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An independent set J_c of a graph G is called critical if
 $|J_c| - |N(J_c)| = \max_c \{|J| - |N(J)| : J \text{ is an independent set of } G\}$
and a vertex subset U_c is called critical if
 $|U_c| - |N(U_c)| = \max_c \{|U| - |N(U)| : U \text{ is a subset of } V(G)\}$.
In this paper, we will show that to find a critical independent set and a critical vertex subset of a graph are solvable in polynomial time.

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**FINDING CRITICAL INDEPENDENT SETS
AND
CRITICAL VERTEX SUBSETS
ARE
POLYNOMIAL PROBLEMS**

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ABSTRACT

An independent set J_c of a graph G is called critical if $|J_c| - |N(J_c)| = \max\{|J| - |N(J)| : J \text{ is an independent set of } G\}$, and a vertex subset U_c is called critical if $|U_c| - |N(U_c)| = \max\{|U| - |N(U)| : U \text{ is a vertex subset of } G\}$. In this paper, we will show that to find a critical independent set and a critical vertex subset of a graph are solvable in polynomial time.

Key words: Independent set, Polynomial algorithm
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§ 1. INTRODUCTION

It has been proved by mathematicians that finding a maximum independent set in some certain kinds of graphs is solvable in polynomial time (for example, line graphs, bipartite graphs, circle graphs, circular arc graphs and claw free graphs, (see [GJ])). But it is well-known that it is an NP-complete problem for general graphs, (see [GJS]) . In this paper, we will investigate another problem -- finding a certain kind of independent sets in general graphs. An independent set J_c of a graph G is called critical if $|J_c| - |N(J_c)|$ is the maximum of $|J| - |N(J)|$ over all independent sets J of G , where $N(J)$ is the set all vertices of G adjacent to some vertex of J . It will be proved in this paper that finding a critical independent set of a graph is solvable in polynomial time. Let

$$\alpha_c = \max \{ |J| - |N(J)| : J \text{ is an independent set of } G \}$$

which is a parameter of a graph G and is called the critical independence number of G . The critical independence number

α_c of a graph plays the central role in the study of fractional independence functions and fractional matching functions of graphs [GZ]. (It is proved in [GZ] that the fractional independence number and the fractional matching

number of a graph G are $\frac{n - \alpha_c}{2}$ and $\frac{n + \alpha_c}{2}$, respectively, where $n = |V(G)|$).

Some related problems and parameters of graphs will also be investigated in this paper. A vertex subset U_c of a graph $G = (V, E)$ is called critical if $|U_c| - |N(U_c)|$ is the maximum of $|U| - |N(U)|$ over all vertex subsets U of G . Let

$$\mu_c = \max \{ |U| - |N(U)| : U \text{ is a vertex subset of } G \}$$

which is a parameter of a graph G . Some similar parameters of graphs have been studied by Woodall [WD] and Mohar [MB].



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The binding number $b(G)$ [WD] and the isoperimetric number $i(G)$ [MB] of a graph G are defined as the following:

$$b(G) = \min \left\{ \frac{|N(U)|}{|U|} : U \subset V(G), U \neq \emptyset \text{ and } N(U) \neq V(G) \right\};$$

$$i(G) = \min \left\{ \frac{|\partial(U)|}{|U|} : U \subset V(G), U \neq \emptyset \text{ and } |U| \leq \frac{|V(G)|}{2} \right\}.$$

(where $\partial(U)$ is the number of edges of G with one endvertex in U). Later in this paper, it will be proved that $\alpha_c = \mu_c$.

