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INFORMATION MANUAL

SMALL AND GREAT CIRCLE ROUTINES

Victoria Booth

A Division of CNA Hudson Institute

CENTER FOR NAVAL ANALYSES

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ABSTRACT

This information manual documents routines in the CNA mapping software library that calculate the points of small and great circles on the earth. The equations upon which the routines are based are described and a user's walk-through of the programs is given.

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ILLUSTRATIONS

INTRODUCTION

Routines that calculate the points of small and great circles on the earth have been added to the CNA mapping software library. This manual first defines small and great circles and discusses various uses for these routines. Next, the equations upon which the routines are based are briefly described and instructions given for plotting the circles using the mapping software. Finally, a user's walk-through of both programs is given. The appendix contains examples of the output of the routines used with the mapping software.

BACKGROUND OF ROUTINES

SMALL AND GREAT CIRCLES

A small circle is the intersection of a sphere and a plane that does not contain the center of the sphere. All latitude lines, except the equator, are small circles on the earth. Figure 1 shows several small circles.

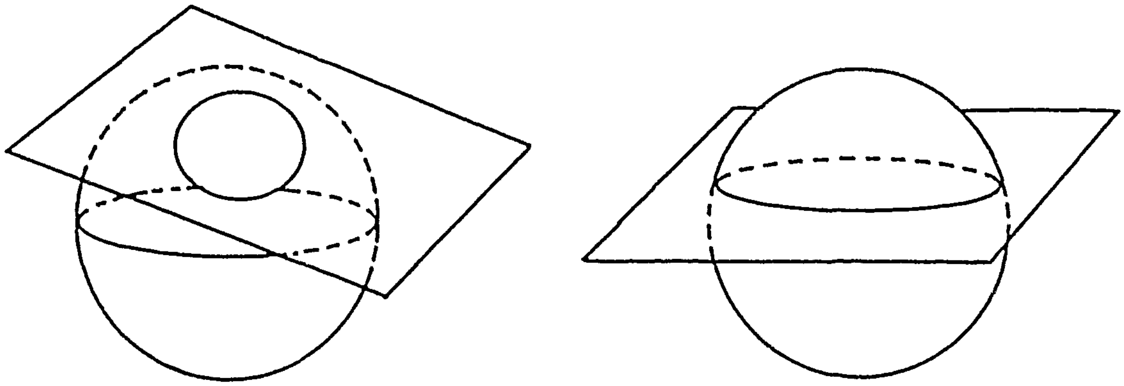

Figure 1. Small circles

A great circle is the intersection of a sphere and a plane that contains the center of the sphere. All longitude lines and the equator are great circles on the earth. Usually a great circle is considered to be cocentric with the sphere and have a radius equal to that of the sphere. However, it can also be viewed as a small circle with radius equal to one-fourth of the circumference of the sphere having two diametrically opposite centers on the surface of the sphere. Figure 2 shows several great circles.

Great circles also define the shortest distance, on the surface, between two points on a sphere. Ideally, then, ships should navigate by great circle courses. They do not, however, because following a great circle course requires continual direction changes. Also great circles often do not avoid land masses and dangerous routes. Ships usually follow a variation of a great circle course. Aircraft, however, follow great circle courses more closely.

USES FOR THE ROUTINES

Programs for calculating small and great circles were developed under the Antiair Warfare Master Plan (AAWMP). An original use was to display radar coverage sectors. The small circle routine was also used to show ranges of threat aircraft, and the great circle program was used to generate possible threat air routes. Other uses include defining threat contours of aircraft and of surface and subsurface units and to easily display distances on charts. Examples are contained in the appendix.

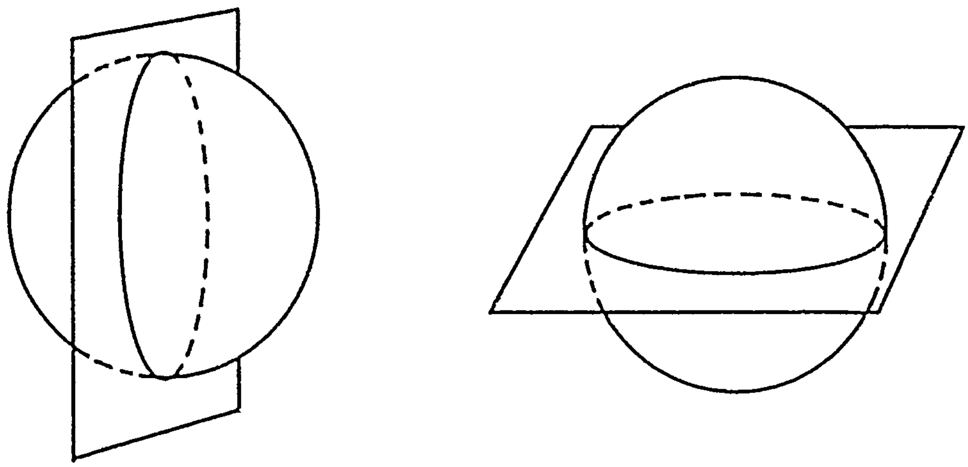

Figure 2. Great circles

EQUATIONS BEHIND ROUTINES

The equations upon which the small and great circle routines are based were derived as part of a surveillance study for the AAWMP.[1] The calculations are in spherical coordinates on the surface of the earth. The polar angle θ is measured from the North Pole and the azimuthal angle ϕ is referenced from the international date line. The conversion equations from latitude and longitude to spherical coordinates are:

