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LOGARITHMIC TRANSFORMATIONS AND STOCHASTIC CONTROL

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1. Introduction. We are concerned with a class of problems described in a somewhat imprecise way as follows. Consider a linear operator of the form $L + V(x)$, where L is the generator of a Markov process x_t and the "potential" $V(x)$ is some real-valued function on the state space Σ of x_t. We are interested in probabilistic representations for solutions $\phi(s,x)$ to ~~the~~ a certain backward equation

$$\text{(1.1)} \qquad \text{phi} \qquad \frac{d\phi}{ds} + L\phi + V(x)\phi = 0, \quad s \le T,$$

with data $\phi(T,x) = \Phi(x)$ at a final time T. It is well known that, under suitable assumptions,

$$\text{(1.2)} \qquad \phi(s,x) = E_{sx}\{\Phi(x_T)\exp\int_s^T V(x_t)dt\} \qquad \text{(KR)}$$

gives such a representation. For instance, if $x_t = x_s + w_t - w_s$, with w_t a brownian motion, then (1.2) is just the Feynman-Kac formula. We seek a different kind of probabilistic representation for $I = -\log\phi$, if $\phi(s,x)$ is a positive solution to (1.1). In this representation the generator L is replaced by another generator $L^{\underline{u}}$ of a Markov process ξ_t (possibly time inhomogeneous.) The operator $L^{\underline{u}}$ is chosen to solve an optimal stochastic control problem of the following kind. The logarithmic transformation $I = -\log\phi$ changes (1.1) into the nonlinear equation

$$\text{(1.3)} \qquad \frac{dI}{ds} + H(I) - V(x) = 0, \text{ where}$$

$$\text{(1.4)} \qquad H(I) = -e^{I}L(e^{-I}).$$

The function H is concave. For a fairly wide class of Markov processes, we wish to write (1.3) as the dynamic programming equation associated with a suitable optimal stochastic control problem for Markov processes. The stochastic control problem is specified by giving: (a) a suitable control space U; for each constant control $u \in U$, the generator L^u of a Markov process; and (c) a cost function $k(x,u)$ associated with constant control u and state x. See [6, Chap. VI]. It is re-

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quired that

$$H(I)(x) = \min_{u \in U} [L^u I(x) + k(x,w)], \quad x \in \Sigma . \tag{1.5}$$

Then (1.3) becomes a dynamic programming equation:

$$\frac{dI}{ds} + \min_{u \in U} [L^u I + k(x,u) - V(x)] = 0. \tag{1.6}$$

Time and state dependent controls $\underline{u}(s,x)$, in feedback form, with values in the control space U are allowed. The stochastic control problem is to find a feedback $\underline{u}$ minimizing

$$J(s,x;\underline{u}) = E_{sx} \int_s^T [k(\xi_t,u_t) - V(\xi_t)]dt + \Psi(\xi_T) \quad , \tag{1.7}$$

where ξ_t is the (controlled) Markov process with generator $L^{\underline{u}}$, $\xi_s = x$, and

$$u_t = \underline{u}(t,\xi_t), \quad \Psi = -\log \Phi .$$

The Verification Theorem of optimal stochastic control theory [6, p.159] asserts that if I is a "well behaved" solution to (1.3) with $I(T,x) = \Psi(x)$ and if certain other technical conditions hold, then

$$I(s,x) = \min_{\underline{u}} J(s,x;\underline{u}) . \tag{}$$

Moreover, an optimal feedback control $\underline{u}(s,x)$ is found by minimizing $L^u I(s,x) + k(x,u)$ over the control space U.

In this paper we take $\Sigma \subset R^n$, a subset of n-dimensional euclidean space. In §2 we review the case when x_t is a diffusion process on R^n. For nondegenerate diffusions, an appropriate stochastic control problem is immediately suggested by the form of equation (1.3). In §3 we consider jump Markov processes x_t, and associated stochastic control problems. The choice of an appropriate control problem is less immediate for jump processes than for diffusions. In his Ph.D. thesis S-J Sheu [15] uses a different control formulation, valid for a wide class of generators L(§4). The optimal control in his sense leads to the change of probability measures described in (4.5). In §5 we give a formal derivation indicating why stochastic control methods can be used to obtain asymptotic estimates for exit probabilities for a family x_t^ε of nearly deterministic jump processes. The results are not new (see [1][12]); the interest is in the stochastic control method. Rigorous proofs are given in [11] using such methods.

