

In general, possibilistic methods, which are strongly rooted on *fuzzy set theory* [41], provide assessments such as "the profit will be adequate," indicating that the predicted value of the profit will have a similarity greater than zero (sometimes possibilistic techniques produce specific lower bounds) to a value that is a good example of "adequate gain." Often it is also said that these vague statements describe the degree of ease by which the concept "adequate" matches the situation at hand. The ability to represent vague concepts by possibility distributions — attained by indicating that a value of a variable matches the vague concept to a degree — is central to fuzzy set theory, which was conceived as a basis for the formal treatment of linguistic utterances as they are commonly found in everyday discourse.

In summary, we may say that the approach to the analysis of imprecise and uncertain information that is used by any approximate reasoning methodology is based on the solution of a problem that is related to but different from, the unsolvable problem of determining, without ambiguity, the truth of a hypothesis. In the probabilistic case, the answers provided consist of estimates of frequency of the truth of the hypothesis in similar cases as determined by prior observation (objectivist interpretation) or degree of commitment in a gamble based on the actual truth of the hypothesis (subjectivist interpretation). In the possibilistic case, in contrast, the answers provided assert that a related, similar, hypothesis is true.

2 Probabilistic Reasoning

Probabilistic reasoning methods focus on the description of the relative proportions of the occurrence of truth or falsehood of certain hypotheses under certain evidential constraints. These constraints, representing available evidence $\mathcal{E}$, conditions the probabilities $P(X = x|\mathcal{E})$ describing the frequency of occurrence of the value x of the state variable X when $\mathcal{E}$ is true. Using again the Carnapian characterization, we may describe these techniques as being concerned with the determination of the probability of some subsets of the Carnapian universe on the basis of the probability of related subsets.

If possible worlds in the Carnapian universe correspond to individual combinations of the values of n state variables $X_1, X_2, \ldots, X_n$, that is,

$$p_\alpha \equiv (X_1 = x_1) \wedge (X_2 = x_2) \wedge \ldots \wedge (X_n = x_n)$$

then, in general, probabilistic reasoning problems require the determination of either the joint probability distribution

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n|\mathcal{E})$$

or, alternatively, one of its marginal distributions on the bases of information consisting of related marginal and conditional probability distributions.

2.1 Conventional Probabilistic Reasoning

Classical probabilistic techniques rely on a calculus that is directly derived from the axioms of probability theory and that, in addition, assumes that all required numerical probability values are available, either as the result of prior empirical observation (i.e., frequencies of occurrence) or as the result of elicitation of personal commitment to gambling outcomes ("degrees of belief").

The rules used for this derivation include the additivity axiom of probability

$$P(A) + P(B) = P(A \cap B) + P(A \cup B),$$

and the celebrated identity of Bayes-Laplace

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$

which is a direct consequence of the definition of conditional probability.

The bane of all methods relying on the use of classical probability procedures is the lack of sufficient information about the required values of conditional and marginal (a priori) probabilities. Even when assumptions of independence between variable values, i.e.,

$$P((X = x) \wedge (Y = y)) = P(X = x)\,P(Y = y),$$

and conditional independence between variable values, i.e.,

$$P(X = x|Y = y, Z = z) = P(X = x|Y = y),$$

are used to simplify the required computations [27], the number of variables involved in a typical approximate reasoning problem lead to the need to estimate a large number (usually exponentially related to the number of variables) of marginal and conditional probability distributions.

The difficulties inherent in such estimation required early efforts, such as the development of PROSPECTOR [9], to use a combination of probabilistic procedures in combination with ad hoc or heuristic techniques to overcome problems associated with lack of probabilistic information and to resolve some inconsistencies that occurred whenever estimated information overconstrained some probability distributions.

Some of these methodological problems can also be traced to the desire to generalize the network-based, goal-oriented procedures of classical expert systems to situations where the traditional truth values of classical logic (i.e., **true** and **false**) were generalized to a continuous scale by equating truth-value with probability. The difficulties involved in such a generalization were soon apparent, as, for example, the transitivity of implication valid in conventional inference, that is,

If X implies Y, and if Y implies Z, then X implies Z fails to hold for probabilities; that is, $P(Y|X)$ may be high, $P(Z|Y)$ may be high, but $P(Z|X)$ may be zero. Current methodologies based on the use of classical probability theory to compute the values of a joint probability distribution [22,25] have solved these methodological problems but, in spite of their deft exploitation of independence assumptions in ***probabilistic networks*** [27], they still face the combinatorial explosion difficulties that are typical of multivariable problems.

2.2 The Estimation of Probability Distributions

If a purely objectivist viewpoint is taken, it is clear that the probability distributions required to determine the probability of a hypothesis given available evidence may not be available. In this view, which we hold, probability can only be the result of experience accumulated through previous observation, and while, theoretically, absent values may derivable by empirical means, it is often the case that the required experiments are unfeasible or impractical. This is particularly true in problems involving systems that are not easy to manipulate or observe (e.g., evaluation of building damage due to earthquakes) or when the required information is actively denied or obscured by adversaries (e.g., in military situation-assessment problems).

The orthodox subjectivist view of probability claims, on the other hand, that it is impossible to ignore the values of probability distributions, as they are always statements of the degree of belief that certain agents have about the truth of hypotheses. The rationale supporting the representation of such beliefs by numerical functions having the properties of a probability function is based on the famous "dutch book" argument [6]. If an agent is to engage in a gamble involving the truth or falsehood of a certain hypothesis, it will be irrational for him to choose a combination of bets where he will be sure to lose (a dutch-book) regardless of the outcome of the gamble turns. Under such conditions, it can be shown that his personal beliefs (assumed to be numbers) on truth and falsehood of hypotheses must satisfy the axioms of probability.

Other personalistic axiomatic systems have also been proposed to support the contention that personal beliefs on hypothetical truth can always be estimated using a single numerical value [33]. These axiomatic systems have, however, been subject to considerable criticism both on the basis of their naturality or rationality [37,21] and on the basis of observation of the actual behavior of rational agents under controlled circumstances [2,10].

Perhaps more controversial is the so-called "pragmatic necessity" argument proposed by some decision scientists to justify their choice of probability values in the absence of relevant knowledge. The essential point of this argument emphasizes the decision-oriented nature of most approximate reasoning problems. It is said that if a decision must be made, when all empirical information has been considered, then any missing probability values (consistent with such knowledge) may be chosen because something, after all, must be done. While not claiming that this procedure replaces objectively determined probability values, it is said that ignorance of such quantities is inconsequential.[4] Such light dismissal of required probability values may have, of course, significant undesirable consequences.

Metaphysical principles, such as the *principle of insufficient reason* or the *maximum entropy principle*, that seek to formalize the choice of single distributions on purportedly "rational" bases other than empirical knowledge are vulnerable to the same criticism. Regardless of whatever claims some may make invoking pragmatic needs or metaphysics to develop AI tools to assess complex situations, scientific practice — fundamentally interested in understanding the world and interacting with it — eschews these practices, relying instead on experiment-based, hypothesis-testing paradigms.

When it is accepted, at least, that sometimes probability values may not be either observable or capable of being elicited, it is clear that probabilistic reasoning techniques must proceed beyond classical probability calculus and develop alternative computation schemes that do not assume such informational availability. This generalization does not require, as it is claimed by some, to abandon either the axioms of probability or Bayes' rule as essential elements of the underlying calculus. Instead, we are simply extending our computational — rather than our conceptual schemes to determine the effects of our ignorance on the results of probabilistic analyses.

3 Generalized Probabilistic Reasoning

Current approaches that generalize the calculus of probabilities are, as stated above, based on generalization of computational rather than conceptual schemes. As such, the qualifier "non-Bayesian" that is sometimes associated with them, is basically incorrect; its validity is limited to the current skepticism, among orthodox subjectivists (often called Bayesians), about their necessity. All of these schemes are based on variations of the same idea: the determination of intervals [36] where unknown probability values must lie.

3.1 Interval-valued Probabilities

General formalisms for the representation and manipulation of interval probability bounds have been investigated by Kyburg [20], who also studied issues germane to the relations between this general formulation and the calculus of evidence of Dempster-Shafer [19]. The central notion in his treatment of probabilistic knowledge is that of "convex probabilities" used to describe the set of probability values in multidimensional space where possible values of the underlying distributions lie.

Although general interval-valued probability is preferable to other schemes, which are limited by their theoretical representation capabilities, the corresponding calculus of intervals is hampered by the difficulties associated with the storage and processing of a large

[4]It is important to point out, however, that many decision scientists rely, under these circumstances, on analyses of the sensitivity of their results to such convenient assumptions.

number of probability bounds. If m ground propositions are identified as the initial generators of a Carnapian universe, it may be necessary to store and manipulate 2^{2^m} bounds corresponding to all subsets of this universe. These difficulties have effectively limited the application of interval-based approaches in practice.

Practical schemes that are amenable to computer-based implementation, on the other hand, do not have the same generality. In general, these approaches rely on manipulation of intervals that have been generated by knowledge of probability values for *some* subsets that are then used to determine interval bounds for the probabilities of subsets of interest (i.e., inner or lower probabilities). Among such schemes relying on the use of lower probabilities, the calculus of evidence of Dempster-Shafer has found the largest acceptance in the approximate reasoning community.

3.2 Evidential Reasoning

Evidential reasoning is the name of the methodology based on the Dempster-Shafer calculus of evidence.[5] The basic structures of the calculus of evidence were introduced by Dempster in 1966 [7]. Shafer [34] proposed in 1976 the use of those constructs to represent and manipulate evidence. The methodology was first applied to the solution of approximate reasoning problems in artificial intelligence at SRI International [12,23]. Although the calculus of evidence is often regarded as being non-Bayesian (meaning primarily nonprobabilistic), its original derivation by Dempster is fully consistent with conventional probability theory. Recent results by Ruspini [30,31] have further supported this contention.

Evidential reasoning is based on the representation of probabilistic evidence by means of *mass functions* or *basic probability assignments*. Mass functions assign a nonnegative mass value to every subset in a space of possible solutions (or possible worlds). The sum of all these mass assignments over the set of all such subsets (called the *power set*) is always 1.

Evidential reasoning is advantageous in that it allows representation of the degree of support provided by evidence toward the truth of a hypothesis without requiring that such support be split among more specific propositions implying that hypothesis. For example, in a criminal investigation case, evidence may indicate that the perpetrator is blonde without actually identifying his or her identity. In such a case, a mass function that assigns a mass of 1 to the set of all blonde suspects and 0 to all other subsets is used to represent the evidential weight. Note that in this case the sum of the masses for all sets consisting of a single blonde suspect (0) is different from the mass assigned to the set of all blonde suspects (1). Had masses corresponded to actual probabilities of guilt, those two quantities should have been the same.[6]

Closely associated with the notion of mass are the *belief* and *plausibility* functions defined

[5]The reader must be warned about a recent tendency in the literature to use the expression "evidential reasoning" as a synonym of "approximate reasoning."

[6]For this and other reasons it has been claimed that evidential reasoning is non-Bayesian or nonprobabilistic. As we will see below, this assessment is based on incorrect interpretation of the meaning of mass functions.

by

$$\mathrm{Bel}(A) = \sum_{B \subseteq A} m(B),$$

and

$$\mathrm{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(A).$$

The belief function is a measure of the total support provided by evidence toward the truth of a particular proposition, while the plausibility function measures the degree by which the evidence fails to refute it.

3.2.1 Logical Bases for Evidential Reasoning

Our possible-worlds approach to the description of probabilistic reasoning may be extended to develop a formal foundation for the basic functions and structures of evidential reasoning. This extension is based on the use of a form of modal logic, called *epistemic logic*, introduced to deal with issues that are relevant to the states of knowledge of rational agents. The insight provided by this characterization has helped to clarify a number of fundamental issues in evidential reasoning, notably in the areas of semantic characterization of the notion of evidential independence and in the derivation of schemes for the combination of dependent and conditional evidence.

Epistemic logic is, like conventional Boolean logic, a two-valued logic where each proposition is assigned one and only one of the classical truth values, i.e., **true** or **false**. In epistemic logic, however, propositions may be not only true or false, but may also be *known* to be true or false, or, alternatively, they may not be known to be either true or false. Rather than introducing new scales of truth, as is done in multivalued logic [29], epistemic logic resorts to a representation scheme where knowledge of a proposition is represented by means of another, related, proposition.

A rational agent's state of knowledge about the truth of a proposition is represented by means of a special operator $\mathbf{K}$, used as a prefix to symbols describing other propositions. For example, knowledge of the truth of a proposition p is denoted $\mathbf{K}p$, while $\neg\mathbf{K}p$ symbolizes lack of such knowledge.[7] The discussion of epistemic systems also requires differentiation between propositions that describe certain properties of the real world (*objective* propositions) and propositions that include one or more epistemic operators (*epistemic* propositions).

In our investigation, we have employed a particular form of epistemic logic proposed by Moore [24] to deal with problems of reasoning and planning in artificial intelligence applications. The axiom schemata for such a modal system is:

A1. Axioms of the ordinary propositional calculus.

A2. $\mathbf{K}p \rightarrow p$ (If a proposition is known to be true, then it is true.)

[7] The meaning of the notation $\neg\mathbf{K}p$ should not be confused with ignorance about the truth of p represented by $\neg\mathbf{K}p \wedge \neg\mathbf{K}(\neg p)$, i.e., neither p nor its negation is known to be true.

A3. $\mathbf{K}p \rightarrow \mathbf{KK}p$ (*Positive introspection:* If a proposition is known to be true, then it is known that it is known to be true.)

A4. $\mathbf{K}(p \rightarrow q) \rightarrow (\mathbf{K}p \rightarrow \mathbf{K}q)$ (*Consequential omniscience:* If it is known that p implies q, then knowledge of the truth of p implies knowledge of the truth of q.)

A5. If p is an axiom, then $\mathbf{K}p$ is true.

A6. $\neg\mathbf{K}p \rightarrow \mathbf{K}\neg\mathbf{K}p$ (*Negative Introspection*: If the truth value of a proposition is unknown, then such a state of ignorance is known.)

The set of all possible truth assignments to the sentences of a modal propositional system that satisfy these axioms is called an *epistemic universe* (Figure 3) — a concept that generalizes that of the Carnapian universe. Each member of this universe is a *possible*

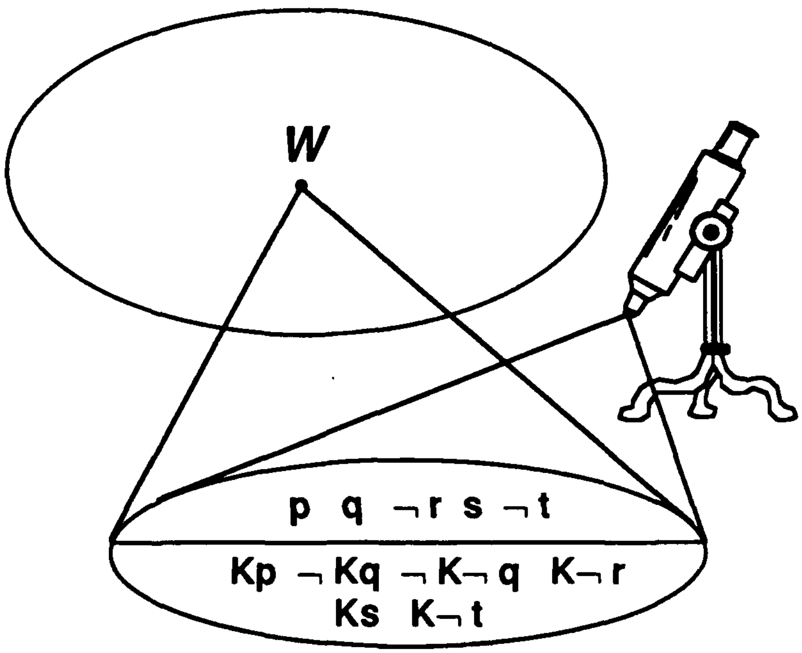

Figure 3: The Epistemic Universe.

world that represents both a particular state of the world and the state of knowledge that certain rational agents have about it. In this universe two classes of subsets are of special importance.

The first class consists of subsets of possible worlds where some objective proposition p is true. These subsets are called *truth sets*. The truth set for a proposition p is denoted $\mathrm{t}(p)$.

The second class consists of subsets having as members possible worlds where some objective proposition p is *known* to be true. These subsets are called *support sets*, with $\mathbf{k}(p)$ denoting the support set for the objective proposition p.

Closely related to support sets are the *epistemic sets*, which partition the epistemic universe into subsets characterized by the same knowledge pattern. Each such epistemic set may be associated with a proposition p that represents the *best* or *most specific* knowledge available in each possible world within that epistemic set (this proposition is the conjunction of all known propositions in each world). Epistemic subsets are identical to the elements of the quotient space of the epistemic universe by the *accessibility relation*. The accessibility relation captures the informal notion that, for all we know in a possible world w, we might just as well be in an *accessible* or *conceivable* world w'. The epistemic set corresponding to an objective proposition p is denoted $\mathrm{e}(p)$.

Several important set-theoretic relations, illustrated in Figure 4, exist between members

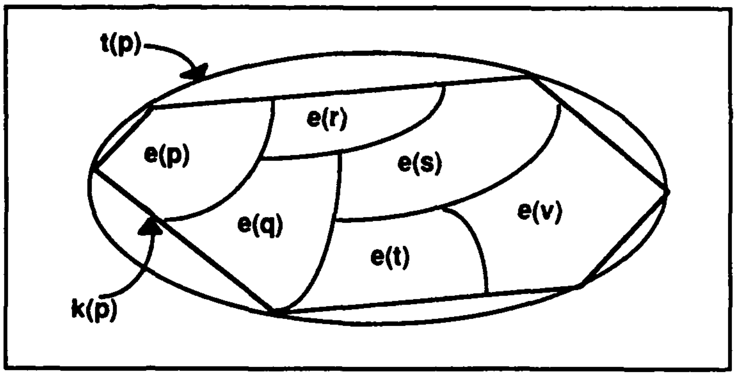

Figure 4: Relations between Special Sets in the Epistemic Universe.

of these classes:

- The support set for a proposition p is the union of the (disjoint) epistemic sets corresponding to propositions q that imply p, i.e.,

$$\mathbf{k}(p) = \bigcup_{q \to p} \mathrm{e}(q) .$$

  In plain words, if p is known to be true, it is either because that is the "best available knowledge," or because such "most specific knowledge" is that another proposition q, that implies p, is true.

- The support set $\mathbf{k}(p)$ is the largest support set (in fact, it is the largest arbitrary union of epistemic sets) included in the truth set $\mathrm{t}(p)$.

Because epistemic and support sets are always uniquely associated with an objective proposition, their probabilities may be thought of also as measures that assign a unique nonnegative value to each such objective proposition.

