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Some Operational Formulations For Computational Solutions Of Underwater Acoustic Scattering Problems

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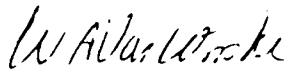
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Naval Underwater Systems Center
Newport, Rhode Island/New London, Connecticut

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Preface

This document was prepared for the Office of the Associate Technical Director for Technology by Dr. David Middleton.

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Scattering from ocean wave surfaces, volumes, and bottoms remains a perennial problem not only in underwater acoustics but also in the use of acoustical methods in remote sensing and modeling of the medium itself. Although analytic solutions are sometimes available, cf. [1], [6], in many important cases, particularly those involving multiple scatter and non-isogradient ($\nabla c \neq 0$) phenomena including the rôle of both stationary and moving arrays, and general geometries, effective modeling and quantitative, predictive results ultimately require computational methods. This is true for both deterministic and random media.

It is the aim of this introductory study to present a combined analytical-operational framework from which computational solutions to underwater acoustic scattering problems, including relevant ocean boundary interactions, can be initiated and carried out. It is emphasized that here explicit computational procedures are *not* developed. Rather, the specific aim is to present (cont'd)

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the "macro-algorithms" from and to which appropriate software may be designed and applied.
This is done by providing formal operational structures (with some special cases in which analytic
solutions are obtainable). A collateral aim of this work is to entice interest in pursuing
computational formulations, based on the various formal operational "solutions" derived here for
both deterministic and general random acoustical (i.e., scalar) fields, $\alpha(R,t)$.

For inhomogeneous deterministic media ($\nabla c \neq 0$; $c = c(r,t) = c_0 + \hat{c}(r,t)$, with $\hat{c}$
nonrandom, for example), operational solutions may be derived directly: all representations of the
medium (and boundaries) are equivalent. The resulting solutions provide a possible alternative
approach to the current "standard" methods involving partial or parabolic equations (PE)
developments [7], [8], at the same time including general arrays, geometries, and signals.

On the other hand, most of the scattering mechanisms in the ocean are essentially random.
Accordingly, in these cases, scattering "solutions" are based on the appropriate Langevin equation
governing propagation, namely, on the ensemble of representations $\{\alpha(R,t)\}$ for which various
appropriate probability measures (often nongaussian) are developed, based on both the physics,
geometry, and the random mechanisms involved. Thus, practical solutions here are the various
moments $<\alpha>$, $<\alpha_1\alpha_2>$, etc., and pdf's $w_1(\alpha)$, $w_1(\alpha_1,\alpha_2)$, etc., which are needed, for example,
in the calculation of scattering cross sections and subsequent signal processing for target detection
and remote sensing [2]–[4].

The desired framework is obtained by recognizing that both the deterministic and random
scattered fields in these linear media are representable as four-dimensional feedback systems, with
the inhomogeneous portions (including the ocean interfaces) constituting the feedback operations.
New features of this work include a combination of diagram methods, perturbational operational
series, and equivalent (i.e., deterministic) media operators, which then provide the needed
structure for both the direct and moment solutions α, or $<\alpha_1>$, etc., independently, with the latter
based on the resultant Dyson and Bethe-Salpeter forms, for $<\alpha>$, $<\alpha_1\alpha_2>$, etc.[2], [3]. Various
analytical, iterative, and approximative examples and procedures are briefly presented, to illustrate
the desired formal solutions and to provide elements of a framework for computational
implementation.

The interfacing between these formally analytic solutions, their software realizations, and
the acquisition of (nearly) on-line numerical results remain a major challenge.

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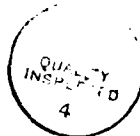
Some Operational Formulations for Computational Solutions of Underwater Acoustic Scattering Problems[*]

by

David Middleton[**]

Abstract

Scattering from ocean wave surfaces, volumes, and bottoms remains a perennial problem not only in underwater acoustics but also in the use of acoustical methods in remote sensing and modeling of the medium itself. Although analytic solutions are sometimes available, cf. [1], [6], in many important cases, particularly those involving multiple scatter and non-isogradient ($\nabla c \neq 0$) phenomena including the rôle of both stationary and moving arrays, and general geometries, effective modeling and quantitative, predictive results ultimately require computational methods. This is true for both deterministic and random media.

It is the aim of this introductory study to present a combined analytical-operational framework from which computational solutions to underwater acoustic scattering problems, including relevant ocean boundary interactions, can be initiated and carried out. It is emphasized that here explicit computational procedures are *not* developed. Rather, the specific aim is to present the "macro-algorithms" from and to which appropriate software may be designed and applied. This is done by providing formal operational structures (with some special cases in which analytic solutions are obtainable). A collateral aim of this work is to entice interest in pursuing computational formulations, based on the various formal operational "solutions" derived here for both deterministic and general random acoustical (i.e., scalar) fields, $\alpha(R,t)$.

For inhomogeneous deterministic media ($\nabla c \neq 0$; $c = c(r,t) = c_0 + \hat{c}(r,t)$, with $\hat{c}$ *nonrandom*, for example), operational solutions may be derived directly: all representations of the medium (and boundaries) are equivalent. The resulting solutions provide a possible alternative approach to the current "standard" methods involving partial or parabolic equations (PE) developments [7], [8], at the same time including general arrays, geometries, and signals.

On the other hand, most of the scattering mechanisms in the ocean are essentially random. Accordingly, in these cases, scattering "solutions" are based on the appropriate Langevin equation governing propagation, namely, on the ensemble of representations $\{\alpha(R,t)\}$ for which various appropriate probability measures (often nongaussian) are developed, based on both the physics,

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geometry, and the random mechanisms involved. Thus, practical solutions here are the various moments $\langle\alpha\rangle$, $\langle\alpha_1\alpha_2\rangle$, etc., and pdf's $w_1(\alpha)$, $w_1(\alpha_1,\alpha_2)$, etc., which are needed, for example, in the calculation of scattering cross sections and subsequent signal processing for target detection and remote sensing [2]–[4].

The desired framework is obtained by recognizing that both the deterministic and random scattered fields in these linear media are representable as four-dimensional feedback systems, with the inhomogeneous portions (including the ocean interfaces) constituting the feedback operations. New features of this work include a combination of diagram methods, perturbational operational series, and equivalent (i.e., deterministic) media operators, which then provide the needed structure for both the direct and moment solutions α, or $\langle\alpha_1\rangle$, etc., independently, with the latter based on the resultant Dyson and Bethe-Salpeter forms, for $\langle\alpha\rangle$, $\langle\alpha_1\alpha_2\rangle$, etc.[2], [3]. Various analytical, iterative, and approximative examples and procedures are briefly presented, to illustrate the desired formal solutions and to provide elements of a framework for computational implementation.

The interfacing between these formally analytic solutions, their software realizations, and the acquisition of (nearly) on-line numerical results remain a major challenge.

1. Introduction

Scattering from ocean wave surfaces, volumes, and bottoms is an enduring, central problem not only in underwater acoustics but also in the use of acoustical methods in remote sensing and modeling of the medium itself. Fortunately, in a number of important applications (where first-order Born approximations are adequate), analytic solutions are available (see, for example, recent work of the author [1], [4]).

In many cases, however, strictly analytic methods fail, primarily because of multiple scatter phenomena, which though linear, destroy the reciprocal character of conventional solutions (for ideal media). Moreover, the requirement of effective modeling imposes further complexities: the field nature of the solutions must include the rôle of the transmitting and receiving arrays whereby the system sensors and signal processing itself are suitably coupled to the medium in question, cf. Fig. 1.1 ff. In addition, doppler and gradient effects may also be important. For short ranges velocity gradients in the medium can often be neglected (i.e., $\nabla c \doteq 0$), but for longer ranges they must be carefully taken into account ($\nabla c \neq 0$). Furthermore, geometry plays a critical rôle: platform motion, beam size, structure, and direction, as well as location in the medium, all dictate potential interactions with the wave-surface and bottom boundaries, as Fig. 2.2 indicates, along with the inherently random inhomogeneous effects of the volume itself [3], [4].

We distinguish two main areas of interest: **A**. *deterministic inhomogeneous media*, where

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random variations are largely ignorable vis-à-vis the large-scale deterministic variations, and **B.**
random inhomogeneous media, for which significant media variations are stochastic, with and
without deterministic components. The former class (A) appears to be so far the one which has
received the greatest attention with respect to numerical methods [7], [8], and [9], [10], with
attention largely centered on the purely spatially dependent velocity gradient $\nabla c(r,z) = c_0 + \tilde{\alpha}(r,z)$.
Here, however, in advance of current efforts, we shall direct our attention to the more complex
situation (B), where in addition to deterministic variations of ∇c, there are important random ones
as well. In fact, our treatment will proceed in a general canonical way, allowing for both the
deterministic and random variations in the medium, including its important boundaries.

1.1 Aims of the Paper

Before numerical procedures can be applied, however, it is necessary to develop the
appropriate analytical forms of solution from which in most cases numerical results must be
extracted. The aims of this paper, therefore, are to outline an analytical-operational framework for
initiating computational solutions to underwater acoustic scattering problems, in particular, which
include the relevant ocean surface and bottom boundaries, as well as volume effects.

