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PANCONNECTIVITY OF LOCALLY CONNECTED $K_{1,3}$-FREE GRAPHS

by

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ABSTRACT

A locally connected $K_{1,3}$-free graph is panconnected if and only if the graph is 3-connected.

1. INTRODUCTION

We use [1] for basic terminology and notation. Let $G = (V,E)$ be a graph with vertex set V and edge set E. If V′ is a subset of V, then the induced subgraph is denoted by G(V′). The set of all vertices adjacent to some vertex of a subset V′ is denoted by N(V′). A vertex v is called k-locally connected if the induced subgraph G(N(v)) is k-connected. A graph is called k-locally connected if every vertex of the graph is k-locally connected. A graph is called $K_{1,3}$-free if there is no induced subgraph in G isomorphic to $K_{1,3}$. It is well known that every line graph is $K_{1,3}$-free.

Let $P = v_0 \cdots v_p$ be a path. The segment of P between v_i and v_j is denoted by $v_i\ P\ v_j$ if $i \leq j$, or $v_i\ \overline{P}\ v_j$ if $i \geq j$. The distance between two vertices x and y in a graph is denoted by $d_{x,y}$.

Since the concept of locally connected $K_{1,3}$-free graph was introcuded by Chartrand, Gould and Polimeni [3], many results about cycles and paths of locally connected $K_{1,3}$-free graphs have been declared by mathematicians.

Theorem A. (Oberly and Sumner [6].)

Every connected locally connected $K_{1,3}$-free graph contains a Hamilton cycle.

Theorem B. (Clark [6].)

Every connected locally connected $K_{1,3}$-free graph contains cycles of all possible lengths.

Theorem C. (Zhang [7].)

Every vertex of a connected quasi-locally connected $K_{1,3}$-free graph is contained in cycles of all possible lengths.

A graph is called <u>quasi-locally</u> <u>connected</u> if every vertex-cut of it contains a locally connected vertex. It is clear that every locally connected graph is quasi-locally connected. Also, an example of $K_{1,3}$-free graph which is quasi-locally connected but not locally connected was illustrated in [7].

Theorem D. (Chartrand, Gould and Polimeni [3].)

Every pair of distinct vertices of a 3-locally connected $K_{1,3}$-free graph are joined by a Hamilton path.

And it was asked in [3].

Question E.

Whether Theorem D is the best possible?

The following theorem answered the question.

Theorem F. (Kanetkar and Rao [5].)

Every pair of distinct vertices x and y of a 2-locally connected $K_{1,3}$-free graph of order n are joined by a path of length h for any integer $h: d_{x,y} \leq h \leq n-1$.

An improvement of Theorem D and F has been expected and a conjecture was proposed by Broersma and Veldman.

Conjecture G. (Broersma and Veldman [2].)

Every pair of distinct vertices x and y of a *3-connected* locally connected $K_{1,3}$-free graph of order n are joined by a path of length h for any integer $h: d_{x,y} \leq h \leq n-1$.

Note that the condition of 3-connectivity is necessary since the vertices of a 2-vertex cut of a graph cannot be joined by a Hamilton path. It was proved in [3] that the condition of 2-locally connectivity implies the 3-connectivity of the graph. It was proved in [2] that the conjecture is true if the graph is a line graph.

The main theorem in this paper will verify Conjecture G and therefore gives a final answer to Question E.

Main Theorem.

> Any pair of vertices x and y of a 3-connected locally connected $K_{1,3}$-free graph of order n, are joined by a path of length h for any integer h: $d_{x,y} \leq h \leq n-1$.

In Section 3, a stronger version (Theorem 3.3) of the main theorem will be obtained. One may expect that the main theorem could be generalized to quasi-locally connected $K_{1,3}$-free graphs. Unfortunately, it is impossible and a counterexample is illustrated in Figure 1. (The vertices x and y in Figure 1 are not joined by any path of length two while the distance between x and y is only one.) If the condition of locally connectivity is replaced by some weaker conditions, the conclusion of pan-connectivity may not hold in some case. However, a Hamiltonian-connectivity property could be obtained. (See Theorem 3.5.)

2. LEMMAS

Lemma 2-1.

Let G be a $K_{1,3}$-free graph, v be a vertex of G, and Z be a subset of N(v), then the diameter of any component of the induced subgraph G(Z) is at most three.

Proof.

Suppose that the diameter of some component of the induced subgraph G(Z) is at least four. Then there must be a pair of vertices x and y in this component such that the length of a shortest path Q joining x and y in G(Z) is at least four. Let $Q = v_o \cdots v_q$ where $q \geq 4$, $v_o = x$, and $v_q = y$. By the choice of Q, the vertex subset $\{v_0, v_2, v_4\}$ is an independent set of G and $\{v_0, v_2, v_4, v\}$ induce a $K_{1,3}$ subgraph. This is a contradiction.

