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# On Free Products of Torsion Free Abelian Groups

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# ON FREE PRODUCTS OF TORSION FREE ABELIAN GROUPS

## 1. INTRODUCTION

In a previous paper [1], the quotient groups of the lower central series  $\bar{G}_n = G_n / G_{n+1}$  were studied. There the group  $G$  was assumed to be a free product of a finite number of finitely generated Abelian groups and  $G_n$  denoted the  $n^{\text{th}}$  subgroup of the lower central series of  $G$ . Here we give an improved proof of a complicated lemma which first appeared in [1] (in particular, Lemma 4.4[1]). The proof given here, especially for property (iii) of the conclusion of that lemma, is a significant simplification of that which appears in [1]. We observe that one of the consequences of Lemma 4.4[1] is to give a set of free generators for the lower central quotients in the case where the free factors are torsion free (i.e.,  $G = J$  in the terminology of [1]. Moreover the free generators are the  $J$ -basic commutators also using the terminology of Definition 4.1 of [1].). The author only thought of this simplification after the publication of [1]. Furthermore, the improved proof uses results from [2] and [3].

In this paper, notation, terminology, results, references, and equations of [1] will be employed. Furthermore, the numbering, a.b, of any definition, equation, etc., of [1] will correspond here to a.b-I. For example, Lemma 2.1-I will mean Lemma 2.1 of [1].

## 2. PRELIMINARIES

In order to carry out our goal, we need to give some preliminary machinery.

**Lemma 2.1.** (cf. Corollary 5.1 [3]) Let  $S$  be the subgroup of

F generated by a finite set  $\Sigma$  of F-simple commutators (see Definition 3.6-I) of weight  $> 1$ . Let  $U = \{u_1, u_2, \dots, u_m\} \subseteq \Sigma$ . Let  $V = \{v_1, v_2, \dots, v_m\}$  be a set of m nontrivial elements  $v_i$  of F such that  $W(v_i) \geq W(u_i)$  for  $1 \leq i \leq m$ . (Here we are using the notation of Definition 2.1-I for weight.) Let  $\psi$  be the mapping of  $\Sigma$  given by

$$\begin{aligned} u_i &\longrightarrow v_i \text{ if } u_i \in U \\ c_i &\longrightarrow c_i \text{ if } c_i \in U. \end{aligned}$$

Then  $\psi$  extends to a homomorphism, also denoted by  $\psi$ , of S into F such that if  $u \in S$  and  $W(u) = \omega > 0$ , then its image  $v = \psi(u) \in F_\omega$ .

We also will require

**Lemma 2.2.** (cf. Theorem 1.1[2]) Let  $c$  be the F-simple basic commutator of weight  $n > 1$ ,

$$c = (c_{j_1}, c_{j_2}, \dots, c_{j_n}). \quad (2.1)$$

Let  $d$  be a generator of F such that  $d < c_{j_n}$ . Let  $v =$

$(c_{j_2}, d, e_1, e_2, \dots, e_{n-1})$  be such that  $e_1 = c_{j_1}$  for  $n = 2$ , but

$e_1 \leq e_2 \leq \dots \leq e_{n-1}$  is a rearrangement of  $c_{j_1}, c_{j_3}, \dots, c_{j_n}$ , for

$n > 2$ . (Note that  $v$  is F-simple of weight  $n + 1$  if  $d < c_{j_2}$ .) Then

the following identity holds in F

$$(c, d) = \left\{ \begin{array}{ll} \Pi_1 M(c, d) \Pi_2 & \text{if } d \geq c_{j_2} \text{ (Case I)} \\ \Pi_1 v^{-1} \Pi_2 M(c, d) \Pi_3 & \text{if } d < c_{j_2} \text{ (Case II)} \end{array} \right\} \quad (2.2)$$

where the  $\Pi_j$  ( $j = 1, 2, 3$ ) are either  $= 1$  or are words in finitely many F-simple commutators  $u_i$  such that

(i)  $1 < W(u_i) < n + 1$ ;

(ii) if  $u_i = (v_1, v_2, \dots, v_t)$ , then  $v_1, v_2, \dots, v_t$  is a rearrangement of a subsequence of  $c_{j_1}, \dots, c_{j_n}, d$ .