If $\mathbf{P}$ is such a probability, the functions

$$\begin{aligned} m(p) = &\quad \mathbf{P}(\mathrm{e}(p)) , \\ \mathrm{Bel}(p) = &\quad \mathbf{P}(\mathbf{k}(p)) , \end{aligned}$$

are related by the basic identity

$$\mathrm{Bel}(q) = \sum_{p \Rightarrow q} m(p)\,,$$

which is central to the calculus of evidence [34].

Probabilities over the epistemic algebra (and their associated functions) represent the effect of uncertain evidence on a rational agent's state of knowledge. The corresponding probabilities defined on the *truth algebra* of the truth sets $\mathbf{t}(p)$ can be interpreted as the degrees of likelihood (usually unknown) of objective propositions.

Because the largest member of the epistemic algebra that is contained in the truth set $\mathbf{t}(p)$ is the support set $\mathbf{k}(p)$, it follows (from standard results on lower- and upper-probability functions) that any extension of a probability $\mathbf{P}$, defined over the epistemic algebra, to a probability $\widehat{\mathbf{P}}$ defined over the truth algebra must satisfy the inequality

$$\mathrm{Bel}(p) \leq \widehat{\mathbf{P}}(\mathbf{t}(p)) \leq \mathrm{Pl}(p)\,,$$

where $\mathrm{Pl}(p)$ is the *plausibility function* of the Dempster-Shafer calculus of evidence. Furthermore, these bounds are the best possible and cannot be improved. In other words, knowledge of actual probability values over some subsets provides bounds, which may not be improved except by incorporation of additional evidence — on the probability values of other sets.

Issues related to the combination of evidence are readily modeled by considering another, more complex, set of possible worlds called the *product epistemic universe*. The members of this set are, as was the case in previous epistemic universes, possible worlds, that is, functions that assign conventional binary truth values (i.e., **true** or **false**) to certain propositions of interest. The difference in this case consists in the use of multiple epistemic operators $\mathbf{K}_1, \mathbf{K}_2, \ldots$ representing the knowledge possessed by several rational agents about the truth of objective propositions or of other epistemic propositions.

Constraining ourselves momentarily to situations involving two different rational agents $\mathbf{A}_1$ and $\mathbf{A}_2$, each ignorant of the knowledge of the other, their *common* (or integrated) knowledge may be modeled by introduction of a third, nonindexed, epistemic operator $\mathbf{K}$. It is assumed that the knowledge available to this third agent is the sole and exclusive result of the combination of the knowledge available to $\mathbf{A}_1$ and $\mathbf{A}_2$ without any other additional sources of information. This assumption is formally modeled by means of the following *knowledge combination axiom*:

(KC) $\mathbf{K}p$ *is true if and only if there exist sentences* p_1 *and* p_2 *such that* $\mathbf{K}_1 p_1$ *and* $\mathbf{K}_2 p_2$ *are true, and, in addition, such that* $p_1 \wedge p_2 \Rightarrow p$.

If the epistemic sets corresponding to the operators $\mathbf{K}$, $\mathbf{K}_1$, and $\mathbf{K}_2$ are denoted by $\mathbf{e}(p)$, $\mathbf{e}_1(p)$ and $\mathbf{e}_2(p)$, respectively, the following important set-equation, relating all types of epistemic sets, is the basis for the derivation of a variety of combination formulas:

$$\mathbf{e}(p) = \bigcup_{p_1 \wedge p_2 = p} (\mathbf{e}_1(p_1) \cap \mathbf{e}_2(p_2))\,,$$

from which, under certain assumptions of probabilistic independence, the Dempster combination formula

$$m(p) = \lambda \sum_{p_1 \wedge p_2 = p} m_1(p_1) m_2(p_2),$$

is readily derived.

3.2.2 Semantic Issues of Evidential Reasoning

Using an objectivist interpretation of the concept of probability, the author has formulated a Kripke-type model [17] that explicates basic probability assignments as the principal output estimated by a *generalized statistical experiment*. This model-theoretic formalism also sheds light on the general character and nature of probabilistic knowledge and on the mechanisms used to capture it. Rather than providing a formal characterization of the Kripkean formulation, we will informally describe a general model of a statistical experiment that provides insight into the nature of the theoretical structures discussed further below.

The informal model that serves as our point of departure is illustrated in Figure 5, which presents the typical steps involved in the collection of statistics about the behavior of a real-world system. A *statistical experiment*, as illustrated, commences with a mechanism

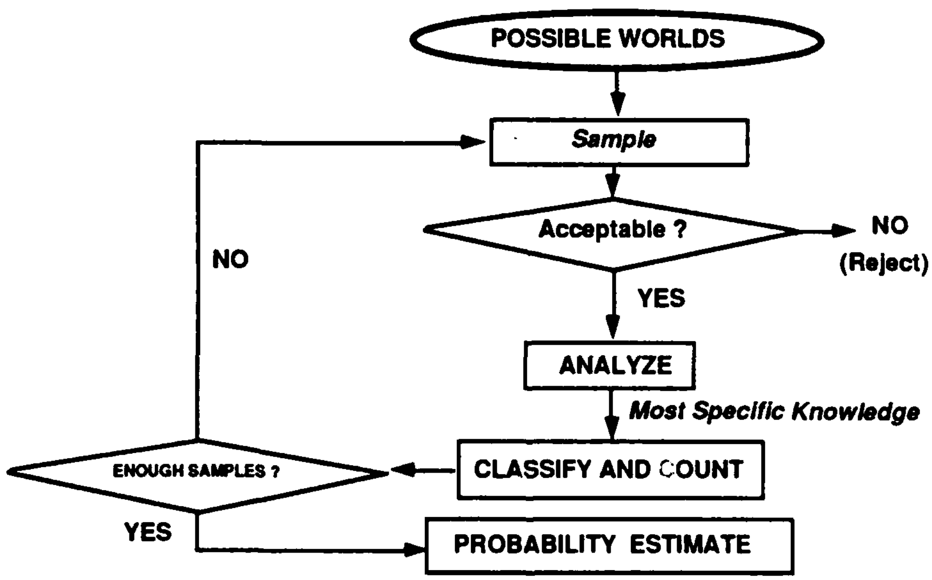

Figure 5: The General Statistical Experiment.

for the generation of samples (i.e., sequences of possible worlds that reflect the relative frequency of occurrence of such states of affairs in actual experience).

Each such sample is then examined for compliance with some experimental criteria used to determine if the corresponding possible world satisfies the criteria used for the generation of the desired statistical distribution. In other words, we are interested in estimating a conditional probability, and this test determines whether or not the condition is met. Future usage of the generated statistical values is valid *solely* if available evidence (i.e., a true proposition about the world) corresponds exactly to the condition used in the generation of the statistics.

It should be noted that the nature of the device (sensor) used to make this determination is of extreme importance in determining whether the generated statistics correspond to an epistemic probability over the truth set $\mathbf{t}(p)$ (e.g., if the sensor is capable of reliable binary discrimination between samples where p is true and samples where p is false), or over a support set $\mathbf{k}(p)$ corresponding to a rational agent that may or may not be that involved in the next analysis step (e.g., some sensor, not necessarily that used to further analyze the sample, is used to determine if p is valid; its failures, however, do not mean that p is false).

If the sample satisfies the conditions defining the statistical distribution being estimated, then the next step consists in the determination of properties (i.e., propositions that are true) in this particular possible world. The conjunctions of these propositions are the "most specific knowledge" available for that sample. In classical statistical setups, the analyzing devices that perform such a determination are designed so as to determine if the sample falls into one of several exclusive categories. For example, in clinical trials, the result of each trial is typically classified on the basis of its success into several disjoint sets (e.g., "success" or "failure"). In more general experiments, however, the ability to determine "most specific knowledge" may be severely limited and the sample will be placed into one of several classes that may be overlapping. For example, if the samples correspond to medical patients having certain types of afflictions (e.g., the "condition" is that they have a renal or a hepatic disorder), available knowledge may indicate that a particular patient has a disorder within a certain class (e.g., kidney disease), while failing to determine a specific disease.

If each sample is so classified and the results of successive analysis are tabulated as frequencies, the resulting distribution is a mass distribution in the sense of Shafer rather than a conventional probability distribution. When the differences between probability distributions and their sample-based estimates (which are often the source of second-order probability distributions) are ignored, the computed frequencies may be considered to be the same as a nonconventional distribution that corresponds to an epistemic probability. The rational agent in this distribution is the *statistical experimenter* who has a "most specific knowledge" for each possible world (actually for a relevant sample of such worlds).[8]

The knowledge of the approximate reasoner, on the other hand, is limited to knowledge of (aggregated) results of the statistical experiment coupled with knowledge of the condition validating the use of the statistical (epistemic) distribution (i.e., the condition used to determine if the samples were acceptable). Note that this distribution generally induces

[8]Note that in classical experimental setups, where the conditions of the experiment may be closely controlled, the most specific knowledge corresponds to the determination of the actual possible world where the sample lies. In those cases, the sample frequencies estimate probability values for an actual probability distribution.

bounds on the probability of truth sets. The latter, however, are needed to solve typical decision-making problems.

In closing our description of the calculus of evidence, it is important to point out that, in addition to our objectivist model, subjectivist interpretations of belief and mass functions have been proposed by Smets [35] and Jaffray [16]. The formulation of Jaffray is particularly attractive in that it provides a simple, direct generalization of the basic results of DeFinetti [6] on the probabilistic nature of degrees of belief.

4 Possibilistic Reasoning

Possibilistic approaches produce, as is the case with their probabilistic counterparts, solutions to problems that are a modified formulation of the impossible (or, at least, very difficult) task of determining hypothesis validity. The emphasis, however, is not on determination of the frequency of instances where, under similar conditions, the hypothesis will be true or false. Possibilistic methods seek to produce unequivocal answers to *other* questions that are *similar* in some sense to those of interest to the system analyst.

For example, in a medical diagnosis problem, a probabilistic method may answer the question *"Does the patient have disease D?"* by means of a probability value that fails to indicate whether the disease exists or not but that allows evaluation of the chances of successful treatment. A possibilistic method, on the other hand, may answer the same question by responding unequivocally (i.e., true or false) to the modified query *"Does the patient have a disease of type D^*?"* where D^* stands for a class of diseases that are similar, in some sense, to the disease D.

Similarity between propositions (sometimes regarded as the "degree of ease" by which a proposition describes a particular state of affairs) may be used as the basis for explaining the basic concepts and structures of fuzzy set theory and its logic-oriented extensions.

A fuzzy set f [41] is defined by its *membership function* mapping elements from a universe $\mathcal{U}$ to the $[0,1]$ interval of the real line

$$\mu_f : \mathcal{U} \mapsto [0,1] .$$

The concept of membership function generalizes the notion of characteristic function of a conventional set. For a particular element x of $\mathcal{U}$, the value $\mu_f(x)$ represents the degree of membership of x to the fuzzy set f. Unlike conventional sets where elements either belong or do not belong to a set, fuzzy sets — representing vague concepts — admit partial membership ranging from 0 (nonmembership) to 1 (full membership).

Fuzzy sets may also be described by means of their $\alpha-cuts$ consisting of all members with a degree of membership greater than or equal to a value α:

$$f(\alpha) = \{x : \mu_f(x) \geq \alpha\} .$$

Using this important concept, fuzzy sets may also be regarded, from a logical viewpoint, as a set of related indexed propositions representing different levels of conceptual applicability to a particular state of affairs.

The set-theoretic operations (union, intersection, complementation), originally proposed by Zadeh [41], generalize the corresponding operations for conventional sets:

$$
\begin{aligned}
\mu_{f\cap g}(x) &= \min[\mu_f(x), \mu_g(x)]\,, \\
\mu_{f\cup g}(x) &= \max[\mu_f(x), \mu_g(x)]\,, \\
\mu_{\overline{f}}(x) &= 1 - \mu_f(x)\,,
\end{aligned}
$$

where x is a member of the universe $\mathcal{U}$.

An important concept in fuzzy set theory is that of *fuzzy relation*, which generalizes the conventional set-theoretic notion of relation. If $\mathcal{U}$ and $\mathcal{V}$ are universes, then a fuzzy relation between $\mathcal{U}$ and $\mathcal{V}$ is a fuzzy set in the set of all pairs (u, v) (or *cartesian product*), where u is an element of $\mathcal{U}$ and where v is an element of $\mathcal{V}$. One of the main reasons for the importance of fuzzy relations is their role in the representation of vague relationships between variables, e.g.,

If u is high, then v is small.

Approximate reasoning systems used in possibilistic systems use fuzzy relations to represent inferential rules in their knowledge bases.

4.1 Possibility Theory

Possibility theory is based on the representation of vague information as *elastic* constraints on the possible values that may be attained by a variable. For example, if information is available indicating that "James is rich," a possibilistic approach represents this fact as a ***possibility distribution*** on the values of a variable describing James's wealth (called here ***James-net-worth***) in the form

$$\Pi_{James-net-worth} = rich$$

where rich is a fuzzy set defined over the real numbers intended to describe for each possible value of ***James-net-worth*** the degree of ease by which the concept "rich" agrees with that particular net worth.

In general, if a variable X takes values over a universe $\mathcal{U}$, then a linguistic expression of the form "X is F" will be formally translated by a possibilistic assignment $\Pi_X = F$, such translation being denoted as

$$X \text{ is } F \rightarrow \Pi_X = F\,,$$

meaning that the values that may be attained by X are constrained as specified by the fuzzy set F. Because vague statements in natural language are translated, in possibility theory, into formal statements that assign a fuzzy value to a variable (as opposed to assigning a precise value as would be the case for a *precise* statement), such a variable is called a *linguistic variable*.

Other translation rules are used to derive representations for more complicated linguistic statements, such as "X is F and Y is G" or "Q Xs are F" (where Q is a generalized quantifier such as "most"), are the basis of an uncertainty calculus that is complemented by certain inferential rules that allow derivation of possibilistic constraints for certain variables as a function of constraints on related variables. Among these rules the most important is the "generalized modus ponens" that produces an approximate conclusion

$$\Pi_Y = G',$$

meaning "Y is G'", from knowledge that

$$\Pi_X = F',$$

meaning that "X is F'", and that

$$\Pi_{Y/X} = (F \rightarrow G),$$

i.e., "If X is F, then Y is G".

The qualifier "generalized" is used to indicate the important fact that, unlike classical modus ponens, this inference rule allows a rule to be used even when available facts, F', do not match *precisely* the antecedent of the rule (i.e., F). The conclusion G' in such a case differs also, in general, from the consequent of the rule, being a *more general* or *less specific* constraint than G.

4.2 Similarity Relations and Possible Worlds

A similarity relation in a set X is a function that assigns a real value between 0 and 1 to every pair of objects from X. Similarity relations play an important role, recently investigated in detail by the author [32], in the interpretation of the basic concepts and structures of possibility theory. The results of this research show that the notion of possibility may be explained in terms of a similarity function defined over a universe of possible worlds. This similarity defines a metric that quantifies the extent of resemblance between pairs of states (as evaluated from the viewpoint of the particular problem being considered). For example, in a planning problem, the planner may use such measures to describe the extent by which the plan's effects resemble some planning goal or objective.

The value $S(w, w')$ that a similarity relation assigns to a pair of worlds (w, w') in a universe $\mathcal{U}$ is a numerical[9] measure of the extent by which propositions that are true at w may be expected to hold true at w'. A similarity value of 1 for $S(w, w')$ (the highest possible) indicates that, from the point of view of the propositions used to construct our

[9]The requirement that similarities be numerical may be relaxed considerably. We shall confine our exposition, however, to $[0, 1]$-valued similarities for the sake of clarity.

universe, both worlds are *indiscernible*, i.e., that the same propositions are true in w and in w'. A value of 0, in contrast, tells us that knowledge of propositional truth in w does not have any predictive value over truth-values in w' (and vice versa).

Unlike probability values that represent the behavior of a system and, as such, are a property of the system (the same may be said, under an subjectivist interpretation, of degrees of belief as a property of a rational agent), similarity functions are arbitrarily defined (but not necessarily subjective) scales that facilitate the description of the degree by which an object has some property. Thus, similarities are as useful (and arbitrary) as any other metric scale; their utility is essentially a function of the degree by which the scale distinguishes between different states of a system and the degree by which similarity scales that are associated with different properties (e.g., the pressure and volume of a perfect gas) are related to each other by means of actual physical laws (or facilitate the expression of such laws).

Simply stated, similarities provide the measurement sticks that must be employed to characterize, in an approximate fashion, the state of the real world. Correspondingly, approximate inference rules describe how similarity from some respect (e.g., resemblance of the actual state, *pressure = 80 kg/m²*, to some prototypical situation, *pressure > 100 kg/m²*), relates to similarity from another viewpoint (e.g., *temperature* $> 200°C$), by means of a fuzzy relation (e.g., *"If the pressure is high, then the temperature is high"*).

4.2.1 Properties of Similarities: Triangular Norms

A similarity function S defined on a possible-world universe $\mathcal{U}$ may be regarded as a generalization of the modal-logic notion of *accessibility* or *conceivability* [15], by introduction of multiple binary relations R_α between possible worlds (one for each value of α between 0 and 1), defined by

$$R_\alpha(w, w') \text{ if and only if } S(w, w') \geq \alpha\,.$$

Using these relations, we may say that conditions in w are *possible* to some degree in w' on the basis of the value of $S(w, w')$ (generalizing the classical definition of the modal operator for possible truth).

To assure that the function S has the properties of a similarity function, a number of properties must be required to assure that S is truly a measure of a resemblance between objects. Among these, the requirements that $S(w, w) = 1$ (i.e., the similarity between any world and itself is as high as possible), and that $S(w, w') = S(w', w)$ (i.e., w resembles w' as much as w' resembles w) are rather natural.

Less obvious than those properties is a form of transitivity that may be motivated by noting that if S were to assign values of similarity to the pairs (w, w') and (w', w'') that make both w and w' highly similar and w' and w'' also highly similar, then it would be surprising if w and w'' did not resemble each other at all. Any function claiming to measure resemblance must be such, therefore, that the similarity value $S(w, w'')$, is bounded by below by a function of $S(w, w')$ and $S(w', w'')$, expressed by means of a binary operation $\circledast$

in the form

$$S(w, w'') \geq S(w, w') \circledast S(w', w''),$$

which is graphically illustrated in Figure 6.

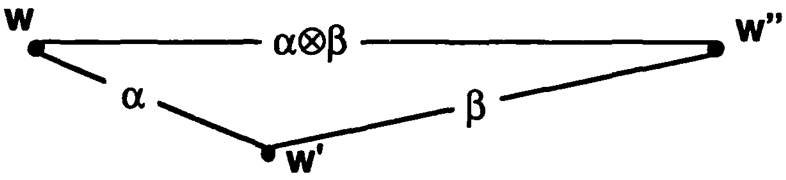

Figure 6: Transitivity of the Similarity Relation.

