Efficient Smoothing for Boundary Value Models*

Howard L. Weinert

Department of Electrical and Computer Engineering

The Johns Hopkins University

Baltimore, MD 21218

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Abstract

In this report, we investigate the smoothing problem for two-point boundary value models. We show that by triangularizing the relevant Hamiltonian system, we can derive a new smoothing algorithm which is twice as fast as the one which has been obtained by diagonalizing the Hamiltonian.

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1. Introduction

When modeling physical systems with a spatial independent variable, the *a priori* information is usually given in the form of boundary conditions rather than initial conditions. Consider the following linear, two-point boundary value model for $t_0 \le t \le t_1$:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$r = V_0 x(t_0) + V_1 x(t_1) \tag{2}$$

where u is a zero-mean, unit intensity white noise with m components, r is a zero-mean random vector with n components and covariance Q, and is uncorrelated with u, x is the n-component state vector, and A, B, V_0, V_1 are appropriate matrices. We have continuous measurements on the interval $[t_0, t_1]$ given by

$$y(t) = Cx(t) + v(t)$$

where v is a zero-mean, unit intensity white noise with p components and is uncorrelated with u and r.

As is customary, we will assume that the model is *well-posed*; i.e., that u and r give rise to a unique x. This will be the case if and only if the matrix $V_0 + V_1\Phi(t_1, t_0)$ is nonsingular, in which case we can assume, without loss of generality, that

$$V_0 + V_1\Phi(t_1, t_0) = I$$

where Φ is the state transition matrix of (1).

The problem of developing stable recursive algorithms for the linear least-squares smoothed estimate of the state x given the measurements y, and for the associated error covariance, has been addressed by several authors [1]-[5]. In these references, the Hamiltonian equations characterizing the smoothed estimate are solved by the use of a diagonalizing Riccati transformation. This transformation requires the solution of two Riccati equations. One can also solve the Hamiltonian equations by triangularizing them, a procedure that requires the solution of only one Riccati equation. Although triangularization was mentioned in [1] as an option, the necessary equations and a comparison with diagonalization were not provided. It turns out that triangularization leads to substantial savings in computation. In fact, as we will show, the number of required floating point operations is cut in half.

2. Triangularization and Shooting

The Hamiltonian equations for the smoothed estimate $\hat{x}$ are derived in various equivalent forms in [1]-[5]. The form we will use is given below:

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & BB' \\ C'C & -A' \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -C'y(t) \end{bmatrix}, \quad t \in [t_0, t_1] \tag{3}$$

$$\begin{bmatrix} V_0 & -Q \\ 0 & V_1' \end{bmatrix} \begin{bmatrix} \hat{x}(t_0) \\ \lambda(t_0) \end{bmatrix} + \begin{bmatrix} V_1 & Q\Phi'(t_1,t_0) \\ 0 & I - V_1'\Phi'(t_1,t_0) \end{bmatrix} \begin{bmatrix} \hat{x}(t_1) \\ \lambda(t_1) \end{bmatrix} = 0 \tag{4}$$

Now let N be the solution to the following Riccati equation:

$$\dot{N}(t) = -N(t)A - A'N(t) + N(t)BB'N(t) - C'C, \quad N(t_1) = 0 \tag{5}$$

The coordinate change

$$\begin{bmatrix} \hat{x}(t) \\ \rho(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ N(t) & I \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \lambda(t) \end{bmatrix}$$

transforms eqns. (3)-(4) into

$$\begin{bmatrix}\dot{\hat{x}}(t)\\ \dot{\rho}(t)\end{bmatrix}=\begin{bmatrix}A-BB'N(t) & BB'\\ 0 & -(A-BB'N(t))'\end{bmatrix}\begin{bmatrix}\hat{x}(t)\\ \rho(t)\end{bmatrix}+\begin{bmatrix}0\\ -C'y(t)\end{bmatrix},\ t\in[t_0,t_1] \tag{6}$$

$$\begin{bmatrix}V_0+QN(t_0) & -Q\\ -V_1'N(t_0) & V_1'\end{bmatrix}\begin{bmatrix}\hat{x}(t_0)\\ \rho(t_0)\end{bmatrix}+\begin{bmatrix}V_1 & Q\Phi'(t_1,t_0)\\ 0 & I-V_1'\Phi'(t_1,t_0)\end{bmatrix}\begin{bmatrix}\hat{x}(t_1)\\ \rho(t_1)\end{bmatrix}=0 \tag{7}$$

Eqns. (6)-(7) constitute a triangularized two-point boundary value problem which can be solved by a standard shooting procedure. Of course, shooting could be applied directly to eqns. (3)-(4), but the resulting system would not be stable. The point of using a Riccati transformation is to produce stable differential equations. If (A, B) is stabilizable and (A, C) is detectable, then $(A\text{-}BB'N)$ will be stable in the direction of increasing t.