### 3. STATEMENT AND PROOF OF THE MAIN RESULT

We are now ready to state and prove our main result

**Theorem 3.1.** (See Lemma 4.4-I.) Let  $c$  be F-simple as in (2.1). Suppose that  $c$  satisfies Criterion 1 but not Criterion 3. (See Definition 4.1-I.) Let  $t$  be the largest integer such that  $c_{j_t}$  occurs in  $c$  and  $(c_{j_t}, c_{j_1}) = 1$  in  $J$ . (The hypothesis that  $c$  does not satisfy Criterion 3 guarantees that such a  $t$  exists,  $t \geq 3$ , and that  $j_t > j_1$ .) Let the F-simple commutator  $d(c)$  be defined by

$$d(c) = (c_{j_t}, d_2, d_3, \dots, d_n) \quad (3.1)$$

where  $d_2, d_3, \dots, d_n$  is that rearrangement of  $c_{j_1}, \dots, c_{j_{t-1}}, c_{j_{t+1}}, \dots, c_{j_n}$  for which  $d_2 \leq \dots \leq d_n$ . Then the following holds in  $J$

$$c = H_1 d(c) H_2 \quad (3.2)$$

where the  $H_j$  ( $j = 1, 2$ ) are words in  $J$ -simple commutators  $u_i$  such that (i)  $1 < W(u_i) < n$ ; (ii) if  $u_i = (v_1, \dots, v_k)$ , then  $v_1, \dots, v_k$  is a rearrangement of a subsequence of  $c_{j_1}, \dots, c_{j_n}$ ; (iii)  $W(H_1 d(c) H_2) \geq n$  in  $F$ .

**Proof.** We proceed by induction on the place of  $c$  in our lexicographic ordering of basic commutators. Clearly,  $W(c) = n \geq 3$ . For  $n = 3$ , we have  $c = (c_{j_1}, c_{j_2}, c_{j_3})$  with  $(c_{j_1}, c_{j_2}) = 1$  in  $J$  and  $c_{j_3} > c_{j_1}$ . Thus equation (3.20)-I yields that in  $J$

$$c = (c_{j_1}, c_{j_2})^{-1} (c_{j_3}, c_{j_2}) (c_{j_3}, c_{j_2}, c_{j_1}) \cdot (c_{j_1}, c_{j_2}) (c_{j_3}, c_{j_2})^{-1}. \quad (3.3)$$

This shows that we have (3.2) with properties (i) and (ii) for  $n = 3$ . In order to show that the right hand side of (3.3) also satisfies property (iii), we let  $a = (c_{j_1}, c_{j_2})$ ,  $b = (c_{j_3}, c_{j_2})$ , and  $A = (c_{j_3}, c_{j_2}, c_{j_1})$ . Then (3.3) becomes

$$c = a^{-1} b A a b^{-1}$$

and we need to show that  $W(a^{-1} b A a b^{-1}) \geq 3$  in  $F$ . But in  $F$

$$\begin{aligned} a^{-1} b A a b^{-1} &= (a^{-1} b A b^{-1} a) \cdot (a, b^{-1}) \\ &= R \cdot (a, b^{-1}) \end{aligned}$$

where  $R = a^{-1} b A b^{-1} a$ . Evidently by (2.3a)-I,  $W(a, b^{-1}) \geq 4$  in  $F$  so that

$$W(a^{-1} b A a b^{-1}) = W(R(a, b^{-1})) \geq W(R) \geq 3$$

in  $F$ . This proves the result for  $n = W(c) = 3$ .

Next, suppose that we have already proven the result for all  $c < c_k$  with  $W(c_k) > 3$ . We then proceed to the smallest commutator  $c$  which satisfies the hypothesis of our theorem and is such that  $c \geq c_k$ . (Thus  $n = W(c) \geq W(c_k) > 3$ .)

We now define  $\delta = n - t$  and we treat two cases:  $\delta > 0$  and  $\delta = 0$ . For  $\delta > 0$ , let us write  $e = c^L$  and we note that, in  $J$ ,  $e = (c_{j_1}, \dots, c_{j_{n-1}})$  is by the induction hypothesis a word of the form (3.2), i.e.,

$$e = H_{11} d(e) H_{12},$$

where  $H_{1j}$  ( $j = 1, 2$ ) are words in  $J$ -simple basic commutators  $z_i$  satisfying (i) and (ii) of our conclusion, and  $W(H_{11} d(e) H_{12}) \geq n - 1$  in  $F$ . In particular every  $z_i$  has  $1 < W(z_i) < n - 1$  and also has  $z_i^R \leq c_{j_{n-1}}$  (recall by hypothesis that  $c_{j_1} < c_{j_t} \leq c_{j_{n-1}}$ ).