Since the empty set is an independent set and the set of all vertices of a graph G is also a vertex subset of G , it is trivial that

$$\alpha_c \geq 0 \text{ and } \mu_c \geq 0$$

for any graph G . Note that the empty set and the entire graph are critical vertex subsets of some connected graph G if $\mu_c(G) = 0$. If we are to avoid these two trivial vertex subsets \emptyset and $V(G)$, we may consider the following parameter of a graph G :

$$\mu'_c = \max \{ |U| - |N(U)| : U \subset V(G), U \neq \emptyset \text{ and } U \neq V(G) \}.$$

But for any connected graph G and a vertex v of G , we have that $N(V(G) \setminus \{v\}) = V(G)$ and therefore the parameter $\mu'_c(G)$ still have a lower bound -1 . In order to get more information about graphs, we prefer to only consider those vertex subsets U of a graph G such that $U \neq \emptyset$ and $N(U) \neq V(G)$ which is similar to the definition of the binding number of graphs. A vertex subset U of G is called proper if $U \neq \emptyset$ and $N(U) \neq V(G)$. A proper vertex subset U_{pc} of G is called critical if $|U_{pc}| - |N(U_{pc})|$ is the maximum of $|U| - |N(U)|$ over all proper vertex subsets U of G . The parameter μ_{pc} of a graph G is defined as the following:

$$\mu_{pc} = \max \{ |U| - |N(U)| : U \subset V(G), U \neq \emptyset \text{ and } U \neq V(G) \}$$

The problems will be proved to be solvable in polynomial time in this paper are listed as the following:

INSTANCE. let $G=(V,E)$ be a graph with the vertex set V and the edge set E and k be an integer.

PROBLEM 1. Is there an independent set J of G such that

$$|J| - |N(J)| \geq k?$$

PROBLEM 1*. Find a critical independent set J_c and the critical independence number α_c of G . (That is to find

$$\begin{aligned} \alpha_c &= |J_c| - |N(J_c)| \\ &= \max \{ |J| - |N(J)| : J \text{ is an independent set of } G \}. \end{aligned}$$

PROBLEM 2. Is there a vertex set U of G such that

$$|U| - |N(U)| \geq k?$$

PROBLEM 2*. Find a critical vertex subset U_c and the parameter μ_c of G . (That is to find

$$\begin{aligned} \mu_c &= |U_c| - |N(U_c)| \\ &= \max \{ |U| - |N(U)| : U \text{ is a vertex subset of } G \}. \end{aligned}$$

PROBLEM 3. Is there a proper vertex subset U of G such that

$$|U| - |N(U)| \geq k?$$

PROBLEM 3*. Find a critical proper vertex subset U_{pc} and the parameter μ_{pc} of G (That is to find

$$\begin{aligned} \mu_{pc} &= |U_{pc}| - |N(U_{pc})| \\ &= \max \{ |U| - |N(U)| : U \subset V(G), U \neq \emptyset \text{ and } U \neq V(G) \}. \end{aligned}$$

§ 2. MAIN RESULTS

THEOREM 1.

Problem 3 and 3* are solvable in polynomial time.

Before we prove the Theorem 1 we would like to consider the following problems first. And the Theorem 1 will be a corollary of the Theorem 2.

INSTANCE. Let $G=(V,E)$ be a graph with the vertex set V and the edge set E , $\{u,v\}$ be an ordered pair of nonadjacent vertices of G and k be an integer.

PROBLEM 4. Is there a vertex subset U of G such that
 $u \in U, v \in N(U)$ and
 $|U| - |N(U)| \geq k$?

PROBLEM 4*. Find a vertex subset U_0 of G such that
 $|U_0| - |N(U_0)|$

$$= \max \{ |U| - |N(U)| : U \subset V(G), u \in U \text{ and } v \in N(U) \}.$$

The vertex subset U_0 found in problem 4* is called (u,v) -critical subset of G .

THEOREM 2.

The problems 4 and 4* are solvable in polynomial time.

The following lemmas will be used in the proof of the Theorem 2.

LEMMA 3. (Hall's Theorem [HP])

Let $B=(V_1, V_2; E)$ be a bipartite graph. The graph B has a matching covering all vertices of V_2 if and only if $|U| \leq |N(U)|$ for any subset U of V_2 .