$$\theta = \pi/2 - \pi \cdot L/180$$

$$\phi = \pi + \pi \cdot M/180 \ , \qquad (1)$$

where L is the latitude and M is the longitude, expressed in decimal degrees. In the calculations, L is positive when latitude is north and M is positive when longitude is east.

The equations for small and great circles are derived from the equation for the distance along a great circle between two points on the surface of the earth. Two arbitrary points on the surface of the earth, (θ_1, ϕ_1) and (θ_2, ϕ_2), and the North Pole form the vertices of a spherical

1. CNA analyst Barry McCoy derived these equations.

triangle, as shown in figure 3. The sides of the triangle are great circles because as noted above the shortest distance on the surface of a sphere between two points is along a great circle. The sides also define angles subtended at the center of the earth. The angular magnitudes of the sides adjacent to the North Pole are θ_1 and θ_2, respectively. The angle between these two sides is $\phi_2 - \phi_1$ if the triangle does not contain the date line or $\phi_2 + (2\pi - \phi_1)$ if it does. The law of cosines for spherical triangles gives the angular magnitude of the third side, a, by:

$$\cos a = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2) \cdot \cos(\phi_2 - \phi_1) \ . \tag{2}$$

Note that this equation is true whether or not the triangle contains the date line, because $\cos(\phi_2 - \phi_1) = \cos(\phi_2 + 2\pi - \phi_1)$. However, the distance between points 1 and 2 along the third side, A, equals $Re \cdot a$, where Re is the radius of the earth. Substitution yields:

$$A = Re \cdot \cos^{-1} [\cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2) \cdot \cos(\phi_2 - \phi_1)] \ . \tag{3}$$

The distance A is in nautical miles if the radius of the earth is given in nautical miles. The above equation also gives all the points (θ_2, ϕ_2) on the surface of the earth that are a distance A away from (θ_1, ϕ_1). In other words, equation 3 will give the equation for a small circle on the earth. Solving equation 3 for ϕ_2 yields:

$$\phi_2 = \phi_1 \pm \cos^{-1} \left\{ \frac{[\cos(A/Re) - \cos(\theta_1) \cdot \cos(\theta_2)]}{[\sin(\theta_1) \cdot \sin(\theta_2)]} \right\} \ . \tag{4}$$

This equation is valid providing neither θ_1 nor θ_2 is 0 or an integer multiple of π. Equation 4 has two branches, each describing one-half of the circle. The branches meet when

$$\cos(A/Re) = \cos(\theta_1) \cdot \cos(\theta_2) + \sin(\theta_1) \cdot \sin(\theta_2)$$

or

$$A/Re = \pm(\theta_1 - \theta_2) \ . \tag{5}$$

Thus, the appropriate limits of θ_2 are $\theta_1 + A/Re$ and $\theta_1 - A/Re$. In summary, equation 4 describes a small circle centered at (θ_1, ϕ_1) of radius A when θ_2 is taken between $\theta_1 + A/Re$ and $\theta_1 - A/Re$, with the provision mentioned above. The increment of θ_2 depends on the desired number of points of the circle.

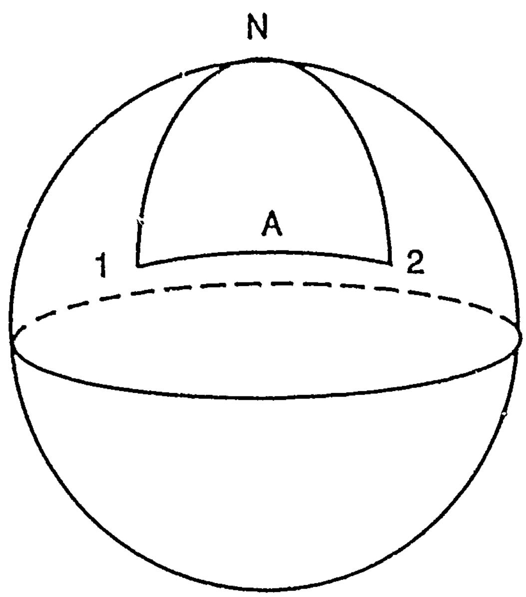

Figure 3. Spherical triangle

When θ_1 is 0 or an integer multiple of π, the circle is centered at one of the poles and is a latitude curve. There are easier ways to determine a latitude curve than by equations of the above sort, so this case will not be considered. When θ_2 is 0 or an integer multiple of π, the circle passes through the North and/or South Pole. Other exceptional cases to equation 4 appear if the circle contains one or both of the poles. These cases will be examined separately.

