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Some Remarks on Solutions of Fermat's Last Equation in Terms of Wright's Hypergeometric Function

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Solutions of Fermat's last equation are expressed in terms of Wright's generalized hypergometric function. Some of the properties of these solutions are discussed and an equivalent form of Fermat's conjecture is given in terms of them.

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SOME REMARKS ON SOLUTIONS OF FERMAT'S LAST EQUATION IN TERMS OF WRIGHT'S HYPERGEOMETRIC FUNCTION

INTRODUCTION

In the seventeenth century the French mathematician and lawyer, Pierre de Fermat (1601–1665) stated that he had a proof of a result which is now known as

Fermat's Last Theorem: If n is an integer greater than 2, then the equation

$$x^n + y^n = z^n \tag{1}$$

has no solution in positive integers.

Fermat did not give a proof of this result and, to this day, no one else has either.

Recently it has been shown [1,2] that if Eq. (1) has a solution in positive real numbers with $x < y$, then the exponent in Eq. (1) is given by

$$n = \log_{y/x}[\lambda\Psi(\lambda)] \ , \tag{2}$$

where

$$\lambda = \log_{x/z}(y/z) \ , \tag{3}$$

and

$$\Psi(\lambda) = {}_1\Psi_1 \left[\begin{matrix} (\lambda, \lambda) & ; \\ (\lambda+1, \lambda-1) & ; \end{matrix} \ -1 \right] , \quad 0 < \lambda < 1 \ . \tag{4}$$

The function ${}_1\Psi_1$ in Eq. (4) is a special case of Wright's generalized hypergeometric function ${}_p\Psi_q$ whose series representation is given by

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \ldots, (\alpha_p, A_p) & ; \\ (\beta_1, B_1), \ldots, (\beta_q, B_q) & ; \end{matrix} \ z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + A_i n)}{\prod_{i=1}^{q} \Gamma(\beta_i + B_i n)} \frac{z^n}{n!} \ . \tag{5}$$

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In this paper we shall give an equivalent form of Fermat's last theorem in terms of the Wright function $\Psi(\lambda)$ and the related Wright function $\hat{\Psi}(\lambda)$ defined by

$$\hat{\Psi}(\lambda) \equiv {}_1\Psi_1 \left[\begin{matrix} (1,\lambda) & ; \\ (2,\lambda-1) & ; \end{matrix} \; -1 \right] , \quad 0 < \lambda < 1 . \tag{6}$$

In addition, we shall give some elementary properties of the functions $\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$.

MAIN RESULT

Theorem: The following are equivalent:

(a) Fermat's Last Theorem is false.

(b) There is a transcendental number $\lambda \in (0,1)$ and a positive integer $n > 2$ such that $\lambda\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ are simultaneously n-th powers of rational numbers in (0,1).

In order to prove this theorem we prove the following two lemmas.

Lemma 1: If Eq. (1) has a solution in positive integers with $x < y$ and $\gcd(x,y,z) = 1$, then there is a unique transcendental number λ in the interval $(0,1)$ such that

$$\left(\frac{x}{z}\right)^{\lambda} = \frac{y}{z} . \tag{7}$$

Proof. The function $f(\lambda) \equiv (x/z)^{\lambda}$ is a decreasing continuous function of λ on [0,1], since $x/z < 1$. Moreover, $f(1) = x/z < y/z < 1 = f(0)$, so by the intermediate value theorem there is a unique λ in the interval (0,1) such that

$$f(\lambda) = \left(\frac{x}{z}\right)^\lambda = \frac{y}{z} \ .$$

Since $\gcd(x, y, z) = 1$, the integers x, y, z are pairwise relatively prime. Suppose $\lambda = a/b$ is rational, where a and b are relatively prime positive integers. Then by Eq. (7),

$$\left(\frac{x}{z}\right)^{a/b} = \frac{y}{z}$$

or

$$x^a z^b = y^b z^a \ . \tag{8}$$

Since x, y, z are pairwise relatively prime, there must be a prime p that divides y but divides neither x nor z. So p divides the right side of Eq. (8) but not the left side, and we have a contradiction. Thus λ must be irrational.

This corrects the proof that λ is irrational given in [1, p.7]. In [1,2] a second proof of this result is given.

In order to show the transcendence of λ we shall need the result of Gelfond and Schneider [3, p. 21] which states that if α and β are algebraic, $\alpha \neq 0, 1$ and β is irrational, then α^β is transcendental.

From Eq. (7), since x/z is algebraic, $0 < x/z < 1$, and λ is irrational, then if λ is also algebraic, y/z would be transcendental, which it is not. Hence λ must be transcendental and the lemma is proved. Note that Eq. (7) is just another form of Eq. (3).