LEMMA 4.

Let $B = (V_1, V_2; E)$ be a bipartite graph. Assume that there is no matching of B covering all vertices of V_2 . We will have the following conclusions:

- (i). There is a subset U of V_2 such that $|N(U)| < |U|$;
- (ii). Let U_0 be a subset of V_2 such that $|U_0| - |N(U_0)|$ is as great as possible, then there is a matching of the induced bipartite subgraph $(U_0, (N(U_0); E[U_0, N(U_0)]))$ covering all vertices of $N(U_0)$.

Note, if A and B are a pair of disjoint vertex subset of a graph G, the set of all edges joining A and B is denoted by $E(A,B)$.

PROOF

The conclusion of (i) is an immediate corollary of Hall's Theorem.

Let U be a subset of V_2 such that $|U| - |N(U)|$ is as great as possible. Let $B'=(U,N(U); E[U,N(U)])$ be the subgraph of B induced by $U \cup N(U)$. We claim that $|X| \leq |N(X) \cap U|$ for any subset X of $N(U)$. If not, let $X \subseteq N(U)$ such that

$$|X| > |N(X) \cap U|.$$

We will consider the subset $Y = U \setminus N(X)$. Note that

$$N(Y) = N(U \setminus N(X)) \subseteq N(U) \setminus X.$$

And

$$\begin{aligned} |Y| - |N(Y)| &\geq |U \setminus N(X)| - |N(U) \setminus X| \\ &= [|U| - |U \cap N(X)|] - [|N(U)| - |X|] \\ &= [|U| - |N(U)|] + [|X| - |U \cap N(X)|] \\ &> |U| - |N(U)|. \end{aligned}$$

This contradicts the choice of U that $|U| - |N(U)|$ is maximum. So by Hall's theorem, there is a matching in B' which covers all vertices of $N(U)$.

###

PROOF OF THE THEOREM 2.

We are only to prove that problem 4* is solvable in polynomial time. Let $G=(V,E)$ be a graph with the vertex set $V=\{1,2,3,\dots,n\}$ and the edge set E. We will consider the ordered pair of vertices (1,2) of G and find a (1,2)-critical vertex subset.

Define a bipartite graph $B = (X,Y; E_B)$ where

$$X = \{x_1, \dots, x_n\},$$

$$Y = \{y_1, \dots, y_n\}$$

and $E_B = \{(x_i, y_j) : (i, j) \text{ is an edge of } G\}$.

Let V' be a subset of $V(G)$, then the corresponding subsets in X and Y are denoted by $X(V')$ and $Y(V')$, respectively. For

example, if $V' = \{i_1, \dots, i_t\}$, then

$$X(V') = X(\{i_1, \dots, i_t\}) = \{x_{i_1}, \dots, x_{i_t}\} \text{ and}$$

$$Y(V') = Y(\{i_1, \dots, i_t\}) = \{y_{i_1}, \dots, y_{i_t}\}. \text{ (Here } X \text{ and } Y \text{ can be}$$

considered as bijections mapping $\{1, 2, \dots, n\}$ onto $\{x_1, \dots, x_n\}$

and $\{y_1, \dots, y_n\}$). If $W = \{x_{i_1}, \dots, x_{i_t}\} \subset X$ (or

$W = \{y_{i_1}, \dots, y_{i_t}\} \subset Y$), then the corresponding subset $\{i_1, \dots, i_t\}$

of $V(G)$ is denoted by $X^{-1}(W)$ (or $Y^{-1}(W)$, respectively). The

set of all neighbors of a vertex u in B is denoted by $N_B(u)$.