In §6 we consider briefly the Donsker-Varadhan formula for the dominant eigenvalue λ_1 of $L+V$, from a control viewpoint. For nondegenerate diffusions the stochastic control representation obtained for λ_1 is the same as Holland's [9].

2. Diffusion processes. Let x_t be a diffusion in n-dimensional R^n, with generator

$$Lf = \frac{1}{2} \operatorname{tr} a(x) f_{xx} + b(x) \cdot f_x \tag{2.1}$$

$$\operatorname{tr} a(x) f_{xx} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j},$$

and with f_x the gradient. In this case,

$$H(I) = \frac{1}{2} \operatorname{tr} a(x) I_{xx} + b(x) \cdot I_x - \frac{1}{2} I_x' a(x) I_x \tag{2.2}$$

We may take $U = R^n$, $u = (u_1, \cdots, u_n)$,

$$L^u I = \frac{1}{2} \operatorname{tr} a(x) I_{xx} + u \cdot I_x . \tag{2.3}$$

$$k(x,u) = \frac{1}{2}(b(x)-u)' a^{-1}(x)(b(x)-u). \tag{2.4}$$

For a feedback control $\underline{u}$, the drift coefficient $b(x)$ in (2.1) is changed to drift coefficient $\underline{u}(s,x)$ in the operator $L^{\underline{u}}$.

The stochastic control representation (1.8) was used in [3] to give a stochastic control proof of results of Ventsel-Freidlin type for some large deviations problems for nearly deterministic diffusions. In those results $a(x)$ is replaced by $\varepsilon a(x)$, ε small. In [4] the logarithmic transformation was used to obtain stochastic representations for positive solutions to the heat equation with a potential term, and to obtain the "classical mechanical limit." In [5] [10] the same logarithmic transformation was applied to solutions to the pathwise equation of nonlinear filtering. Large deviations results for the nonlinear filter problem are obtained by Hijab [8] elsewhere in this volume.

In [7] Hernandez-Lerma obtained similar results for certain degenerate diffusions, for which the matrix $(a_{ij}(x))$, $i,j=1,\cdots,m < n$ is positive definite and $a_{ij}(x)=0$ if $i > m$ or $j > m$.

3. Jump processes. To motivate our choice of stochastic control problem, let us begin with a simple special case in which the process x_t jumps only by a fixed increment y (as for example for a Poisson process.) In this case the generator L takes the form

$$Lf(x) = a(x)[f(x+y)-f(x)].$$

From (1.4)

$$H(1)(x) = a(x)(1 - \exp [I(x) - I(x+y)]).$$

The dual function to the convex function e^r is $u - u \log u$ $(u > 0)$:

$$e^r = \max_{u>0} [u - u \log u + ur] . \tag{3.1}$$

The max occurs when $\log u = r$. Let

$$L^u I(x) = ua(x)[I(x+y)-I(x)], \quad u > 0 \tag{3.2}$$

$$k(x,u) = a(x)(u \log u - u+1). \tag{3.3}$$

By taking $r = I(x)-I(x+y)$ in (3.1) and changing signs (to replace max by min), we get the required form (1.5) for $H(I)$. In this special case the control u is scalar, with $u > 0$. A constant control u changes the jumping rate from $a(x)$ to $ua(x)$. A feedback control $\underline{u}(s,x)$ changes the rate at time s and state x from $a(x)$ to $\underline{u}(s,x)a(x)$. If $I(s,x) = -\log \phi(s,x)$ as in §1, then the optimal feedback control is $\underline{u}^*(s,x) = \phi(s,x)^{-1}\phi(s,x+y)$.