In terms of accessibility relations, this condition is a generalization of the classical expression for the transitivity of R, i.e.,

$$R \subseteq R \circ R,$$

to the form

$$R_{\alpha \circledast \beta} \subseteq R_\alpha \circ R_\beta, \quad \text{for all } 0 \leq \alpha, \beta \leq 1,$$

involving the multiple relations R_α.

Imposition of reasonable requirements upon the operation $\circledast$ immediately shows it to be a ***triangular norm***, introduced here by means of arguments related to metrics and similarity, but of extreme importance, otherwise, in multivalued logic [38]. Important examples of this operation include the functions

$$a \circledast b = \min(a, b), \qquad a \circledast b = \max(a + b - 1, 0), \qquad \text{and } a \circledast b = ab,$$

called the ***Zadeh***, ***Lukasiewicz***, and ***product*** triangular norms, respectively.

If a function δ is defined, between pairs of possible worlds, by means of the relation

$$\delta = 1 - S,$$

then it may be seen that when $\circledast$ is the triangular norm of Lukasiewicz, δ is an ordinary metric or distance, satisfying the well-known triangular inequality

$$\delta(w, w'') \leq \delta(w, w') + \delta(w', w'').$$

When $\circledast$ is the Zadeh triangular norm, however, the transitivity property is equivalent to the more stringent condition

$$\delta(w, w'') \leq \max(\delta(w, w'), \delta(w', w'')),$$

stating that δ is an ***ultrametric*** distance.

4.2.2 Logic and Metrics: The Generalized Modus Ponens

Metric structures, introduced by means of similarity relations, provide a mechanism for the characterization of logical relations by means of structures that stress proximity rather than subset-membership relations between possible worlds.

If a typical "conditional" proposition in Boolean logic, i.e., " If q, then p," is thought of as a statement that every world where q is true is one where also p is true, then it is clear that implications are equivalent, as is well known, to a relationship of inclusions between possible worlds: the subset of q-worlds is a subset of the set of p-worlds.

Statements of inclusion between subsets of possible worlds may, however, also be characterized in metric terms by stating that every q-world has a p-world (i.e., itself) that is as similar as possible to it. Logic structures, however, allow us only to say that either q implies p, or that q implies its negation $\neg p$, or that neither of those statements is true. Similarity relations, by contrast, permit the measurement of the amount by which a set must be "stretched" (as illustrated in Figure 7) in order for an inclusion relation to hold.

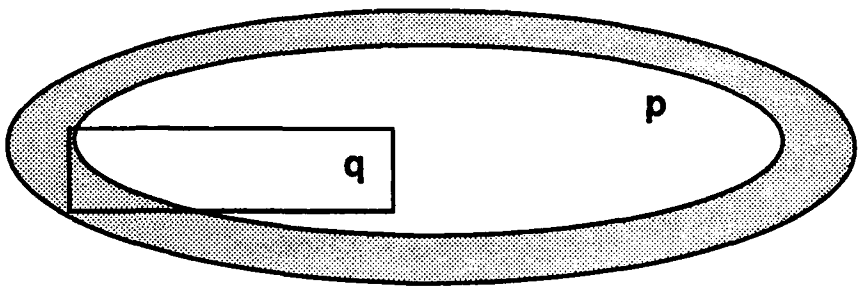

Figure 7: Extended Set Inclusion.

One such measure of inclusion is provided by the function **I** (called the *degree of implication*), defined for pairs of propositions p and q by the expression

$$\mathbf{I}(p \mid q) = \inf_{w' \vdash q} \; \sup_{w \vdash p} S(w, w'),$$

which is related to the well-known Hausdorff-distance, introduced in metric space theory to measure distance between subsets as a function of the distance between their elements.

Note, in particular, $\mathbf{I}(p \mid q) = 1$, then every q-world is similar a p-world that is logically "indistiguishable" from it (i.e., implication), while if both $\mathbf{I}(p \mid q)$ and $\mathbf{I}(q \mid p)$ are equal to 1, then p and q are logically equivalent.

From this perspective, if inferential rules, such as the *modus ponens*, are thought of as the tools of an "implicational" calculus, i.e., "If q is a subset of p, and r is a subset of q, then r is a subset of pr", then possibility theory generalizes such calculus by deriving relations between neighborhoods of certain subsets of possible worlds (actually between their sizes).

The ***generalized modus ponens*** of Zadeh [39] is a direct consequence of the transitivity property

$$\mathbf{I}(p \mid r) \geq \mathbf{I}(p \mid q) \circledast \mathbf{I}(q \mid r),$$

of the degree-of-implication function, which is illustrated in Figure 8.

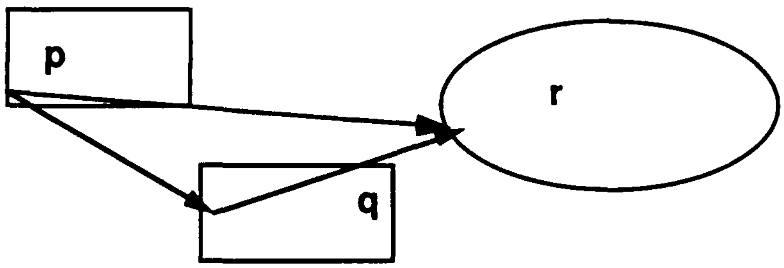

Figure 8: The Generalized Modus Ponens.

Derivation of the actual form of the generalized modus ponens from similarity-based structures, which involve possibility distributions, is outside of the scope of this paper. It will suffice to say here that possibility distributions measure the similarity, from the restricted viewpoint (called *marginal similarity*) of one or more *variables*, between certain subsets of possible worlds, and that fuzzy inference rules provide metric knowledge about inclusion relations between such subsets.

In closing, it is important to stress that similarity relations justify the use of possibilistic logic as a form of "logical extrapolation" exploiting similarities between possible worlds. The topological and metric structures that are introduced to enhance our basic Carnapian universe are of a substantially different nature than the set measures exploited by probability theory that, typically, measure the "sizes" of the complementary subsets of possible worlds where a proposition is true or false, respectively.

5 Nonmonotonic Logic and Commonsense Reasoning

Nonmonotonic logic and commonsense reasoning are also concerned with the problems caused by lack of the information that is required to deduce the truth value of certain hypotheses. As is the case with approximate reasoning methodologies, these concerns go beyond considerations about the theoretical ability to produce the required knowledge, encompassing also the practical issues involved in such production. To use a most famous example, to deduce that a particular bird flies requires knowledge that such bird is not a penguin or ostrich (at least, a nonflying ostrich), that he is not sick, dead, and so forth. The production and storage of this information imposes heavy burdens on both users and systems.

5.1 Nonmonotonic Logic

Faced with the impossibility of collecting such information, nonmonotonic logic systems [28, 13,8] are also forced to deal with a subset of possible solutions. Rather than relying on descriptions of *extensive* properties of such set, as done by approximate reasoning methods, nonmonotonic procedures choose one of its members. If subsequent information eliminates that choice as a candidate, then one or more of the "defeasible" assumptions are retracted. Use of the term *nonmonotonic* to characterize this type of reasoning is intended to reflect both the nature of the variation of truth values and the corresponding changes in the set of true statements as the consequence of the assimilation of new information (classical logic methods always *add* new truths to the set of existing theorems, thus leading to "smaller" sets of possible worlds).

The majority of nonmonotonic logic techniques rely on minimality arguments to choose possible worlds among a set of potential solutions. The general idea of these methods consists in the identification of a "least exceptional" world, that is, a world where the only objects that satisfy certain predicates are precisely those that are known to do so. Recent work [3] has extended these ideas to the approximate reasoning domain by consideration of numerical degrees of exceptionality.

Similar commonsense reasoning techniques [28], notably *default reasoning*, are also related to probabilistic reasoning. Default assumptions (such as the hypothesis that, by default, birds fly) can be thought of as stating that the assumption, given our current state of knowledge, has a high probability of being true. Known characteristics of default reasoning, notably the lack of transitivity of the modus ponens, have equivalent counterparts in probabilistic reasoning.

Studies of problems where knowledge is expressed by high probability statements [26,1] and developments in possibilistic reasoning techniques concerned with the manipulation of certain generalized quantifiers (e.g., "most") [40] and with linguistic statements of probability (e.g., "usually") [43] have also shown substantial similarities between default and probabilistic reasoning.

5.2 Qualitative Process Theory

A number of recent research efforts [11,14,18] have been oriented toward the development of methods and techniques for the description of qualitative aspects of system behavior. The basic idea of these qualitative or "naïve" physics approaches is the development of a computer-assisted understanding of the major behavioral characteristics of systems of major practical interest.

These efforts have emphasized the use of imprecise descriptions in order to avoid unnecessary numerical detail that, according to their proponents, would complicate rather than aid understanding of causal relationships and system behavior. This concern is similar to that whicht originally motivated the introduction of fuzzy set theory, which sought to provide tools to produce understandable descriptions of large and complex systems by avoidance of unnecessary descriptive detail.

The relationships between the theories go considerably beyond their common goals and objectives as qualitative process theory has made substantial use of imprecise scalar-variable scales that recognize three possible classes of values: *negative*, *zero*, and *positive*. These values are special cases of *linguistic variables*, introduced in fuzzy set theory [42], which provide for the qualitative description of scalar variables using formal representations of linguistic qualifiers such as *large*, *very large*, and *small*. The relationship between the theories is the current object of substantial attention.

6 Conclusions

Possible-world semantics provides a perspective into approximate reasoning problems and methods that helps clarify many of the fundamental issues surrounding the nature and usefulness of different methodologies.

Through use of constructs based in possible-world formalisms, it is easy to see that all existing techniques produce correct and sound descriptions of the properties of the subset of possible worlds that are consistent with observed evidence rather than, as sometimes thought, ad hoc characterizations of an ambiguously relaxed notion of truth.

Furthermore, these formalizations underscore the basic relations between probabilistic techniques showing that the Dempster-Shafer calculus of evidence is fully consistent with the theory of probability. By contrast, these models also reveal basic, substantial differences between probabilistic and possibilistic methods — the former related to set measures that characterize the frequency of occurrence of some event, and the latter linked to notions of similarity between possible situations. From this viewpoint it is evident that possibilistic and probabilistic techniques should not be regarded as competing tools but, rather, as complementary techniques seeking to describe different properties of sets of possible worlds.

Finally, it is important to point out that possible-world semantics also helps to clarify the characteristics and purposes of nonmonotonic and commonsense approaches to deductive inference.

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ON THE SEMANTICS OF FUZZY LOGIC

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Abstract

This note presents a formal semantic characterization of the major concepts and constructs of fuzzy logic in terms of notions of distance, closeness, and similarity between pairs of possible worlds. The formalism is a direct extension (by recognition of multiple degrees of accessibility, conceivability, or reachability) of the major modal logic concepts of possible and necessary truth.

Given a function that maps pairs of possible worlds into a number between 0 and 1, generalizing the conventional concept of an equivalence relation, the major constructs of fuzzy logic (i.e., conditioned and unconditional possibility distributions) are defined in terms of this generalized similarity relation using familiar concepts from the mathematical theory of metric spaces. This interpretation is different in nature and character from the typical, chance-oriented, meanings associated with probabilistic concepts, which are grounded on the mathematical notion of set measure. The similarity structure defines a topological notion of continuity in the space of possible worlds (and in that of its subsets, i.e., propositions) that allows a form of logical "extrapolation" between possible worlds.

This logical extrapolation operation corresponds to the major deductive rule of fuzzy logic —the compositional rule of inference or generalized modus ponens of Zadeh—an inferential operation that generalizes its classical counterpart by virtue of its ability to be utilized when propositions representing available evidence only match approximately the antecedents of conditional propositions. The relations between the similarity-based interpretation of the role of conditional possibility distributions and the approximate inferential procedures of Baldwin are also discussed.

A straightforward extension of the theory to the case where the similarity scale is symbolic rather than numeric is described. The problem of generating similarity functions from a given set of possibility distributions, with the latter interpreted as defining a number of (graded) discernibility relations and the former as the result of combining them into a joint measure of distinguishability between possible worlds, is briefly discussed.

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To my friends Nadal Battle, Francesc Esteva, Ramón López de Mántaras, Enric Trillas, and Llorenç Valverde.

— En noblea són quatre coses especials e singulars.
Primerament que lo cavaller sia clar ens sos fets.
La segona que sia verdader.
La terça que sia fort de cor.
La quarta que haja coinexença,
car fort és odiosa descoinexença a Déu.

— *Joanot Martorell* [Martí Joan de Galba], *TIRANT LO BLANC* (CC).

1 INTRODUCTION

This note presents a semantic characterization of the major concepts and constructs of fuzzy logic in terms of notions of similarity, closeness, and proximity between possible states of a system that is being reasoned about. Informally, a "possible state" (to be formalized later using the notion of "possible world") is an assignment of a well-defined truth-value (i.e., either **true** or **false**) to all relevant declarative knowledge statements about that system.

The primary goal that guided the research leading to the results presented in this work has been one of conceptual clarification. A great deal of energy has been directed in past few years to debating the methodological necessity and relative merits of various approximate reasoning methodologies. As a result of these exchanges, the need to consider certain nonclassical approaches, has been questioned on a variety of bases.

Recognizing the need for the development of sound semantic formalisms that shed light on the nature of different approaches, the author has pursued, in the past few years, a line of theoretical research seeking to describe various approximate reasoning methodologies using a common framework. These investigations have recently shown the close connection between the Dempster-Shafer calculus of evidence [35] and epistemic logics. This relationship was elucidated by straightforward application of conventional probabilistic concepts to models of knowledge-states that distinguish between the truth of a proposition and knowledge (by rational agents) of that truth. Central to this development is the notion of "possible world" used by Carnap [6] to develop logical bases for probability theory.

The same central notion of possible state of affairs is also the conceptual basis of the results presented in this note, which is aimed at establishing the semantic bases of possibilistic logic with emphasis on the study of its possible relations and differences, if any, with probabilistic reasoning.

The results of this investigation clearly show that possibilistic logic can be interpreted in terms of nonprobabilistic concepts that are related to the notions of continuity and proximity. The major functional structures of fuzzy logic, i.e., possibility and necessity distributions,[1] may be defined in terms of the more primitive notion of similarity between possible states of a system using constructs that are the direct extension of well-known concepts in the theory of metric spaces. The topological metric structure that is so defined may be used to derive a sound inferential rule that is a form of logical "extrapolation." This rule is also shown to be the compositional rule of inference or generalized modus ponens proposed by Zadeh [53]. Conversely, possibility distributions—expressing resemblance from some specific regard—may be used to derive the actual similarity functions—discerning between possible worlds from the joint viewpoint of several respects.

The constructs that are used to derive the interpretation presented in this note are formally, structurally, and conceptually different from those that explain probabilistic reasoning, in either its objective or subjective interpretations, irrespective of methodological reliance on interval-based approaches to represent ignorance. The latter class of methods—measuring the relative proportion

[1] It is important to remark that the scope of this work is limited to the most fundamental concepts and constructs of fuzzy logic without examining related notions such as, for example, generalized quantifiers.

of (either observed or believed) occurrence of some event—are based on the mathematical notion of set measure, while the former—seeking to establish similarities between situations that may be used for analogical reasoning—are related to the theory of distances and metric spaces.

This presentation of the relationships between similarity-based concepts and possibilistic notions, while grounded on a formal treatment that is based on rigorous logical and mathematical formalisms, will be kept at a level that is as informal as possible. The purpose of this presentation style is to facilitate comprehension of major ideas without the clutter that would need to be otherwise introduced to keep matters strictly precise. For this reason, we will refrain from formal introduction of structures and axiom schemata, that, although correct and proper, may encumber understanding of the basic concepts.

Before we proceed to the detailed consideration of semantic models, I must briefly remark on the epistemological implication of these developments. The present interpretation is not claimed to be the only one that may be advanced to define the notion of possibility in terms of simpler concepts, nor do I claim that it may not be sometimes possible, even desirable, to model possibilistic structures from other bases. My intent is not to prove the conceptual superiority of one approach over another or to argue about the relative utility of different technologies. Rather, I hope that these results have contributed to establish the basic conceptual differences to the treatment of imprecise and uncertain information that are inherent in probabilistic and possibilistic methods; the former oriented toward quantifying believed or measured frequency of occurrence, and the latter seeking to determine propositions—implied by the evidence—that are similar, in some sense, to a hypothesis of interest. In other words, beyond accidental domain-specific relations, both types of methods are needed to analyze and clarify the significance of imprecise and uncertain information.

2 APPROXIMATE REASONING AND POSSIBLE WORLDS

Our point of departure is the model-theoretic formalisms of modal logics. Let us assume that declarative statements about the state, situation, or behavior of a real-world system under study are symbolically represented by the letters of some alphabet

$$\mathscr{A} = \{p, q, r, \ldots\},$$

which are combined in the customary way using the logical operators $\neg, \vee, \wedge, \rightarrow$ and $\leftrightarrow$ (to be interpreted with their usual meanings) to derive a language $\mathscr{L}$ (i.e., a collection of sentences). Furthermore, we augment this language by use of two unary operators N and Π, called the *necessity* and *possibility* operators, respectively, having usage governed by the rule

If ϕ is a sentence, then $N\phi$ and $\Pi\phi$ are also sentences,

introducing the ability to represent different modalities for the truth of propositions.

A *model* for this propositional system is a structure consisting of three components:

1. A nonempty set of possible worlds $\mathcal{U}$ introduced to represent states, situations, or behaviors of the system being modeled by our sentences. In what follows we will refer to this set as the *universe of discourse, or universe, for short.*

   We will also need to consider a nonempty subset $\mathscr{E}$ of the universe $\mathcal{U}$, which is introduced to model the set of conceivable worlds that are consistent with observed evidence. This set (possibly equal to the whole universe $\mathcal{U}$) will be called the *evidential set.* Throughout this note, we will assume that evidence about the world is always given by means of conventional propositions that allow to determine, without ambiguity, whether a possible world either is or is not a member of the evidential set.[2]

2. A function (called a *valuation*) that assigns one and only one of the truth values **true** or **false** to every possible world w in the universe $\mathcal{U}$ and every sentence ϕ in the language. Assignment of the truth-value **true** to a pair (w, ϕ) will be denoted $w \vdash \phi$ (i.e., ϕ is true in the world w).

   In what follows, we will use the same symbols to describe subsets of possible worlds and the propositions that are true only in worlds that are members of such subsets. For example, the symbol $\mathscr{E}$ will be used to denote both the evidential set and the proposition that asserts the validity of the corresponding evidential observations. Using this notation, for example, we will write $w \vdash \mathscr{E}$ to indicate that the world w is compatible (i.e., logically consistent) with the evidence $\mathscr{E}$.