We emphasize that this paper does not intend to provide explicit computational procedures
and solutions: it is rather aimed at providing the "macro-algorithms" from and to which appropriate
software may be designed and applied. Our aims include providing formal operational structures,
with some special cases in which analytic solutions are obtainable, and which consequently require
a lower order of computational effort and sophistication. A collateral aim of this work is to entice
interest in pursuing a computational formulation, based on the various formal operational
"solutions" derived here for the composite deterministic and random fields $\alpha(R,t)$, namely $<\alpha>$,
cf. (2.33), and higher-order moments $<\alpha_1 \alpha_2>$, etc. Our approach here is a "top-down" approach,
in which we start with as general a formulation as possible, consistent with the broad class of
problems embodied in the propagation and processing schema of Fig. 1.1 and the geometries of
Fig. 2.1 ff. We proceed then to more specialized cases. This reveals the embedding of the latter in
the generality of the former, and thus provides a "perspective" for specific medium models.

1.2 New Elements

Among the new features of the present paper are

(1) Formal operational formulations which can provide the basis for a large-scale
computational attack on these four-dimensional scattering problems, Sec. 2.

(2) A feedback operational method for obtaining the (first-order) equivalent medium
renormalization operator, cf. (2.34), by means of a converging series of estimates,
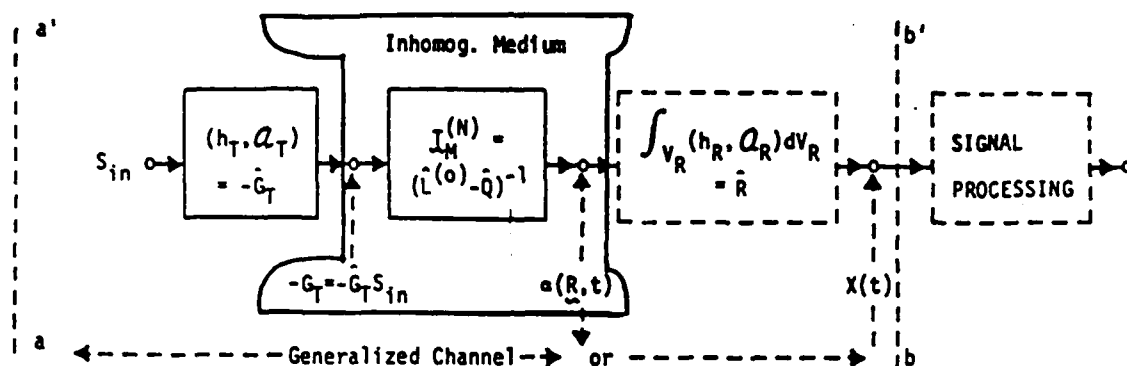Sec. 3.3.

Fig. 1.1   *Operational schematic of an inhomogeneous (linear)medium, with source and receiver coupling. Here $\hat{Q}$ is the inhomogeneity operator associated with this general inhomogeneous (linear) medium.*

(3)   Some fully analytic solutions in important special cases, e.g., infinite or "weakly-bounded" media, where in effect there is negligible interaction between sources, boundaries, and among the different scattering modes (Surface, Volume, Bottom), Sec. 2.7.

(4)   Various explicit physical scattering models for the inhomogeneity operator $\hat{Q}$, including the important wind-wave scattering cases, Eqs. (2.31).

(5)   A hierarchy of approximations for the equivalent deterministic inhomogeneity operator $\hat{Q}^{(d)}$, which apply in different physical situations, Sec. 2.5.

1.3 Organization

Accordingly, the present paper is organized in the following general but concise fashion: Section 2 introduces formal operational solutions, which form the basis for the treatment of both deterministic and random media. In Section 3 various diagram methods and solution equivalents are presented, first for the deterministic cases and then for the many situations involving random media components. Useful approximations are next briefly described in this section, along with a possible method for estimating the renormalization operators. Section 4 completes the paper with a

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discussion of the interfacing problems between the analytic/operational and computational formulations by reviewing via the text the main steps leading up to the as yet unachieved software realizations.

2. Formal Operational Solutions

Accordingly, the above discussion indicates the critical need, at some point, for numerical procedures to obtain the needed quantitative description of the received acoustic field, for whatever operational or oceanographic purpose at hand.

What are the needed descriptors of this received field? This will, of course, depend on whether or not there are significant random, as well as deterministic inhomogeneities in the medium. Here we shall assume a (linear) combination of both types, which permits us to focus on one or the other as limiting cases. Thus, in general we can write formally for the inhomogeneity operator $\hat{Q}$, cf. Fig. 1.1

$$\hat{Q} = \hat{Q}^{(h)} + \hat{Q}^{(I)}, \tag{2.1}$$

where $\hat{Q}^{(h)}$, $\hat{Q}^{(I)}$ are respectively associated with the deterministic and random parts. For example, $\hat{Q}^{(h)}$ usualy dominates in a nonzero-gradient ocean with small surface interactions (cf. some of the current treatments [7]–[10]), while $\hat{Q}^{(I)}$ is most significant in surface or bottom scatter situations, at comparatively short ranges and high frequencies, where $\nabla c \doteq 0$ often, cf. [1], [4].

Now, since the medium, including its boundaries, is both spatially and temporally stochastic in general, "solutions" to the appropriate propagation equations for the input, incident, and scattered fields are necessarily *statistics* of these fields, namely, means, variances, covariances, higher-order moments, and finally, various orders of the *probability distributions* of these fields. Thus, if $\alpha(R,t)$ represents the (acoustic) field at point (R,t) in this random, inhomogeneous medium, one seeks $<\alpha>$, $<\alpha_1\alpha_2>$ (= $<\alpha(R_1,t_1)\alpha(R_2,t_2)>$), $<\alpha_1\alpha_2\alpha_3>$, etc., and more completely, $w_1(\alpha_1)$, $w_2(\alpha_1,\alpha_2)$, etc. The lower order moments, for example, $<\alpha_1>$, $<\alpha_1\alpha_2>$, are needed in obtaining the scattering cross-sections of the "illuminated" wave-surface or bottom, cf. Fig. 2.1, while $w_1(\alpha_1)$ is required for signal processing (cf. Part II of [4]). What makes the latter critically non-trivial is the fact that α is often strongly *nongaussian* (cf. VI of [4]).. Even in the cases where the Central Limit Theorem applies (cf. Sec. 7.7 of [5], for example), so that the field is gaussian, one still needs the lower-order moments $<\alpha>$, $<\alpha_1\alpha_2>$, in order to specify the gaussian process, $\alpha(R,t)$.

2.1 Formal Development

Clearly, for realistic solutions, many complicating effects must be included, which almost always at some point makes a computational effort essential. Initially, one starts with a
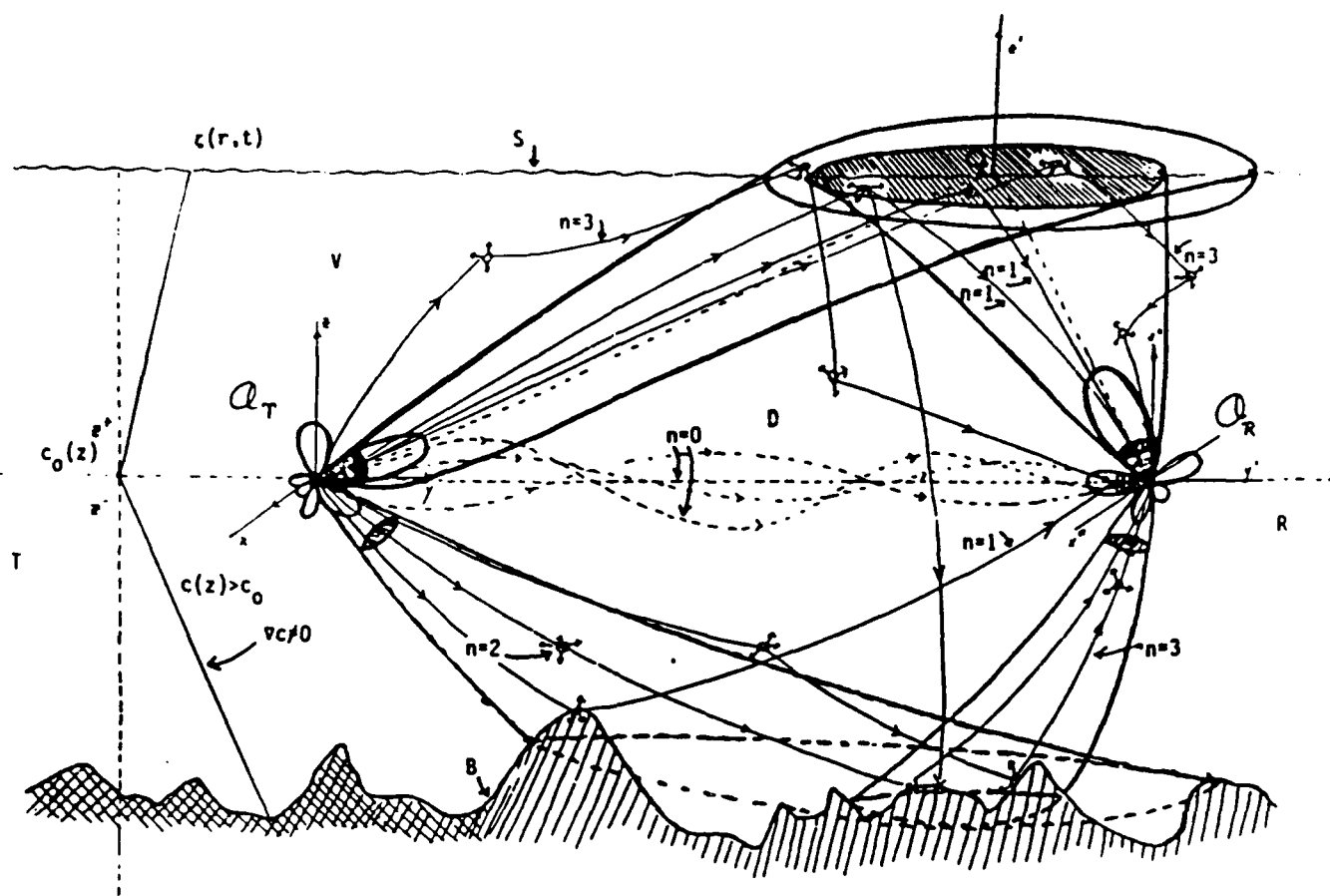
Fig. 2.1 *Schematic scatter channel geometry of source (T), medium (V), and boundaries (S,B),
with receiver (R), D = unscattered, or "direct" path, and n is the order of the interactions.
Sufficiently high frequencies are assumed to permit illustrative ray paths.*

propagation equation which at least implicitly contains the major physical phenomena governing
the generation and development of both the scattered and unscattered fields. Formally, for the
linear media assumed here, the propagated field $\alpha(R,t)$ obeys a partial integro-differential equation
of the form

$$(\hat{L}^{(0)} - \hat{Q})\alpha = -G_T + [\text{b.c.'s} + \text{i.c.'s}].$$

(2.2)