If i is a vertex of G , then

$$N_B(x_i) = \{y_j \in Y : (x_i, y_j) \in E_B\} = \{y_j \in Y : (i, j) \in E(G)\} = Y(N(i)).$$

A weight $w: X \cup Y \rightarrow [0, 2]$ is called a (1,2)-proper weight of B if

$$w(x_1) = 2,$$

$$w(y_2) = 1,$$

$$1 \leq w(x_i) \leq 2 \quad \text{for each vertex } x_i \in X,$$

$$0 \leq w(y_i) \leq 1 \quad \text{for each vertex } y_i \in Y$$

$$\text{and } 0 \leq w(x_i) + w(y_i) \leq 2 \quad \text{for each edge } (x_i, y_i) \in E_B.$$

The total weight $\sum_{u \in X \cup Y} w(u)$ of B is denoted by $w(B)$. A (1,2)-

proper weight w_m of B is call optimum if

$$w_m(B) = \max\{w(B) : w \text{ is a (1,2)-proper weight of } B\}.$$

It is obvious that finding an optimum (1,2)-proper weight of B is a linear programming problem. Hence it is solvable in

polynomial time. The purpose of the investigation of an

optimum (1,2)-proper weight w_m of B is to prove that the

vertex subset $\{i \in V(G) : w_m(x_i) > 1\}$ is a (1,2)-critical subset

of G .

I. Let w_m be an optimum $(1,2)$ -proper weight of B and U_0 be a $(1,2)$ -critical vertex subset of G . We claim that

$$w_m(B) \geq 2n + |U_0| - |N(U_0)| = 2n + \alpha_c \quad (1)$$

Consider the following weight w_1 of B :

$$w_1(x_i) = \begin{cases} 2 & \text{if } i \in U_0 \\ 1 & \text{otherwise} \end{cases}$$

and $w_1(y_j) = \begin{cases} 0 & \text{if } j \in N(U_0) \\ 1 & \text{otherwise} \end{cases}$.

It is easy to see that w_1 is a $(1,2)$ -proper weight of B and

$$\begin{aligned} w_1(B) &= 2|U_0| + (n - |U_0|) + (n - |N(U_0)|) \\ &= 2n + |U_0| - |N(U_0)|. \end{aligned}$$

By the choice of w_m , we have verified the inequality (1).

II. Let

$$X_b = \{x_i : w_m(x_i) > 1\}$$

$$Y_s = \{y_i : w_m(y_i) < 1\}$$

$$X_{b'} = \{x_i : 1 < w_m(x_i) < 2\}$$

and

$$Y_{s'} = \{y_i : 0 < w_m(y_i) < 1\}.$$

By the definition of $(1,2)$ -proper weight, it is obvious that $N_B(X_b) \subseteq Y_s$ and $N_B(Y_{s'}) \subseteq [X \setminus X_b] \cup X_{b'}$.

III. Case 1. Suppose that there is a matching M in the induced subgraph $B(X_{b'} \cup Y_{s'})$ covering all vertices of $Y_{s'}$. We claim that X_b is a $(1,2)$ -critical vertex subset in this case. We are to adjust the weight w_m so that the new weight of each vertex in $X_{b'}$ is two and the new weight of each vertex in $Y_{s'}$ is one. If we can verify that this new weight is optimum, by the inequality (1), it can be shown that $X^{-1}(X_b)$ is a $(1,2)$ -critical vertex set of G .

If (u,v) is an edge of M , then let $u=M(v)$ and $v=M(u)$. And the sets of vertices of $X_{b'}$ covered and not covered by M is denoted by $M(Y_{s'})$ and $X_{b'} \setminus M$, respectively. Thus $X_{b'} \cup Y_{s'} = Y_{s'} \cup M(Y_{s'}) \cup (X_{b'} \setminus M)$ since M covers all vertices of $Y_{s'}$.