   Furthermore, we will use the symbol $\mathscr{L}$, introduced above as a set of well-formed sentences, to denote also the power set of the universe $\mathcal{U}$. Rigorously, subsets of $\mathcal{U}$ strictly correspond to the classes of equivalence of the sentence set $\mathscr{L}$ that are obtained by equating logically equivalent sentences. In the same simplifying vein, we will drop also the customary distinction

[2] For the sake of simplicity, fuzzy evidential facts such as "Tom is rich," usually considered in fuzzy logic, will not be treated in this note. The meaning of such assertions will be discussed in a forthcoming paper.

between sentences—the linguistic expressions of something that may be true or false—and propositions—the actual things being asserted.

3. A binary relation R, between possible worlds, called the *accessibility*, *conceivability*, or *reachability* relation, introduced to model the semantic of the modal operators N and Π.

It is not necessary to review here the well-known axioms [21] that restrict the assignment of truth values to well-formed sentences according to the rules of propositional logic. To facilitate comprehension of our formalism, we need to recall solely the rules that constrain assignment of truth values to sentences formed by prefixing other valid expressions with the modal operators, i.e.,

1. The sentence ϕ is *necessarily true* in the possible world w (i.e., $w \vdash \mathrm{N}\phi$) if and only if it is true in every world w' that is related to the world w by the relation R.

2. The sentence ϕ is *possibly* true in the possible world w (i.e., $w \vdash \Pi\phi$) if and only if it is true in some world w' that is related to the world w by the relation R.

If, for example, the relation R relates worlds that share the same (possibly empty) subset of true sentences of the prespecified set of expressions

$$\mathcal{F} = \{\phi_1, \phi_2, \dots\},$$

i.e., $R(w, w')$ if and only if any sentence ϕ in $\mathcal{F}$ is either true in both w and w' or it is false in both w and w', then the resulting system has an "epistemic" interpretation that regards related possible worlds as "being possible for all we know" (i.e., observed evidence, corresponding to a subset of $\mathcal{F}$ is the same for both worlds). In this case, the necessity operator N corresponds the epistemic operator K of epistemic logics, with the corresponding system having the properties of the modal system S5, which was used—in the context of probability theory—as the semantic basis for the Dempster-Shafer calculus of evidence [35].

If, on the other hand, the original interpretation of logical necessity—corresponding to a relation R that is equal to $\mathcal{U} \times \mathcal{U}$, i.e., that relates every pair of possible worlds—is given to the operator N, then a proposition is necessarily true if and only if it is true in every possible world.

If the relation R is chosen as

$$R = \mathcal{E} \times \mathcal{E},$$

then this interpretation may be used to characterize approximate reasoning problems as those where a hypothesis of interest is neither necessarily true nor necessarily false in worlds in the evidential set $\mathcal{E}$, reflecting the inability of conventional deductive techniques to unambiguously determine the truth-value of the hypothesis.[3]

In those problems, in spite of this fundamental impossibility, we may resort to approximate reasoning methods to describe various properties of the evidential set $\mathcal{E}$. For example, the probabilistic structures utilized by various probabilistic reasoning approaches typically characterize relations of the form

$$\mu(H \wedge \mathcal{E}) : \mu(\neg H \wedge \mathcal{E}),$$

between the "measures" of the subsets of the evidential set $\mathcal{E}$ where a hypothesis H is true or false, respectively.

[3] The notion of approximate reasoning problem is often extended to encompass situations where deductive techniques cannot always be used because of practical limitations on computational resources.

Our aim will be to study how other structures, defining a *metric* or *distance* in the universe $\mathcal{U}$, may be used to describe the nature of the evidential set. To do so, we will assign a different meaning to the accessibility relation, giving it an interpretation that regards related worlds as "similar" or "close" in some sense. We will require, however, a scheme that is richer than that provided by a single relation so that we can extend modal notions and derive semantics bases for fuzzy logic, which relies on concepts of degrees of matching or closeness expressed by real numbers between 0 and 1.

In what follows we will use the symbols $\Rightarrow$ and $\Leftrightarrow$ to denote strong implication and equivalence, respectively. A proposition q *strongly implies* p (denoted $q \Rightarrow p$) if and only if p is true in any world where q is. Similarly, p is *logically equivalent* to q (denoted $p \Leftrightarrow q$) if and only if p and q are true in the same subset of worlds of $\mathcal{U}$.

Following traditional terminology, we will say also that a proposition p *is satisfiable* if there exists a possible world p such that $w \vdash p$.

3 EXTENDED MODALITIES

We turn first our attention to the problem of generalizing modal logic formalisms to explain the structures and functions of fuzzy logic.

A number of authors have studied various relations between fuzzy and modal logics. Lakoff [24], Murai et al. [28], and Schocht [36] have proposed graded generalizations of basic modal constructs. Dubois and Prade [13,14] have also explored analogies between these nonstandard logics. In a recent paper [12], they have developed, in addition, a modal basis for possibility theory by means of the introduction of fuzzy structures into modal frameworks with the goal of deriving proof mechanisms that may be used in possibilistic reasoning.

The goal for the model presented in this note is somewhat different from the objectives guiding those efforts. We will seek explanations for possibilistic constructs on the basis of previously existing notions rather than generalizations of modal frameworks by means of fuzzy constructs. The model presented here is not based on the use of graded notions of possibility and necessity as primitive —and, by implication, easy to understand—structures. The foundation for this model is provided by a generalization of the accessibility relation, which is given a simple interpretation as a measure of resemblance and proximity between possible worlds.

We will extend the notion of accessibility relation to encompass a family of nonempty binary relations R_α that are indexed by a numerical parameter α between 0 and 1. These relations, which are nested, i.e.,

$$R_\alpha \subseteq R_\beta, \quad \text{whenever} \quad \beta \leq \alpha\,,$$

are introduced to represent different degrees of similarity, using a scheme that is akin to that used by Lewis in his study of counterfactuals [25]. The family of accessibility relations introduced here differs from that proposed by Lewis, however, in its use of numerical indexes[4] and in the nature of the overall modeling goals that, in Lewis' formalism, are intended to represent changes of scale induced by consideration of different restrictive statements.

3.1 Similarity Relations

To facilitate the definition of a family of accessibility relations we introduce a *similarity function*

$$S : \mathcal{U} \times \mathcal{U} \longmapsto [0,1]\,,$$

assigning to each pair of possible worlds (w, w') a unique *degree of similarity* between 0 (corresponding to maximum dissimilarity) to 1 (corresponding to maximum similarity).

With the help of this function, we will then say that w and w' are related to the degree α, denoted $R_\alpha(w, w')$, if and only if $S(w, w') \geq \alpha$. In this way, the relations R_α have the required nesting property with R_0 corresponding to the whole Cartesian product $\mathcal{U} \times \mathcal{U}$ (or, every possible world is at least similar in a degree zero to every other possible world).

[4] We will later see that similarities may be measured using more general, nonnumeric, scales. For simplicity reasons, we will avoid at this point the introduction of more general schemes that unnecessarily complicate the exposition.

Some properties are required to assure that the function S has the required semantics of a metric relationship capturing the intuitive notion of similarity or "proximity." It is first necessary to demand that the degree of similarity between any world and itself be as high as possible, i.e.,

$$S(w, w) = 1, \quad \text{for all } w \text{ in } \mathcal{U}.$$

This property assures that every one of the accessibility relations R_α will be *reflexive* and, following the nomenclature introduced by Zadeh for fuzzy relations [52], we will also say that the similarity relation is reflexive.

Next, we will call for the function S to be *symmetric*, i.e.,

$$S(w, w') = S(w', w), \quad \text{for any worlds } w \text{ and } w' \text{ in } \mathcal{U}.$$

This is a very natural requirement of any relation intended to represent a relation of resemblance between objects.

Finally, and most importantly, we will impose a form of *transitivity* requirement upon the similarity function S that turns it into a generalized equivalence relation. The purpose of this restriction is to assure that S has a reasonable behavior as a metric in the universe of possible worlds. It would certainly be surprising if, for some similarity S, we were to be told that w and w' are very similar and that w' and w'' are also very similar, but that w does not resemble w'' at all. Clearly, there should be a lower bound on the possible values of $S(w, w'')$ that may be expressed as a function of the values of $S(w, w')$ and $S(w', w'')$. We will express such a constraint using a numeric operation, denoted $\circledast$, that takes as arguments two real numbers between 0 and 1 and that returns another number in the same range, i.e.,

$$\circledast\colon\ [0,1] \times [0,1] \longmapsto [0,1],$$

in the form of the inequality

$$S(w, w'') \geq S(w, w') \circledast S(w', w''),$$

assumed valid for any worlds w, w' and w'' in the universe $\mathcal{U}$. Recurring again to a modal terminology, the above transitivity constraint, which will be called *$\circledast$-transitivity*, may be rewritten in relational form as

$$R_{\alpha \circledast \beta} \subseteq R_\alpha \circ R_\beta, \quad \text{for all } 0 \leq \alpha, \beta \leq 1,$$

making obvious its generalization of the conventional definition of transitivity for ordinary binary relations, i.e.,

$$R \subseteq R \circ R.$$

Since the role of $\circledast$, through recursive application, is that of providing a lower bound for the similarity between the two end members w_1 and w_n of a chain of possible worlds $[w_1, w_2, \ldots, w_n]$, it is obvious that the operation $\circledast$ should be commutative and associative. Furthermore, it should also be nondecreasing in each argument, as it is reasonable to ask that the desired lower bound be a monotonic function of its arguments. Finally, it is also desirable to ask that

$$\alpha \circledast 1 = 1 \circledast \alpha = \alpha,$$

i.e., that the values of the similarities of two indistinguishable objects to a third should be the same. These requirements are equivalent to demanding that the operation $\circledast$ be a *triangular norm* [37], or *T-norm*, for short.

Triangular norms, originally introduced in the theory of probabilistic metric spaces to treat certain statistical problems, play a distinguished role in $[0,1]$-multivalued logics [1,11,17,31] as the result of imposing reasonable requirements upon operations that produce the truth value of the conjunction of two expressions as a function of the truth values of the conjuncts. Furthermore, generalized similarity relations (called B-R relations by Zadeh [54]) also have an important function, to be examined further later in this note, in the generalization of the inferential rule of *modus ponens* [43,10]. Our axiomatic derivation for the requirement that $\circledast$ be a T-norm is based, however, solely on metric considerations, applied here to a space of possible worlds, but is valid in general metric spaces.

From the axioms of triangular norms, it is easy to see that

$$\alpha \circledast \beta \leq \min(\alpha, \beta),$$

showing that the minimum function, itself a T-norm, is the largest element in this class of operations. Its minimal element, on the other hand, is the noncontinuous function $\circledast$ defined by

$$\alpha \circledast \beta = \begin{cases} \alpha, & \text{if } \beta = 1, \\ \beta, & \text{if } \alpha = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Every symmetric and reflexive relation is $\circledast$-transitive for this triangular norm, which is, therefore, of little practical utility.

In what follows, we will also impose a most reasonable additional assumption of continuity of $\circledast$ with respect to its arguments (i.e., why should there be a jump in the value of a lower bound provided by $\circledast$ when the values of its arguments are slightly changed?). The class of continuous T-norms does not have a minimal element, although under certain additional assumptions (requiring T-norms to be also *J-copulas* [37]), the inequality

$$\max(\alpha + \beta - 1, 0) \leq \alpha \circledast \beta$$

also holds true, showing that certain important continuous T-norms lie between that of the $\aleph_1$-logic of Lukasïewicz [17] and that of the original fuzzy logic proposed by Zadeh [53].

Continuous triangular norms play a significant part in the theories of pattern recognition and automatic classification. The author [33] proposed the use of generalized similarity relations based on the T-norm of Lukasïewicz to generalize existing classification techniques—based on the mapping of a similarity function into a conventional equivalence relation—to the fuzzy domain—by mapping these T-norms (called *likeness relations* by Ruspini) into generalized fuzzy partitions. Bezdek and Harris [3] independently studied axiomatic approaches to cluster analysis based on the use of several continuous T-norms.

The author has also studied [34] the possible relation between the multivalued logic and similarity related aspects of T-norms, and suggested that the degrees of similarity between two objects A and B may be regarded as the "degree of truth" of the vague proposition

"A is similar to B."

Having argued that S should have the structure of a generalized equivalence relation, we will assume, mainly for reasons of simplicity, that the function S is the dual of a "true" distance, i.e., that

$$S(w, w') = 1 \text{ if and only if } w = w'.$$

This restriction, which is not substantial, is introduced primarily to assure that different possible worlds may be distinguished by means of the function S. Otherwise, the equivalence relation that relates two worlds w and w' if and only if $S(w, w') = 1$ may be used to partition our universe $\mathcal{U}$ into "indistinguishable" nonintersecting classes—indicating that our metric cannot discriminate between significant differences in system state.

Before closing our presentation of generalized similarity relations, it is important to remark upon the close relation between the notion of similarity and that of distance. If a function δ is defined in terms of a similarity function S by the simple relation

$$\delta = 1 - S\,,$$

then it is easy to see that the function δ has the properties of a metric or distance. This is evident if the operation $\circledast$ corresponds to the T-norm of Lukasïewicz, since the transitivity condition is equivalent to the well-known triangular inequality, i.e.,

$$\delta(w, w'') \leq \delta(w, w') + \delta(w', w'')\,.$$

If other T-norms are used, even stronger inequalities hold, with the so-called "ultrametric inequality"

$$\delta(w, w'') \leq \max\left(\delta(w, w'), \delta(w', w'')\right)$$

being valid for the T-norm of Zadeh. In this case, each of the relations in the family R_α (known in fuzzy set theory as the α-cut[5] of the similarity S) is a conventional equivalence relation. This fact was exploited, prior to the introduction of fuzzy set theory and fuzzy cluster analysis, by a variety of clustering procedures of the "single-link" type [22,40].

3.2 Possible and Necessary Similarity

Our semantic formalization needs require the introduction of constructs to indicate the extent by which a concept exemplifies, illustrates, or is an adequate model of another concept. Our interpretations shall, therefore, be oriented toward characterization of the degree by which a concept can be said to be a good example of another concept with the purpose of defining vague concepts by means of measures of proximity between defined and defining concepts. In our treatment, each of the multiple "definiens" will be a conventional proposition corresponding to a subset of possible worlds. It is conceivable, however, that new vague concepts might also be described by indicating their metric relations to other vague concepts.

The required constructs are based on the idea that whenever p and q are propositions such that $p \Rightarrow q$, then any p-world is an "example" of a q-world. This basic notion will be generalized by the introduction of modal structures that define to what degree possible worlds that satisfy a certain proposition q fit a vague concept. Some of those possible worlds are "paradigmatic" of the vague concept, i.e., they fit it to a degree equal to 1 in the same sense that we may say, for example, in an absolute (i.e., nongraded) sense that somebody whose height is 7 ft is definitely "tall." If we use a notion of graded fitness, however, certain worlds will fit the concept to a degree, i.e., they resemble (or are similar) to some paradigmatic example of the vague concept.

The conventional interpretation of possibility needs to be modified, therefore, to capture the idea that a particular possible world is similar in some degree to another world that satisfies a "reference" proposition.

[5]The α-cut of a fuzzy set $\mu\colon \mathcal{U} \mapsto [0,1]$ is the conventional set of all points w such that $\mu(w) \geq \alpha$. A similar concept is defined for relations as subsets of a product space $\mathcal{U} \times \mathcal{V}$.

More generally, however, we will be interested in relations of similarity between *pairs of subsets* of possible worlds rather than between pairs of possible worlds. This requirement complicates matters considerably since we will be forced to consider both the "validity" of a proposition p in *some* world where another proposition q is true, as well as its applicability in *every* world where q is true. In the former case, we will care about the existence of q-worlds that are *similar to some degree to some* p-world, while in the latter we will be concerned with the size of the minimum neighborhood of p (as a subset of the universe $\mathcal{U}$) that fully encloses the subset q.

This dual concern for what may possibly apply and what must necessarily hold—an essential aspect of modal logic—is typical of situations where relationships between ensembles of objects are described in terms of relations between their members. In the probability calculus, for example, knowledge of probabilities over certain families of subsets provides "sharp" upper and lower bounds (called *inner* and *upper probabilities*, respectively) for the probabilities of other subsets—an important fact in the extension of set measures to larger domains [19]. The role and properties of these bounds in the Dempster-Shafer calculus of evidence is well-known, having been described in the original paper of Dempster [8], related to concepts of modal logic by Ruspini [35], and being also the subjects of considerable formal study [7] as mathematical structures.

Analogies between the role of probabilistic bounds (i.e., bounds for probability values) and possibility/necessity distributions—shown below to have play a similar part with respect to metric structures—have been the source of much of the confusion about the need for possibilistic schemes. Each upper/lower-bound pair, however, leads to a substantially description of the nature of a subset of possible worlds, being, in either case, measures that arise naturally when pointwise properties are extended to set partitions. General properties of these measures have been studied by Dubois and Prade [11] in the context of approximate reasoning and in other regards by Pavlak [30].

Our generalizations of the notions of possibility and necessity are related to the so-called *de re* [21] interpretation of the statement "If q, then p is possible" as the modal propositional relation

$$q \Rightarrow \Pi p\,.$$

We will say that the proposition q *implies*, or is a *necessary model* of, the proposition p *to the degree* α if and only for every q-world w there exists a p-world w' that is at least α-similar to it, (i.e., $S(w, w') \geq \alpha$), or equivalently, whenever

$$q \Rightarrow \Pi_\alpha p\,.$$

Similarly, we will say that the proposition q *is consistent with*, or is a *possible model* of, the proposition p *to the degree* α [6] if and only there exist a q-world w and a p-world w' that are at least α-similar, or equivalently, whenever

$$\neg(p \Rightarrow \neg\Pi_\alpha q)\,.$$

The similarity function that we have introduced in the universe $\mathcal{U}$ provides us with a simple mechanism to quantify both the extent of "inclusion" and that of the "intersection" between pairs of subsets of possible worlds.[7]

[6] Note that our characterizations of both possibility and necessity distributions are based in the modal possibility operators Π_α.

[7] For reasons that by now should be evident, we will not need to introduce a concept of "unconditioned possibility" although it would be easy to do so using $q = \mathcal{U}$. Being concerned with the power of certain propositions to exemplify other conditions, we will not have much occassion to deal with the strength of tautologies in that regard.

3.3 Possibilistic Implication and Consistence

The notion of subset inclusion and its related concept of set identity are of central importance in deductive logic, since subsets of possible worlds are formally equivalent to propositions with subset inclusion and identity corresponding to logical implication and equivalence, respectively. These propositional relationships are the basis of derivation rules such as the modus ponens. The notion of intersection plays a similar role in modal analyses because of its ability to express the potential validity of a statement.