Lemma 2-2.

Let G be a $K_{1,3}$-free graph and $P = v_0 \cdots v_p$ be a path of length p in G. Assume that there is no (v_0, v_p)-path of length p+1 and containing all vertices of P in G. If there is a vertex $v_i \in \{v_1, \cdots, v_{p-1}\}$ such that $N(v_i) \cap [V(G) \backslash V(P)] \neq \varphi$, then the vertices v_{i-1} and v_{i+1} must be adjacent in G.

Proof.

Let w be a vertex of $N(v_i) \cap [V(G)\backslash V(P)]$. Since $\{v_{i-1}, v_{i+1}, w, v_1\}$ cannot induce a $K_{1,3}$ subgraph, there must be an edge joining some vertices of $\{v_{i-1}, v_{i+1}, w\}$. The vertex w cannot be adjacent to any vertex of $\{v_{i-1}, v_{i+1}\}$ because otherwise the path P can be extended by inserting w between v_i and v_{i-1} or v_{i+1}. Therefore, we must have that v_{i-1} and v_{i+1} are adjacent in G.

Lemma 2-3.

Let G be a $K_{1,3}$-free graph, v be a vertex of G, and Z be a subset of N(v). If the induced subgraph G(Z) is not connected, then

i. G(Z) is the union of two cliques B′ and B".

ii. For any $w \in N(v)$, either $\{w\} \bigcup B'$ or $\{w\} \bigcup B''$ induces a clique.

Proof.

i. Since the independence number of G(N(v)) and G(Z) are at most two, the disconnected induced subgraph G(Z) only can have two components. Let B′ and B" be the two components of G(Z). If B′ is not a clique, let w′ and $w'' \in B'$ and $w^* \in B''$ such that (w′, w") is not an edge of G. Then {w′, w", w*} is an independent set of G(Z) and {w′, w", w*, v} induce a $K_{1,3}$ subgraph. It is a contradiction and therefore both B′ and B" are cliques.

ii. If $w \in Z$, we have done by (i). Assume that $w \notin Z$. If neither $\{w\} \cup B'$ nor $\{w\} \cup B''$ induces a clique, then there is a vertex w′ of B′ and a vertex w" of B" such that (w, w′), (w, w") $\notin$ E(G). Thus {w′, w", w} is an independent set of G(Z) and {w′, w", w, v} induce a $K_{1,3}$ subgraph. It leads a contradiction.

Lemma 2-4.

Let G be a $K_{1,3}$-free graph, v be a vertex of G, and Z be a subset of N(v). If the induced subgraph G(Z) is not 2-connected, then G(Z) is the union of two disjoint cliques B_1, B_2 and some edges between B_1 and B_2.

Proof.

If G(Z) is not connected, we are done in Lemma 2-3. So we assume that G(Z) has a cut-vertex w. Let B′ and B" be two components of G(Z\w). By (i) of Lemma 2-3, both B′ and B" are cliques. By (ii) of Lemma 2-3, either $\{w\} \cup B'$ or $\{w\} \cup B''$ induces a clique.

3. MAIN THEOREMS

Lemma 3-1.

Let G be a $K_{1\text{-}3}$-free graph and $P = v_0 \cdots v_p$ be a path of length p in G. Assume that $N(v_0)\backslash V(P) \neq \phi$ (or $N(v_p)\backslash V(P) \neq \phi$). If the vertex v_1 (or v_{p-1}, respectively) and some vertex $w \in N(v_0)\backslash V(P)$ (or $w \in N(v_p)\backslash V(P)$, respectively) are in the same component of the induced subgraph $G(N(v_0)\backslash\{v_p\})$ (or $G(N(v_p)\backslash\{v_0\})$, respectively), then P is contained in a (v_0, v_p)-path of length p+1.

Proof.

The lemma will be proved by contradiction. Assume that there is a path joining some vertex of $N(v_0)\backslash V(P)$ and $\{v_1\}$ in $G(N(v_0)\backslash\{v_p\})$, and

(*) P is not contained in any (v_0, v_p)-path of length p+1.

Let $Q = u_1 \cdots u_q$ be a shortest path in $G(N(v_0)\backslash\{v_p\})$ joining $N(v_0)\backslash V(P)$ and $u_1 = v_1$. Let $u_q = w$ which is the only vertex of Q not contained in P. By Lemma 2-1, Q is of length at most three, that is $q \leq 4$. It is clear that the length of Q is at least two because otherwise the path P can be extended by inserting w between v_0 and v_1.