Consider the following weight w_2 of B :

$$w_2(v) = \begin{cases} 2 & \text{if } v \in X_b, \\ 0 & \text{if } v \in Y_s, \\ w_m(v) & \text{otherwise} \end{cases} .$$

Since any vertex of Y adjacent to a vertex x_i of X_b must be in Y_s in which the weight of each vertex is zero, w_2 is a $(1,2)$ -proper weight of B . Note that w_m is optimum and the total weight $w_m(B)$ cannot be less than $w_2(B)$. We claim that w_2 is also an optimum $(1,2)$ -proper weight of B by proving that $w_m(B) \leq w_2(B)$. Since

$$\{v \in X \cup Y : w_m(v) \neq w_2(v)\} = X_b \cup Y_s,$$

we must have that

$$\begin{aligned} & w_m(B) - w_2(B) \\ &= \sum_{v \in X_b} [w_m(v) - w_2(v)] + \sum_{v \in Y_s} [w_m(v) - w_2(v)] \\ &= \sum_{v \in Y_s} \{ [w_m(v) - w_2(v)] + [w_m(M(v)) - w_2(M(v))] \} + \\ & \quad + \sum_{v \in X_b \setminus M} [w_m(v) - w_2(v)]. \end{aligned}$$

$$\begin{aligned} \text{Here} \quad & [w_m(v) - w_2(v)] + [w_m(M(v)) - w_2(M(v))] \\ &= [w_m(v) - w_m(M(v))] - [w_2(v) - w_2(M(v))] \\ &\leq 2 - (0+2) \\ &= 0 \end{aligned}$$

for any $v \in Y_s$, and

$$w_m(v) - w_2(v) < 2 - 2 = 0$$

for any $v \in X_b \setminus M$. This implies that $w_m(B) - w_2(B) \leq 0$.

Therefore w_2 is also an optimum $(1,2)$ -proper weight of B and $w_m(B) = w_2(B)$. The total weight of w_2 is

$$w_2(B) = 2|X_b| + |X \setminus X_b| + |Y \setminus Y_s|.$$

Since $N_B(X_b) \subseteq Y_s$, we must have that

$$\begin{aligned} w_m(B) = w_2(B) &\leq 2|X_b| + (n - |X_b|) + (n - |N(X_b)|) \\ &= 2n + |X_b| - |N(X_b)| \\ &= 2n + |X^{-1}(X_b)| - |N(X^{-1}(X_b))| \\ &\leq 2n + \alpha_c \quad (\text{by the definition} \end{aligned}$$

of α_c).

By (1), all equalities hold and therefore $X^{-1}(X_b) = \{i \in V(G) : w_m(x_i) > 1\}$ is a (1,2)-critical vertex subset of G .

IV. Case 2. If there is no matching of $B(X_{b'} \cup Y_{s'})$ covering all vertices of $Y_{s'}$. By Lemma 4, there is a subset Y_0 of $Y_{s'}$ such that (i). $|Y_0| > |N_B(Y_0) \cap X_{b'}|$ and, (ii). there is a matching M' in the induced bipartite subgraph $B(Y_0 \cup [N_B(Y_0) \cap X_{b'}])$ covering all vertices of $N_B(Y_0) \cap X_{b'}$. We are to adjust the weight w_m so that the new weight of each vertex in $Y_0 \cup [N_B(Y_0) \cap X_{b'}]$ is one. We will find that the new weight is greater than w_m . It will contradict that w_m is optimum.

Consider the following weight w_3 :

$$w_3(v) = \begin{cases} 1 & \text{if } v \in Y_0 \cup [N_B(Y_0) \cap X_{b'}] \\ w_m(v) & \text{otherwise} \end{cases}$$