Classical accounts, however, recognize only two "degrees" of inclusion corresponding to the cases when either a set q is a subset of another set p or it is not, with a similar dichotomy applying to degrees of intersection. Our generalization exploits the metric structures defined between sets of possible worlds by introducing measures that describe a subset as enclosed in a *neighborhood* (of some size) of another set while intersecting another of its neighborhoods (of "smaller" size).[8] The problem of measuring the "size" of those neighborhoods is the subject of our immediate considerations.

3.3.1 Degree of Implication

Our definition of partial implication between propositions was based on conditions that determine whether, given two propositions p and q, one of them implies the other to the some value α. In particular, since every world w is always similar in a degree that is at least equal to zero to any other world w', it is always true that any proposition q implies any other proposition p to the degree zero. It is often the case, however, that the degree of implication between p and q is at least equal to some certain positive value α.

If we want to generalize procedures based on inclusion relationships, such as the modus ponens, in an efficient fashion, we will need measure the "optimal" (or maximum) value of the parameter α such that q implies p to the degree α. This value is a measure of the degree by which the set of all p-worlds must be "stretched" to encompass the set of all q-worlds. The least upper bound of the values of the similarities between any q-world w' and some p-world w (depending, in general, from w') is given by the *degree of implication* function:

Definition: The *degree of implication* of p by q is the value

$$\mathbf{I}(p\,|\,q) = \inf_{w' \vdash q} \sup_{w \vdash p} S(w, w') .$$

Defined in this way, the degree of implication $\mathbf{I}(p\,|\,q)$ is a measure of the "minimal amount" of stretching required to reach a p-world from any q-world, in the sense that if $\beta < \mathbf{I}(p\,|\,q)$, then

$$q \Rightarrow \Pi_\beta p .$$

Furthermore, α is the largest real value for which the above statement may be made.

As the following theorem makes clearer, this function provides the bases for the generalization of the modus ponens. This truth-derivation procedure may be thought of as an expression of the nesting relationships that hold between the sizes of neighborhoods of such subsets.

[8]It is important to recall that, due to our reliance on similarity rather than on the dual notion of dissimilarity or distance, high values of α correspond to low values of "stretching" or to smaller set neighborhoods.

Theorem: The degree of implication function,

$$\mathbf{I}\colon \mathscr{L} \times \mathscr{L} \mapsto [0,1],$$

has the following properties:

(i) If $p \Rightarrow r$, then $\mathbf{I}(p \mid q) \leq \mathbf{I}(r \mid q)$

(ii) If $q \Rightarrow r$, then $\mathbf{I}(p \mid q) \geq \mathbf{I}(p \mid r)$

(iii) $\mathbf{I}(p \mid q) \geq \mathbf{I}(p \mid r) \circledast \mathbf{I}(r \mid q)$

where p, q and r are any satisfiable propositions.

Proof: The first two properties are an immediate consequence of the definition of degree of implication. To prove the third, observe that by definition of similarity

$$S(w, w') \geq S(w, w'') \circledast S(w'', w')$$

for any worlds w, w', and w''.

Taking the supremum on both sides of this inequality with respect to all worlds $w \vdash p$, it follows, because $\circledast$ is continuous, that

$$\sup_{w \vdash p} S(w, w') \geq \left[\sup_{w \vdash p} S(w, w'')\right] \circledast S(w'', w').$$

Since this expression is true, in particular, for all worlds $w'' \vdash r$, it is true that

$$\begin{aligned} \sup_{w \vdash p} S(w, w') &\geq \left[\inf_{w'' \vdash r} \sup_{w \vdash p} S(w, w'')\right] \circledast S(\hat{w}, w') \\ &= \mathbf{I}(p \mid r) \circledast S(\hat{w}, w'), \end{aligned}$$

where $\hat{w}$ is any world such that $\hat{w} \vdash r$.

From this inequality, it follows, since $\circledast$ is continuous, that

$$\sup_{w \vdash p} S(w, w') \geq \mathbf{I}(p \mid r) \circledast \left[\sup_{\hat{w} \vdash r} S(\hat{w}, w')\right].$$

Taking now the infimum on both sides of this expression over all worlds w' such that $w' \vdash q$, it is easy to see, using again the continuity of $\circledast$, that

$$\inf_{w' \vdash q} \sup_{w \vdash p} S(w, w') \geq \mathbf{I}(p \mid r) \circledast \left[\inf_{w' \vdash q} \sup_{\hat{w} \vdash r} S(\hat{w}, w')\right],$$

proving the $\circledast$-transitivity of $\mathbf{I}$. ∎

Note, that since $\mathbf{I}(q \mid q) = 1$ for any proposition q, the following statement is also true:

Corollary. If p and q are propositions in $\mathscr{L}$, then

$$\mathbf{I}(p \mid q) = \sup_{r} \left[\mathbf{I}(p \mid r) \circledast \mathbf{I}(r \mid q)\right].$$

Notice also that if $\mathbf{I}(p \mid q) = 1$, then

$$\sup_{w \vdash p} S(w, w') = 1\,, \qquad \text{for all } w' \vdash q\,.$$

Under minimal assumptions (assuring that the supremum operation is actually a maximization), this relation is equivalent to stating that q strongly implies p, or that any q-world is also a p-world.

The nonsymmetric function I measures the extent by which every world w' in a certain class resembles some world w (dependent of w') in a reference class, possibly explicating the nature of the nonsymmetric assessments [45] found in psychological experimentation when subjects are asked to evaluate the degree by which an object "resembles" another. The results obtained in those experiments suggest that human beings, when assessing similarity between objects, use one of them (or a class of similar objects) as a reference landmark to describe the other. Such assymmetries might be explained by noticing that, in general, $\mathbf{I}(p \mid q) \neq \mathbf{I}(q \mid p)$, indicating that the stronger stimulus might generally be used to construct a reference class, which is then used to describe other stimuli.

The degree of implication of one proposition by another can be readily used to generate a measure of similarity between propositions that generalizes our original measure of similarity between possible worlds:

$$\widehat{S}(p, q) = \min\left[\mathbf{I}(p \mid q), \mathbf{I}(q \mid p)\right],$$

quantifying the degree by which the propositions p and q are equivalent.

It may be readily proved [44], from its definition and from the transitivity property of I that $\widehat{S}$ is a reflexive, symmetric, and $\circledast$-transitive function between subsets of possible worlds. This similarity function is the dual of the well-known *Hausdorff distance*, defined between subsets of a metric as a function of the distance between pairs of their members [9], which *is given by the expression*

$$\hat{\delta}(A, B) = \max\left[\left(\sup_{x \in A} \inf_{y \in B} \delta(x, y)\right), \left(\sup_{x \in B} \inf_{y \in A} \delta(x, y)\right)\right].$$

The result expressed by the transitive property of the degree of implication may be stated using modal notation in the form

$$q \Rightarrow \Pi_\alpha r \;\text{ and }\; r \Rightarrow \Pi_\beta q \quad \text{imply that} \quad q \Rightarrow \Pi_{\alpha \circledast \beta}\, p\,,$$

as the simplest form of the generalized modus ponens rule of Zadeh.

The relationship between this rule and the classical modus ponens is easier to perceive if it is remembered that classical conditional propositions of the form "If q, then p," simply state that the set of q-worlds is a subset of the set of p-worlds. Such relationships of inclusion may also be described in metric terms by saying that every q-world has a p-world (i.e., itself) that is as similar as possible to it.

Logic structures, however, only allow us to say that either q implies p or that q implies its negation $\neg p$, or that neither of those statements is true. By contrast, similarity relations allow measurement of the amount by which a set must be "stretched" (as illustrated in Figure 1) to enclose another set. Using such metrics, we may describe the generalized modus ponens as a relation between the stretching required to reach p from any point of the set r, the stretching required to reach r from any point of the set q, and the stretching required to reach p from any point of the set q.

In Section 5 we will derive alternative expressions for the generalized modus ponens that allow to propagate both measures characterizing degree of implication and *degree of consistence*; a dual

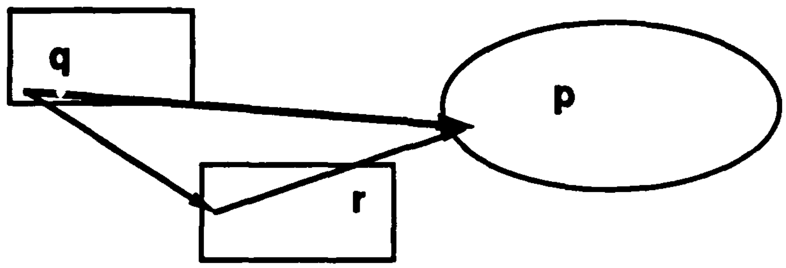

Figure 1: The Generalized Modus Ponens.

concept that plays, with respect to the notion of possibility, the function that is fulfilled by the degree of implication function with respect to necessity. In those derivations, by introduction of sharper bounds for certain conditional concepts, we will also be able to improve the quality of the bounds provided by generalized modus ponens rules while being closer in spirit to its usual fuzzy-logic formulation.

3.3.2 Degree of Consistence

A notion that is dual to that of degree of implication is given by a function that measures the point-wise proximity between pairs of possible worlds from an "optimistic" point of view characterizing the degree by which statements that are true in some worlds *may* apply on others. *By contrast*, the degree of implication measures the extent by which statements that are true in p-worlds *must* hold in q-worlds.

Definition: The *degree of consistence* of p and q is the value

$$\mathbf{C}(p \mid q) = \sup_{w' \vdash q} \; \sup_{w \vdash p} S(w, w') .$$

An immediate consequence of this definition that $\mathbf{C}(\cdot \mid \cdot)$ is a symmetric function that is increasingly monotonic in both arguments (with respect to the $\Rightarrow$). It is also easy to see that the values of the degree of consistence function are never smaller than the corresponding values of the degree of consistence function,

$$\mathbf{I}(p \mid q) \leq \mathbf{C}(p \mid q) ,$$

as the amount of stretching required to reach p from some "convenient" q-world is smaller (i.e., higher values of S) than that required to reach p from any q-world. In general, however, the degree of consistence function is not transitive, preventing the statement of a "compatibility" counterpart of the generalized modus ponens rule. Its relationship with the degree of implication function expressed by the expression

$$\mathbf{C}(p \mid q) = \sup_{w' \vdash q} \; \mathbf{I}(p \mid w') = \sup_{w \vdash p} \; \mathbf{I}(q \mid w)$$

will permit us, nonetheless, to derive a useful bound-propagation expression.

4 POSSIBILITY AND NECESSITY DISTRIBUTIONS

This section presents interpretations of the major constructs of fuzzy logic —possibility and necessity distributions—in terms of similarity-based structures. Possibility and necessity distributions are functions that measure the proximity of either all or some of the worlds in the evidential set to worlds in other sets that are employed as reference landmarks.

The role played by possibility and necessity distributions is similar to that performed by lower and upper bounds of probability distributions (or by the belief and plausibility functions of the Dempster-Shafer calculus of evidence) with respect to probability distributions. The essential difference between these bounds and those provided by possibility/necessity pairs lies in the fundamentally dissimilar character of what is being bound—metric structures relating pairs of worlds in one case; measures of set size, on the other. Furthermore, in the model of possibilistic structures that is presented in this note necessity (possibility) distributions are any lower (upper) bounds of certain metric functions rather than its "best" or "sharpest" bounds. The operations of fuzzy logic allow computation of bounds for some of these measures as a function of bounds of other measures.

4.1 Inverse of a Triangular Norm

When working in ordinary metric spaces, it is often convenient to express the conventional statement of the triangular inequality, i.e.,

$$\delta(w, w') \leq \delta(w, w'') + \delta(w'', w'),$$

in the equivalent form

$$\delta(w, w') \geq |\, \delta(w, w'') - \delta(w', w'')\, |,$$

which utilizes a form of inverse (i.e., the substraction operator $-$) of the function used to express the original inequality (i.e., the addition operator $+$). This notion of inverse may be directly generalized [37] to provide us with the tools required to define possibility and necessity functions and to derive useful forms of the generalized modus ponens involving either type of these constructs.

Definition: If $\circledast$ is a triangular norm, its *pseudoinverse* $\oslash$ is the function defined over pairs of numbers in the unit interval of the real line, by the expression

$$a \oslash b = \sup\{\, c : \, b \circledast c \leq a \,\}.$$

From this definition it is clear that $a \oslash b$ is nondecreasing in a and nonincreasing in b. Furthermore, $a \oslash 0 = 1$ and $a \oslash 1 = a$ for any a in $[0, 1]$. Other important properties of the pseudoinverse function are given in the works of Schweizer and Sklar [37], Trillas and Valverde [43], and Valverde [44].

Examples of the pseudoinverses of important triangular norms are given in Table 1 together with the corresponding conorms.

Table 1: Triangular Norms, Conorms, and Pseudoinverses

Name	T-Norm $a \circledast b$	Conorm $a \oplus b$	Pseudoinverse $a \oslash b$
Lukasiewicz	$\max(a+b-1,0)$	$\min(a+b,1)$	$\min(1+a-b,1)$
Product	ab	$a+b-ab$	a/b, if $b > a$ 1, otherwise
Zadeh	$\min(a,b)$	$\max(a,b)$	a, if $b > a$ 1, otherwise

4.2 Unconditioned Necessity Distributions

We introduce first a family of functions that bound by below the value of the similarity between any evidential world in $\mathscr{E}$ to some world where another proposition p is true. These unconditioned *necessity* distributions are lower bounds for values of the degree of implication $\mathbf{I}(p \mid \mathscr{E})$, which measures the extent by which statements that are true in a reference set (i.e., the subset of p-worlds) must hold in the evidential set.

As observed before, whenever $\mathbf{I}(p \mid \mathscr{E}) = 1$, it is true, under minimal assumptions, that the evidential subset $\mathscr{E}$ is a subset of the set of all p-worlds, or that p necessarily holds in $\mathscr{E}$. If, on the other hand, $\mathbf{I}(p \mid \mathscr{E}) = \alpha < 1$, then p *must* be stretched a certain amount—with smaller α corresponding to larger stretching—in order for one of its neighborhoods to encompass $\mathscr{E}$.

Definition: If $\mathscr{E}$ is an evidential set, then a a function $\mathbf{Nec}(\cdot)$ defined over propositions in the language $\mathscr{L}$ is called an *unconditioned necessity distribution* for $\mathscr{E}$ if

$$\mathbf{Nec}(p) \le \mathbf{I}(p \mid \mathscr{E}).$$

4.3 Unconditioned Possibility Distributions

The dual counterpart of the unconditioned necessity distribution is provided by upper bounds of the degree of consistence $\mathbf{C}(p \mid \mathscr{E})$. Whenever $\mathbf{C}(p \mid \mathscr{E}) = 1$, it is easy to see that, under minimal assumptions, there exists a p-world w that is in the evidential set $\mathscr{E}$ or, equivalently, that p (for all we know) is possibly true. If, on the other hand, $\mathbf{C}(p \mid \mathscr{E}) = \alpha < 1$, then there exists a neighborhood (of "size" α) of *some* p-world that intersects the evidential set.

Definition: If $\mathscr{E}$ is an evidential set, then a function $\mathbf{Poss}(\cdot)$ defined over propositions in the language $\mathscr{L}$ is called an *unconditioned possibility distribution* for $\mathscr{E}$ if

$$\mathbf{Poss}(p) \ge \mathbf{C}(p \mid \mathscr{E}).$$

Since the value $\mathbf{Poss}(p)$ of any possibility function $\mathbf{Poss}(\cdot)$ is an upper bound of the value $\mathbf{C}(p \mid \mathscr{E})$ of the degree of consistence, while the corresponding value $\mathbf{Nec}(p)$ of any necessity function $\mathbf{Nec}(\cdot)$ is a lower bound of $\mathbf{I}(p \mid q)$, it follows that values of a possibility function can never be smaller than the corresponding values of any necessity function, i.e., that

$$\mathbf{Nec}(p) \le \mathbf{Poss}(p).$$

4.4 Properties of Possibility and Necessity Distributions

In this subsection we will develop similarity-based interpretations for some basic formulæ of possibilistic calculus. These expressions may be thought of as mechanisms that allow the extension of a partially known possibility distribution. For example, the property that

$$\max(\mathbf{Poss}(p), \mathbf{Poss}(q)) \geq \mathbf{C}(p \vee q \mid \mathscr{E}),$$

which is proved below, is the similarity interpretation of the standard rule that allows computation of the value of the possibility value of a disjunction in fuzzy logic, i.e.,

$$\mathbf{Poss}(p \vee q) = \max(\mathbf{Poss}(p), \mathbf{Poss}(q)).$$

Theorem: If p and q are propositions, and if the quantities $\mathbf{Poss}(p), \mathbf{Poss}(q), \mathbf{Nec}(p)$, and $\mathbf{Nec}(q)$ are such that

$$\begin{aligned} \mathbf{Nec}(p) &\leq \mathbf{I}(p \mid \mathscr{E}), & \mathbf{Nec}(q) &\leq \mathbf{I}(q \mid \mathscr{E}), \\ \mathbf{Poss}(p) &\geq \mathbf{C}(p \mid \mathscr{E}), & \mathbf{Poss}(q) &\geq \mathbf{C}(q \mid \mathscr{E}), \end{aligned}$$

then the following statements *(similarity-based interpretations of the basic laws of fuzzy logic)* are valid:

$$\begin{aligned} \max(\mathbf{Nec}(p), \mathbf{Nec}(q)) &\leq \mathbf{I}(p \vee q \mid \mathscr{E}), \\ \max(\mathbf{Poss}(p), \mathbf{Poss}(q)) &\geq \mathbf{C}(p \vee q \mid \mathscr{E}), \\ \min(\mathbf{Poss}(p), \mathbf{Poss}(q)) &\geq \mathbf{C}(p \wedge q \mid \mathscr{E}). \end{aligned}$$

Proof: Note first that since $\mathbf{C}(\cdot \mid \cdot)$ is nondecreasing (with respect to the $\Rightarrow$ order) in its arguments, it is true that

$$\begin{aligned} \mathbf{Poss}(p) &\geq \mathbf{C}(p \mid \mathscr{E}) \geq \mathbf{C}(p \wedge q \mid \mathscr{E}), \\ \mathbf{Poss}(q) &\geq \mathbf{C}(q \mid \mathscr{E}) \geq \mathbf{C}(p \wedge q \mid \mathscr{E}), \end{aligned}$$

whenever $p \wedge q$ is satisfiable, from which it is easy to see that

$$\min(\mathbf{Poss}(p), \mathbf{Poss}(q)) \geq \mathbf{C}(p \wedge q \mid \mathscr{E}).$$

The corresponding result is obvious when $p \wedge q$ is nonsatisfiable.