Applying the trivial identity (3.19)-I to the computation of  $c = (e, c_{j_n}) = (H_{11} d(e) H_{12}, c_{j_n})$ , we find that in  $J$

$$c = \Pi_1 (d(e), c_{j_n}) \Pi_2 \quad (3.4)$$

where the  $\Pi_j$  ( $j = 1, 2$ ) are words in  $z_i, (z_i, c_{j_n})$ , and  $d(e)$ . (Note  $d(e)$  only occurs in  $\Pi_1$ .) Now  $d(c) = (d(e), c_{j_n})$  by (3.1) since  $\delta > 0$ . Here we denote the word on the right hand side of (3.4) by

w and note that w is a word in the F-simple basic commutators  $d(e)$ ,  $z_i$ ,  $(z_i, c_{j_n})$ , and  $d(c)$ , i.e.,

$$w = \Pi_1 d(c) \Pi_2 \quad (3.4a)$$

where the  $\Pi_j$  ( $j = 1, 2$ ) are as in (3.4). Moreover, we note that  $W(z_i, c_{j_n}) \leq n - 1$ . So that either the F-simple commutators  $(z_i, c_{j_n})$  are all J-simple or we may use the induction hypothesis to write those  $(z_i, c_{j_n})$  which are not J-simple as words of the form (3.2) in J-simple commutators. After making the replacements for those  $(z_i, c_{j_n})$  in (3.4a) which are not J-simple, we find that in J

$$c = H_1 d(c) H_2 \quad (3.5)$$

where we have demonstrated that the  $H_j$  ( $j = 1, 2$ ) are words in J-simple commutators satisfying conditions (i) and (ii) of our conclusion.

To see that our expression for c in (3.5) also satisfies condition (iii) of our conclusion, we first note that the word w as given in (3.4a) has  $W(w) \geq n$  in F. (This is true since  $w = (H_{11} d(e) H_{12}, c_{j_n})$  in F and  $W(H_{11} d(e) H_{12}) \geq n - 1$ .) Thus if all the  $(z_i, c_{j_n})$  satisfy Criterion 3, i.e., are J-simple, we are done. Otherwise let  $\Sigma$  be set of all the distinct F-simple basic commutators which occur in w as given by (3.4a). Let  $U = (u_1, \dots, u_m) \subseteq \Sigma$  be those  $(z_i, c_{j_n})$  in (3.4a) which are not

J-simple and let  $\{v_1, \dots, v_m\}$  be those words in J-simple commutators which are obtained by applying the induction hypothesis to the  $(z_i, c_{j_n})$ . Thus  $u_i = v_i$  in J and  $W(u_i) \leq W(v_i)$  in F for all  $i = 1, \dots, m$ . Now to go from (3.4) to (3.5), we replace the  $u_i$  by the  $v_i$  in  $w$  as given by (3.4a), i.e., we apply the map  $\psi : u_i \rightarrow v_i$ . Thus since  $w \in F_n$ , Lemma 2.1 implies that

$$\psi(w) = H_1 d(c) H_2$$

in (3.5) is such that  $(H_1 d(c) H_2) \in F_n$ , i.e.,  $W(H_1 d(c) H_2) \geq n$ .

This completes the proof of our result in the case  $\delta > 0$ .

It remains to consider the case  $\delta = 0$  for which we write

$(c^L)^L = A$ ,  $(c^L)^R = c_{j_n} = g$ , and  $c^R = h$ . Then identity

(3.20)-I gives

$$c = (A, g, h) = (A, g)^{-1} (A, h)^{-1} A^{-1} (h, g)$$

$$A(A, g)(A, h)(A, h, g)(h, g)^{-1}. \quad (3.6)$$

Either  $(h, g) = 1$  in J or it is J-simple. Also all the commutators  $A$ ,  $(A, g)$ , and  $(A, h)$  are F-simple of weight  $< n$ . Hence any such commutator is either J-simple or can be written in J by the induction hypothesis as a word in J-simple commutators satisfying conditions (i), (ii), and (iii) of our conclusion. Letting

$$Z = (A, h, g)$$

in (3.6), we claim that is sufficient to prove that Z can be written in J as a word in J-simple commutators in the form

$$Z = H_{11} d(c) H_{12} \quad (3.7)$$

where the  $H_{1j}$  ( $j = 1, 2$ ) are words in  $J$ -simple commutators,  $u_i$ , satisfying (i) and (ii) of our conclusion, and

$$W(H_{11} d(c) H_{12}) \geq n.$$

To see this, suppose we substitute (3.7) and the expressions in terms of  $J$ -simple commutators (for those among  $A$ ,  $(A, g)$ , and  $(A, h)$  which are not  $J$ -simple) into (3.6). This writes  $c$  in  $J$  as

$$c = H_1 d(c) H_2 \quad (3.8)$$

where the  $H_j$  ( $j = 1, 2$ ) are words in  $J$ -simple commutators. Clearly, conditions (i) and (ii) of our conclusion are satisfied by (3.8), according to the induction hypothesis and our claim.