Lemma 2: For all $\lambda \in (0, 1)$ we have the identity

$$\lambda \Psi(\lambda) + \hat{\Psi}(\lambda) = 1 \ . \tag{9}$$

Proof. Using Eq. (5) we see that since $\Gamma(1+z) = z\Gamma(z)$,

$$
\begin{aligned}
{}_1\Psi_1\left[\begin{array}{ll} (1,\lambda) & ; \\ (2,\lambda-1) & ; \end{array} -1\right] &= \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(2+(\lambda-1)n)} \frac{(-1)^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{\Gamma(1+\lambda n)}{\Gamma(2+(\lambda-1)n)} \frac{(-1)^n}{\Gamma(1+n)} \\
&= 1 + \sum_{n=0}^{\infty} \frac{\Gamma(1+\lambda+\lambda n)}{\Gamma(\lambda+1+(\lambda-1)n)} \frac{(-1)^{n+1}}{\Gamma(2+n)} \\
&= 1 - \sum_{n=0}^{\infty} \frac{\lambda(1+n)\Gamma(\lambda+\lambda n)}{\Gamma(\lambda+1+(\lambda-1)n)} \frac{(-1)^n}{(1+n)\Gamma(1+n)} \\
&= 1 - \lambda\sum_{n=0}^{\infty} \frac{\Gamma(\lambda+\lambda n)}{\Gamma(\lambda+1+(\lambda-1)n)} \frac{(-1)^n}{n!} .
\end{aligned}
$$

Now using Eqs. (4)–(6) we have the result Eq. (9).

Proof of Theorem: If Fermat's Theorem is false, then Eq. (1) holds for some $n > 2$ and positive integers $x < y < z$. By Lemma 1 there is a unique transcendental number $\lambda \in (0,1)$ such that Eq. (2) holds, i.e.,

$$\lambda\Psi(\lambda) = (y/z)^n$$

and from Lemma 2,

$$
\begin{aligned}
\hat{\Psi}(\lambda) &= 1 - \lambda\Psi(\lambda) \\
&= 1 - (y/z)^n
\end{aligned}
$$

so that

$$\hat{\Psi}(\lambda) = (x/z)^n . \tag{10}$$

(We shall derive Eq. (10) in another way in the next section.)

Conversely, if for some transcendental $\lambda \in (0,1)$, both $\lambda\Psi(\lambda)$ and $\hat{\Psi}(\lambda)$ are n-th ($n > 2$) powers of some rational numbers, say

$$\lambda\Psi(\lambda) = (u_1/v_1)^n \,, \quad \hat{\Psi}(\lambda) = (u_2/v_2)^n \,,$$

then from Lemma 2 we have

$$(u_1v_2)^n + (u_2v_1)^n = (v_1v_2)^n$$

and the theorem is proved.

ANOTHER DERIVATION OF EQ. (10)

From Eq. (3) we have $y = x^\lambda z^{1-\lambda}$ which when substituted into Eq. (1) gives

$$(x/z)^n + (x/z)^{\lambda n} - 1 = 0 \,.$$

Setting $\xi = (x/z)^n$ we have

$$\xi + \xi^\lambda - 1 = 0 \,, \quad 0 < \lambda < 1 \,. \tag{11}$$

In order to solve Eq. (11) for ξ as a function of λ we shall need the following result. The positive root of the trinomial equation

$$\xi^p + \mu\xi^q - 1 = 0 \,, \quad p > q > 0 \tag{12}$$

is given by

$$\xi = \frac{1}{p}\,{}_1\Psi_1\left[\begin{matrix}(1/p,\ q/p) & ; \\ (1+1/p,\ q/p-1) & ;\end{matrix}\ -\mu\right] \,, \tag{13}$$

for real μ such that

$$|\mu| < (q/p)^{-q/p}(1 - q/p)^{q/p-1} \leq 2 \; . \tag{14}$$

For integers p and q, Mellin in 1915 [4] gave the series solution, Eq. (13), of the trinomial Eq. (12). However, his result is valid for real p and q. Lagrange circa 1768 [5] derived essentially the same result in terms of a series of binomial coefficients. In 1758 [6] Lambert gave the solution of trinomial equations. Ramanujan [7, pp. 71, 307] studied and derived solutions of trinomials in Chapter 3 of his notebooks (1903–1914) and in his first quarterly report (1913).

Now set $p = 1$, $q = \lambda$, $\mu = 1$ in Eqs. (12)–(13). Then the condition Eq. (14) is easily checked and using the *definition of* $\hat{\Psi}(\lambda)$ given by Eq. (6) we arrive at Eq. (10).

ADDITIONAL OBSERVATIONS

From Eq. (10) we see that the exponent in Eq. (1) is also given by

$$n = \log_{x/z} \hat{\Psi}(\lambda) \; . \tag{15}$$

This result is somewhat less complex than the result Eq. (2). We may also write

$$\begin{aligned}
\hat{\Psi}(\lambda) &= \lambda \sum_{n=0}^{\infty} (-1)^n {}_2F_1[-n,\ (1-\lambda)(n+2);\ 2;\ 1] \; , \\
\hat{\Psi}(\lambda) &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \binom{\lambda(2+n)-1}{n} \; , \quad 0 < \lambda < 1 \; .
\end{aligned}$$

The latter equations follow directly from Eq. (9) and [1, Eqs. (19)-(20)]. In addition from Eqs. (2), (3), (9), and (15) we observe that

$$[\hat{\Psi}(\lambda)]^{\lambda} = 1 - \hat{\Psi}(\lambda) ,$$

$$[1 - \lambda\Psi(\lambda)]^{\lambda} = \lambda\Psi(\lambda) .$$

CONCLUSION AND ACKNOWLEDGEMENT

Fermat's Last Theorem may be stated in equivalent form using Wright's generalized hypergeometric function ${}_1\Psi_1$.

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