The weight w_3 is (1,2)-proper since $x_1 \notin N(Y_{s'})$ and any vertex adjacent to a vertex of Y_0 must be in $[X \setminus X_b] \cup [N_B(Y_0) \cap X_{b'}]$ in which the weight of each vertex is one. Note that w_m is an optimum (1,2)-proper weight of B , thus the total weight of w_m cannot be less than the total weight of w_3 . Since

$$\begin{aligned} \{v \in V(B) : w_m(v) \neq w_3(v)\} &\subseteq Y_0 \cup [N_B(Y_0) \cap X_{b'}] \\ &= [N_B(Y_0) \cap X_{b'}] \cup M' [N_B(Y_0) \cap X_{b'}] \cup [Y_0 \setminus M'], \end{aligned}$$

where $M' [N_B(Y_0) \cap X_{b'}]$ and $[Y_0 \setminus M']$ are the sets of vertices of Y_0 covered and not covered by M' , respectively. Thus we must have that

$$\begin{aligned} w_m(B) - w_3(B) &= \sum_{v \in N_B(Y_0) \cap X_{b'}} [w_m(v) - w_3(v)] + \sum_{v \in Y_0} [w_m(v) - w_3(v)] \\ &= \sum_{v \in N_B(Y_0) \cap X_{b'}} \{ [w_m(v) - w_3(v)] + [w_m(M(v)) - w_3(M(v))] \} + \\ &\quad + \sum_{v \in Y_0 \setminus M'} [w_m(v) - w_3(v)]. \end{aligned}$$

But

$$[w_m(v) - w_3(v)] + [w_m(M(v)) - w_3(M(v))]$$

$$\begin{aligned}
&= [w_m(v) - w_m(M(v))] - [w_3(v) - w_3(M(v))] \\
&\leq 2 - (1+1) \\
&= 0
\end{aligned}$$

for any $v \in N_B(Y_0) \cap X_{b'}$, and

$$w_m(v) - w_3(v) < 1 - 1 = 0$$

for any $v \in Y_0 \setminus M'$. Since $|Y_0| > |N_B(Y_0) \cap X_{b'}|$, the set $Y_0 \setminus M'$ is not empty and we have that $w_m(B) - w_3(B) < 0$. This contradicts that w_m is optimum and completes the proof of the theorem.

###

Proof of the Theorem 1.

For any ordered pair of non-adjacent vertices $\{u, v\}$ of G , by Theorem 2, we can find a (u, v) -critical subset in polynomial time. The (u, v) -critical subset is denoted by $U_C(u, v)$. Then choose a (u_0, v_0) -critical vertex subset $U_C(u_0, v_0)$ such that

$$\begin{aligned}
&|U_C(u_0, v_0)| - |N(U_C(u_0, v_0))| \\
&= \max\{|U_C(u, v)| - |N(U_C(u, v))| : (u, v) \text{ are an ordered}
\end{aligned}$$

pair of nonadjacent vertices of $G\}$ which is a proper critical subset of G desired in problem 3*. The total cost of finding a critical proper vertex subset is polynomial since the cost of finding a (u, v) -critical vertex subset is polynomial and the number of pairs of non-adjacent vertices in G is at most $\binom{n}{2}$.

###

Theorem 5.

Problem 2 and 2* are solvable in polynomial time.

Proof.

Let $G=(V, E)$ be a graph. Consider a new graph G' by adding two isolated vertices x, y to G . Let U_C be a (x, y) -

critical subset of G' . Obviously, $x, y \in U_c$ and it is clear that $U_c \setminus \{x, y\}$ is a critical subset of G .

Alternating proof of the Theorem: Define a bipartite graph $B = (X, Y; E_B)$ where $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$ and $E_B = \{(x_i, y_j) : (i, j) \text{ is an edge of } G\}$. And assign a weight w to the vertex set of B such that $w: X \cup Y \rightarrow [0, 2]$ and

$$1 \leq w(x_i) \leq 2 \quad \text{for each vertex } x_i \in X,$$

$$0 \leq w(y_i) \leq 1 \quad \text{for each vertex } y_i \in Y$$

$$\text{and} \quad 0 \leq w(x_i) + w(y_i) \leq 2 \quad \text{for each edge } (x_i, y_i) \in E_B.$$

The total weight $\sum_{u \in X \cup Y} w(u)$ is denoted by $w(B)$. Let

$$w_m(B) = \max\{w(B) : w \text{ is a weight of } B \text{ satisfies the above definition}\}.$$

By an argument similar to the proof of Theorem 2, we can prove that the set $V_b = \{i \in V(G) : w_m(x_i) > 1\}$ is an critical vertex subset of G .