For condition (iii), we note that (3.6) is an identity in  $F$ . Thus the right hand side of (3.6) has weight  $n$  in  $F$  since  $c$  is  $F$ -simple of weight  $n$ . Now let  $(h, g) = d$ ,  $(A, g) = a$ , and  $(A, h) = b$ . Then (3.6) becomes

$$\begin{aligned} c &= a^{-1} b^{-1} A^{-1} d A a b Z d^{-1} \\ &= (a, b) (A a b, d^{-1}) (d Z d^{-1}). \end{aligned} \quad (3.6a)$$

By (2.3a-I),  $a \in F_{n-1}$ ,  $b \in F_{n-1}$ ,  $A \in F_{n-2}$ ,  $Z \in F_n$ , and  $d \in F_2$ .

(Either  $d = 1$  or  $W(d) = 2$ .) Let  $\tilde{a}$ ,  $\tilde{b}$ ,  $\tilde{A}$ ,  $\tilde{Z}$  be the words in  $J$ -simple commutators such that  $a = \tilde{a}$ ,  $b = \tilde{b}$ ,  $Z = \tilde{Z}$  in  $J$  (i.e.,  $\tilde{Z} = H_{11} d(c) H_{12}$  as in (3.7)). We know by the induction hypothesis

and our claim for  $Z$  that  $\tilde{a} \in F_{n-1}$ ,  $\tilde{b} \in F_{n-1}$ ,  $\tilde{A} \in F_{n-2}$ ,  $\tilde{Z} \in F_n$ .

Also  $\tilde{d} \in F_2$  since  $d$  either is 1 in  $J$  or it is a  $J$ -simple

commutator of weight 2. (Here  $\tilde{d} = d$  if  $d \neq 1$  in  $J$  but  $\tilde{d} = 1$  if  $d = 1$  in  $J$ .) Furthermore, the mapping of  $F \rightarrow J$  is a homomorphism; also the  $J$ -simple commutators are free generators of the subgroup of  $F_2$  which they generate by Corollary 4.1-I. Thus

$$\tilde{c} = (\tilde{a}, \tilde{b}) (\tilde{A} \tilde{a} \tilde{b}, \tilde{d}^{-1}) (\tilde{d} \tilde{Z} \tilde{d}^{-1})$$

is according to (2.3a)-I such that  $(\tilde{a}, \tilde{b}) \in F_{2n-2} \subseteq F_n$  (since

$2n - 2 > n$  in our case  $n > 3$ ),  $(\tilde{A} \tilde{a} \tilde{b}, \tilde{d}^{-1}) \in F_n$ , and

$\tilde{d} \tilde{Z} \tilde{d}^{-1} \in F_n$ . Thus  $W(\tilde{c}) \geq n$  in  $F$  where  $\tilde{c}$  is the unique word in

$J$ -simple commutators which =  $c$  in  $J$ . (Here  $\tilde{c} = H_1 d(c) H_2$  as in (3.8).)

Thus we now focus our attention on  $Z = (A, h, g)$ , and prove the claim for  $Z$  as given by (3.7). As already mentioned,  $b = (A, h)$  is, by the induction hypothesis, a word,  $\tilde{b}$ , in  $J$  of the form (3.2) in  $d(b)$  and  $J$ -simple commutators,  $s_i$ , of weight  $< n-1$  with all the special properties of our conclusion. Therefore  $Z = (b, g)$  can be written by the trivial identity (3.19)-I in  $J$  in the form

$$Z = (\tilde{b}, g) = H_{01} (d(b), g) H_{02} \quad (3.9)$$

where the  $H_{0j}$  ( $j = 1, 2$ ) are words in  $d(b)$ ,  $s_i$ , and  $(s_i, g)$ . By the

induction hypothesis  $W(\tilde{b}) \geq n - 1$ , hence (2.3a)-I implies that the right side of (3.9),  $H_{01}(d(b), g)H_{02}$ , has weight at least  $n$  in  $F$ .