A similar argument shows, for necessity functions, that

$$\max(\mathbf{Nec}(p), \mathbf{Nec}(q)) \leq \mathbf{I}(p \vee q \mid \mathscr{E}).$$

To prove the disjunctive law for possibilities, notice that if f is any function mapping elements of a general domain D into real numbers, then

$$\sup\{\, f(d) : d \in A \cup B \,\} = \max\Big[\sup\{\, f(d) : d \in A \,\}, \sup\{\, f(d) : d \in B \,\}\Big].$$

From this equality, it is easy to see that if $\mathbf{Poss}(p)$ and $\mathbf{Poss}(q)$ are upper bounds of $\mathbf{I}(p \mid \mathscr{E})$ and $\mathbf{I}(q \mid \mathscr{E})$, respectively, then

$$\max(\,\mathbf{Poss}(p), \mathbf{Poss}(q)\,) \geq \mathbf{C}(p \vee q \mid \mathscr{E})\,,$$

completing the proof of the theorem. ∎

Note, however, that another law commonly given as an axiom for necessity functions does not hold valid in our interpretation. As illustrated in Figure 2, the distance from a point to the intersection of two sets may be strictly larger than the distances to either set (i.e., the similarity will be strictly smaller). In general, therefore, it is

$$\min(\,\mathbf{Nec}(p), \mathbf{Nec}(q)\,) \not\leq \mathbf{I}(p \wedge q \mid \mathscr{E})\,,$$

making invalid, under this interpretation, the conjunctive law for necessities [11]

$$\mathbf{Nec}(p \wedge q) = \min\left(\mathbf{Nec}(p), \mathbf{Nec}(q)\right).$$

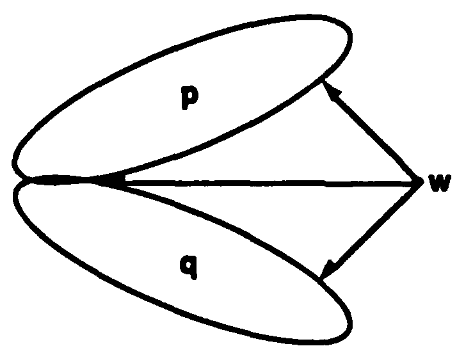

Figure 2: Failure of Conjunctive Necessity.

We may also note in this regard that the similarity-based model that is discussed here does not make use of the notion of negation either as a mechanism to generate dual concepts or on its own right as an important logical concept. It is the intent of the author to study, in the immediate future, alternative models where notions of negation and maximal dissimilarity play more substantive roles.

4.5 Conditional Possibilities and Necessities

The concepts of conditional possibility and necessity are closely related to the previously introduced unconditioned structures. These structures may be thought of as a characterization of the proximity of a world w to some or all of the worlds where a proposition p is true, *given that w is similar in the degree* 1 *to the evidential set* $\mathscr{E}$ (i.e. $w \vdash \mathscr{E}$). With this fact, in mind, we could have used the somewhat baroque formulation

$$\mathbf{C}(p \mid \mathscr{E}) = \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(p \mid w) \oslash \mathbf{I}(\mathscr{E} \mid w)\right]$$

to define unconditioned possibility distributions—a rather unnecessary effort if we consider that $\mathbf{I}(\mathscr{E} \mid w) = 1$ whenever $w \vdash \mathscr{E}$, showing its obvious equivalence to the simpler form used in Section 3.3.2 above. In spite of such observation, the above identity is important in understanding the purpose of the definitions given below. Those definitions interpret conditional possibilities and necessities as a measure of the proximity of worlds on the evidential set $\mathscr{E}$ to (some or all) worlds satisfying a (conditioned) proposition p relative to their proximity to (some or all) the worlds that satisfy another (conditioning) proposition q.

The mechanism used to specify that relationship, which is closely related in spirit to results of Valverde [44] on the structure of indistinguishability relations, is based on the pseudoinverse function introduced in Section 4.1. The basic idea used by these definitions is also illustrated in Figure 3, where, from the perspective of the evidential world w, the similarity between the p-world u and the q-world v is estimated by means of an inequality that generalizes the "absolute value" form of the triangular inequality, i.e.,

$$\delta(u, v) \geq |\, \delta(u, w) - \delta(v, w)\, |\,,$$

to its similarity-based form

$$S(u, v) \leq \min\left[\, S(u, w) \oslash S(v, w),\; S(v, w) \oslash S(u, w)\,\right].$$

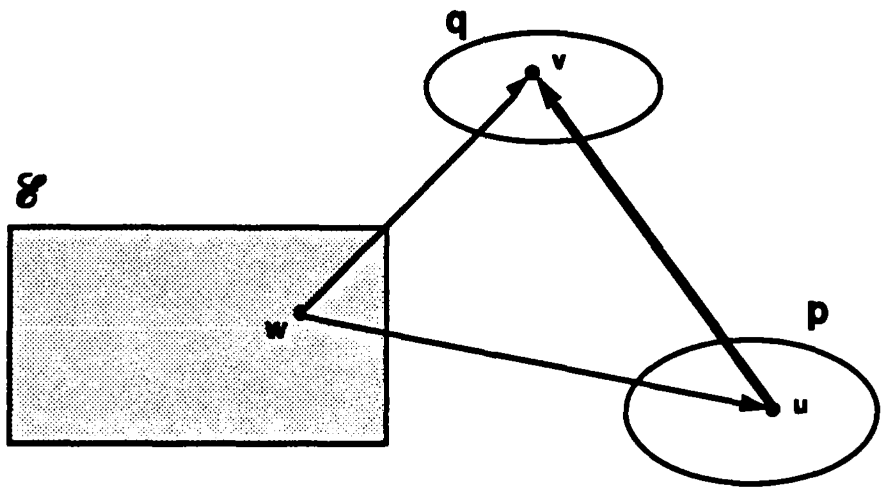

Figure 3: Similarities as Viewed from the Evidential Set.

The required interplay between similarities to conditioning and conditioned sets is captured by the following definitions.

Definition: Let $\mathscr{E}$ be an evidential set. A function $\mathbf{Nec}(\cdot|\cdot)$ mapping pairs of propositions in the language $\mathscr{L}$ into $[0, 1]$ is called a *conditional necessity distribution* for $\mathscr{E}$ if

$$\mathbf{Nec}(q|p) \leq \inf_{w \vdash \mathscr{E}} \left[\, \mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w)\,\right],$$

for any propositions p and q in $\mathscr{L}$.

Definition: Let $\mathscr{E}$ be an evidential set. A function $\mathbf{Poss}(\cdot|\cdot)$ mapping pairs of propositions in the language $\mathscr{L}$ into $[0,1]$ is called a *conditional possibility distribution* for $\mathscr{E}$ if

$$\mathbf{Poss}(q|p) \geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w) \right],$$

for any propositions p and q in $\mathscr{L}$.

It is easy to see, from these definitions, that the values of a conditional necessity distribution are never larger than the corresponding values of any conditional possibility distribution, i.e.,

$$\mathbf{Nec}(q|p) \leq \mathbf{Poss}(q|p).$$

Furthermore, since $\mathbf{I}(\cdot \mid \cdot)$ is $\circledast$-transitive, then

$$\mathbf{I}(q \mid w) \geq \mathbf{I}(q \mid p) \circledast \mathbf{I}(p \mid w).$$

From this inequality and the definition of pseudoinverse of a triangular norm, it is easy to see that any necessity function satisfies the inequality

$$\mathbf{Nec}(q|p) \geq \mathbf{I}(q \mid p),$$

i.e., the bounds for necessity functions provided by the evidential-set perspective are stronger than those that can be obtained by direct use of the degree of implication function.[9]

Note also that if $\mathbf{Nec}(p) = 1$, indicating that $\mathbf{I}(p \mid \mathscr{E}) = 1$, and if $\mathbf{Nec}(q|p) = 1$, then the above definition of conditional necessity shows that $\mathbf{I}(q \mid \mathscr{E}) = 1$, indicating that $\mathbf{Nec}(q)$ may be taken to be equal to 1, thus generalizing the well-known axiom (consequential closure) of certain modal systems (e.g., the system $\mathbf{T}$, as discussed in Hughes and Creswell [21])

$$\text{If } \mathbf{N}p \text{ and } \mathbf{N}(p \rightarrow q), \text{ then } \mathbf{N}q.$$

The definitions above can also be further interpreted as a way to compare the similarities between evidential worlds and those in the conditioning and conditioned sets by noting that whenever

$$\mathbf{I}(q \mid w) \geq \mathbf{I}(p \mid w),$$

for every evidential world $w \vdash \mathscr{E}$, then $\mathbf{Nec}(q|p)$ may be chosen to be equal to 1. Similarly, if there exists some world $w \vdash \mathscr{E}$ where this inequality holds, then it is $\mathbf{Poss}(q|p) = 1$. In either case, however, the maximum value for the conditional distribution (i.e., 1) is reached when the proximity of one evidential world w—in the case of possibilities—or of every one of them—in the case of necessities—to a world w_q in the conditioned set exceeds the proximity of w to the conditioning set p. In either case, once again recurring to an apparent notational overkill, we may state this fact by means of the identity function τ in the unit interval:

$$\tau\colon [0,1] \mapsto [0,1] : \alpha \mapsto \alpha,$$

in the form

$$\mathbf{I}(q \mid w) \geq \tau\big(\mathbf{I}(p \mid w)\big),$$

[9] A dual inequality for possibilities involving $\mathbf{C}(q \mid p)$ does not hold in general. It is easy to see, however, that $\mathbf{C}(q \mid \mathscr{E}) \oslash \mathbf{I}(p \mid \mathscr{E})$ is a possibility function for q given p.

for some $w \vdash \mathscr{E}$ in the case of possibilities, with the same inequality holding for every $w \vdash \mathscr{E}$ in the case of necessities. We may, however, conceive of other functions

$$\gamma\colon [0,1] \mapsto [0,1]\colon \alpha \mapsto \gamma(\alpha),$$

with $\gamma(\alpha) \geq \alpha$ to specify a stronger form of implication, as illustrated in Figure 4, i.e.,

$$\mathbf{I}(q \mid w) \geq \gamma(\mathbf{I}(p \mid w)).$$

Similarly, one may also conceive of functions ψ with $\psi(\alpha) \leq \alpha$ that may be used to model weaker forms of implication.

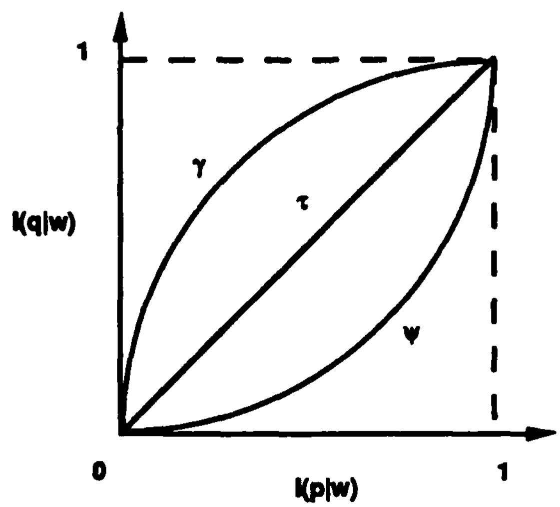

Figure 4: Examples of Possible Similarity Relationships between Conditioning and Conditioned Sets.

Possibilistic calculi based on the propagation of truth-mappings of this type, first proposed by Baldwin [2], are utilized in the RUM [4,5] and MILORD [18] expert systems. The particular case when $\gamma = \tau$, stating that every α-cut of the conditioning proposition p is fully enclosed (in the conventional sense) in the α-cut of the conditioned proposition q, has been called the *truth* mapping in the fuzzy logic literature.

The primary purpose of conditional distributions, however, is to provide a quantitative measure of the strength by which one proposition may be said to imply another with a view to extend inferential procedures by means of structures that superimpose the topological notion of continuity upon a logical framework concerned with propositional validity.

5 GENERALIZED INFERENCE

The major inferential tool of fuzzy logic is the *compositional rule of inference* of Zadeh [53], which generalizes the corresponding classical rule of inference by its ability to infer valid statements even when a perfect match between facts and rule antecedent does not exist, i.e.,

$$\text{from} \quad \frac{\begin{array}{l} p \\ p \;\rightarrow\; q \end{array}}{q} \quad \text{to its “approximate” version} \quad \frac{\begin{array}{l} p' \\ p \;\rightarrow\; q \end{array}}{q'} \,,$$

where p' and q' are similar to p and q, respectively. In this sense, the generalized modus ponens operates as an “interpolation” (or, more precisely, as an “extrapolation”) procedure in possible-world space.

Unlike the interpolation procedures of numerical analysis, however, which yield estimates of function value, this extrapolation procedure approximates truth in the sense that it produces a proposition that is both more general than the consequent of the inferential rule and resembles it to some degree (which is a function of the degree by which p' resembles p). The “extrapolated conclusion,” however, is a correctly derived proposition, i.e., the result of a sound logical procedure rather than of an approximate heuristic technique.

5.1 Generalized Modus Ponens

The theorems that are proven below are based on the use of a family $\mathcal{P}$ of propositions that partitions the universe of discourse $\mathcal{U}$ in the sense that every possible world will satisfy at least one proposition in $\mathcal{P}$.

Definition: If $\mathcal{P}$ is a subset of satisfiable propositions in $\mathcal{L}$ such that if w is a possible world in the universe $\mathcal{U}$, then there exists a proposition p in $\mathcal{P}$ such that $w \vdash p$, then the family $\mathcal{P}$ is called a *partition* of $\mathcal{U}$.

These results make use of information such as the values of the unconditioned necessity (resp., possibility) distributions for antecedent propositions p in the family $\mathcal{P}$ together with the values $\mathbf{Nec}(q|p)$ (resp., $\mathbf{Poss}(q|p)$) to “extend” the unconditioned distributions to the “consequent” proposition q. In this sense, these findings interpret, in the same spirit used in the theorem of Section 4.4 for other basic laws, the generalized modus ponens laws of fuzzy logic:

$$\begin{aligned} \mathbf{Nec}(q) &= \sup_{\mathcal{P}} \left[\, \mathbf{Nec}(q|p) \circledast \mathbf{Nec}(p) \,\right], \\ \mathbf{Poss}(q) &= \sup_{\mathcal{P}} \left[\, \mathbf{Poss}(q|p) \circledast \mathbf{Poss}(p) \,\right]. \end{aligned}$$

Theorem *(Generalized Modus Ponens for Necessity Functions)*: Let $\mathscr{P}$ be a partition of $\mathcal{U}$ and let q be a proposition. If $\mathbf{Nec}(p)$ and $\mathbf{Nec}(q|p)$ are real values, defined for every proposition p in the partition $\mathscr{P}$, such that

$$\begin{aligned} \mathbf{Nec}(p) &\leq \mathbf{I}(p \mid \mathscr{E}), \\ \mathbf{Nec}(q|p) &\leq \inf_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w) \right], \end{aligned}$$

then the following inequality is valid

$$\sup_{\mathscr{P}} \left[\mathbf{Nec}(q|p) \circledast \mathbf{Nec}(p) \right] \leq \mathbf{I}(q \mid \mathscr{E}).$$

Proof: Note first that since $\oslash$ is nonincreasing in its second argument and since

$$\mathbf{I}(p \mid \mathscr{E}) \leq \mathbf{I}(p \mid w)$$

for every evidential world w, it is

$$\mathbf{Nec}(q|p) \leq \inf_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w) \right] \leq \inf_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathscr{E}) \right].$$

It follows then from the monotonicity and continuity of $\circledast$ with respect to its arguments that

$$\begin{aligned} \mathbf{Nec}(p) \circledast \mathbf{Nec}(q|p) &\leq \mathbf{I}(p \mid \mathscr{E}) \circledast \inf_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathscr{E}) \right] \\ &= \inf_{w \vdash \mathscr{E}} \left[\mathbf{I}(p \mid \mathscr{E}) \circledast \left(\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathscr{E}) \right) \right] \\ &\leq \inf_{w \vdash \mathscr{E}} \mathbf{I}(q \mid w) \\ &= \mathbf{I}(q \mid \mathscr{E}) \end{aligned}$$

since

$$\mathbf{I}(p \mid \mathscr{E}) \circledast \left(\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid \mathscr{E}) \right) \leq \mathbf{I}(q \mid w),$$

because of the definition of $\oslash$ and the continuity of $\circledast$.

Since the above inequality is valid for any proposition p in $\mathscr{P}$, the theorem follows. ∎

A dual result also holds for possibility functions.

Theorem *(Generalized Modus Ponens for Possibility Functions)*: Let $\mathscr{P}$ be a partition of $\mathcal{U}$ and let q be a proposition. If $\mathbf{Poss}(p)$ and $\mathbf{Poss}(q|p)$ are real values, defined for every proposition p in $\mathscr{P}$, such that

$$\begin{aligned} \mathbf{Poss}(p) &\geq \mathbf{C}(p \mid \mathscr{E}), \\ \mathbf{Poss}(q|p) &\geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q \mid w) \oslash \mathbf{I}(p \mid w) \right], \end{aligned}$$

then the following inequality is valid

$$\sup_{\mathscr{P}} \left[\mathbf{Poss}(q|p) \circledast \mathbf{Poss}(p) \right] \geq \mathbf{C}(q \mid \mathscr{E}).$$

Proof: Note first that if w is an evidential world, then

$$\mathbf{C}(p \mid \mathscr{E}) \geq \mathbf{I}(p \mid w).$$

It follows then from the nonincreasing nature of $\oslash$ with respect to its second argument that

$$\begin{aligned}\mathbf{Poss}(q|p) &\geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q\,|\,w) \oslash \mathbf{I}(p\,|\,w)\right] \\ &\geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q\,|\,w) \oslash \mathbf{C}(p\,|\,\mathscr{E})\right],\end{aligned}$$

and, therefore, that

$$\mathbf{Poss}(q|p) \circledast \mathbf{Poss}(p) \geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q\,|\,w) \oslash \mathbf{C}(p\,|\,\mathscr{E})\right] \circledast \mathbf{C}(p\,|\,\mathscr{E}).$$

Taking now, in the above expression, the supremum with respect to all propositions p in $\mathscr{P}$, it is

$$\sup_{\mathscr{P}} \left[\mathbf{Poss}(q|p) \circledast \mathbf{Poss}(p)\right] \geq \sup_{\mathscr{P}} \left[\sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q\,|\,w) \oslash \mathbf{C}(p\,|\,\mathscr{E})\right] \circledast \mathbf{C}(p\,|\,\mathscr{E})\right]. \tag{1}$$

Note, however, that since $\mathscr{P}$ is a partition, there always exists a proposition $\hat{p}$ in $\mathscr{P}$ such that $\mathbf{C}(\hat{p}\,|\,\mathscr{E}) = 1$ (i.e., $\hat{p}$ "intersects" $\mathscr{E}$) and, therefore,

$$\begin{aligned}\sup_{\mathscr{P}} \left[\sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q\,|\,w) \oslash \mathbf{C}(p\,|\,\mathscr{E})\right] \circledast \mathbf{C}(p\,|\,\mathscr{E})\right] &\geq \sup_{w \vdash \mathscr{E}} \left[\mathbf{I}(q\,|\,w) \oslash \mathbf{C}(\hat{p}\,|\,\mathscr{E})\right] \circledast \mathbf{C}(\hat{p}\,|\,\mathscr{E}) \\ &= \sup_{w \vdash \mathscr{E}} \mathbf{I}(q\,|\,w) \\ &= \mathbf{C}(q\,|\,\mathscr{E}).\end{aligned} \tag{2}$$

The thesis follows at once by combination of the inequalities (1) and (2). ∎

Finally, notice also that, although the theorems above have been characterized as duals, it is not necessary that $\mathscr{P}$ be a partition for the generalized modus ponens for necessities to hold, while the proof of its possibilistic counterpart relies on such assumption. It should be clear, however, that richer propositional collections $\mathscr{P}$ would lead to better lower bounds for values of the degree of implication $\mathbf{I}(q\,|\,\mathscr{E})$.