If the length of Q is two, let $u_2 = v_i$ which is a vertex of P other than v_0, v_1, and v_p. By Lemma 2-2, the vertices v_{i-1} and v_{i+1} are adjacent in G. Then the (v_0, v_p)-path v_0 w v_i v_1 P v_{i-1} v_{i+1} P v_p is of length p+1 and contains all vertices of P. It contradicts the assumption (*) and therefore the length of Q must be three. (See Figure 2.)

Let $U_2 = v_i$ and $U_3 = v_j$ be two vertices of P other than v_0, v_1, and v_p. (See Figure 3.) Since $\{v_i, v_{j+1}, w\}$ is a subset of $N(v_j)$ and $\{v_j, v_i, v_{j+1}, w\}$ cannot induce a $K_{1,3}$ subgraph, either $|\{v_i, v_{j+1}, w\}| < 3$ or there is an edge joining a pair of vertices of $\{v_i, v_{j+1}, w\}$. If $|\{v_i, v_{j+1}, w\}| < 3$. We must have $v_i = v_{j+1}$. But the (v_0, v_p)-path v_0 w v_j $\overline{P}$ v_1 v_i P v_p is of length p+1 and contains all vertices of P. It contradicts the assumption (*) (See Figure 4.) so there must be an edge joining a pair of vertices of $\{v_i, v_{j+1}, w\}$. It is obvious that the vertices w and v_{j+1} cannot be adjacent since otherwise the path P can be extended by inserting w between v_j and v_{j+1}. If the vertices v_i and w are adjacent, then v_1 v_i w is a path in $G(N(v_0)\backslash\{v_p\})$ joining $\{v_1\}$ and $N(v_0)\backslash V(P)$ which is shorter than Q. It contradicts the choice of Q. Therefore we must have that the vertices v_i and v_{j+1} are adjacent in G. Since $\{w, v_1, v_{j+1}\}$ is a subset of $N(v_i)$ and $\{v_i, w, v_1, v_{j+1}\}$cannot induce a $K_{1,3}$ subgraph, either $|\{w, v_1, v_{j+1}\}| < 3$ or there is an edge joining a pair of vertices of $\{w, v_1, v_{j+1}\}$. Since $v_0 \neq v_j$, it is impossible that $v_1 = v_{j+1}$. Thus $\{w, v_1, v_{j+1}\}$

contains precisely three vertices and cannot be an independent set. It is clear that neither v_1 nor v_{j+1} is adjacent to w since otherwise the P can be extended by inserting the vertex w between v_0 and v_1, or v_j and v_{j+1}. The only remaining case is that the vertices v_1 and v_{j+1} are adjacent in G. But the (v_1, v_p)-path $v_0\ w\ v_j\ \overline{P}\ v_1\ v_{j+1}\ P\ v_p$ is of length p+1 and contains all vertices of P. It contradicts the assumption (*) and completes the proof of the Lemma. (See Figure 5.)

Lemma 3-2.

Let G be a $K_{1,3}$-free graph and $P = v_0 \cdots v_p$ be a path of length p in G. If there is a locally connected vertex $v_i \in V(P) \setminus \{v_0, v_p\}$ such that $N(v_i) \setminus V(P) \neq \phi$, then there is a (v_0, v_p)-path of length p+1 containing all vertices of P.

Proof.

The Lemma will be proved by contradiction. Assume that v_i is a locally connected vertex of $V(P) \setminus \{v_0, v_p\}$, and

(*) P is not contained in any (v_0, v_p)-path of length p+1.

Let $Q = u_1 \cdots u_q$ be a shortest path in the induced subgraph $G(N(v_i))$ joining $\{v_{i-1}, v_{i+1}\}$ and $N(v_i) \setminus V(P)$. It is clear that

$u_q = w$ is the only vertex of Q not contained in P. By Lemma 2-1, Q is of length at most three.

i. The vertices u_1 and w cannot be adjacent since otherwise the path P can be extended by inserting the vertex w between v_i and $u_1 \in \{v_{i-1}, v_{i+1}\}$. Thus the length of Q cannot be one.