Let us now consider a jump process x_t with generator of the form

$$Lf(x) = a(x) \int_{R^n} [f(x+y) - f(x)]\cdot\pi(x,dy). \tag{3.4}$$

Here $f \in B(R^n)$, the space of bounded Borel measurable functions on R^n. We assume that $a \in B(R^n)$ and that $\pi(x,\cdot)$ is a probability measure with $\pi(\cdot,A)$ Borel measurable for each Borel set A and $\pi(x,\{0\}) = 0$. Additional conditions on a and π need to be imposed later. Motivated by the special case above, we control the jumping distribution, replacing $a(x)\pi(x,dy)$ by $a(x)\underline{u}(s,x;y)\pi(dy)$. To formalize this idea, we introduce the control space

$$U = \{u(\cdot)\colon u,u^{-1} \in B(R^n),\ u(y) > 0 \text{ for all } y \in R^n \}. \tag{3.5}$$

Suitable $L^{u(\cdot)}$ and $k(x,u(\cdot))$ are obtained by integrating (3.2), (3.3) with respect to $\pi(x,dy)$:

$$L^{u(\cdot)}I(x) = a(x) \int_{R^n}[I(x+y)-I(x)]u(y)\pi(x,dy) \tag{3.6}$$

$$k(x,u(\cdot)) = a(x) \int_{R^n} [u(y)\log u(y)-u(y)+1]\pi(x,dy). \tag{3.7}$$

We get as in equation (1.5)

$$H(I)(x) = \min_{u(\cdot)\in U} [L^{u(\cdot)}I(x)+k(x,u(\cdot))] \tag{3.8}$$

If $\phi(s,x)$ is a positive solution to (1.1) and $I = -\log \phi$, then the optimal feedback control is

$$(3.9) \qquad \underline{u}^*(s,x;\cdot) = \frac{\phi(s,x+\cdot)}{\phi(s,x)} \quad .$$

As outlined in the next section, it is sometimes more convenient to consider instead a related control problem. In particular, the formulation in §4 is the one used in [11] to give control method proofs of the results on the exit problem mentioned in §5.

4. <u>The Sheu formulation</u>. In [11] another kind of control problem is considered. Let L be a bounded linear operator on $C(\Sigma)$, the space of continuous bounded functions on Σ, such that L obeys a positive maximum principle. (In particular, L may be of the form (3.4) above.) For $w = w(\cdot)$ a positive function with $w, w^{-1} \in C(\Sigma)$, define the operator $\tilde{L}^w$ by

$$(4.1) \qquad \tilde{L}^w f = \frac{1}{w}[L(wf) - fLw].$$

In addition, define $K^w(x)$ by

$$(4.2) \qquad K^w = \tilde{L}^w(\log w) - \frac{1}{w}L(w).$$

For unbounded L, additional restrictions on w are needed in order that $\tilde{L}^w$ and K^w be well defined.

From the duality (3.1) between e^r and $u \log u - u$, it is not difficult to show [11] that for $I \in C(\Sigma)$

$$(4.3) \qquad H(I) = \min_w [\tilde{L}^w I + K^w] \; .$$

The minimum is attained for $w = \exp(-I)$. For L the generator of a jump process, the two formulations are related by $\tilde{L}^w = L^{\underline{u}}$, where $\underline{u}$ is the (stationary) feedback control defined by

$$(4.4) \qquad \underline{u}(x;y) = \frac{w(x+y)}{w(x)} \quad .$$

Moreover, $K^w(x) = k(x,\underline{u}(x;\cdot))$.

In Sheu's formulation, the control problem is to choose $w_t(\cdot)$ for $s \le t \le T$ to minimize

$$\mathcal{J}(s,x;w) = E_{sx}\left\{\int_s^T [K^{w_t}(\xi_t) - V(\xi_t)]dt + \Psi(\xi_T)\right\} ,$$

where ξ is a Markov process with generator $\tilde{L}^{w_t}$ and with $\xi_s = x$. Here we assume that L is the generator of a Markov process x_t which implies in particular $L1 = 1$.

Suppose that ϕ is a positive solution to (1.1), with $\phi(s,\cdot), \phi(s,\cdot)^{-1} \in C(\Sigma)$ and with $V \in C(\Sigma)$. We can use (4.3) together with the Verification Theorem in stochastic control to conclude that $I(s,x) \leq \mathcal{J}(s,x;w)$ with equality when $w_t^* = \phi(t,\cdot)$. Thus the control $w_t^* = \phi(t,\cdot)$ is optimal in this sense. For jump processes this agrees with (3.9), according to (4.4).