Here G_T is the source function, e.g., $G_T = \hat{G}_T S_{in}$, cf. Fig. 1.1, and $\hat{L}^{(0)}$ is a linear (scalar) partial
differential operator, where $\hat{L}^{(0)}$ is associated with the homogeneous portion of the medium. In

(2.2), $\hat{Q}$ is, generally, a (scalar, linear) integro-differential scattering operator, which describes the interaction of the incident, i.e., homogeneous field with the differential, or local, scattering elements of the inhomogeneous portion of the medium. (The possible integral operator component of $\hat{Q}$ stems from local scatterer doppler effects, for example.) Here, *b.c.'s* = boundary conditions, and *i.c.'s* = initial conditions. Since, formally from (2.2) $\alpha = (\hat{L}^{(0)} - \hat{Q})^{-1}(-G_T)|_{b.c.+i.c.}$, from $\alpha = T_M^{(N)}(-G_T)$, cf. Fig. 1.1, one has directly

$$ T_M^{(N)} = (\hat{L}^{(0)} - \hat{Q})^{-1} \tag{2.3} $$

for the medium operator $T_M^{(N)}$, now expressed in terms of the homogeneous and inhomogeneous components of the medium.

Because of the usual statistical nature of $\hat{Q}$, Eq. (2.2) is really an ensemble of equations, which is called the (associated) *Langevin Equation* [cf. Chapter 10 of [5]], which can be written as

$$ \{(\hat{L}^{(0)} - \hat{Q})\alpha = -G_T + (\text{b.c.'s} + \text{i.c.'s})\}, \tag{2.4} $$

where the brackets { } emphasize the ensemble character of this Langevin equation. In subsequent discussion the brackets may be omitted for convenience, it being noted that $\hat{Q}$, and therefore α, are stochastic. (See Sec. IIB of [4] for remarks.)

An important example for acoustic propagation in underwater media is the case of inhomogeneous absorptive media [3], for which (2.2) is specifically [11]

$$ \left\{ \left(1 + \tau_x(r,t)\frac{\partial}{\partial t}\right)\nabla^2 - \frac{1}{c_0^2}[(1 + \varepsilon(R,t)]\frac{\partial^2}{\partial t^2} \right\} \alpha = -G_T \tag{2.5a} $$

where

$$ \therefore \hat{L}^{(0)} \equiv \left(1 + \tau_{ox}\frac{\partial}{\partial t}\right)\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} $$

$$ \hat{Q} \equiv \frac{\varepsilon}{c_0^2}\frac{\partial^2}{\partial t^2} - \tau_{ox}\gamma_x\frac{\partial}{\partial t}\nabla^2 \tag{2.5b} $$

with $\tau_x \equiv \tau_{ox}(1 + \gamma_x[r,t])$ generally. When $\tau_{ox} = 0$, (2.5a) reduces to the more familiar "extended" Helmholtz equation

$$ \left\{ \nabla^2 - \frac{1}{c_0^2}[(1 + \varepsilon(R,t)]\frac{\partial^2}{\partial t^2} \right\} \alpha = -G_T \tag{2.6} $$

with

$$\hat{Q} = \frac{\varepsilon}{c_0^2}\frac{\partial^2}{\partial t^2}; \quad \hat{L}^{(0)} \equiv \nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}, \qquad (2.6a)$$

where

$$\mathbf{R} = \hat{i}_x x + \hat{i}_y y + \hat{i}_z z, \qquad (2.7)$$

and rectangular coördinates are assumed throughout, so that $\nabla \cdot \nabla = \nabla^2$, etc. Specifically, $c_0 =$ (constant) wavefront eed, ε embodies the effects of velocity gradients, internal wave phenomena, and/or (k) local turbulence, while τ_x represents the effects of relaxation absoprtion (due to Mg_2SO_4 and other salts).

Similarly, in the more restricted situations where (2.6) is replaced by a purely deterministic model in which $\varepsilon(\mathbf{R},t) \rightarrow \varepsilon(\mathbf{R})$ is nonrandom, this Helmholz equation can be approximated (at a given frequency f_0) by a basic form of the *parabolic equation* (PE), which is espressed by (Eq. (2.34), [8]), in cylindrical coördinates [$\mathbf{R} = (r,z) = (r,\theta,z)$]:

$$\frac{\partial \alpha_0}{\partial r} = \left(A + B\frac{\partial^2}{\partial z^2}\right)\alpha_0 \; ; \quad \alpha_0 = \alpha_0(\mathbf{R}), \qquad (2.8)$$

where specifically

$$A \equiv \frac{i}{2}k_0\,[n^2(r,z) - 1]; \quad B \equiv \frac{i}{2k_0} \; ; \quad k_0 = \omega_0/c_0 = 2\pi f_0/c_0, \qquad (2.8a)$$

$$n^2 = (1 + \varepsilon). \qquad (2.8b)$$

Comparing (2.2) and (2.8) gives at once here

$$\hat{L}^{(0)} = B\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial r}; \quad \hat{Q} = A_\varepsilon \hat{I} = [-ik_0\varepsilon/2]\hat{I} \qquad (2.9)$$

where $\hat{I} =$ unit operator, $\hat{I}C = C$, etc., and $\hat{L}^{(0)}\,\alpha_0 = 0$ is an equation of the Schrödinger type since B is purely imaginary.

The formal solution of the Langevin equation (2.4) is obtained from the equivalent operational form

$$(\hat{I} - \hat{M}\hat{Q})\alpha = \hat{M}(-G_T) \; ; \quad \alpha = \alpha_H + \alpha_I; \quad \alpha_H \equiv \hat{M}(-G_T), \qquad (2.10)$$

where α_H is the homogeneous (i.e., non-scattered) component of the total field α, with α_I the associated inhomogeneous (scattered) field. $\hat{M}$ is $\hat{L}^{(0)-1}$, with the boundary and initial conditions; in fact, $\hat{M}$ is the integral Green's function operator

$$\hat{M}(R,t \mid R',t') \equiv - \int_{-\infty}^{\infty} dt' \int_{(\infty)} dR' g(R,t \mid R',t') \; (\;)_{R',t'} \tag{2.11}$$

obtained from $\hat{L}^{(0)}g = -\delta(R-R') \; \delta(t-t')$. The formal operational solution of (2.10) now becomes

$$\boxed{\alpha = (\hat{1} - \hat{\eta})^{-1}\alpha_H, \quad \hat{\eta} \equiv \hat{M}\hat{Q}.} \tag{2.12}$$

Here $\hat{\eta}$ is the *field renormalization operator* (FRO), and (2.12) represents the *Feedback Operational Solution* (FOS) for α, c.f. remarks following Eq. (2.35) et seq. Expansion of (2.12) yields the *Perturbation Theoretical Series Solution* (PTSS)

$$\boxed{\alpha = \alpha_H + \sum_{n=1}^{\infty} \hat{\eta}^{(n)}\alpha_H \; ; \quad \|\hat{\eta}\| < 1,} \tag{2.13}$$

where $\hat{\eta}^{(n)} = [\hat{\eta}\hat{\eta}...\hat{\eta}]_n$ is the nth-iteration of $\hat{\eta}$ and where the operator formalism used here is represented explicitly by

$$(\hat{A})^{(n)} = \hat{A}^{(n)} \equiv \hat{A}(R,t \mid R',t') \; \hat{A}(R',t' \mid R'',t'') \; ... \; \hat{A}(R^{(n-1)},t^{(n-1)} \mid R^{(n)}t^{(n)}) \tag{2.13a}$$

which is generally distinct from *the direct operator product*

$$\hat{A}^n \equiv \hat{A}(R,t \mid R',t') \; \hat{A}(R,t \mid R'',t'') \; ... \; \hat{A}(R,t \mid R^{(n)},t^{(n)}) \tag{2.13b}$$

and its more general form

$$\hat{A}_1 ... \hat{A}^n = \hat{A}(R,t \mid R',t') \; ... \; \hat{A}(R_n,t_n \mid R^{(n)},t^{(n)}). \tag{2.13c}$$

We have also the conventions

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$$\hat{A}_1\hat{B}_1 = \hat{A}(\mathbf{R}_1,t_1|\mathbf{R}',t')\hat{B}(\mathbf{R}_1,t_1|\mathbf{R}'',t'')$$

$$\left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\} \quad (2.13d)$$

$$\hat{A}_1\hat{B}_2 = \hat{A}(\mathbf{R}_1,t_1|\mathbf{R}',t')\hat{B}(\mathbf{R}_2,t_2|\mathbf{R}'',t'')$$

so that clearly $\hat{A}_1\hat{B}_2 \neq \hat{B}_2\hat{A}_1$, i.e., $\hat{A}_1$, $\hat{B}_2$ do not commute, unless the operand is symmetric.