Case A: $\theta_1 - A/Re = 0$ and $\theta_1 + A/Re < \pi$

In this case, the circle passes through the North Pole. For all θ_2 except 0, equation 4 is a valid representation of the circle. At the point where $\theta_2 = 0$, ϕ_2 could be any value. For the sake of continuity, ϕ_2 will be chosen so that it is a continuous function of θ_2. Taking the limit of equation 4 as θ_2 approaches zero yields

$$\lim_{\theta_2 \to 0} \phi_2 = \phi_1 \pm \pi/2 \ .$$

Thus, when $\theta_2 = 0$, $\phi_2 = \phi_1 \pm \pi/2$ to conserve continuity of the function $\phi_2(\theta_2)$. Whether $\pi/2$ is added or subtracted depends on which direction θ_2 approaches zero.

Case B: $\theta_1 - A/Re > 0$ and $\theta_1 + A/Re = \pi$

This case is similar to Case A, but here the circle passes through the South Pole. A procedure similar to Case A yields

$$\lim_{\theta_2 \to 0} \phi_2 = \phi_1 \pm \pi/2 \ .$$

As in Case A, equation 4 is valid for all θ_2, such that

$$0 < \theta_1 - A/Re \leq \theta_2 < \pi \ .$$

When $\theta_2 = \pi$, $\phi_2 = \phi_1 \pm \pi/2$, depending on which direction θ_2 approaches π.

Case C: $\theta_1 - A/Re = 0$ and $\theta_1 + A/Re = \pi$

In this case, the circle passes through both the North and South Poles. In other words, the circle is a great circle and defines two longitude lines. It has two diametrically opposite centers on the surface of the earth with θ-components of $\pi/2$ and a radius of $Re \cdot \pi/2$. Making these substitutions for θ_1 and A in equation 4 yields the equation for this great circle:

$$\phi_2 = \phi_1 \pm \pi/2 \ \text{for} \ 0 \leq \theta_2 \leq \pi \ .$$

Case D: $\theta_1 - A/Re < 0$ and $\theta_1 + A/Re < \pi$

In this case, the circle contains the North Pole. Equation 4 is still valid but the limits of θ_2 are different:

$$A/Re - \theta_1 \leq \theta_2 \leq \theta_1 + A/Re \ .$$

Case E: $\theta_1 - A/Re > 0$ and $\theta_1 + A/Re > \pi$

Here, the circle contains the South Pole. As in Case D, equation 4 is valid but the limits of θ_2 have changed:

$$\theta_1 - A/Re \leq \theta_2 \leq 2\pi - A/Re - \theta_1 \ .$$

Case F: $\theta_1 - A/Re < 0$ and $\theta_1 + A/Re > \pi$

Here, both Cases D and E occur; the circle contains both the North and South Poles. To counter this problem, the solutions of both cases are applied. Equation 4 is valid and the limits of θ_2 are

$$A/Re - \theta_1 \le \theta_2 \le 2\pi - A/Re - \theta_1 \ .$$

As shown in Case C above, equation 4 can give the equation of a great circle when A is $Re \cdot \pi/2$. This substitution yields:

$$\phi_2 = \phi_1 \pm \cos^{-1}[-\cot\theta_1 \cdot \cot\theta_2]$$

or

$$\theta_2 = \cot^{-1}\left[\frac{-\cos(\phi_2 - \phi_1)}{\cot\theta_1}\right] \tag{6}$$

for $0 \le \phi_2 \le 2\pi$. Equation 6 is valid for $\cot\theta_1 \ne 0$, i.e., $\theta_1 \ne \pi/2$. Case C gives the equation when $\theta_1 = \pi/2$.

However, the center of a great circle (θ_1, ϕ_1) is not often known, but two points through which the circle passes may be. If (θ_A, ϕ_A) and (θ_B, ϕ_B) are the two points, (θ_1, ϕ_1) must satisfy equation 6 for each:

$$\theta_A = \cot^{-1}\left[\frac{-\cos(\phi_A - \phi_1)}{\cot\theta_1}\right] \tag{7}$$

$$\theta_B = \cot^{-1}\left[\frac{-\cos(\phi_B - \phi_1)}{\cot\theta_1}\right] \ . \tag{8}$$