To solve eqns. (6)-(7), first solve the following equations:

$$\dot{\hat{x}}_0(t)=(A-BB'N(t))\hat{x}_0(t)+BB'\rho_0,\quad \hat{x}_0(t_0)=0 \tag{8}$$

$$\dot{\rho}_0(t)=-(A-BB'N(t))'\rho_0(t)-C'y(t),\quad \rho_0(t_1)=0 \tag{9}$$

Then it can easily be checked by differentiation that

$$\hat{x}(t)=\hat{x}_0(t)+\Psi(t,t_0)\hat{x}(t_0)+M(t)\Psi'(t_1,t)\rho(t_1) \tag{10}$$

$$\rho(t)=\rho_0(t)+\Psi'(t_1,t)\rho(t_1) \tag{11}$$

where Ψ is the state transition matrix of $(A\text{-}BB'N)$ and

$$\dot{M}(t)=(A-BB'N(t))M(t)+M(t)(A-BB'N(t))'+BB',\quad M(t_0)=0 \tag{12}$$

Plugging $\hat{x}(t_1)$ and $\rho(t_0)$ obtained from eqns. (10)-(11) into eqn. (7) gives

$$\left[I+\begin{bmatrix}-Q & V_1\\ V_1' & 0\end{bmatrix}\begin{bmatrix}-N(t_0) & \Psi'(t_1,t_0)-\Phi'(t_1,t_0)\\ \Psi(t_1,t_0)-\Phi(t_1,t_0) & M(t_1)\end{bmatrix}\right]\begin{bmatrix}\hat{x}(t_0)\\ \rho(t_1)\end{bmatrix}=\begin{bmatrix}Q\rho_0(t_0)-V_1\hat{x}_0(t_1)\\ -V_1'\rho_0(t_0)\end{bmatrix} \tag{13}$$

The solution of this equation is then used in eqn. (10) to yield the smoothed state estimate. The well-posedness of eqns. (1)-(2) and the uniqueness of linear least-squares estimates guarantee that the coefficient matrix in eqn. (10) is nonsingular.

3. Error Covariance

The smoothing error $\tilde{x}$ satisfies [1], [5]:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ -\dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & BB' \\ C'C & -A' \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ -\lambda(t) \end{bmatrix} + \begin{bmatrix} Bu(t) \\ C'\upsilon(t) \end{bmatrix}, \ t \in [t_0, t_1] \tag{14}$$

$$\begin{bmatrix} V_0 & -Q \\ 0 & V_1' \end{bmatrix} \begin{bmatrix} \tilde{x}(t_0) \\ -\lambda(t_0) \end{bmatrix} + \begin{bmatrix} V_1 & Q\Phi'(t_1,t_0) \\ 0 & I - V_1'\Phi'(t_1,t_0) \end{bmatrix} \begin{bmatrix} \tilde{x}(t_1) \\ -\lambda(t_1) \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{15}$$

After the coordinate transformation

$$\begin{bmatrix} \tilde{x}(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ N(t) & I \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ -\lambda(t) \end{bmatrix}$$

eqns. (14)-(15) become triangularized as follows:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} A - BB'N(t) & BB' \\ 0 & -(A - BB'N(t))' \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} Bu(t) \\ N(t)Bu(t) + C'\upsilon(t) \end{bmatrix}, \ t \in [t_0, t_1] \tag{16}$$

$$\begin{bmatrix} V_0 + QN(t_0) & -Q \\ -V_1'N(t_0) & V_1' \end{bmatrix} \begin{bmatrix} \tilde{x}(t_0) \\ \xi(t_0) \end{bmatrix} + \begin{bmatrix} V_1 & Q\Phi'(t_1,t_0) \\ 0 & I - V_1'\Phi'(t_1,t_0) \end{bmatrix} \begin{bmatrix} \tilde{x}(t_1) \\ \xi(t_1) \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{17}$$