2.2 Some Homogeneous Methods and Results

For the usual cases where there is negligible interaction of the medium and boundaries upon the *source*, we can replace $\hat{M}$ by $\hat{M}_\infty$, where $\hat{M}_\infty$ is the integral Green's function operator for the associated infinite homogeneous medium. Then, also, we have $\hat{\eta} \rightarrow \hat{\eta}_\infty = \hat{M}_\infty\hat{Q}$, cf. (2.12).

Next, to proceed further it is clear from (2.12) that we must determine the homogeneous field α_H, $\hat{\eta}_\infty = 0$, or $\hat{Q} = 0$, now. In the light of the fact that media boundaries or interfaces can be equivalently considered as (distributed) inhomogeneities in an otherwise homogeneous medium, cf. Sec. 2.4 ff., to obtain solutions in the homogeneous cases we must therefore consider *unbounded* media, along with the condition $\hat{Q} = 0$. More precisely, by "unbounded" we mean the infinite spatial domain exclusive of sources.

The general method of solution here (including bounded media) involves two steps:

I. Obtain the associated Green's function, g, from (2.2), ($\hat{Q} = 0$);

II. Construct the corresponding GHP (Generalized Huygen's Principle) [20],

$$\left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\} \quad (2.14)$$

and thereby obtain the field, α. For unbounded homogeneous media this is easily done. Thus, the Green's function g_∞ is found from the defining relation

$$\boxed{\hat{L}^{(0)}g_\infty = -\delta(t-t')\delta(\mathbf{R}-\mathbf{R}'),} \quad (2.14a)$$

or

$$g(\mathbf{R},t \mid \mathbf{R}',t')_\infty = - \hat{M}_\infty\delta_{t't''}\delta_{\mathbf{R}'\mathbf{R}''}, \quad (2.14b)$$

where $\hat{M}_\infty$ is the inverse (integral) operator to $\hat{L}^{(0)}$ + i.c.'s, with $\hat{M}_\infty^{-1} = \hat{L}^{(0)}$. Integrating over $(\mathbf{R}',t')$ shows at once that $\hat{M}_\infty$ and g_∞ are related here by

$$\boxed{\begin{array}{c}\hat{M}(\mathbf{R},t \mid \mathbf{R}',t')_\infty = - \int_{-\infty}^{\infty}dt' \int_{(\infty)}d\mathbf{R}'g(\mathbf{R},t|\mathbf{R}',t')_\infty (\ )_{\mathbf{R}',t'} \\[2mm] = 0, \quad t \leq t'- ,\end{array}} \quad (2.15)$$

where, by causality, $g_\infty = 0$, $t \le t'-$, which represents the initial condition (i.c.) on the "point"-source in (2.14a). Thus, $\hat{M}_\infty$ is now the *Green's function operator*, with the kernel g_∞.

The desired field $\alpha(R,t)_\infty$ is found directly from (2.2) ($\hat{Q}$, b.c.'s = 0), viz.:

$$\hat{L}^{(0)}\alpha_\infty = - G_T, \text{ or} \tag{2.16a}$$

$$\alpha(R,t)_\infty = \hat{M}_\infty(-G_T) = \int_{-\infty}^{\infty}dt' \int_{V_T}G_T(R',t')g(R,t \mid R',t')_\infty \, dR' \tag{2.16b}$$

$$= \int_{-\infty}^{\infty}dt' \, \hat{G}_T(g_\infty S_{in}) = \int_{-\infty}^{\infty}T_A T S_{in}dt', \tag{2.16c}$$

where $G_T = 0$, $t' \le t_0-$, $R \ne V_T$: the source function is initiated at $t' = t_0$ and occupies the domain $V_T(<\infty)$. The relations (2.16b,c) are the simplest form of GHP here for homogeneous unbounded media (outside the source domain).

Various transform equivalents of $\hat{M}_\infty$ prove useful. These may be expressed as follows, from (2.16b):

$$\hat{M}_\infty = -\mathcal{F}_s^{-1} \left\{ \int_{-\infty}^{\infty}dt' \int_{(\infty)}Y_0(R,s \mid R't')_\infty \, (\,)_{R',t'} \, dR' \right\} \equiv -\mathcal{F}_s^{-1} \{Y_{0,\infty}\} \tag{2.17a}$$

$$= -\mathcal{F}_k\mathcal{F}_s^{-1} \left\{ \int_{-\infty}^{\infty}dt' \int_{(\infty)}Y_0(k,s \mid R't')_\infty \, (\,)_{R',t'} \, dR' \right\} \equiv -\mathcal{F}_k\mathcal{F}_s^{-1} \{\hat{\mathcal{Y}}_{0,\infty}\} \tag{2.17b}$$

$$= -\mathcal{F}_k\mathcal{F}_s^{-1} \left\{ \mathcal{Y}_0(k,s)_\infty \int_{-\infty}^{\infty}dt' \int_{(\infty)}e^{-ik\cdot R'-st'} \, (\,)_{R',t'} \, dR' \right\} \tag{2.17c}$$

where (2.17a–c) define the transformed operators $\hat{Y}_{0,\infty}$, $\hat{\mathcal{Y}}_{0,\infty}$ $(= \hat{\mathcal{Y}}_0(k,s \mid R',t')_\infty)$ respectively, and where $\mathcal{F}_t\{\hat{M}_\infty\} = -\hat{Y}_{0,\infty}$, $\mathcal{F}_R^{-1}\mathcal{F}_t\{\hat{M}_\infty\} = -\hat{\mathcal{Y}}_0(k,s \mid R',t')$ are the corresponding inverses.

The general transform relations are explicitly

$$a(t-t') = \mathcal{F}_s^{-1} \{e^{-st'}A(s)\} = \int_{Br_1=-\infty i+d}^{\infty i+d(>0)}A(s)e^{s(t-t')}\frac{ds}{2\pi i} \ne 0, \quad t > t'-$$
$$= 0, \quad t \le t'- \tag{2.18a}$$

where s is a complex (angular) frequency, and provided all the singularities of A(s) lie in Re(s) < d; a = 0, t ≤ t'– is, of course, an initial condition (i.c.). The inverse of (2.18a) is

$$e^{-st'}A(s) = \mathcal{F}_t\{a(t-t')\} = \int_{-\infty}^{\infty} e^{-st}a(t-t')dt = e^{-st'}\int_{-\infty}^{\infty}e^{-s\tau}a(\tau)d\tau, \quad Re(s) > 0. \quad (2.18b)$$

In the steady-state régime (t' → –∞) we choose d = 0 and use s = iω = 2πif, where (ω,f) are real frequencies, with an appropriate indentation of the contour (Br₁, d=0) for any singularities on Re(s) = 0. The contour BR_1 is a Bromwich contour [26]. Similarly, $\mathcal{F}_R^{-1}$, $\mathcal{F}_k$ are spatial Fourier transforms, e.g.,

$$B(k) = \mathcal{F}_R^{-1}\{b(R)\} = \int_{[R]} e^{-ik\cdot R}b(R)dR ;$$

$$b(R) = \mathcal{F}_k\{B(k)\} = \int_{(k)} e^{ik\cdot R}B(k)\frac{dk}{(2\pi)^3} \quad (2.18c)$$

where, as before, cf. (2.7), $k = \hat{i}_xk_x + \hat{i}_yk_y + \hat{i}_zk_z$ (= 2πν) is a *vector wave number* (and ν is a *vector spatial frequency*). Multiple transforms of quantities like C(R,t) are handled in the same way, cf. (2.17b,c).

For those unbounded media—unbounded in the sense that a propagating wave front never encounters a bounding surface—the evaluation of the Green's function g_∞ from (2.14a) and the general field from (2.16) is readily accomplished. From (2.14a) one has

$$\mathcal{F}_R^{-1}\mathcal{F}_t\hat{L}^{(0)}\{\mathcal{F}_s^{-1}\mathcal{F}_k\{g_{0,\infty}\}\} = -e^{-ik\cdot R'-st'}, \quad (2.19a)$$

$$\therefore \; g_{0,\infty} = g_0(k,s\mid R',t')_\infty = -L_0(k,s)^{-1}e^{-ik\cdot R'-st'} , \quad (2.19b)$$

where $L_0(k,s)$ is the result of applying $\hat{L}^{(0)}$ to $\mathcal{F}_s^{-1}\mathcal{F}_k$, remembering that $\nabla e^{ik\cdot R} = ike^{ik\cdot R}$, $\nabla^2 e^{ik\cdot R} = -k\cdot ke^{ik\cdot R} = -k^2e^{ik\cdot R}$, $\frac{\partial}{\partial t}e^{st} = se^{st}$, etc. In fact, one has

$$\boxed{-L_0(k,s)^{-1} = \mathcal{Y}_0(k,s)_\infty} \quad (2.20)$$

cf. (2.17b,c) so that, from (2.19b),

$$g(R,t\mid R't')_\infty = \mathcal{F}_s^{-1}\mathcal{F}_k\{g_{0,\infty}\} = \int_{Br_1} e^{s(t-t')}\frac{ds}{2\pi i}\int_{[k]} e^{ik\cdot(R-R')}\{-L_0(k,s)^{-1}\}\frac{dk}{(2\pi)^3} . \quad (2.21a)$$