ii. If the length of Q is two, we first claim that $u_2 \neq v_0, v_p$. Without loss of generality, suppose that $u_2 = v_0$. Since $w = u_3$ is adjacent to v_0, by Lemma 3-1, $\{v_1\}$ and $\{w\}$ must be in the deferent components of the induced subgraph $G(N(v_0)\backslash\{v_p\})$. By Lemma 2-3, there are two components of $G(N(v_0)\backslash\{u_p\})$ both of which are cliques. If the vertex $u_1 \in \{v_{i-1}, v_{i+1}\}$ is not the vertex v_p, then the subset $\{u_1, v_i, w\}$ of $N(v_0)$ must be contained in the same clique of $G(N(v_0)\backslash\{u_p\})$ since $\{u_1, v_i, w\}$ induce a connected subgraph in $G(N(v_0))$. This contradicts the result proved in (i) that u_1 and w cannot be adjacent and therefore $u_1 = v_p$. Since $\{w, v_1, v_p = u_1\}$ is a subset of $N(v_0)$ and cannot be an independent set, there must be an edge in the induced subgraph $G(\{w, v_1, v_p\})$. It is clear that the vertex w cannot be adjacent to v_1 and $u_1 = v_p$ because otherwise the path P can be extended by inserting the vertex w between v_0 and v_1, or $v_i = v_{p-1}$ and $u_1 = v_p$. So (v_1, v_p) must be an edge of G and $v_0\ w\ v_i\ \overline{P}\ v_1\ V_p$ is a (v_0, v_p)-path of length p+1. (See Figure 6.) It contradicts the assumption (*) and therefore follows our claim that $u_2 \neq v_0$. And it is similar that $u_2 \neq v_p$.

Let $u_2 = v_h$ be a vertex of P other than v_0 and v_p. By Lemma 2-2, (v_{h-1}, v_{h+1}) is an edge of G. And

$$[P\backslash\{\text{vertex } v_h, \text{ edge } (v_i, u_1)\}] \bigcup \{(v_{h-1}, v_{h+1})\}$$

is a union of two paths joining $\{v_0, v_p\}$ and $\{v_i, u_1\}$. Joining these two paths by $u_1\ v_h\ w\ v_i$, we obtain a (v_0, v_p) path of length p+1 and containing all vertices of P. It contradicts the assumption (*) and therefore the length of Q must be three. (See Figure 7.)

iii. Let $u_2 = v_h$, $u_3 = v_k$ be vertices of P.

Case one. $u_3 = v_k \notin \{v_0, v_p\}$.

Without loss of generality, let $u_2 = v_h \neq v_p$. Then $\{v_i, v_h, v_k\} \subseteq V(P)\backslash\{v_p\}$. By Lemma 2-2, (v_{i-1}, v_{i+1}), (v_{k-1}, v_{k+1}) are edges of G. Let $\sigma \in \{i,k\}$. We claim that $v_{\sigma+1} \neq v_h$. Suppose that $v_{\sigma+1} = v_h$. If $\sigma = i$, then $v_{i+1}\ v_k\ w$ is a path in $G(N(v_i))$ shorter than Q. It contradicts the choice of Q. If $\sigma = k$, then $v_h = v_{k+1}$ and

$$[P\backslash\{\text{vertex } v_i, \text{ edge } (v_k, v_h)\}] \bigcup \{\text{edge } (v_{i-1}, v_{i+1})\}$$

is a union of two paths joining $\{v_0, v_p\}$ and $\{v_k, v_h\}$. Joining these two paths by $v_k\ w\ v_i\ v_h$, we obtain a (v_0, v_p)-path of length p+1 and containing $V(P)$. It contradicts the assumption (*) and follows the claim. (See Figure 8.)

Since $\{w, v_{\sigma+1}, v_h\}$ is a subset of $N(v_\sigma)$ containing three distinct vertices, it cannot be an independent set. If $(w, v_{\sigma+1})$ is an edge of G, then the path P can be extended by inserting the vertex w between v_σ and $v_{\sigma+1}$. If (w, v_h) is an edge of G, then the path Q has a chord $(v_h, w) = (u_2, u_4)$. It contradicts the choice of Q which is a shortest path joining u_1 and $u_4 = w$. Therefore, we have that v_h is adjacent to both v_{k+1} and v_{i+1}.

Note that $\{v_k, v_i, v_h\} \subseteq V(P)\setminus\{v_p\}$. Since $\{v_{k+1}, v_{i+1}, v_{h+1}\}$ is a subset of $N(v_h)$, there must be an edge joining a pair of vertices of $\{v_{k+1}, v_{i+1}, v_{h+1}\}$. If (v_{k+1}, v_{i+1}) is an edge of G, then

$$[P\setminus\{(v_k, v_{k+1}), (v_i, v_{i+1})\}] \bigcup \{(v_{k+1}, v_{i+1}), (v_k, w), (w, v_i)\}$$

is a (v_0, v_p)-path of length p+1 and containing all vertices of P. (See Figure 9.) If $(v_{\sigma+1}, v_{h+1})$ is an edge of G for any $\sigma \in \{i,k\}$, then let $\sigma' = \{i,k\}\setminus\{\sigma\}$ and

$$[P\setminus\{\text{vertex } v_{\sigma'}, \text{ edges } (v_h,v_{h+1}), (v_\sigma,v_{\sigma+1})\}] \bigcup \{(v_{\sigma'-1}, v_{\sigma'+1})\}$$

is a union of three paths joining $\{v_0, v_{h+1}, v_{\sigma+1}\}$ and $\{v_h, v_\sigma, v_p\}$. Joining these paths by the edge $(v_{\sigma+1}, v_{h+1})$ and a path $v_h\ v_{\sigma'}\ w\ v_\sigma$, we obtain a (v_0, v_p)-path of length p+1 and containing all vertices of P. It contradicts the assumption (*) and completes the proof of the Lemma in this case. (See Figure 10.)