The change of generator from L to $\tilde{L} = \tilde{L}^{w_t^*}$ corresponds to a change of probability measure, from P to $\tilde{P}$, as follows:

$$\tilde{E}_{sx} f(\xi_t) = \frac{E_{sx}[f(x_t)\phi(x_T)]}{E_{sx}\phi(x_T)}, \quad s \leq t \leq T, \quad f \in C(\Sigma). \tag{4.5}$$

This is seen from the following argument. The denominator of the right side is $\phi(s,x)$. Let

$$\psi(s,x) = E_{sx}[f(x_t)\phi(x_T)] = E_{sx}[f(x_t)\phi(t,x_t)] .$$

Since ϕ and ψ both satisfy (1.1) with $V = 0$, the quotient $v = \psi\phi^{-1}$ satisfies

$$\frac{\partial v}{\partial s} = -\left[\frac{L\psi}{\phi} - \frac{\psi L\phi}{\phi^2}\right] = -\frac{1}{\phi}\left[L(v\phi) - v\, L\phi\right],$$

$$\frac{\partial v}{\partial s} + \tilde{L}v = 0, \quad s \leq t, \tag{4.6}$$

with $v(t,x) = f(x)$ as required.

The author wishes to thank M. Day for a helpful suggestion related to (4.5).

5. Asymptotic estimates for exit probabilities.

Let x_t^ε be a family of Markov processes, $s \leq t \leq T$, depending on a small parameter $\varepsilon > 0$, such that x_t^ε tends (in a suitable sense) to a deterministic limit x_t^0 as $\varepsilon \to 0$. Let ϕ^ε denote the probability that x^ε belongs to a set Γ of trajectories which does not include trajectories "near" x^0. Typically ϕ^ε is exponentially small. Its asymptotic rate of decay to 0 can be found from the theory of large deviations [1][12][13]. In the exponent a constant I^0 appears, which is the minimum of a certain action functional over a set of smooth paths.

In many instances these asymptotic estimates can also be obtained by introducing a stochastic control problem of the kind indicated in previous sections, for each $\varepsilon > 0$ [3] [11]. With this method a (stochastic) optimization problem appears for each

$\varepsilon > 0$, not just in the limit as $\varepsilon \to 0$.

Let us consider the special case when ϕ^ε is an exit probability:

$$\phi^\varepsilon(s,x) = P_{sx}(\tau^\varepsilon \leq T),$$

where τ^ε is the exit time of x_t^ε from a bounded, open set $D \subset R^n$, and where $x_t^0 \in D$ for $s \leq t \leq T$. We consider nearly deterministic jump processes, as follows. Nearly deterministic diffusions were considered in [3] [7]. Following Vent'cel [12] let us rescale the jump process in §3, replacing y by εy and $a(x)$ by $\varepsilon^{-1}a(x)$ to obtain the generator for x_t^ε:

$$L_\varepsilon f(x) = \varepsilon^{-1}a(x)\int_{R^n}[f(x+\varepsilon y)-f(x)]\,\pi(x,dy). \tag{5.1}$$

Fix $x_s^\varepsilon = x$. For $s \leq t \leq T$, the path x^ε tends in probability as $\varepsilon \to 0$ (D-metric) to x^0, where x_t^0 satisfies

$$\frac{dx_t^0}{dt} = a(x_t^0)\int_{R^n} y\pi(x_t^0,dy), \quad s \leq t \leq T, \tag{5.2}$$

with $x_s^0 = x$. The exit probability $\phi^\varepsilon(s,x)$ is a positive solution to

$$\frac{\partial\phi^\varepsilon}{\partial s} + L_\varepsilon\phi^\varepsilon = 0 \tag{5.3}$$

in $(-\infty,T) \times D$. The logarithmic transformation $I^\varepsilon = -\varepsilon \log \phi^\varepsilon$ changes (5.3) into

$$\frac{\partial I^\varepsilon}{\partial s} + \varepsilon H_\varepsilon(\varepsilon^{-1}I^\varepsilon) = 0, \tag{5.4}$$

where $H_\varepsilon(I) = -e^I L_\varepsilon(e^{-I})$. Then

$$\varepsilon H_\varepsilon(\varepsilon^{-1}I) = a(x)\int_{R^n}\left(1 - \exp\left[\frac{I(x)-I(x+\varepsilon y)}{\varepsilon}\right]\right)\pi(x,dy) \tag{5.5}$$