This can be put in the more convenient equivalent form for evaluation with the help of spherical coordinates: $k = (k, w\,[=\cos\theta], \phi)$, viz.:

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$$g(R,t \mid R't')_\infty = \int_{Br_1} e^{s(t-t')} \frac{ds}{2\pi i} \frac{1}{(2\pi)^3} \int_0^\infty k^2 dk \int_{-1}^1 dw \cdot \int_0^{2\pi} [-L_0(k,s)^{-1}] e^{ik|R-R'|w} d\phi \ . \qquad (2.21b)$$

A. *Conditions*:

Various conditions and properties of (2.21b) are now noted:

(i). *"Spatial causality"*: this is the *radiation condition* [15]

$$\lim_{R\to\infty} \left(\left\{ \begin{matrix} R \dfrac{\partial}{\partial R} \\ \\ Rik \end{matrix} \right\} + \dfrac{R}{R} \left\{ \begin{matrix} \dfrac{1}{c_0} \dfrac{\partial}{\partial t} \\ \\ \dfrac{s}{c_0} \end{matrix} \right\} \right) \left\{ \begin{matrix} g_\infty \\ \\ Y_{0,\infty} \end{matrix} \right\} = 0, \quad R = |R| , \qquad (2.22)$$

which ensures that only outgoing waves from the source (εV_T) are propagated. This spatial causality condition is used in the specific evaluation of $g_{0,\infty}$, (2.21b), by appropriate modifications C_k of the k-contour ($-\infty,\infty$) to include only singularities in the complex k-plane which yield the required outgoing waves.

(ii). *"Temporal causality"*: this is the well-known condition that propagation cannot occur before the source initiating it is activated, cf. (2.15), (2.18), or more precisely, that events cannot occur (macroscopically) at some point P(R,t) before the source has been initiated and the field generated in the medium has reached the point in question. For causality in time one has the *Paley-Wiener criterion* [5], pp. 96, 97, 102 and Refs. Here ($s \to i\omega$):

$$J \equiv \int_0^\infty \frac{\log|Y_0(R,i\omega|R')_\infty|^2}{1 + \omega^2} \, d\omega \ < \infty \ , \qquad (2.23)$$

where specifically

$$Y_0(R,s|R')_\infty = \mathcal{F}_t \{g(R,s|R',t')_\infty e^{st'}\} = Y_0(R,s|R',t')_\infty e^{st'}, \qquad (2.23a)$$

$$= \frac{1}{(2\pi)^3} \int_0^\infty k^2 dk \int_{-1}^1 dw \int_0^{2\pi} [-L_0(k,s)]^{-1} e^{ik|R-R'|\phi} \, d\phi, \qquad (2.23b)$$

from (2.21b), (2.17a). This condition is obeyed generally, from the initial condition on g, cf.

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(2.15). *The combination of (2.22) and (2.23) is the full, space-time causality condition, which when axiomatically invoked (as it is here) ensures that only time-related solutions of (2.16) are possible.*

(iii). *Reciprocity*: the point source [in (2.14a)] and the observer [at P(R,t)] are interchangeable in the Green's function, provided temporal causality [(ii) above] is also obeyed. Thus, one has $R' \rightarrow R$, $R \rightarrow R'$ *and* $t \rightarrow -t'$, $t' \rightarrow -t$, so that $g(R',-t'|R,-t)$. From (2.21) it is at once evident that reciprocity for these homogeneous media is obeyed.

(iv). *"Regularity at infinity"*: [15] this ensures that $|g_\infty|$ $(\leq \infty)$, i.e., is bounded as $R \rightarrow \infty$; (this is *not* necessarily guaranteed by (2.22)). This condition is a reflection of the fact that the (point-) source has bounded energy and consequently its field (g_∞) must have also, all R.

Not too surprisingly from the above, we may now regard these homogeneous linear media as being, in effect, *linear space-time filters*. In the language of modern circuit theory the Green's function, g_∞, represents the medium's *weighting function*; $Y_{0,\infty}$, $\mathcal{Y}_{0,\infty}$ correspond to the frequency-, and frequency-wave number *system functions*, or "filter" *spectral densities*, of this causal, space-time filter, where the governing space-time causality condition is the combination of (i) and (ii) above, cf. (2.22), (2.23). However, unlike conventional, "lumped-element" circuit models, these "circuits" (i.e., the medium) are in effect composed of a continuum of elements, distributed throughout space. The dynamical ("circuit") equations of the latter are *partial* differential equations, obeying boundary conditions (here as $R \rightarrow \infty$, cf. (2.22)) as well as initial conditions, while the former are limited to ordinary differential equations, subject only to initial conditions; (both obey appropriate stability conditions, (iv), for example), [27]. Further critical differences appear when the medium becomes inhomogeneous, and random [cf. Sec. 2.4 ff.].

For the Helmholtz case (2.6), (2.6a), $\hat{Q} = 0$, it is easily seen that (2.11) becomes (2.15), and that following the methods described in (2.16)–(2.18) one gets specifically for with the Green's function now

$$g_\infty = \delta(\Delta t - \rho/c_0)/4\pi\rho \ ; \quad \rho = |R - R'|, \quad \Delta t = t-t' \ . \tag{2.24}$$

Similarly, it is also found [3] that when $\hat{Q} = 0$, $\hat{M} \rightarrow \hat{M}_\infty$, the solution of (2.5a) reduces to

$$g(R,t \mid R',t')_\infty = F_k F_s^{-1} \ \{ \ \hat{\mathcal{Y}}_{0,\infty} \} = F_s^{-1} \ \{\hat{Y}_{0,\infty}\}, \tag{2.25a}$$

where F, etc. denote Fourier transforms, with

$$\hat{\mathcal{Y}}_{0,\infty} = \mathcal{Y}_0(k,s) \int dt' \int_{(\infty)} e^{-ik \cdot R'-st'}(\ )_{R',t'} dR' = - F_R^{-1} \ F_t \ \{\hat{M}_\infty\}, \tag{2.25b}$$

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$$\mathscr{G}_0(k,s)_\infty = [k^2(1+\tau_{ox}s) + s^2/c_0^2]^{-1} .$$

(2.25c)

It can be shown that here [6]

$$\hat{Y}_{0,\infty} = \frac{e^{-(\rho s/c_0)(1+\tau_{ox}s)^{-1/2}-st'}}{4\pi\rho(1+\tau_{ox}s)}$$

(2.26a)

and that now, explicitly,

$$g_\infty = \frac{1}{4\pi\rho\tau_{ox}} \left\{ 1 - \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n(1/2)_n}{(2n+1)!} \left(\frac{\rho}{c_0\sqrt{\tau_{ox}\Delta t}}\right)^{2n+1} e^{-\Delta t/\tau_{ox}} {}_1F_1(-1-2n;\tfrac{1}{2}-n;\Delta t/\tau_{ox}) \right\}$$

(2.26b)

for the Green's function of the associated unbounded medium, where ${}_1F_1$ is a confluent hypergeometric function.

Finally, in the case of the PE approximation (2.8) above, one can show that the Green's function associated with (2.9) is

$$g(r,z|r',z')_\infty = \frac{-ik_0}{2\pi\Delta r} e^{ik\Delta z^2/2\Delta r}, \quad \Delta r = r-r'.$$

(2.27)

With a *gaussian beam* for the source (along z) for these narrow angles, e.g.,

$$G_T = A_0 e^{-z^2/a^2} ,$$

(2.28)

the associated homogeneous field α_{oH} is found to be [19]

$$\alpha_{oH}(\Delta r,z) = \int g(r,z|r',z')_\infty \, G_T(z')dz' ,$$

(2.29a)

$$= \frac{ak_0A_0}{(4\Delta r^2+k_0^2a^4)^{1/2}} \exp\left[-\frac{k_0^2a^2z^2}{4\Delta r^2+k_0^2a^2} + i\left(\frac{2\Delta rk_0z^2}{4\Delta r^2+k_0^2a^4} - \tan^{-1}\frac{2\Delta r}{k_0^2a^2}\right)\right] ,$$

(2.29b)

which shows the expected $(\Delta r)^{-1}$ fall-off as $\Delta r \to \infty$.

2.3 The FOR

The result (2.12) above can be expressed formally in terms of a generalized *feedback loop*, as shown in Fig. 2.2. Thus, (2.12) can be regarded as an extension of conventional control theory, now to *deterministic* inhomogeneous media for the moment, cf. Sec. 2.4 ff. We call Fig.

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2.2 the *feedback operational representation* (FOR) for the FOS, (2.12). Equation (2.13) above is the associated PTSS.
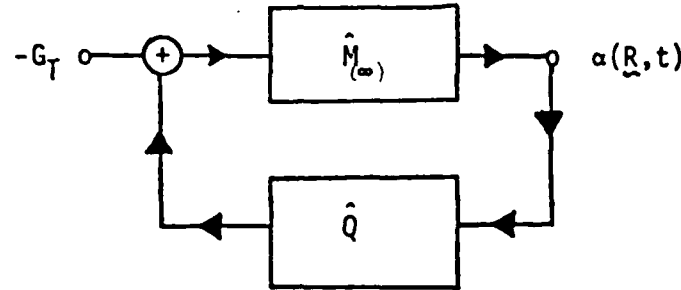
Fig. 2.2   Feedback Operational Representation (FOR) for the propagated field α, Eq. (2.12), for linear inhomogeneous media.