###

Theorem 6.

Problem 1 and 1* are solvable in polynomial time.

Before the proof of the Theorem 6, we will prove a Theorem by which Theorem 6 is an immediate corollary.

Theorem 7.

Let $G = (V, E)$ be a graph.

(i). Let U_c be a critical vertex subset of G and T_1, \dots, T_h be all non-trivial components of the induced subgraph $G(U_c)$. Then $J = V(U_c) \setminus [V(T_1) \cup \dots \cup V(T_h)]$ is a critical independent set of G and $|J| - |N(J)| = |U_c| - |N(U_c)|$.

(ii). $\max \{ |J| - |N(J)| : J \text{ is an independent set of } G \}$
 $= \max \{ |U| - |N(U)| : U \text{ is a vertex subset of } G \}$.

That is, any critical independent set is also a critical vertex subset of G and therefore $\alpha_c = \mu_c$.

Proof.

It is obvious that

$$0 \leq \alpha_c = \max \{ |J| - |N(J)| : J \text{ is an independent set of } G \}$$

$$\leq \max \{ |U| - |N(U)| : U \text{ is a vertex subset of } G \} = \mu_c$$

.....(2)

since any independent set is a vertex subset of G .

Let U_c be a critical vertex subset of G . The Theorem is trivial if U_c is an empty set. Assume that U_c is a counterexample to the Theorem containing minimum number of vertices. By the assumption U_c cannot be an independent set of G . Let T be a non-trivial component of the induced subgraph $G(U_c)$. It is clear that $V(T) \subseteq N(T)$ since T is not a singleton. Thus

$$|U_c \setminus T| - |N(U_c \setminus T)| \geq [|U_c| - |T|] - [|N(U_c) \setminus T|]$$

$$= |U_c| - |N(U_c)|$$

It implies that $U_c \setminus T$ is also a critical vertex subset of G which contains less number of vertices than U_c . It contradicts the choice of U_c and therefore completes the proof of the Theorem.

###

Proof of the Theorem 6.

Let U_c be a critical vertex subset of G . Let T_1, \dots, T_t be all non-trivial components of the induced subgraph $G(U_c)$. Then by Theorem 7, $U_c \setminus \{T_1, \dots, T_t\}$ is a critical independent

set of G . Since finding a critical vertex subset U_c and deleting all vertices of U_c incident with some edge of $G(U_c)$ only need polynomial cost, finding a critical independent set is solvable in polynomial time.

Alternating proof of the Theorem (also see [GZ]):

Consider a weight $w: V(G) \rightarrow [0,1]$ such that

$0 \leq w(v) \leq 1$ for each vertex v of G
 and $w(u) + w(v) \leq 1$ for each edge (u,v) of G

The total weight $\sum_{u \in V(G)} w(u)$ is denoted by $w(G)$. Let

$w_m(G) = \max\{w(G) : w \text{ is a weight of } G \text{ satisfies the above definition}\}.$

Obviously finding w_m is a linear programming problem. By an argument similar to the proof of Theorem 2, we can prove that the set $V_b = \left\{ v \in V(G) : w_m(v) > \frac{1}{2} \right\}$ is an critical independent set of G .

(Note that the weight w defined above is called a fractional independence function of a graph G which was introduced by Domke, Hedetniemi, Laskar in [DHL] and by Grinstead, Slater in [GS2], and were studied by Grinstead, Slater in [GS1] and by Zhang in [CG].)

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Note, S. Poljak (personal communication) recently suggested an alternative proof of Theorem 2 by the theory of integer programming.

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