To proceed, we again use shooting. As before, we can write

$$\tilde{x}(t) = \tilde{x}_0(t) + \Psi(t,t_0)\tilde{x}(t_0) + M(t)\Psi'(t_1,t)\xi(t_1) \tag{18}$$

$$\xi(t) = \xi_0(t) + \Psi'(t_1,t)\xi(t_1) \tag{19}$$

where

$$\dot{\tilde{x}}_0(t) = (A - BB'N(t))\tilde{x}_0(t) + BB'\xi_0(t) + Bu(t), \ \ \tilde{x}_0(t_0) = 0 \tag{20}$$

$$\dot{\xi}_0(t) = -(A - BB'N(t))'\xi_0(t) + N(t)Bu(t) + C'v(t), \quad \xi_0(t_1) = 0 \tag{21}$$

Plugging $\tilde{x}(t_1)$ and $\xi(t_0)$ obtained from eqns. (18)-(19) into eqn. (17) gives

$$F\begin{bmatrix}\tilde{x}(t_0)\\ \xi(t_1)\end{bmatrix} = \begin{bmatrix}r\\ 0\end{bmatrix} + W\begin{bmatrix}\tilde{x}_0(t_1)\\ \xi_0(t_0)\end{bmatrix}$$

where F is the coefficient matrix in eqn. (13) and

$$W = \begin{bmatrix}-V_1 & Q\\ 0 & -V_1'\end{bmatrix}$$

As a result, we can rewrite eqn. (18) as

$$\tilde{x}(t) = \tilde{x}_0(t) + R(t)F^{-1}\left[\begin{bmatrix}r\\ 0\end{bmatrix} + W\begin{bmatrix}\tilde{x}_0(t_1)\\ \xi_0(t_0)\end{bmatrix}\right] \tag{22}$$

where

$$R(t) = \begin{bmatrix}\Psi(t,t_0) & M(t)\Psi'(t_1,t)\end{bmatrix}$$

We are now in a position to determine the error covariance from eqns. (20)-(22).

First, using standard techniques, one can show that if

$$\Sigma_{\xi_0}(t) = E[\xi_0(t)\xi_0'(t)]$$

then

$$\dot{\Sigma}_{\xi_0}(t) = -(A - BB'N(t))'\Sigma_{\xi_0}(t) - \Sigma_{\xi_0}(t)(A - BB'N(t)) - N(t)BB'N(t) - C'C, \quad \Sigma_{\xi_0}(t_1) = 0$$

But in fact, since their difference satisfies a homogeneous linear differential equation with zero initial condition,

$$\Sigma_{\xi_0}(t) \equiv N(t)$$

and thus

$$E[\xi_0(t)\xi_0'(s)] = \begin{cases} \Psi'(s,t)N(s), & t < s \\ N(t)\Psi(t,s), & t > s \end{cases} \tag{23}$$

It is shown in [6] that the process $u(t)+B'\xi_0(t)$ is white. Therefore, if

$$\Sigma_{\tilde{x}_0}(t) = E[\tilde{x}_0(t)\tilde{x}_0'(t)]$$

then, referring to eqn. (12),

$$\Sigma_{\tilde{x}_0}(t) \equiv M(t)$$

and

$$E[\tilde{x}_0(t_1)\tilde{x}_0'(t)] = \Psi(t_1,t)M(t)$$

One can also check, using eqns. (20)-(21), (23) that

$$E[\tilde{x}_0(t)\xi_0'(t_0)] \equiv 0$$

Taking the covariance of both sides of eqn. (22), we get

$$\begin{aligned} P(t) &= E[\tilde{x}(t)\tilde{x}'(t)] \\ &= M(t) + R(t)F^{-1}\begin{bmatrix} Q + QN(t_0)Q + V_1M(t_1)V_1' & -QN(t_0)V_1 \\ -V_1'N(t_0)Q & V_1'N(t_0)V_1 \end{bmatrix}F'^{-1}R'(t) \\ &\quad -R(t)F^{-1}\begin{bmatrix} V_1\Psi(t_1,t)M(t) \\ 0 \end{bmatrix} - [M(t)\Psi'(t_1,t)V_1' \quad 0]F'^{-1}R'(t) \end{aligned} \tag{24}$$

4. Conclusions

We have derived a new smoothing algorithm for two-point boundary value models based on triangularizing the Hamiltonian system. If one carefully tabulates the matrix multiplications needed to determine all intermediate quantities, one finds that for each point t at which the smoothed estimate and its error covariance must be computed, $18n^3$ floating point operations are required (terms of order n^2 or less have been ignored).

Alternatively, for the algorithm detailed in [1], $38n^3$ floating point operations are required. Our algorithm is therefore slightly more than twice as fast.

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