Case two. $u_3 \in \{v_0, v_p\}$. Without loss of generality, let $u_3 = v_0$.

It is clear that $u_2 = v_h$ and $u_4 = w$ are not adjacent since the path Q cannot have a chord. Since $w \in N(v_0)\backslash V(P)$, by Lemma 3-1, the induced subgraph $G(N(v_0)\backslash\{v_p\})$ is disconnected. By Lemma 2-4, let T_1 and T_2 be two disjoint cliques of $G(N(v_0))$ such that $V(T_1 \bigcup T_2) = N(v_0)$ and $T_1\backslash\{v_p\}$, $T_2\backslash\{v_p\}$ are two components of $G(N(v_0)\backslash\{v_p\})$. Let $w \in T_1$ and consequently $v_i \in T_1$. Since w is not adjacent to any of $\{v_1, v_h\}$, the vertices v_1 and v_h must be in the clique T_2. We claim that $v_h = v_p$. If $v_h \neq v_p$, then $\{v_h, v_i, w\}$ must be in the same component $T_1\backslash\{v_p\}$ of $G(N(v_0)\backslash\{v_p\})$ since $\{v_h, v_i, w\}$ is a subset of $N(v_0)\backslash\{v_p\})$ and induces a connected subgraph which contains a path $v_h v_i w$. It contradicts that $v_h \in T_2$ and follows the claim that $v_h = v_p$. Note that the vertices v_1 and v_p are in the clique, (v_1, v_p) is an edge of G.

Since $\{v_1, v_i, v_{p-1}\}$ is a subset of $N(v_p)$, either $|\{v_1,v_i,v_{p-1}\}| < 3$ or $\{v_1,v_i,v_{p-1}\}$ is not an independent set. If $|\{v_1,v_i,v_{p-1}\}| < 3$, then either $v_1 = v_i$ or $v_i = v_{p-1}$. When $v_1 = v_i$, the path P can be extended by inserting w between v_0 and $v_1 = v_i$. When $v_i = v_{p-1}$, we can obtain a (v_0, v_p)-path $v_0\ w\ v_{p-1}\ \overline{P}\ v_1\ v_p$ of length p+1, where v_1 and $v_p = v_h$ are adjacent since they are contained in the clique T_2. (See Figure 11.) Both contradicts the assumption (*) and hence, there must be an edge joining a pair of vertices of

$\{v_1, v_i, v_{p-1}\}$. It is clear that v_1 and v_i are not adjacent since v_1 is in the component $T_2\backslash\{v_p\}$ while v_i is in another component $T_1\backslash\{v_p\}$ of $G(N(v_0)\backslash\{v_p\})$. If u_{p-1} and v_i are adjacent, then $v_0\ w\ v_i\ v_{p-1}\ \overline{P}\ v_{i+1}\ v_{i-1}\ \overline{P}\ v_1\ v_p$ is a (v_0, v_p)-path of length p+1 and containing all vertices of P. It contradicts the assumption (*) (See Figure 12) and therefore v_1 and v_{p-1} must be adjacent, then

$$P' = [P\backslash\{\text{vertices } v_0, v_p, \text{ edge } (v_i, u_1)\}] \bigcup \{(v_1, v_{p-1})\}$$

is a (v_i, u_1)-path of length p-2. Adding paths $v_0\ w\ v_i$ and $u_1\ v_p$ at the ends of the path P′ (note that $v_p = v_h = u_2$) we obtain a (v_0, v_p)-path of length p+1 containing V(P). (See Figure 13.) It contradicts the assumption (*) and completes the proof of the lemma.

Theorem 3-3.

Let G be a connected locally connected $K_{1,3}$-free graph of order n and x,y be a pair of vertices of G such that $G\backslash\{x,y\}$ is connected. Then x and y are joined by a path of length h for any integer $h: d_{x,y} \le h \leq n-1$.

Proof of Theorem 3-3.