2.4 The Inhomogeneous Operators $\hat{Q}$

These operators embody the various inhomogeneities characteristic of the medium, *including* its boundaries. Thus, for a typical propagation scenario, sketched in Fig. 2.1, we may express $\hat{Q}$, (2.1), in more detail for these linear media as a combination of scattering modes:

$$\hat{Q} = \hat{Q}_S + \hat{Q}_V + \hat{Q}_B = [\hat{Q}_S^{(h)} + \hat{Q}_S^{(I)}] + [\hat{Q}_V^{(h)} + \hat{Q}_V^{(I)}] + [\hat{Q}_B^{(h)} + \hat{Q}_B^{(I)}] \qquad (2.30)$$

where S, V, B, respectively, denote the surface, volume, and bottom in this instance, and (h), (I), designate the deterministic and (purely) random components of each associated operator, cf. (2.1).

As examples, we have explicitly from the various acoustic propagation models indicated in Section 2.1, and for interfaces cf. [1], Eq. (15); [18]:

$$\hat{Q}_{SB} \doteq R_o S \; \hat{n} \cdot (\hat{i}_T - \hat{i}_R) \nabla_R \big|_{\Sigma_{S,B}} = R_o S \; \hat{n} \cdot (\hat{i}_T - \hat{i}_R) s/c_o; \quad s = i\omega, \qquad (2.31a)$$

for the Kirchoff approximation to the surface and bottom operator, with Σ = wave or bottom surface, cf. Fig. 2.1. Here R_o, S are the plane wave reflection and shadowing functions, and $\hat{n}$ is the (inwardly drawn) normal to Σ. Also, $\hat{i}_T$, $\hat{i}_R$ are unit vectors in the directions of the incident and scattered wavefronts. A far-field or Fraunhöfer condition is assumed. For volume inhomogeneities we have

$$Q_V = \text{Eq. (2.5b), or Eq. (2.6a); Eq. (2.9) for the PE approx. (2.8), etc.} \qquad (2.31b)$$

The rôle of $\eta^{(n)}$ in (2.13) is to generate multiple scatter components:

$$\hat{\eta}^{(2)} = (\hat{M}_\infty Q)^{(2)} = \mathcal{L}\{(\hat{Q}_S + \hat{Q}_V + \hat{Q}_B)^2\}$$

$$= \mathcal{L}\{\hat{Q}_S^{(2)} + \hat{Q}_V^{(2)} + \ldots \hat{Q}_S\hat{Q}_V + \hat{Q}_V\hat{Q}_S + \ldots\} \qquad (2.32)$$

shows the various second-order multiple scatter contributions among the various types of inhomogeneity.* In most underwater propagation problems inter-modal scattering can be neglected, e.g., terms in $\hat{Q}_S\hat{Q}_V$, etc. are ignorable vis-à-vis the intra-modal contributions $\hat{Q}_S^{(2)}$, $\hat{Q}_V^{(2)}$, which condition can considerably simplify the subsequent calculations. In any case, the structure (2.30), or "M-form," appearing in the FRO, $\hat{\eta}$, provides a useful "bookkeeping" of the various possible interactions.

With inhomogeneous ($\hat{Q} \neq 0$) media, exact conventional techniques fail, largely because the medium is not reciprocal [cf. (iii), Sec. 2.2] and there is no general way of applying and evaluating the initial conditions over the various volume integrals which appear in the development of the Generalized Huygens Principle (GHP) [20] when $\hat{Q} \neq 0$. [For examples of the GHP see Sec. 7.3, [20], and pp. 512, 513, [28].] Even in the simplest cases, which are bounded homogeneous media—the boundary (Σ) being the inhomogeneity here—the usefulness of the results is severely limited by the difficulty in obtaining the corresponding Green's functions, g_Σ, except for boundaries compatible with separable coördinate systems (for which complete sets of eigenfunctions can be obtained). A second difficulty is that the resulting GHP is basically an integral equation in the desired scattered field (i.e., scattered by the boundary surface Σ), α, unless it makes physical sense to specify the field à priori on Σ, an unlikely situation in most applications. A third difficulty is simply the fact that these simple cases are not the ones of principal interest. Finally, a fourth complication is that $\hat{Q}$ is often an *integral operator*, e.g.,

$$\hat{Q}(R,t \mid R',t') = \int dt' \int Q(R,t \mid R',t') (\quad)_{R',t'} dR' , \qquad (2.32a)$$

cf. (2.15) for $\hat{M}_\infty$, for example, formally, expressing the physical fact that the effects of the inhomogeneities may not be local in their interactions.

Another important reason why conventional techniques fail in the treatment of inhomogeneous media is the fact that these media are space-time variable in their inhomogeneities, e.g., $\hat{Q} = \hat{Q}(R,t \mid \ldots)$, cf. (2.5), (2.6). Such media do not generally support space- and/or time-harmonic solutions, exceptions being $\hat{Q}(R \mid \ldots)$, which permits time-harmonic solutions, and

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* Of course, one has an appropriate $\hat{M}_\infty$ preceding each $\hat{Q}$.

$\hat{Q}(t \mid ...)$, which allows space-harmonic solutions. Even the standard perturbational and variational techniques of "classical" approaches (Chapter 9, [20]) are only applicable in the case $\hat{Q}(R \mid ...)$, when $\hat{Q}$ is *deterministic*, which, of course, is often not the physical problem in ocean and atmospheric media, whose local and distributed properties are explicitly space- and time-dependent. Moreover, even these methods break down when the medium is random (i.e., $\hat{Q}(R \mid ...)$ is stochastic). In fact, all the usual deterministic methods become nugatory when the medium, and its boundaries, are random and moving.

Even though exact analytical and standard (i.e., nonrandom) perturbational techniques fail, various useful *approximate* methods are available. The best-known of these is *ray-tracing* (a comparatively high-frequency technique [21], [22]; [29], Parts III, IV), and *modal analysis* (a comparatively low-frequency one [21]). An important recent extension of these ideas, combined with path-integral methods and super-eikonal procedures, has been developed and applied to the study of sound speed fluctuations in nonisotropic oceans by Flatté et al. [22]. All these methods are, of course, necessarily approximate and, in fact, skillful approximation is the successful heart of these approaches. In our present approach to scattering problems we seek to preserve the *exact* formal solutions in a canonical development, reserving approximations to the last possible moment. This has the conceptual advantage of explicitly identifying the disposable portions of the general result, in the contest of the particular problem ultimately to be studied. Approximations are inevitable: we attempt to postpone them as long as possible. Accordingly, in this spirit, we seek formulations which may lead to manageable computational (and occasionally analytic) solutions of (2.2), (2.4), as well as using older methods.

2.5 Statistical Solutions: Dyson and Bethe-Salpeter Equations

In view of the statistical character of the scattering mechanisms in most cases of physical interest, as already noted above at the beginning of Sec. 2 and in Sec. 2.1, $\hat{\eta}$, (2.12), is a *stochastic operator*. Thus, the "solutions" of the Langevin equation (2.4), as mentioned above, are the various *moments* of the total field α, viz.

$$<\alpha> = \alpha_H + < \sum_{n=1} \hat{\eta}^{(n)} > \alpha_H, \qquad (2.33)$$

etc. for $<\alpha_1\alpha_2>$,..., with ($\hat{M} \rightarrow (\hat{M}_\infty$ now, cf. (2.14a). Thus, to get the first moment of α *all* ($n \geq 1$)-order moments of the FRO are required. Similarly, all ($n,n' \geq 1$)-moments of $<\eta^{(n)}\eta^{(n')}>$ are needed for $<\alpha_1\alpha_2>$, etc. This is clearly very cumbersome and also presents questions of convergence-in-probability (CIP), e.g., $\|\hat{\eta}\| < 1 \rightarrow <\|\hat{\eta}\|>$, ... $<\|\hat{\eta}^{(n)}\|>$, ... $<1(?)$.

A way around these difficulties is provided, however, by introducing an equivalent *deterministic* relation, obtained from

$$<\hat{Q}\alpha> \equiv \hat{Q}_1^{(d)}<\alpha> ; \quad \therefore \hat{\eta}_1^{(d)} \equiv \hat{M}_\infty \hat{Q}_1^{(d)} .$$ (2.34)

Rewriting (2.12) as $(\hat{1} - \hat{\eta})\alpha = \alpha_H$ and averaging, using (2.34), then gives the (deterministic) *Dyson equation*

$$<\alpha> = (\hat{1} - \hat{\eta}^{(d)})^{-1}\alpha_H = \alpha_H + \sum_{n=1} \hat{\eta}^{(d)(n)}\alpha_H ,$$ (2.35)

where now $\hat{\eta}^{(d)}$ is the Equivalent Deterministic Field Renormalization Operator (EDFRO).