Solving these equations for (θ_1, ϕ_1) yields:

$$\theta_1^{\pm} = \cot^{-1}\left\{\frac{\pm\sin(\phi_B - \phi_A)}{\sqrt{\cot^2\theta_A + \cot^2\theta_B - 2\cot\theta_A \cot\theta_B \cos(\phi_B - \phi_A)}}\right\} \tag{9}$$

$$\cos \phi_1^{\pm} = \cot(\theta_1^{\pm}) \cdot \left[\frac{\sin \phi_A \cot \theta_B - \sin \phi_B \cot \theta_A}{\sin \phi_B \cos \phi_A - \sin \phi_A \cos \phi_B} \right] \tag{10}$$

$$\sin \phi_1^{\pm} = -\cot(\theta_1^{\pm}) \cdot \left[\frac{\cos \phi_A \cot \theta_B - \cos \phi_B \cot \theta_A}{\sin \phi_B \cos \phi_A - \sin \phi_A \cos \phi_B} \right] . \tag{11}$$

The $\pm$ superscripts indicate the two diametrically opposite centers.

In summary, if two points through which a great circle passes are known, equations 9, 10, and 11 give the centers of the circle. If the θ-components of the centers are not $\pi/2$, then equation 6 describes the points of the great circle. The increment of ϕ_2 is determined by the number of points desired on the circle. Case C in the small circle calculations gives the equation for the circle when the θ-components of the centers are $\pi/2$. In this case, θ_2 is incremented depending on the desired number of points on the circle.

DRAWING CIRCLES WITH THE MAPPING SOFTWARE

The small and great circle routines write the calculated points to a user-named output file, which is in the correct format to send to the MAPREFORM routine of the mapping software. The MAPREFORM routine rewrites the data so it can be read by the ETCH & SKETCH program (see CNA Information Manual 22 for details).[1] MAPREFORM can read latitude and longitude values in two formats, DECIMAL and SECOND. The output of both circle routines is in the DECIMAL format. The commands to send the output file to the two mapping routines are:

```
$@CNA$MAPPING:MAPREFORM

$@CNA$MAPPING:ETCH
```

Both mapping routines are self-explanatory.

1. *Introduction to Map Concepts and Mapping Software*, by Peter J. Meoli, Unclassified, Jun 1988.

USER'S WALK-THROUGH OF THE ROUTINES

SMALL CIRCLE ROUTINE

The small circle routine calculates a specified number of points of an arc centered at a given point on the earth and of a specified radius. Either a complete circle may be calculated or an arc of specified length may be calculated, depending on the degrees given for the beginning and end of the arc. The user-supplied information may be read from a previously prepared input file, or the user will be prompted for it on the screen. The routine will calculate points for as many circles as the user specifies. The calculated points will be sent to a user-named output file. If an incorrect response is encountered during a run, the user will be prompted again for the value.

To run the routine, enter:

$@CNA$MAPPING:SMALLCIRCLE

The user will then encounter the following prompts:

(1) Is data to be read from a file? (Y/N)

Y—indicates that circle information is to be read from a previously prepared file. Figure 4 shows the format of this file. The user will then be prompted for the name of this data file. The extension of this file should be .DAT.

100	(integer from 10 to 500)	(Number of points desired)
30.0	(real between -90.0 and 90.0)	(Latitude of center)
-100.0	real between -180.0 and 180.0)	(Longitude of center)
500.0	(real between 5.0 and 10,820)	(Radius of circle)
0.0	(real between 0.0 and 360.0)	(Beginning degree of arc)
360.0	(real between 0.0 and 360.0)	(Ending degree of arc)
Y	(one character Y/N)	(Indicates retention of the center)
1000.0	(real between 5.0 and 10,820)	(Radius of circle)
0.0	(real between 0.0 and 360.0)	(Beginning degree of arc)
180.0	(real between 0.0 and 360.0)	(Ending degree of arc)
N	(one character Y/N)	(Indicates new center)
47.3	(real between -90.0 and 90.0)	(Latitude of center)
164.2	(real between -180.0 and 180.0)	(Longitude of center)
250.0	(real between 5.0 and 10,820.0)	(Radius of circle)
90.0	(real between 0.0 and 360.0)	(Beginning degree of arc)
270.0	(real between 0.0 and 360.0)	(Ending degree of arc)
.		
.		
.		

Figure 4. Format of input file for small circle routine

N—indicates that circle information is to be entered at the terminal.

(2) Enter name of output file

A file of the given name with a .DAT extension will be created in the user's directory and the points of the arc or circle will be written to that file.

(3) ENTER CTRL/Z OR RETURN TO EXIT PROGRAM

The program will stop at any time if CTRL/Z is typed. If circle information is entered from the terminal, entering RETURN at prompts without a default value will stop the program. If circle information is being read from a file, the program will stop when it comes to the end of the file.