Let P be an (x,y)-path of length P. If P is not a Hamilton path, then $W = G\backslash V(P)$ is not empty. Assume that P contains at least three vertices. Since $G\setminus\{x,y\}$ is connected, there must be a vertex v of $V(P)\setminus\{x,y\}$ adjacent to some vertex of W. Note that v is locally connected. By Lemma 3-2, P is contained in some (x,y)-path of length p+1. Thus we shall assume that P is a single edge. The degree of either x or Y must be at least two because otherwise the graph G is either trivial or disconnected. Without loss of generality, let $d(x) \geq 2$. Since the induced subgraph G(N(x)) is connected and $y \in N(x)$, there must be a vertex w of N(x) adjacent to y. Thus x w y is a path of length 2 = p+1. It completes the proof of the theorem.

Theorem 3-4.

Any pair of vertices x and y of a 3-connected locally connected $K_{1,3}$-free graph of order n are joined by a path of length h for any integer $h: d_{x,y} \leq h \leq n-1$.

Proof.

It is an immediate corollary of Theorem 3-3.

Theorem 3-5.

Let G be a $K_{1,3}$-free graph and x,y be a pair of distinct vertices of G. If each vertex-cut of G contains a locally connected vertex other than x and y, then x and y are joined by a Hamilton path.

Proof.

Let P be a longest (x,y)-path in G. Assume that P is not a Hamilton path. Let W be a component of G\V(P). Since G is connected, there must be some vertex of W adjacent to some vertex of P. If N(W) ∩ V(P) is a proper subset of V(P), then the vertex-cut N(W) ∩ V(P) must contain a locally connected vertex v other than x and y. By Lemma 3-2, the graph G contains an (x,y)-path longer than P. It contradicts the assumption and hence we have that N(W) ∩ V(P) = V(P) is not a vertex-cut of G. But the path P can be extended by inserting a path of W between any pair of adjacent vertices of P. It contradicts the assumption again and therefore completes the proof of the theorem.

4. RELATED PROBLEMS

i. We have seen that Theorem 3-3 and Theorem 3-4 cannot be generalized to quasi-locally connected $K_{1,3}$-free graphs since the graph illustrated in Figure 1 is a counterexample. But we still expect that the Hamiltonian-connected property holds for quasi-locally connected $k_{1,3}$-free graphs.

ii. A graph G is called Chvátal-Erdös connected if the connectivity of G is not less than the independence number of G. This concept was first introduced by Chvátal and Erdös in [4] and it was proved that any Chvátal-Erdös-connected graph contains a Hamilton cycle. A vertex v of a graph G is called a locally Chvátal-Erdös connected if the induced subgraph G(N(V)) is Chvátal-Erdös connected. A graph G is called locally Chvátal-Erdös connected if every vertex of G is locally Chvátal-Erdös connected. Obviously, any 2-locally connected $K_{1,3}$-free graph is locally Chvátal-Erdös connected since the independent index of G(N(v)) is at most two for any vertex v of a $K_{1,3}$-free graph G. The author would like to propose the following conjecture which will generalize Theorem F.

Conjecture.

Any connected locally Chvátal-Erdös connected graph is Hamiltonian-connected.

Note that any connected locally Chvátal-Erdös connected graph is 3-connected. The proof of this claim is quite trivial.

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Fig 1.