5.2 Variables

The $\circledast$-transitivity property of $\mathbf{I}$ is the essential fact expressing the relationships between the degrees of implication of three propositions that were proven in the previous section. The statements of these relations in most works devoted to fuzzy logic are made, however, using special subsets of the universe of discourse that are described through the important notion of *variable*. Introduction of this concept, which is also central to other approximate reasoning methodologies, permits us to make a clearer distinction between similarities defined, in some absolute sense, from the joint viewpoint of several respects and related proximity measures that compare objects (in our case, possible worlds) from the marginal viewpoint of one or more variables.

In what follows, we will assume that only certain propositions, specifying the value of a system variable belonging to a finite set

$$\mathscr{V} = \{\, X, Y, Z, \ldots \,\},$$

will be used to characterize possible worlds.

The propositions of interest are those formed by logical combination of statements of the type

"The value of the variable V is v,"

where V is in the variable set $\mathscr{V}$, and where v is a specific value in the domain $\mathscr{D}(V)$ of the variable V.

We will also assume that, in any possible world, the value of any variable is a member of the corresponding domain of definition of the variable. In the context of our discussion, we will not need to make special assumptions about the scalar or numeric nature of the state variables, using the notion in the same primitive and general sense in which it is customarily used in the predicate calculus.

We will be specially interested in subsets, called *variable-sets*, of the universe $\mathcal{U}$ consisting of worlds where the value of some variable V is equal to a specified value v. We will denote by $[X = x]$ (similarly $[Y = y]$, etc.) the set of all possible worlds where the proposition "The value of the variable X is x" is true. Clearly, the variable-sets in the collection

$$\{\, [X = x] : \; x \text{ is in } \mathscr{D}(X) \,\}$$

partition the universe into disjoint subsets. These collections have recently been used to characterize the concept of *rough sets* [30], of importance in many information-system analysis problems, including some that arise in the context of approximate reasoning. A similar notion has also been used also to describe algorithms for the combination of probabilities and of belief functions [39].

To simplify the notation we will write

$$w \vdash x, \; w \vdash y, \ldots$$

as shorthand for $w \vdash [X = x]$, $w \vdash [Y = y]$, ..., respectively.

5.2.1 Possibilistic Structures and Laws

The usual statements of the laws of fuzzy logic are made, as mentioned before, through the use of variables rather than by means of general symbolic expressions. It is customary, for example, to speak of the possibility of the variable X taking the value x, to describe the value that a possibility function for an evidential set $\mathscr{E}$ attains for the proposition $[X = x]$.

In our model, we will say therefore, that a function

$$\mathbf{Poss}(\cdot) \colon \; \mathscr{D}(X) \mapsto [0, 1]$$

is a possibility function for the evidential set $\mathscr{E}$ and the variable X, whenever

$$\mathbf{Poss}(x) \geq \mathbf{C}\left([X = x] \,|\, \mathscr{E} \right),$$

for all values x in the domain $\mathscr{D}(X)$. Similarly, we will say that $\mathbf{Nec}(\cdot)$ is a necessity function for X whenever

$$\mathbf{Nec}(x) \leq \mathbf{I}\left([X = x] \,|\, \mathscr{E} \right),$$

for all values x in $\mathscr{D}(X)$.

If possibility distributions are point functions defined in this way as point functions in the variable domain $\mathscr{D}(X)$, then it is possible to use the disjunctive laws of fuzzy logic proved in Section 4.4 to extend their definition over the power set of $\mathscr{D}(X)$, i.e.,

$$\begin{aligned}\mathbf{Nec}(A \cup B) &= \max\left[\mathbf{Nec}(A), \mathbf{Nec}(B)\right], \\ \mathbf{Poss}(A \cup B) &= \max\left[\mathbf{Poss}(A), \mathbf{Poss}(B)\right],\end{aligned}$$

where A and B are subsets of the domain $\mathscr{D}(X)$. These equations are usually given as the basic disjunctive laws of possibility distributions.

Note that, using such extensions, both possibility and necessity functions are nondecreasing functions (with respect to the order induced by set inclusion). The value of $\mathbf{Nec}(A)$ measures the extent by which the evidence supports the statement that the variable value necessarily lies in the subset A of its domain of definition, with a dual interpretation being applicable for possibility distributions.

5.2.2 Marginal and Joint Possibilities

The original similarity relation introduced in Section 3.1 may be considered to be a measure of proximity between possible worlds from the joint viewpoint of all system variables. The notion of variable permits, however, the definition of similarities from the restricted viewpoint of some variables or subsets of variables.

These restricted perspectives play a role with respect to the original similarity S that is analogous to that of marginal probability distributions with respect to joint probability distributions. To derive useful expressions that describe similarities between two values x and x' of the same variable X, it should be noted first that the degree of implication $\mathbf{I}(\cdot\,|\,\cdot)$ is transitive. This fact permits the application of a theorem of Valverde [44] to define a function S_X by means of the expression

$$S_X\colon\ \mathscr{D}(X) \times \mathscr{D}(X) \mapsto [0,1]\colon\ (x, x') \mapsto \min\left[\mathbf{I}(x\,|\,x'), \mathbf{I}(x'\,|\,x)\right].$$

Defined in this way as a "symmetrization" of the *preorder* induced by the degree of implication $\mathbf{I}(\cdot\,|\,\cdot)$, the marginal similarity S_X has the properties of a similarity function. Furthermore, the "projection" operation entailed by the use of $\mathbf{I}(x\,|\,x')$, based on the projection of every x'-world into the set of x-worlds), may be considered to be the basic mechanism to transform the original similarity function into one that only discern differences in the values of the variable X.

It must be noted, however, that, unless additional assumptions are made about the nature of the original similarity S, the function S_X fails to satisfy the intuitive requirement

$$S(w, w') \leq S_X(w, w'),$$

whenever $w \vdash x$ and $w' \vdash x'$ i.e., the similarity between two objects from a restricted viewpoint is always higher than their similarity from more general regards that encompass additional criteria of comparison.

Although considerable research remains to identify alternative definitions of marginal similarities that are not hampered by this problem, a basic result of Valverde [44], presented in Section 6.2 below, appears to provide the essential tool that must be employed in to produce the required coarser measures. The role of additional reasonable assumptions that might be demanded from S so as to facilitate the construction of marginal similarities with desirable characteristics is also the object of current investigations of the author.

5.2.3 Conditional Distributions and Generalized Inference

The basic conditional structures of fuzzy logic are usually defined as elastic constraints that restrict the values of a variable given those of another. By simple extension of our previous convention to conditional structures, we will write $\mathbf{Nec}(y|x)$ and $\mathbf{Poss}(y|x)$, as shorthand for

$$\mathbf{Nec}([Y = y] \mid [X = x]) \quad \text{and} \quad \mathbf{Poss}([Y = y] \mid [X = x]),$$

respectively.

If a classical (i.e., Boolean) inferential rule of the type

"If $X = x$, then Y is in $R(x)$"

is thought of as the definition of a relation R defined over pairs (x, y) in the Cartesian product $X \times Y$, then such a relation may be used to define a multivalued mapping that maps possible values of X into possible values of Y as illustrated in Figure 5.

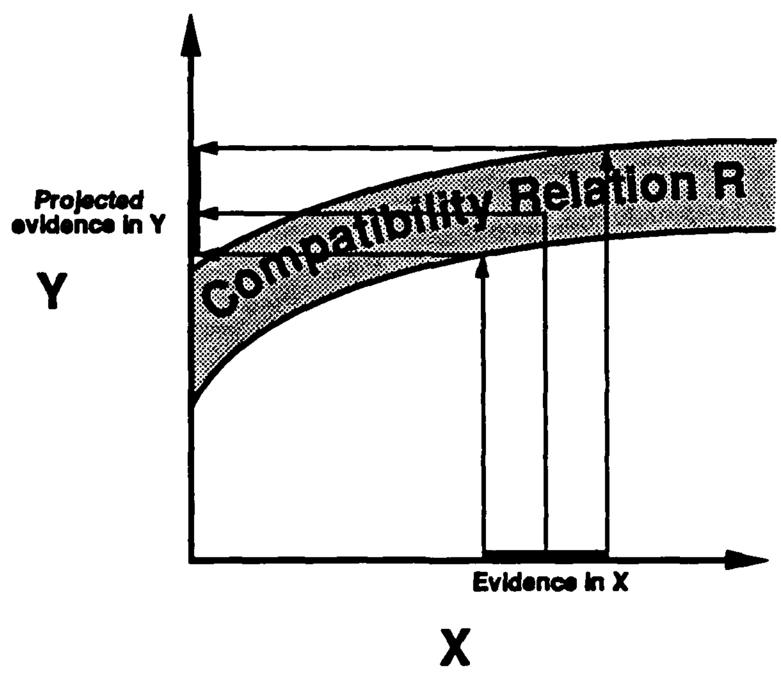

Figure 5: Inference as a Compatibility Relation.

Such a *compatibility relation* perspective was an essential element of the original formulations of both the Dempster-Shafer calculus of evidence [8] where distributions in some space (i.e., the domain of some variable X) are mapped into distributions of another variable (i.e., the domain of another variable Y) by direct transfer of "mass" from individual values to the union of their mapped projections and the compositional rule of inference [51].

Note that, whenever $\mathbf{Poss}(y|x) = 1$, if the bound is actually attained, i.e, if

$$\sup_{w \vdash y} \left[\mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w) \right] = 1 ,$$

then it is possible for an evidential world w in $[X = x]$ (i.e., $\mathbf{I}(x \mid w) = 1$) to be such that $w \vdash y$. Pairs (x, y) such that $\mathbf{Poss}(y|x) = 1$ may be considered to approximate the *core*[10] of a generalized inferential relation that allows to determine bounds for the similarity between evidential worlds and those in the variable set $[Y = y]$ on the basis of knowledge of similar bounds applicable to the variable set $[X = x]$. This relation, which is the fuzzy extension of the classical compatibility mapping R illustrated in Figure 5, may be thought as a descriptor of the behavior, for x-worlds, of the values of the variable Y "near" R. The compatibility relation is itself approximated by (or embedded in) the core of the conditional possibility distribution, i.e., worlds w such that $w \vdash x$ and $w \vdash y$, with $\mathbf{Poss}(y|x) = 1$.

Since the collection of the sets $[X = x]$ partitions the universe $\mathcal{U}$ into disjoint sets, then the generalized modus ponens laws may be readily stated in terms of variable values as

$$\begin{aligned} \mathbf{Nec}(y) &= \sup_x \left[\mathbf{Nec}(y|x) \circledast \mathbf{Nec}(x) \right] , \\ \mathbf{Poss}(y) &= \sup_x \left[\mathbf{Poss}(y|x) \circledast \mathbf{Poss}(x) \right] , \end{aligned}$$

clearly showing the basic nature of the inferential mapping as the composition of relational combination (i.e., $\circledast$-"intersection") and projection (i.e., maximization).

5.2.4 Fuzzy Implication Rules

In this section we will examine proposed interpretations for conditional rules, usually stated in the form

$$\text{If } X \text{ is } A, \text{ then } Y \text{ is } B ,$$

within the context of possibilistic logic. While, in two-valued logic, any such rule simply states that whenever a condition A is true, another condition B also holds, various interpretations have been proposed for rules expressing other notions of conditional truth.

In the case of probabilities, for example, degrees of conditionality have been modeled either by means of conditional probability values $\mathbf{Prob}(A \mid B)$, which measure the likelihood of B given the assumed truth of A, or by the alternative interpretation $\mathbf{Prob}(\neg A \vee B)$, used by Nilsson [29] in his probabilistic logic, which esssentially quantifies the probability that a rule is a valid component of a knowledge base. Either one of these interpretations is valid in particular contexts being, respectively, the probabilistic extensions of the so called "de re," i.e.,

$$p \rightarrow \Pi \, q ,$$

and "de dicto", i.e.,

$$\Pi \, (p \rightarrow q) ,$$

interpretations of conditionals in modal logic.

[10] The core of a fuzzy set μ: $\mathcal{U} \mapsto [0, 1]$ is the set of all points w such that $\mu(w) = 1$, i.e., the points that "fully" belong to μ.

In fuzzy logic, two major interpretations have been advanced to translate conditional rules,[11] with A and B corresponding to the fuzzy sets

$$\mu_A \colon X \mapsto [0,1], \quad \text{and} \quad \mu_B \colon Y \mapsto [0,1].$$

The first interpretation was originally proposed by Zadeh [52], as a formal translation of the statement

If μ_A is a possibility for X, then μ_B is a possibility distribution for Y.

This conditional statement, which may be regarded as a constraint on the values of one variable given those of another, states the existence of a conditional possibility function $\mathbf{Poss}(\cdot|\cdot)$ such that

$$\mu_B(y) \geq \sup_x \left[\mathbf{Poss}(y|x) \circledast \mu_A(x) \right] \geq \mathbf{Poss}(y|x) \circledast \mu_A(x).$$

Recalling now the definition and properties of the pseudoinverse, we may restate this particular interpretation as

$$\mathbf{Poss}(y|x) = \mu_B(y) \oslash \mu_A(x) \geq \mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w),$$

for every world $w \vdash \mathscr{E}$.

In Zadeh's original formulation, made within the context of a calculus based on the minimum function as the T-norm, conditionals were, however, formally translated by means of the pseudoinverse of the Lukasïewicz T-norm. Certain formal problems associated with such a combination were pointed out by Trillas and Valverde [42], who developed translations consistent with the T-norm used as the basis for the possibilistic calculus.

Using the characterization of conditionals introduced in Section 4.5, this relation may also be thought of as a measure of the degree by which a possibility for Y exceeds a fraction (measured by the conditional possibility distribution) of a given possibility distribution for X. In particular, whenever $\mathbf{Poss}(y|x) = 1$, then $\mu_B(y) \geq \mu_A(x)$, indicating the *possible* existence —since $\mathbf{Poss}(y|x)$ is only an upper bound of $\mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w)$ — of an evidential world such that $w \vdash x$ and $w \vdash y$, with x in A and y in B.

As illustrated in Figure 6, where it has been assumed that the underlying metric (i.e., dissimilarity) is proportional to the euclidean distance in the plane, the core of the corresponding conditional possibility distribution is an (upper) approximant of a classical compatibility relation (indicated by the shaded area in the figure) that fans outward from the Cartesian product of the cores of A and B. If this interpretation is taken, whenever several such rules are available, then each one of these rules will lead to a separate possibility distribution. Combination of these upper bounds by minimization results in a sharper possibility estimate that represents the "integrated" effect of the rule set.

The second interpretation of conditional relations, leading to a wide variety of practical applications [41], was utilized by Mamdani and Assilian to develop fuzzy controllers. The basic idea underlying this explanation follows an approach originally outlined by Zadeh [47,48,51]. In this case, a number of conditional statements of the form

$$\text{If } X \text{ is } A_k, \text{ then } Y \text{ is } B_k, \qquad k = 1, 2, \ldots, n,$$

are given as a combined "disjunctive" description of the relation between X and Y, rather than as a set of independently valid rules. The purpose of this rule set is the approximation of the

[11] A rather encompassing account of potential fuzzy reasoning mechanisms can be found in a paper by Mizumoto, Fukami, and Tanaka. [27]

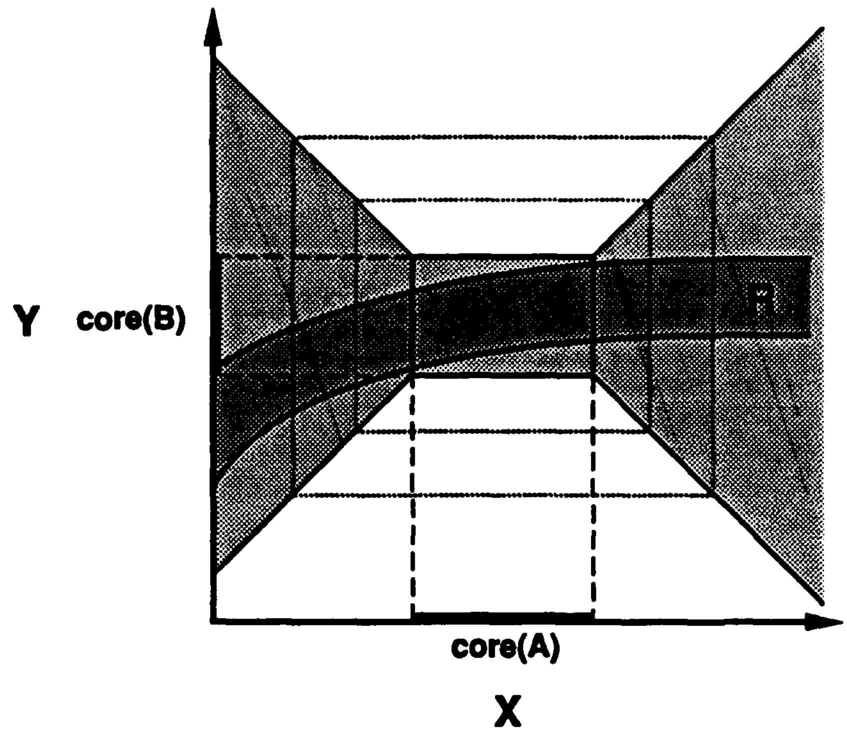

Figure 6: Rules as Possibilistic Approximants of a Compatibility Relation.