Let us now examine  $(d(b), g)$ . From (3.1) and the fact that  $\delta = 0$ ,

$$d(b) = (c_{j_n}, e_2, e_3, \dots, e_{n-1})$$

where  $e_2 \leq e_3 \leq \dots \leq e_{n-1}$  is a rearrangement of

$c_{j_1}, c_{j_2}, \dots, c_{j_{n-2}}$ . (Evidently  $c_{j_2} = e_2$ , since  $c_{j_1} > c_{j_2} \leq c_{j_3} \leq \dots$

$\leq c_{j_n}$ .) We note from Lemma 3.5-I that  $M(d(b), g) = d(c)$ . Next, we

apply Lemma 2.2 to write  $(d(b), g)$  as  $\Pi_1 d(c) \Pi_2$  in  $F$ . (Note that

$g = c_{j_{n-1}} \geq e_2$  so that we are in Case I of Lemma 2.2 and also that if

$g \geq e_{n-1}$ , then  $(d(b), g) = d(c)$  and we just then take  $\Pi_1 = \Pi_2 =$

1.)

Let us also apply Lemma 2.2 to rewrite those  $(s_i, g)$  in (3.9)

which are not basic (i.e.,  $g < s_i^R$  so that Lemma 2.2 applies) in

terms of  $F$ -simple commutators. Thus

$$(s_i, g) = \Pi_{i1} M(s_i, g) \Pi_{i2}$$

in  $F$ . (Here it does not matter which case of (2.2) we are in

because  $W(s_i, g) < n$ .) Thus  $\Pi_1, \Pi_2$ , and  $\Pi_{ij}$  ( $j = 1, 2$ ) are words

in  $F$ -simple commutators,  $z_i$ , all of which have  $1 < W(z_i) < n$

and satisfy condition (ii) of our conclusion. Substituting these

into (3.9), we have that  $Z$  can be written in  $J$  as a word,  $w$ , in

$F$ -simple commutators  $d(c), s_i, (s_i, g)$  (only those which are basic

are included here), and  $z_i$ , i.e., in  $J$  we have

$$Z = w = \Omega_{01} d(c) \Omega_{02} \quad (3.10)$$

where  $\Omega_{0j}$  ( $j = 1, 2$ ) are words in the  $s_i$ ,  $(s_i, g)$  -- only the basic ones -- , and  $z_i$ . We may now use the induction hypothesis to write those  $F$ -simple commutators, the  $(s_i, g)$  and  $z_i$ , which occur in (3.10) and are not  $J$ -simple, as words in  $J$ -simple commutators. Making these replacements in (3.10) gives (3.7). To see this, we note that now  $Z$  is written, in  $J$ , in terms of  $J$ -simple commutators,  $u_i$ , which clearly satisfy (i) and (ii) of our conclusion.

To see that  $W (H_{11} d(c) H_{12}) \geq n$  in  $F$ , we first note that the right hand side of (3.9) is equal to the right hand side of (3.10) in  $F$ , i.e.,

$$w = H_{01} (d(b), g) H_{02}$$

holds in  $F$ . Thus  $w \in F_n$ . We can now continue in a manner exactly analogous to that used in the case of  $\delta > 0$ , because the situation in (3.10) is similar to that of equation (3.4a). In this way Lemma 2.1 shows that the weight of  $H_{11} d(c) H_{12}$  given in (3.7) is at least  $n$ . This completes the proof of our claim for  $Z$  given in (3.7) and with that the proof of our theorem is accomplished.

## CONCLUSION

Theorem 3.1 of this paper shows that in a group which is a free product of torsion free Abelian groups, here denoted by  $J$ , any  $F$ -simple commutator can be expressed in terms of  $J$ -simple commutators. It then follows from well known results [1] that the  $J$ -basic commutators of weight  $n$  form a basis for the quotient group  $J_n / J_{n+1}$ . Thus Theorem 3.1 completely determines these quotients.

## REFERENCES

1. A.M. Gaglione and H.V. Waldinger, Factor groups of the lower central series of free products of finitely generated Abelian groups, *Proc. Groups St. Andrews--1985*, Cambridge University Press, 164-203.
2. \_\_\_\_\_, A theorem in the commutator calculus, *Proc. Singapore Group Theory Conference 1987*, Walter de Gruyter, 367-371.
3. \_\_\_\_\_, The commutator collection process, to appear in *Proc. AMS Special Session on Combinatorial Group Theory*, (1988), Amer. Math. Soc.