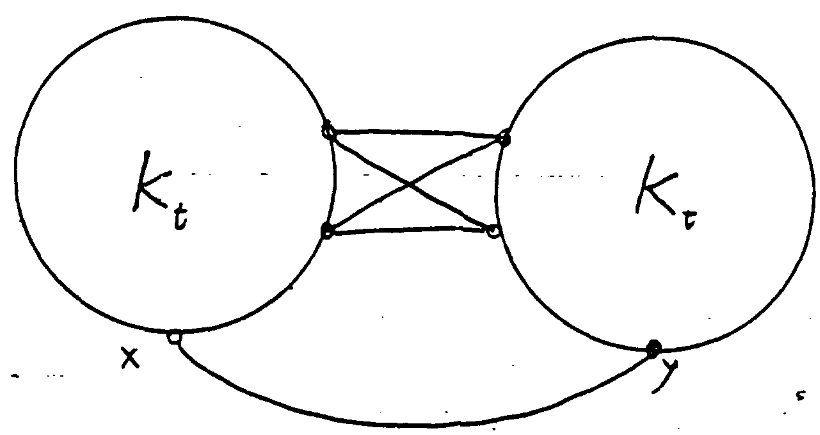

Fig. 2

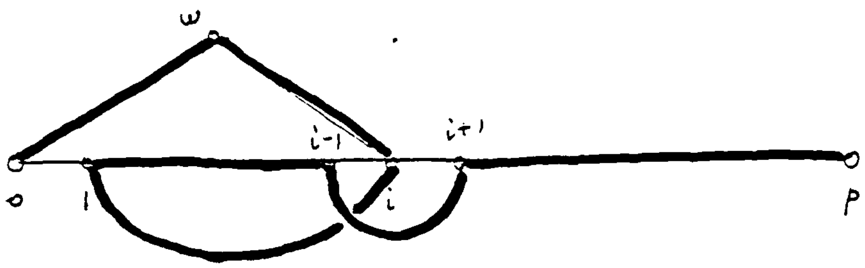

Fig. 3

if i < j

or

if i > j

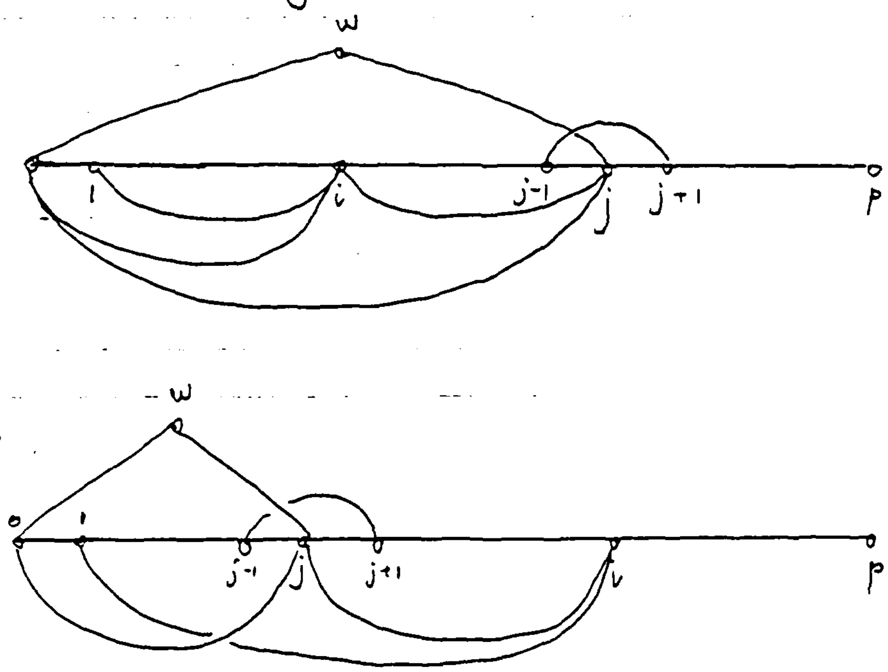

Fig. 4

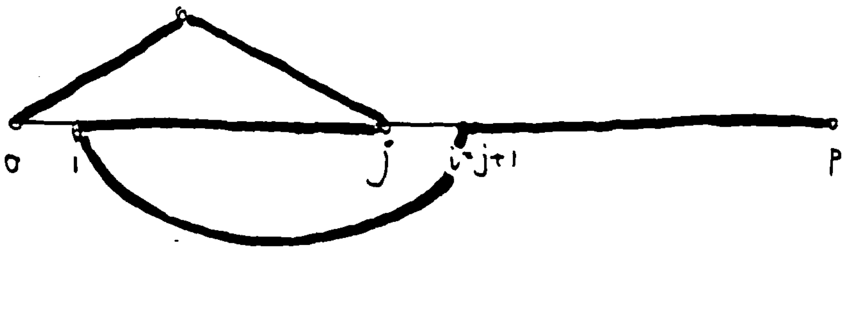

Fig. 5

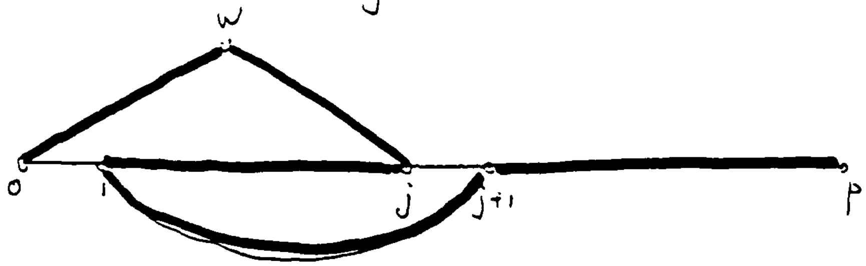

Fig. 6