For $I(x)$ such that I, I_x are continuous, bounded

$$\lim_{\varepsilon\to 0} \varepsilon H_\varepsilon(\varepsilon^{-1}I) = H_0(x,I_x),$$

with I_x the gradient and

$$H_0(x,p) = a(x)\int_{R^n}(1 - e^{-p \cdot y})\pi(x,dy). \tag{5.6}$$

This suggests (but certainly does not prove) that I^ε tends to a limit I^0 as $\varepsilon \to 0$, where I^0 satisfies (perhaps in some generalized sense)

$$\frac{\partial I^0}{\partial s} + H(x, I_x^0) = 0. \tag{5.7}$$

Now (5.7) is the dynamic programming equation for the deterministic control problem with control space U as in §3, with running cost $k(\xi_t, u_t(\cdot))$, and with dynamics

$$\frac{d\xi_t}{dt} = b(\xi_t, u_t(\cdot)), \tag{5.8}$$

$$b(x, u(\cdot)) = a(x) \int_{R^n} y\, u(y)\pi(x, dy).$$

Sheu [11] proved that indeed $I^\varepsilon \to I^0$ as $\varepsilon \to 0$ under the following hypotheses:

(i) $a(\cdot)$ is bounded, positive, and Lipschitz;

(ii) $\pi(x,dy) = g(x,y)\pi_1(dy)$ with π_1 a probability measure, $\pi_1(\{0\}) = 0$, $g(\cdot,y)$ uniformly Lipschitz, and $0 < c_1 < g(x,y) \leq c_2$;

(iii) $\int_{R^n} \exp(\alpha|y|^2)\pi_1(dy) < \infty$ for some $\alpha > 0$;

(iv) the convex hull of the support of π_1 contains a neighborhood of 0.

Condition (iv) insures that $H_0(x,p)$ is the dual of the usual "action integrand" $\Lambda(\xi,\dot\xi)$ in large deviation theory, where for $\xi, \dot\xi \in R^n$

$$\Lambda(\xi,\dot\xi) = \min_{u(\cdot)} \{k(\xi,u(\cdot)) : \dot\xi = b(\xi,u(\cdot))\}\ . \tag{5.9}$$

Then

$$I^0(s,x) = \min_{\xi_\cdot} \int_s^\theta \Lambda(\xi_t,\dot\xi_t)dt, \quad x \in D. \tag{5.10}$$

The minimum is taken among C^1 paths $\xi_\cdot$ with $\xi_s = x$ such that ξ_t first reaches ∂D at time $\theta \leq T$. The requirement in (5.10) that ξ_t exit from D by time T is suggested by the boundary condition $I^\varepsilon(T,x) = +\infty$ for $x \in D$. This corresponds in the limit as $\varepsilon \to 0$ to an infinite penalty for failure to reach ∂D by time T.

In both [3] and [11] the stochastic control method used to show that $I^\varepsilon \to I^0$ depends on comparison arguments involving an optimal stochastic control process when $\varepsilon > 0$ and an optimal $\xi^0_\cdot$ in (5.10) when $\varepsilon=0$.

6. The dominant eigenvalue. In [2] Donsker and Varadhan gave a variational formula [(6.4)below] for the dominant eigenvalue λ_1 of $L + V$. Another derivation of this formula is given in [11], using the family of operators $\tilde L^W$ mentioned in §4.

When L is the generator of a nondegenerate diffusion process, Holland [9] expressed λ_1 as the minimum average cost per unit time in a stochastic control problem. Let us impose strong restrictions on L, and give a short derivation of (6.4).