If data are read from a file, the following commands will appear on the screen, but no response is necessary. If data are entered at the terminal, the user must respond to the following prompts:

(4) Enter number of circle points desired from 10 to 500 (default = 100)

Typing RETURN will enter the default value. The user may type an integer between 10 and 500 to enter another value. The default value of 100 points yields high resolution at most scales; however, very high resolution may be obtained with up to 500 points. Note that the number of points specified here is for the complete circle. The number of points calculated for an arc will be in the same ratio to the number entered here as the length of the arc is to a complete circle. For example, if a semicircle is desired and 100 is entered here, the output will be 50 points.

(5) Enter center latitude (±99.99)

Latitude is in degrees; north is positive and south is negative. The program will exit if RETURN is entered.

(6) Enter center longitude (±999.99)

Longitude is in degrees; east is positive and west is negative. The program will exit if RETURN is entered.

(7) Enter circle radius in n.mi. (9999.99)

The circle radius is in nautical miles and cannot be larger than 1/2 of the circumference of the earth, or approximately 10,820 n.mi. The radius must also be greater than or equal to 5 n.mi., because the calculations are not precise enough to yield a circle with smaller radius. The program will exit if RETURN is entered.

(8) Enter beginning degree of arc; north is 0.0 (default=0.0)

The length of the desired arc is specified at this prompt. The points of the arc are calculated in clockwise direction with north at 0.0 degrees, east at 90.0 degrees, south at 180.0 degrees, and west at 270.0 degrees.

(9) Enter ending degree of arc; north is 360.0 (default=360.0)

The points of the arc are calculated between the degree given above and the degree given at this prompt. A complete circle is computed if RETURN or the same value is entered at both prompts.

The points of the circle specified by the entered data are calculated and sent to the named output file. The next prompt is convenient if more than one circle with the same center is desired.

(10) Does the next circle have the same center as the previous circle? (Y/N) (default=N)

Y—causes the previously entered center latitude and longitude to be retained and prompts (7) through (9) to repeat.

N—causes prompts (5) through (9) to repeat.

The prompts will keep appearing to permit multiple calculations until CTRL/Z is typed or RETURN is entered at a prompt without a default value.

GREAT CIRCLE ROUTINE

The great circle routine calculates a specified number of points on a great circle that passes through two given points. The routine offers the option of computing the points of the complete circle or the points on the segment of the circle between the two given points. As in the small circle routine, the input data for the routine may be read from a previously prepared file or the user may be prompted for it on the screen. The routine can calculate points for as many circles as the user specifies. The calculated points are sent to a user-named output file.

To run the routine, enter:

```
$@CNA$MAPPING:GREATCIRCLE
```

The user will then encounter the following prompts:

(1) Is data to be read from a file (Y/N)?

Y—indicates that circle information is to be read from a previously prepared file. Figure 5 shows the format of this file. The user will then be prompted for the name of this data file. The extension of this file should be .DAT.

N—indicates that circle information is to be entered at the terminal.

100	(Integer from 10 to 500)	(Number of points desired)
60.0	(Real between -90.0 and 90.0)	(Latitude of first point)
-35.95	(Real between -180.0 and 180.0)	(Longitude of first point)
-32.65	(Real between -90.0 and 90.0)	(Latitude of second point)
180.00	(Real between -180.0 and 180.0)	(Longitude of second point)
C	(Character C or S)	(Circle or segment calculated)
-87.5	(Real between -90.0 and 90.0)	(Latitude of first point)
105.9	(Real between -180.0 and 180.0)	(Longitude of first point)
10.0	(Real between -90.0 and 90.0)	(Latitude of second point)
-10.0	(Real between -180.0 and 180.0)	(Longitude of second point)
S	(Character C or S)	(Circle or segment calculated)
Y	(Character Y or N)	(Date line indicator)
.		
.		
.		

Figure 5. Format of input file for great circle routine

(2) Enter output file name

A file with this name and the extension .DAT will be created in the user's directory and the calculated points will be written to it.

(3) ENTER CTRL/Z OR RETURN TO EXIT PROGRAM

The program will stop at any time if CTRL/Z is typed. If the circle information is entered from the terminal, entering RETURN at prompts without a default value will stop the program. If circle information is being read from a file, the program will stop when it comes to the end of the file.

If data are read from a file, the following prompts will appear on the screen, but no response is necessary. However, if data are entered at the terminal, the user must respond to the following prompts:

(4) Enter number of circle points desired from 10 to 500 (default=100)

Typing RETURN will enter the default value. The user may type an integer between 10 and 500 to enter another value. The default value of 100 yields high resolution for most scales, but higher resolution may be obtained with up to 500 calculated points. Note that the number of points specified here is for the complete circle. The number of points calculated for the segment will be in the same ratio to the number entered here as the length of the segment is to a complete circle.

(5) Enter latitude of first point (±9.99)

The latitude of one of the points the great circle passes through is entered here by typing a real number between −90.0 and +90.0; the + sign is optional. The latitude is in degrees, with north as positive and south as negative.