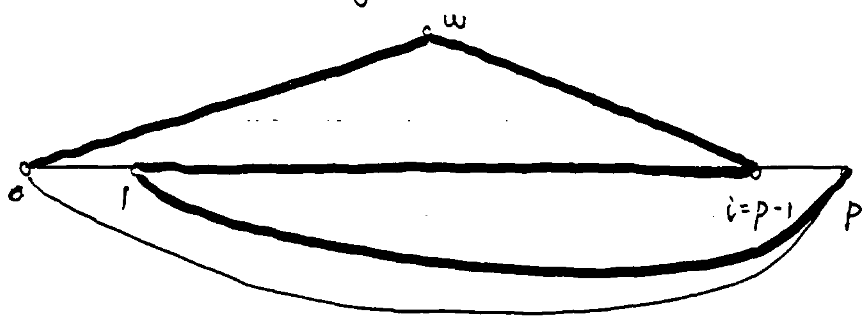

Fig. 7

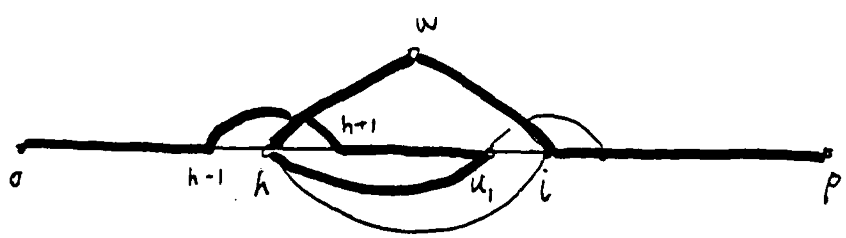

Fig. 8

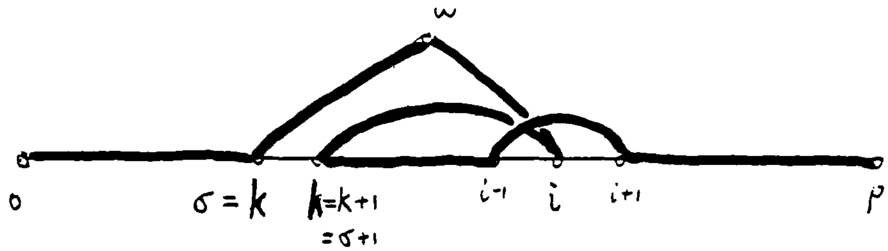

Fig. 9.

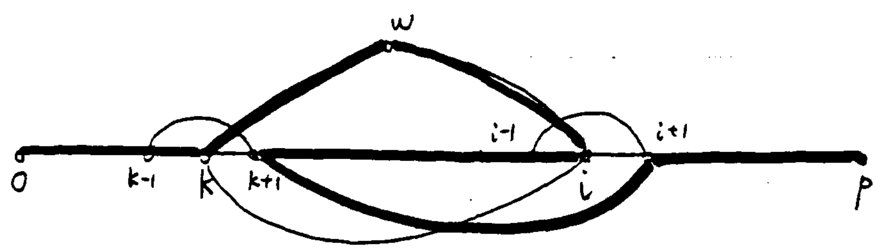

Fig. 10

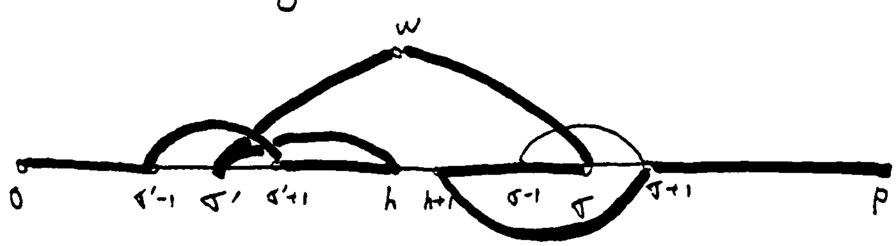

Fig. 11

Fig. 12

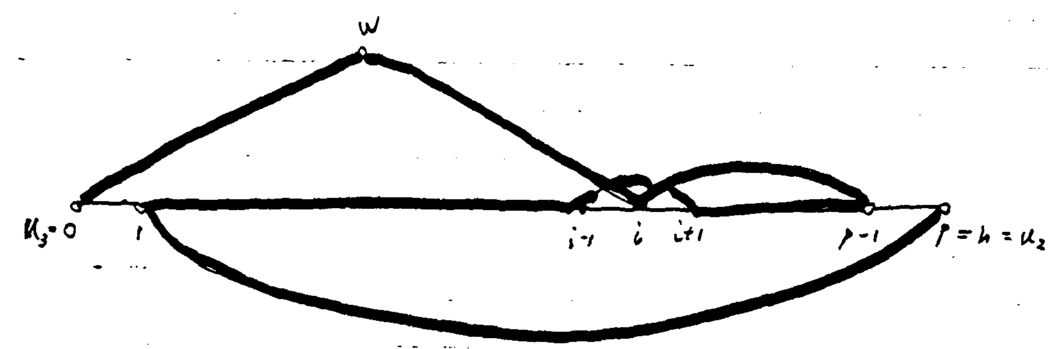

Fig. 13