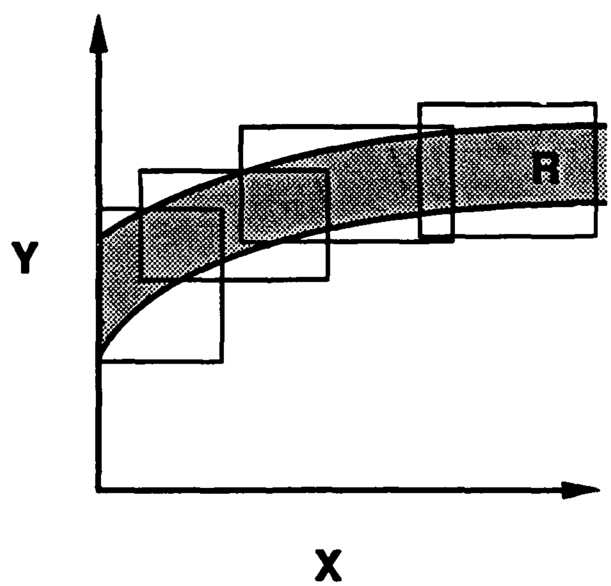

Figure 7: Rule-Sets as Possibilistic Approximants of a Compatibility Relation

compatibility relation by a "fuzzy curve" generated by disjunction of all the rules in the set, as shown in Figure 7.

Recalling the characterization of conditioning as an extension of a classical compatibility relation, we may say that the core of the compatibility relation is approximated by above by the union

$$\bigcup_{k=1}^{n} \left[\operatorname{core}(\mu_{A_k}) \times \operatorname{core}(\mu_{B_k}) \right]$$

of the Cartesian products of the cores of the fuzzy sets for A_k and B_k. In this case the multiple rules are meant to approximate some region of possible (X, Y) values, and the result of application of individual component rules must be combined using maximization to produce a conditional possibility function. We may say, therefore, that under the Zadeh-Mamdani-Assilian (ZMA) interpretation, the function

$$\mathbf{Poss}(y|x) = \sup_k \left[\min(\mu_A(x), \mu_B(y)) \right],$$

is a conditional possibility for Y given X.

It is important to note that the two interpretations of fuzzy rules that we have just examined are based on different approaches to the approximation (by above) of the value

$$\sup_{w \vdash \mathcal{Y}} \left[\mathbf{I}(y \mid w) \oslash \mathbf{I}(x \mid w) \right],$$

being, in the the case of the Zadeh-Trillas-Valverde (ZTV) method, the result of the *conjunction* of multiple fuzzy relations such as that illustrated in Figure 8, while, in the case of the ZMA logic, the construction requires *disjunction* of relations such as that illustrated in Figure 9.

The difference between both approaches when combining several rules is illustrated also in Figures 10 and 11, showing the contour plots for the α-cuts of the fuzzy relations that are obtained in a simple example involving four rules. In these figures, the rectangles with a dark outline correspond to the Cartesian products of the cores of the antecedents A_k and B_k. Darker shades of gray correspond to higher degrees of membership.

The reader should be cautioned, however, about the potential for invalid comparisons that may result from hasty examination of these figures. Each formalism should be regarded as a procedure for the approximation of a compatibility relation that is based on a different approach for the description of relationships between variables. In the case of the ZMA interpretation, the intent is to generalize the interpolation procedures that are normally employed in functional approximation. As such, this approach may be said to be inspired by the methodology of classical system analysis. The ZTV approach, by contrast, is a generalization of classical logical formulations and may be regarded, from a relational viewpoint, as a procedure to describe a function as the locus of points that satisfies a set of constraints rather than as a subset of "fuzzy points" of a Cartesian product.

Figures 10 and 11, while showing that the same rule sets would lead to radically different results, should not be considered, therefore, to discredit interpolative approaches as such techniques, proceeding from a different perspective, should normally be based on rule sets that are different from those utilized when rules are thought of as independent constraints.

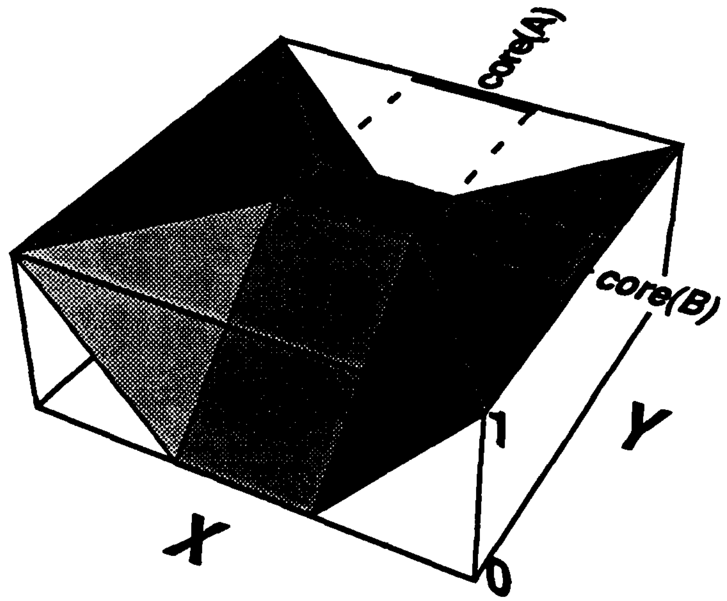

Figure 8: A Possibilistic Conditional Rule (ZTV)

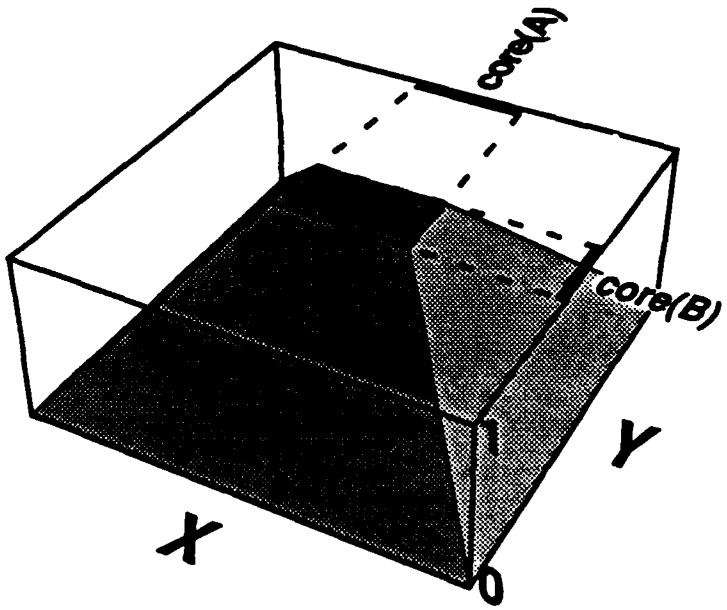

Figure 9: A Component of a Disjunctive Rule Set (ZMA)

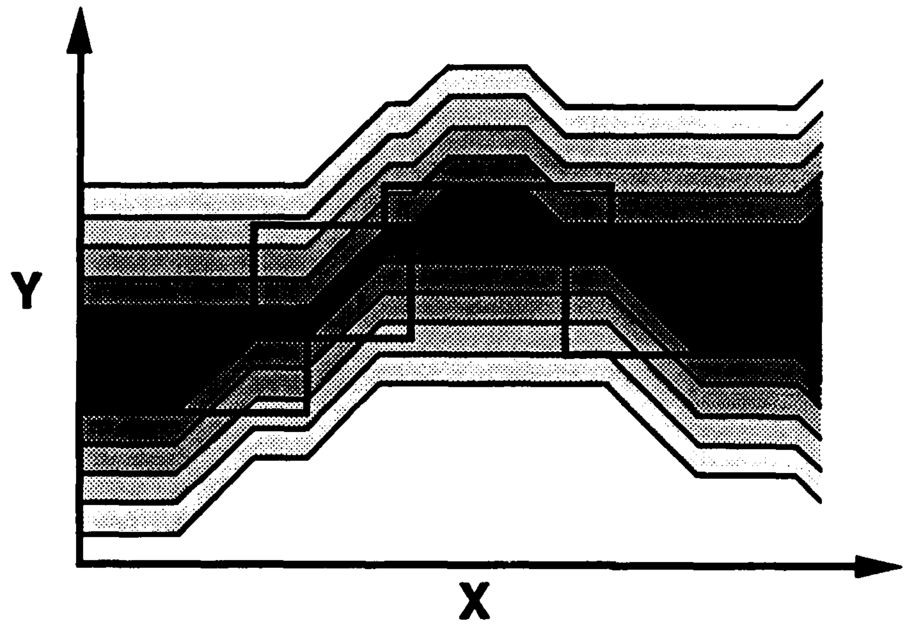

Figure 10: Contour Plots for a Rule Set (ZTV)

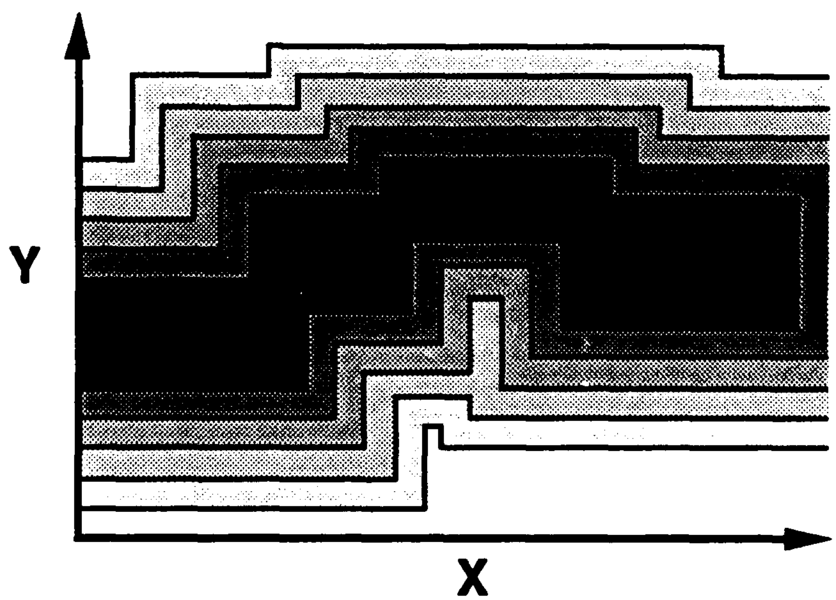

Figure 11: Contour Plots for a Rule Set (ZMA)

6 THE NATURE OF SIMILARITY RELATIONS

In this closing section, we will examine issues that arise naturally from our previous examination of the role of similarities as the semantic bases for possibility theory.

Our discussion focuses on two topics. We look first at the requirements that our theory imposes upon the nature of the scales used to measure proximity or resemblance between possible worlds. Finally, our examination of the interplay between similarities and possibilities turns to issues related to the generation of similarity relations from such sources as domain knowledge that describes significant relations between system variables.

6.1 On Similarity Scales

Our previous interpretation of possibilistic concepts and structures has been based on the use of measures of proximity that quantify interobject resemblance using real numbers between 0 and 1. Our assumptions about the use of the $[0, 1]$ interval as a similarity scale have been made primarily, however, as a matter of convenience so as to simplify the description of our model while being consistent with the customary definitions of possibility and necessity distributions as functions taking values in that interval.

Close examination of the actual requirements imposed upon our similarity scales reveals, however, that our measurement domain may be quite general so as to include symbolic structures such as

$$\{\, \textit{identical}\,, \textit{very similar}\,, \ldots,\ \textit{completely dissimilar} \}\,.$$

Our model is based on the use of a partially ordered set having a maximal and a minimal element that measure identity and complete dissimilarity, respectively. Furthermore, we have assumed the existence of a binary operation (the triangular norm $\circledast$) mapping pairs of possible worlds into real numbers, with certain desirable order-preserving and transitive properties. The concept of triangular norm, however, does not rely substantially on the use of real numbers as its range and may be readily extended to more general partially ordered sets with maximal and minimal elements.

We have also assumed a continuity property for the triangular norm operation. This property, however, simply requires that a notion of proximity also exist among similarity values so as to provide a form of (order-consistent) topology in that space. While, in general, more precise scales will result in more detailed representations of interworld similarity, it is important to stress that the similarity-based model presented here does not rely in "denseness" assumptions such as the existence an intermediate value c between any different values a and b in the similarity-measurement scale.

From a practical viewpoint, the major requirement is to quantify proximity in such a way as to be able to determine that two quantities are similar to some degree (i.e., approximate matching). The degree of precision that such a matching entails is problem-dependent and will be typically the result of conflicting impositions between the desire, on one hand, to keep granularity relatively high to reduce complexity, and the need, on the other, to describe system behavior at an acceptable level of accuracy. The work of Bonissone and Decker [4] is a significant example of the type of systematic study that must be carried out to define similarity scales that are both useful and tractable.

6.2 The Origin of Similarity Functions

The model of fuzzy logic presented in this note is centered on the metric notion of similarity as a primitive concept that is useful to explain the nature of possibilistic constructs and the meaning of possibilistic reasoning. In this formulation, similarities are defined as real functions defined over pairs of possible worlds.

From this perspective, similarities describe relations of resemblance between objects of high complexity, which, typically, result from consideration of a large number of system variables. Reliance on such complex structures has been the direct consequence of a research program that stressed conceptual clarification as its primary objective. In practice, however, it will be generally difficult to define complex measures that quantify similarity between complex objects on the basis of a large number of criteria.

Similarities provide the framework that is required to understand approximate relations of corelevance, usually stated as generalized conditional rules. The practical generation of similarity functions typically proceeds, however, in the opposite direction, from separate statements about limited aspects of system behavior to general metric structures. Once such resemblance measures are defined, they may be used to express and acquire new laws of system behavior determined, for example, from historical experience with similar systems. Furthermore, such similarity notions may be used as the basis for analogical reasoning systems that try to determine system state on the basis of similarity to known cases [23].

Perhaps the simplest mechanism that may be devised to generate complex metrics from simpler ones is that which starts with measures of resemblance that quantify proximity from a limited viewpoint. These metrics are usually derived, using a variety techniques, in unsupervised pattern classification (or clustering) problems [20]. In many important applications, hierarchical taxonomies —a feature of many representation approaches in artificial intelligence—may be used, often in connection with a variety of weighing schemes—quantifying branching importance—to generate metrics that often satisfy the more stringent requirements of an ultrametric [22].

Classification hierarchies such as those may be thought of as sets of general rules, having a particularly useful structure, that specify interset proximity from relevant, but restricted viewpoints, eventually providing measures of similarity between variable values (i.e., the "leaves" of the taxonomical tree). More generally, however, we may expect that sets of possibilistic rules (i.e., a general knowledge base) defining a general semantic network of corelevance relations may be available as the source for the determination of interobject proximity. These possibilistic semantic networks resemble conventional semantic networks in most regards, being more general in that, in addition to specifying knowledge about system behavior in some subsets of state-space,[12] they also specify characteristics of behavior in neighborhoods of those subsets.

We may think, therefore, that the antecedents of implicational rules define general regions in state space where existence of relevant knowledge may increase insight through application of inferential rules. Using Zadeh's terminology, these antecedents define "granules" that identify important regions of state-space and indicate the level of accuracy that is required (or *granularity*) to perform effective system analysis. In this case, the possibilistic granules correspond to fuzzy sets that are used to specify both what is true in the core of the granule and, with decreasing specificity, what is true in a nested set (i.e., the α-cuts) of its neighborhoods. The ability to specify behavior using such a topological structure results in inferential gains that are the direct consequence of our ability

[12] The expression "state-space" is loosely used here to indicate the space defined by all system variables.

to reason by similarity; an ability that is made possible by the approximate matching property of the generalized modus ponens. From another perspective yet, the fuzzy granules identified by possibilistic rules may also be thought of as generalizations of the arbitrary variable sets used in a variety of artificial intelligence efforts aimed at understanding system behavior using qualitative descriptions of reality [16].

A number of heuristics may be easily formulated to integrate "marginal" measures of resemblance into joint similarity relations. More generally, however, we may state the problem of similarity construction as that of defining metric structures on the basis of knowledge of the aspects of system behavior that are important to its understanding—i.e., the previously mentioned granules, which define what must be distinguished. Since generally those granules are fuzzy sets, the relevance to similarity construction of the following representation theorem, due to Valverde, may be immediately seen:

Theorem [Valverde]: A binary function S mapping pairs of objects of a universe of discourse $\mathcal{U}$ into $[0,1]$ is a similarity relation, if and only if there exists a family $\mathcal{H}$ of fuzzy subsets of $\mathcal{U}$ such that

$$S(w,w') = \inf_{\mathcal{H}} \left[\min \left(h(w) \oslash h(w'),\, h(w') \oslash h(w) \right) \right],$$

for all w and w' in $\mathcal{U}$, where the infimum is taken over all fuzzy subsets h in the family $\mathcal{H}$.

Besides its obvious relevance to the generation of similarity relations from knowledge of important sets in the domain of discourse, Valverde's theorem—resulting originally from studies in pattern recognition—is also of potential significance to the solution of knowledge acquisition problems because of the important relations that exist between learning procedures and structure-discovery techniques such as cluster analysis.

7 CONCLUSION

This note has presented a similarity-based model that provides a clear interpretation of the major structures and methods of possibilistic logic using metric concepts that are formally different from the set-measure constructs of probability theory. Regardless of the potential existence, so far unestablished, of probability-based interpretations for possibilistic structures, this metric model makes clear that there are no compelling reasons to confuse two rather different aspects of uncertainty into a single notion simply because one's favorite theoretical framework, in spite of its otherwise many remarkable virtues, fails to fully capture reality.

Succintly stated, being in a situation that resembles a state of affairs S does not make S likely or viceversa. Furthermore, our reference state may not even be possible in the current circumstances —making it completely unlikely—but we may still find it useful as a comparison landmark.This use of "impossible" examples as a way to illustrate system behavior is very prevalent in human culture, being exemplified by such utterances as "he had the strength of a horse and the swiftness of a swallow," even if it is obvious to all that no such beasts exist other than for such metaphorical purposes.

The insight provided by this model makes it rather obvious that very little can be gained by continuing to assert a potential—although never revealed—encompassing probabilistic interpretation for possibilistic structures that, presumably, would render them unnecessary as serious objects of scientific discourse. In addition, and quite beyond whatever understanding theory may provide, the current success of possibilistic logic as the basis for major systems of important human value [41] —often unmatched by other approaches—should be enough to convince those having more pragmatic perspectives as to its utility.

The task for approximate reasoning researchers is to proceed now beyond unnecessary controversy into the study of the issues that arise from models such as the one presented in this note. Among such questions, further studies of the relations between the notions of possibility, similarity, and negation and of those between probability and possibility are of major importance.

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