The evaluation of the *nonrandom* equivalent inhomogeneous operator $\hat{Q}_1^{(d)}$ proceeds from the relation

$$\hat{Q}_1^{(d)} \left< \frac{\hat{1}}{(\hat{1} - \hat{\eta}_\infty)} \right> \alpha_H = \left< \frac{\hat{Q}}{(\hat{1} - \hat{\eta}_\infty)} \right> \alpha_H$$ (2.36)

obtained from (2.12) averaged and (2.35). Expanding $\hat{Q}_1^{(d)}$ by

$$\hat{Q}_1^{(d)} = \sum_{m=0} \hat{A}_m^{(1)}(R,t \mid R',t') \equiv \int\int Q_1^{(d)}(R,t \mid R',t') (\quad)_{R',t'} \, dR'dt'$$ (2.37a)

and applying (2.37a) to (2.36), with the requirement that each term in the resulting PTSS of both sides of (2.36) be of the same operator order, give

$$\hat{A}_m^{(1)} = <\hat{Q}(\hat{M}_\infty \hat{Q})^{(m)}> - \sum_{j=1}^{m} \hat{A}_{m-j}^{(1)} <(\hat{M}_\infty \hat{Q})^{(j)}>, \quad m \geq 1.$$ (2.37b)

Thus, $\hat{Q}_1^{(d)}$ is always an *integral operator*, cf. (2.37a). When $\hat{Q}$ is purely a *fluctuation operator*, e.g., $<\hat{Q}> = 0$, which often occurs in volume scattering when boundary effects can be ignored, usually for geometrical reasons, (2.37b) simplifies considerably. Thus, for example, we find that then

$$\hat{A}_0^{(1)} = 0; \quad \hat{A}_1^{(1)} = <\hat{Q}\hat{M}_\infty \hat{Q}>; \quad \hat{A}_1^{(2)} = <\hat{Q}\hat{M}^{(2)}\hat{Q}^{(2)}>; \dots .$$ (2.37c)

The extension to the second-order cases is made in a similar way ([12], Sec. 4.2). For the second-order moment $<\alpha_1\alpha_2>$, which is part of the general statistical solution to the generic Langevin Equation (2.4), we get

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$$\langle \alpha_1 \alpha_2 \rangle = (\hat{1} - \hat{\eta}_{12,\infty}^{(d)})^{-1} (\hat{1} - \hat{\eta}_{1,\infty}^{(d)} \hat{\eta}_{2,\infty}^{(d)}) \langle \alpha_1 \rangle \langle \alpha_2 \rangle , \tag{2.38}$$

where $\hat{\eta}_{1,\infty}^{(d)} = \hat{M}_{1,\infty} \hat{Q}_1$, etc., and

$$\hat{\eta}_{12,\infty}^{(d)} \equiv \hat{M}_{1,\infty} \hat{M}_{2,\infty} \hat{Q}_{12}^{(d)} \tag{2.38a}$$

and $\langle \alpha_1 \rangle = \langle \alpha(r_1,t_1) \rangle$, etc. Here $\hat{\eta}_{12,\infty}^{(d)}$ is the second-order EDFRO, cf. (2.35) et seq., and $\hat{Q}_{12}^{(d)}$ may be determined by an extension of the approach for $\hat{Q}_1^{(d)}$ above (2.36) et seq. It is found ([12], Sec. 4.2) that in the frequent cases where $\langle \hat{Q} \rangle = 0$,

$$\hat{Q}_{12}^{(d)} = [\langle \hat{Q}_1 \hat{Q}_2 \rangle + \hat{Q}_1^{(d)} \hat{Q}_2^{(d)}] + \langle O(\hat{Q}^{(4)}) \rangle . \tag{2.38b}$$

Writing, cf. (2.37a),

$$\hat{Q}_{12}^{(d)} = \sum_{m=0}^{\infty} \hat{A}_m^{(2)} (R_1,t_1;R_2,t_2 \mid R_1',t_1';R_2',t_2'), \tag{2.39}$$

and using this in the analogue of (2.36), comparing terms of similar order in the resulting PTSS expansions gives finally

$$m = 0: \quad \hat{A}_0^{(2)} = \langle \hat{Q}_1 \hat{Q}_2 \rangle - \langle \hat{Q}_1 \rangle \langle \hat{Q}_2 \rangle + \hat{Q}_1^{(d)} \hat{Q}_2^{(d)} ; \tag{2.39a}$$

$$m = 1: \quad \hat{A}_1^{(2)} = - [\langle \hat{Q}_1 \hat{Q}_2 \rangle - \langle \hat{Q}_1 \rangle \langle \hat{Q}_2 \rangle] + [\hat{M}_{1,\infty} \langle \hat{Q}_1 \rangle + \hat{M}_{2,\infty} \langle \hat{Q}_2 \rangle] ; \tag{2.39b}$$

$$m = 2: \quad \hat{A}_2^{(2)} = \quad \text{etc.,} \tag{2.39c}$$

In the important cases where $\langle \hat{Q} \rangle = 0$, e.g., $\hat{Q}$ is a purely fluctuation operator as is often the case for infinite media, all the odd-order terms in (2.39) vanish, cf. (3.15).

Equation (2.38) is a form of *Bethe-Salpeter Equation*, which is the second-order generalization of the *Dyson Equation* (2.35), and like it, is now completely deterministic. Still higher-order formulations for $\langle \alpha_1 \alpha_2 \alpha_3 \rangle$, ..., $\langle \alpha_1 \alpha_2...\alpha_m \rangle$, ..., may be derived in the same formal way. These, however, appear, in most instances, to be too complex to be useful.

Finally, we note again (cf. Sec. 2.2) that an important feature of the governing Langevin

equation (2.12), (2.13), and the Dyson equation (2.35), is that they also embody a *feedback representation*, as shown in Fig. 2.2. (For the Dyson equation (2.35), one replaces $\hat{Q}$ by $\hat{Q}_1^{(d)}$ and α by $<\alpha>$.) Thus, *these scattering problems may be regarded as an extension of modern control theory*, where now the (linear) ordinary differential equations of the control format are replaced by partial differential equations of propagation in linear (inhomogeneous) random media. Similarly, initial conditions of the former are replaced by both boundary and initial conditions for the latter. This suggests that concepts and methods already available from (linear) control theory can be helpful here in achieving quantitative solutions for $<\alpha>$, $<\alpha_1\alpha_2>$, etc.

2.6 Related Approaches: The Transport Equation

Instead of working with the Dyson equation (2.35), we can achieve formal solutions of the Langevin Equation (2.12), (2.13) by considering propagation of the *field statistics* directly, using the concept above of the equivalent deterministic medium (EDM), as embodied in the operator $\hat{Q}_1^{(d)}$, etc., cf. (2.34). For example, since $\hat{M}_{\infty}^{-1}\hat{\eta}_{1,\infty}^{(d)} = \hat{Q}_1^{(d)}$, we may write (2.4), taking the average of both sides, as

$$<(\hat{L}^{(0)} - \hat{Q})\,\alpha> = <\alpha_H> = \alpha_H,\qquad(2.40a)$$

or

$$(\hat{L}^{(0)} - \hat{Q}_1^{(d)})<\alpha> = \alpha_H,\qquad(2.40b)$$

which is called a *Transport Equation*, or here a propagation equation of the mean (total) field, $<\alpha>$. From (2.37b) et seq., it is evident that (2.40b) is an integro-differential equation in $<\alpha>$. Second- and higher-order generalizations, for the propagation of $<\alpha_1\alpha_2>$, etc., follow in similar fashion. (For a treatment of the propagation of field statistics, see [13], [14], Chapter 14, for instance.)

2.7 An Exact Solution for α and $<\alpha>$

Under certain conditions it is possible to obtain exact solutions for the inhomogeneous (and total) field α_I, α. Physically, this can be done for infinite media where boundary interactions can be ignored, which may be geometrically possible, so that only volume effects occur. We consider the case of deterministic inhomogeneities. It is then required that the (deterministic) inhomogeneity operator $\hat{Q} \to \hat{Q}_\infty$ takes the form

$$\hat{Q}(R,t\,|R',t')_\infty = \int_{(\infty)} dR'\int_{-\infty}^{\infty} dt'\,A(R-R',t-t')(\quad)_{R',t'}$$

$$= \int_{[k]} \frac{dk}{(2\pi)^3}\int_{Br_1} e^{-ik\cdot R+st}\,(\quad)_{k,s}A_0(k,s)\,\frac{ds}{2\pi i},\qquad(2.41)$$

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where $A_0(k,s) = F_R^{-1} F_t\{A(r,t)\}$, and the *kernel, A, of* $\hat{Q}$ *(2.30), is thus of convolutional type in space and time, but* $\neq a\delta(R-R')\delta(t-t')$, and is accordingly *nonlocal*. The result is

$$\alpha(R,t) = F_k F_R^{-1} \{\alpha_0\} = \int_{C_k} \frac{dk}{(2\pi)^3} \int_{Br_1} \frac{G_{0T}(k,s)\mathcal{Y}_0(k,s)_\infty e^{ik\cdot R + si}}{1 - N_0(k,s)} \frac{ds}{2\pi i}$$

$$N_0 = -\mathcal{Y}_{0,\infty} A_0$$

(2.42)

where $\mathcal{Y}_{0,\infty}$ is given by (2.25a,b). Here C_k is an appropriate contour in the complex k-plane to ensure that the radiation condition ("spatial causality") is obeyed [15]. Br_1 denotes a two-sided Laplace transform, or Bromwich contour $(-\infty+id, \infty+id)$, $d \geq 0$, which enables us to handle transient signals and signals of finite duration $(d > 0)$, as well as the steady-state cases $(d = 0)$, which latter reduce to the temporal Fourier transform situation, cf. Sec. 2.2. The evaluation of (2.42) is now reduced to quadratures, which may usually be carried out analytically, at least in an appropriate series form.