(6) Enter longitude of first point (±999.99)

The longitude of the point whose latitude was entered above is entered here by typing a real number between −180.0 and +180.0; the + sign is optional. The longitude is in degrees, with east as positive and west as negative.

(7) Enter latitude of second point (±99.99)

The latitude of the other point the great circle passes through is entered here. The format is the same as that for the latitude of the first point.

(8) Enter longitude of second point (±999.99)

The longitude of the point whose latitude was entered above is entered here. The format is the same as that for the longitude of the first point.

(9) Enter C if entire circle is desired or enter S if segment between points is desired (C/S)

C—Indicates that points of the complete circle are calculated.

S—Indicates that the points of the circle segment between the two points are calculated. The user will then be asked if the desired circle segment crosses the international date line (180.0 or −180.0 longitude). This feature permits calculation of either segment between the points. The user enters Y or N accordingly.

Prompts (5) through (9) will keep appearing to permit multiple calculations until CTRL/Z is typed or RETURN is entered at a prompt without a default value.

APPENDIX A

EXAMPLES

APPENDIX A

EXAMPLES

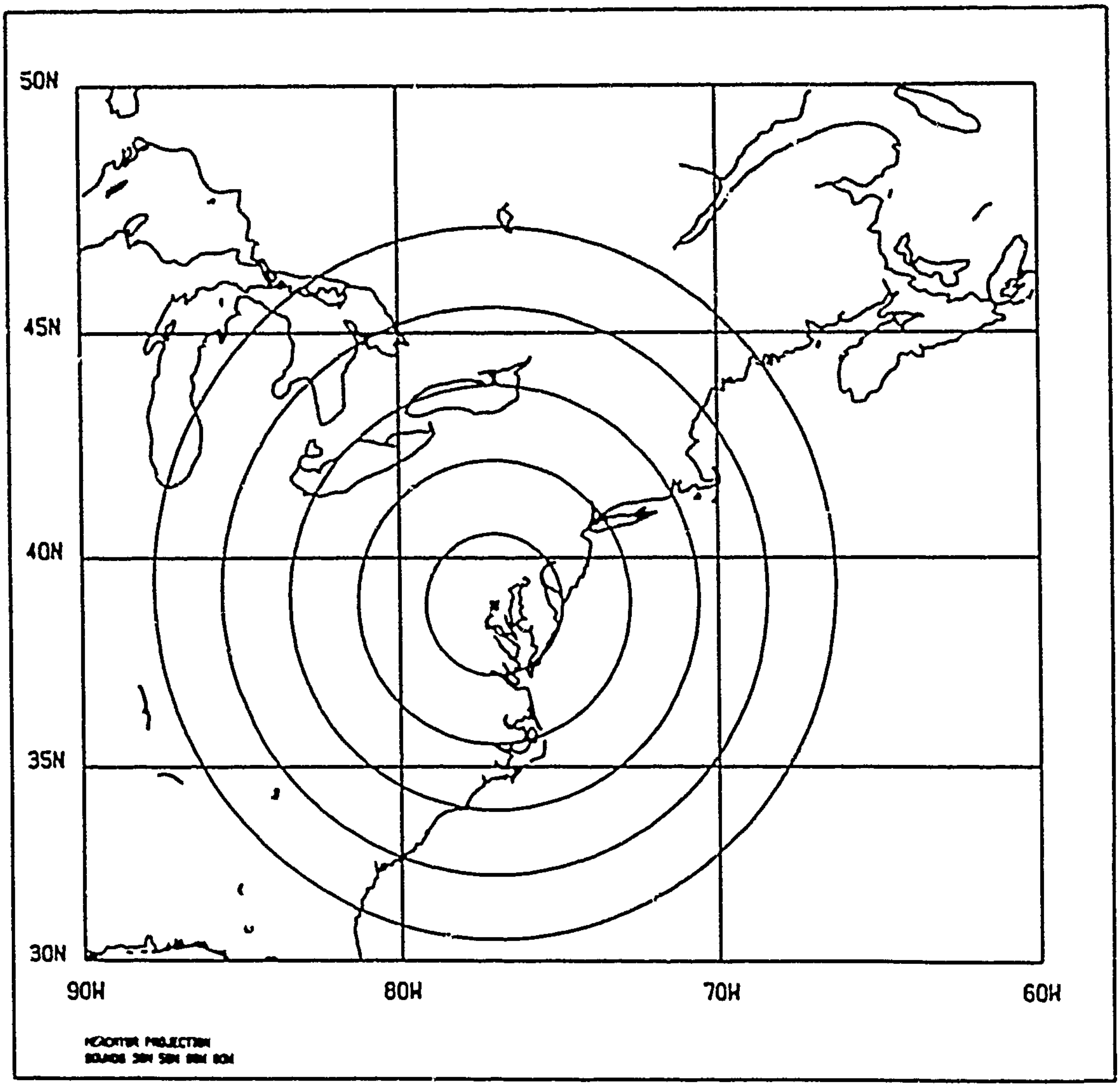

Figure A-1. Small circle routine

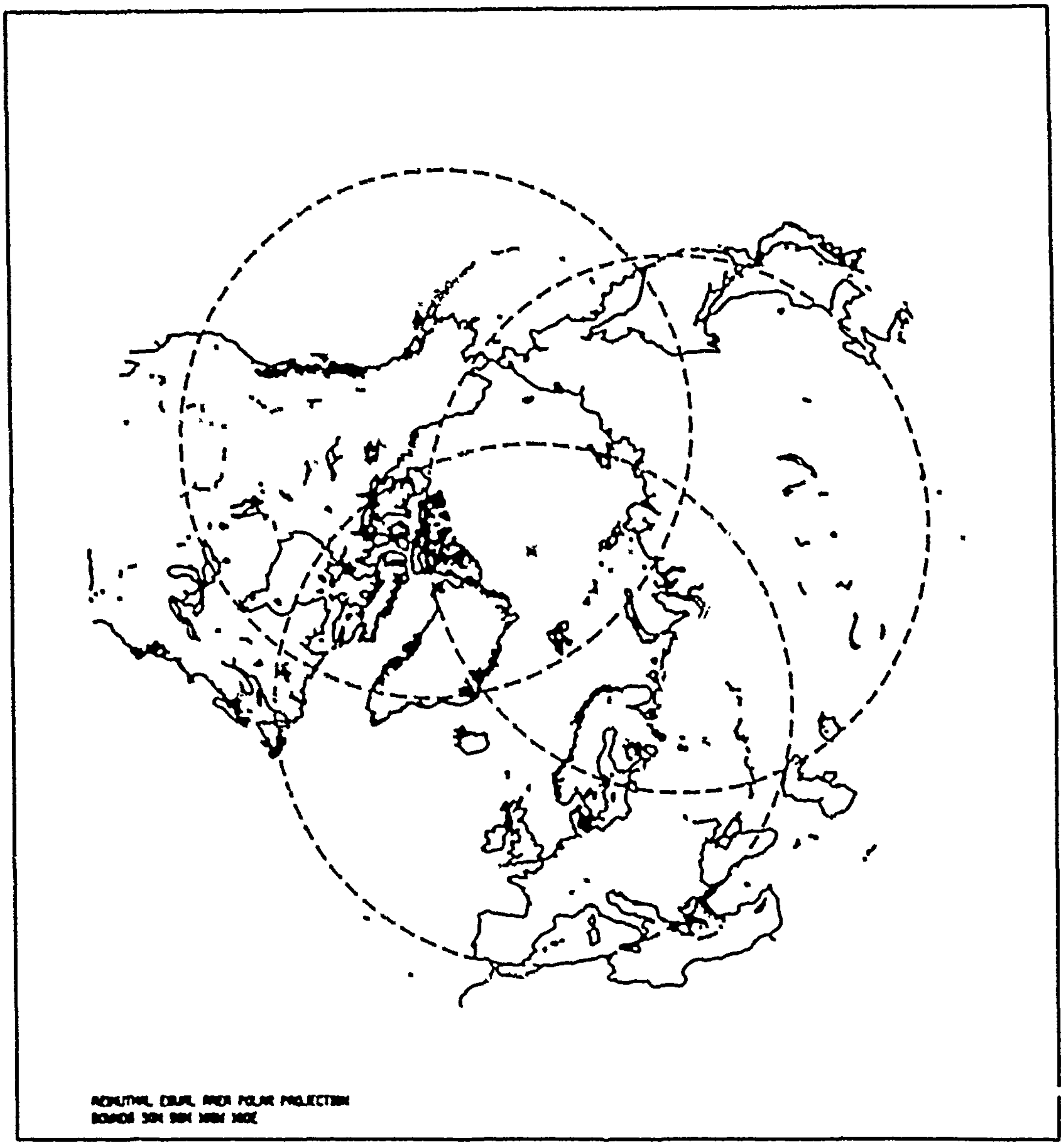

Figure A-2. Small circle routine

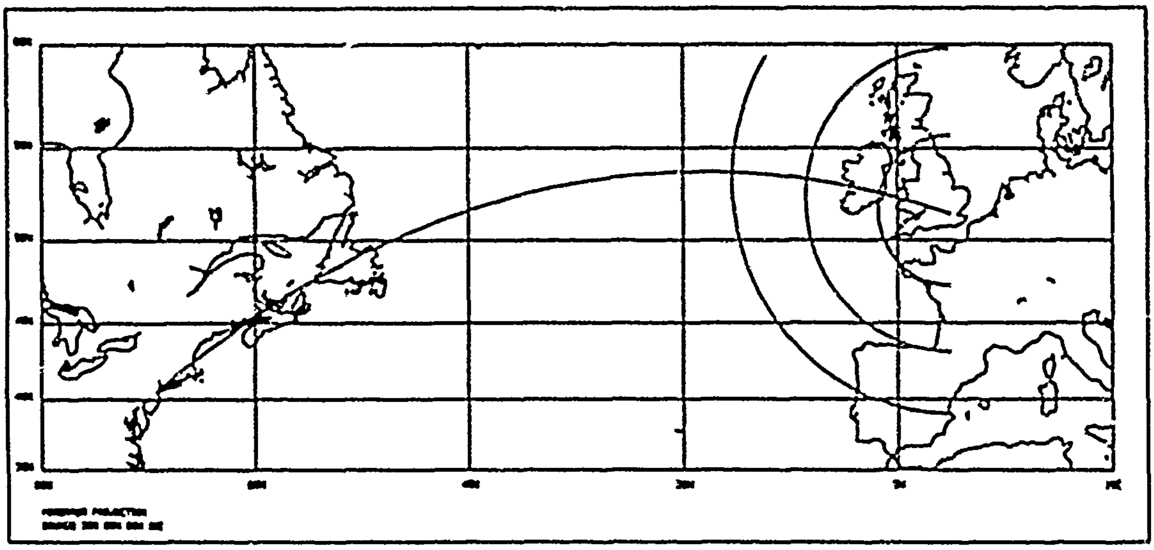

Figure A-3. Small and great circle routines

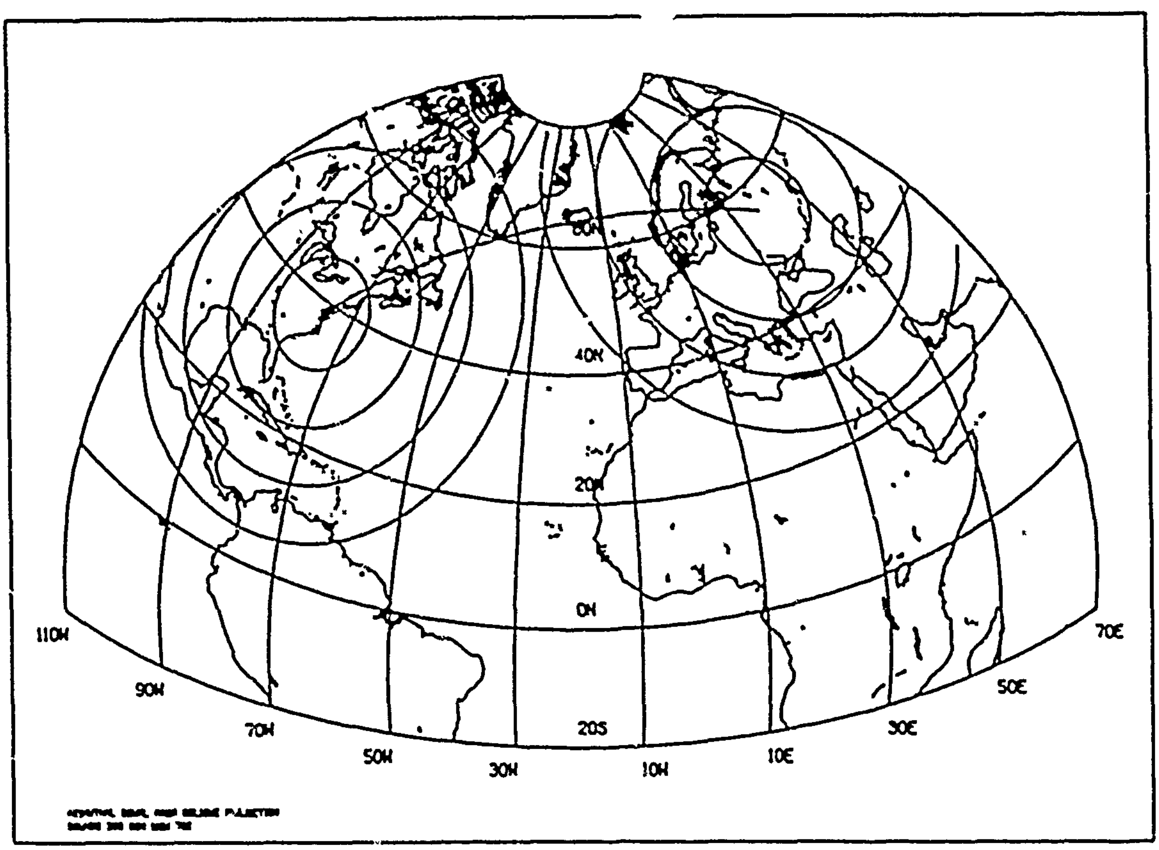

Figure A-4. Small and great circle routines