Assume that $L+V$ has a positive eigenfunction ϕ_1 corresponding to $\lambda_1 : (L+V)\phi_1 = \lambda_1\phi_1$. Let $I_1 = -\log\phi_1$. Then

$$-H(I_1) + V = \lambda_1 . \tag{6.1}$$

Assuming that there is a stochastic control representation (1.5) for $H(I)$, equation (6.1) becomes

$$\min_{u\in U} [L^u I_1(x) + k(x,u)] - V(x) = -\lambda_1. \tag{6.2}$$

Equation (6.2) is the dynamic programming equation for the following average cost per unit time control problem. We admit stationary controls $\underline{u}(\cdot)$ such that the controlled process with generator $L^{\underline{u}}$ has an equilibrium distribution μ . The criterion to be minimized is

$$J(\mu,\underline{u}) = \int_\Sigma [k(x,\underline{u}(x)) - V(x)]d\mu(x). \tag{6.3}$$

(If there is a unique equilibrium distribution $\mu = \mu^{\underline{u}}$ then reference to μ on the left side of (6.3) is unnecessary.). The principle of optimality states that $-\lambda_1 \leq J(\mu,\underline{u})$ with equality provided $\underline{u}^*(x)$ gives the minimum over $u \in U$ of $L^u I_1(x) + k(x,u)$.

Let us now assume that Σ is compact, that the generator L is bounded on $C(\Sigma)$ and $V \in C(\Sigma)$. As in [2] for any probability measure μ on Σ let

$$\mathscr{I}(\mu) = \sup_I \int_\Sigma H(I)d\mu = -\inf_{\phi>0} \int_\Sigma \frac{L\phi}{\phi}\, d\mu ,$$

where $I, \phi \in C(\Sigma)$. The Donsker-Varadhan formula is

$$\lambda_1 = \sup_\mu \left[\int_\Sigma V d\mu - \mathscr{I}(\mu)\right]. \tag{6.4}$$

Let

$$P(I,\mu) = \int_\Sigma [-H(I) + V]d\mu .$$

The function P is convex in I and linear in μ . Formula (6.4) will follow if we can find I_1, μ_1 with the saddle point property:

$$P(I_1,\mu) \leq \lambda_1 \leq P(I,\mu_1) \text{ for all } I, \mu. \tag{6.5}$$

(This idea was known to Donsker and Varadhan a long time ago, and figures in their

From (6.1) we have in fact $P(I_1,\mu) = \lambda_1$ for all probability measures μ on Σ. To get the right hand inequality, choose $\underline{u}^*$ as above and assume that $L^{\underline{u}^*}$ is bounded on $C(\Sigma)$. The corresponding Markov process ξ_t^* has an equilibrium distribution μ_1, and

$$\int_\Sigma (L^{\underline{u}^*} I)\,d\mu_1 = 0, \text{ for all } I \in C(\Sigma). \tag{6.6}$$

(If $L^{\underline{u}^*}$ is unbounded we need to assume the existence of μ_1, and to restrict I to the domain of $L^{\underline{u}^*}$). By taking $u = \underline{u}^*(x)$ in (1.5) we have for $I \in C(\Sigma)$

$$L^{\underline{u}^*} I + k(x,\underline{u}^*) - V \geq H(I) - V.$$

By integrating both sides with respect to μ_1,

$$-\lambda_1 = J(\mu_1,\underline{u}^*) \geq -P(I,\mu_1),\ \lambda_1 \leq P(I,\mu_1),$$

as required.

In order to derive (6.4) in this way we had to impose unnecessarily restrictive hypotheses. In particular, we assumed that λ_1 is a dominant eigenvalue in the strict sense that $(L + V)\phi_1 = \lambda_1\phi_1$, with $\phi_1 > 0$. Actually, (6.4) holds if L is the generator of a strongly continuous, nonnegative semigroup T_t on $C(\Sigma)$, such that $T_t 1 = 1$, L has domain dense in $C(\Sigma)$, and L satisfies the maximum principle [2]. With such assumptions λ_1 is a dominant eigenvalue in the sense that the spectrum of $L + V$ is contained in $\{z: \mathrm{Re}\, z \leq \lambda_1\}$ and $\lambda_1 - (L + V)$ does not have an inverse.

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