For *unbounded random* media the same approach may be followed. We reduce the governing Langevian equation to a Dyson equation, as above, cf. (2.41), where now $\hat{Q}_\infty \rightarrow \hat{Q}_{1,\infty}^{(d)}$ and $\alpha \rightarrow \langle\alpha\rangle$. Equation (2.42) now applies if we replace α by $\langle\alpha\rangle$, $N_0(k,s)$ by $N_{0,1}^{(d)}(k,s) = -\mathcal{Y}_0(k,s)_\infty q_{0,1}^{(d)}(k,s)$, where $q_{0,1}^{(d)}(k,s) \equiv F_R^{-1} F_t\{\hat{Q}_1^{(d)}\}$, with the kernel of $\hat{Q}_1^{(d)}$ having the form $q_1^{(d)}(R-R',t-t')$, cf. (2.41) above. For mixed deterministic and random media, cf. (2.1), a linear combination of operators $\hat{Q}_\infty$, $\hat{Q}_1^{(d)}$ with the above properties may be expected.

2.8 The Received Field, X(t)

So far, we have been considering the (total) field $\alpha(R,t)$, at some point $P(R,t)$, cf. Fig. 2.1. For subsequent signal processing this field must be sensed by an appropriate *receiving aperture*, or array, designated by the (linear) filter operator $\hat{R}$, viz.

$$X = \hat{R}\alpha, \quad \text{or} \quad X = \{X_1, ..., X_J\} = \hat{R}\alpha, \quad (j = m,n); J = MN,$$

(2.43)

where (2.12) now represents the received wave, in continuous or discretely sampled form in space and time. *Beam-forming* is achieved by summing over the spatial inputs, e.g., $X = \{\Sigma_m X_{j=m,n}\}$. See Section IVB of [4].

In fact, whether we are concerned with remote-sensing of the medium or detection and classification, we must inevitably include the receiving sensor (s), embodied in $\hat{R}$. Accordingly, in such cases we must replace α, $\langle\alpha\rangle$, etc., by X, $\langle X\rangle$, etc., in the above and subsequent analyses.

This is formally straightforward, but will introduce additional computational requirements. We remark that from the viewpoint of the transmitter-receiver systems of Fig. 1.1 the fields we are dealing with here are all *signal-dependent*, i.e., they are generated by a designated active source (S_{in}, Fig. 1.1) and can produce random, signal-dependent *noise* (reverberation or incoherent scatter) in the receiver. This is in contrast to *ambient noise* fields, whose mechanisms are quite independent of the designated source S_{in}. For the purpose of the present treatment we regard the always-present ambient noise as essentially negligible; but see Sec. IV, A, [4]: the effects iof ambient noise sources may be formally including if we now replace the source function $-G_T$ by $-G_T-G_A$, when G_A is (also) a distributed source mechanism.
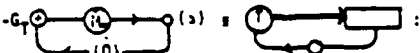
3. Diagram Methods and Approximations

An important aid in integrating and approximating the formal solutions of the field equations here, and in achieving numerical results eventually, are *diagram methods* ([3], [12], Part I, [16]). These enable us, along with the relevant physics, to select the dominant contributions and omit the unimportant ones, and often to choose approximations which considerably reduce the ensuing calculations. We illustrate the approach with first-order results in the case of random media.
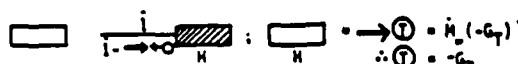
Note that for *deterministic media these r-sults apply as well if we drop the various averages over the Langevin ensemble*, which now contains only one member field representation, α_0.

3.1 Basic Diagrams

We begin with the following *ensemble diagrams*, for the ensemble FOR, FOS, PTSS, etc., and FD (Feynman diagram) equivalents for the field ensemble $\{\alpha\}$ at any allowed point $P(R,t)$:

FOR: cf. Fig. 2.1   (3.1a)

FD: Eqs. (2.10, 2.12)   (3.1b)

FOS: Eq. (2.12)   (3.1c)

PTSS: (3.1d)

Here we have $\rightarrow$ ($\equiv \hat{M}_\infty$) representing the *feedforward* operator, while $\leftarrow$o ($\equiv \{\hat{Q}\}$) denotes the ensemble *feedback* or random scattering operator, $\hat{Q}$. The ensemble Feynman diagram equivalent (FD) of the FOR, (3.1a), is precisely (3.1b), while (3.1c) gives the corresponding ensemble FOS, and (3.1d) the PTSS, which is readily obtained by iterating the FD, (3.1b), or as the unaveraged

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form of (2.33). The (deterministic) first-order Dyson equation, Eq. (2.35) is now easily written:

First-order Dyson Equation:

$$\Longleftrightarrow \; \cdot \; \blacksquare_H \quad \cdot \quad \longrightarrow\!\!\times\!\!\bullet \quad \cdot \quad \Longleftrightarrow \qquad \Bigg\} \qquad (3.2)$$

$$<\alpha> \; = \; \alpha_H \quad + \quad \hat{M}_\infty \hat{Q}^{(d)} \qquad <\alpha>, \; Eq. \; (4.4a),$$

where now we denote the average by $< \; >$, applied to the corresponding ensemble symbol. Our general diagram conventions are to represent ensemble quantities as "open" forms, e.g., $\square$, $\leftarrow$o, o, etc., and average and otherwise deterministic quantities by solid forms, e.g., $\leftarrow\bullet$, $\bullet$, $\bullet$, with field averages denoted by brackets, $< \; >$, e.g., $\Longleftrightarrow = <\alpha>$, etc. The aim, of course, is consistency of notation and simplicity with a minimal evolution of complexity.

Diagrams may be manipulated in the same fashion as are operator equations [cf. (2.35), etc.]: one observes the same positioning rules and conditions of inversion [cf. (2.13)], e.g., for multiplication and "division." Addition and subtraction are also directly equivalent to the addition and subtraction of operators and operator derived (algebraic) quantities, as indicated in the examples below. Finally, *transforms of diagrams are the corresponding diagrams of the transforms*. This permits an explicit diagrammatic representation of the various wave number-frequency (WNF) spectra of the scattered field, particularly the amplitude and intensity spectra in operator form.

Accordingly, the corresponding diagrams for the equivalent medium [cf. Sec. 2.5] become [cf. (3.1)]:

Equiv. Med. FOR:
(2.35)
$$\text{diagram} \quad \cdot \text{cf. (2.35)} ; \qquad (3.3a)$$

Equiv. Med. FOS:
(2.35)
$$\text{diagram} \qquad (3.3b)$$

Equiv. Med. FD:
First-Order Dyson Equations:
(2.35)
$$\Longleftrightarrow \; \cdot \; \blacksquare_H \; \cdot \; \leftarrow\!\!\bullet \; \Longleftrightarrow ; \qquad (3.3c)$$

PTSS:
(2.13)

$$\Leftrightarrow \cdot [i \cdot \longrightarrow \cdot -\overset{\frown}{\longrightarrow} \cdots] \text{///}; \quad \bullet \langle \dot{\Phi} \rangle; \qquad (3.3d)$$

FOS + PTSS:
(2.12), (2.13)

$$\Leftrightarrow \cdot \frac{i}{1 - \overset{\bullet}{\underset{d}{\longrightarrow}}} \text{///} \cdot \hat{U} \cdot \longrightarrow \cdots \cdot (\longrightarrow)^{(n)} \cdots] \text{///}. \qquad (3.3e)$$

Here $\overset{\frown}{\mathcal{M}}_{(n \geq 2)}$ denotes the n-fold average; $\bullet \equiv \langle \hat{Q} \rangle$; $\overset{\frown}{} \equiv \langle \hat{Q}^{(2)} \rangle = \langle \hat{Q}^{(1)} \hat{Q}^{(1)} \rangle$, $\overset{\frown}{\longrightarrow} \equiv \langle \hat{Q}^{(1)} \hat{M}_\infty \hat{Q}^{(1)} \rangle$, etc. Equations (3.3e) follow from the iteration of the FD (3.3c). In the same way we find that the ED$_1$M, cf. (2.34), (2.35) as developed in (2.37) with (2.37a,b), can be expressed diagrammatically as

$$\hat{Q}_1^{(d)} : \underset{d}{\longrightarrow} \cdot [\bullet]_{m=0} \cdot [\overset{\frown}{\longrightarrow} \cdot \longrightarrow]_{m=1} \cdot [\overset{\frown}{\longrightarrow} \cdot \overset{\frown}{\longrightarrow} \cdot \cdots]_{m=2} \cdots \qquad (3.4a)$$

$$\cdot [\bullet]_{m=0} \cdot [\overset{\frown}{\longrightarrow} \cdot \bullet \longrightarrow]_{m=1} \cdot [\overset{\frown}{\longrightarrow} \cdot \overset{\frown}{\longrightarrow} \cdot \cdots]_{m=2} \cdots$$

$$(3.4b)$$

this last in terms of $\hat{\eta}_\infty$ ($\equiv \multimap$), where, for example, $\langle \hat{\eta}^{(2)} \rangle = \langle \multimap \multimap \rangle \equiv \overset{\frown}{}$, etc.

3.2 Diagram Approximations

A major problem, as always for strong-scatter situations, is the approximate evaluation of the FOS or its PTSS equivalent, since exact analytic solutions are possible only in special instances [cf. Sec. 2.7]. Another important problem is our ability to truncate the PTSS in comparatively weak-signal régimes: for example, what is the effect of stopping at a given term in the PTSS? What is the overall contribution of the remaining terms? What criteria of truncation are reasonable, etc.? In any event, various approximations are in order.

We consider briefly here two useful classes of approximation: I. *Series modification*, and II. *Truncation*. In the former, the character of the equivalent deterministic medium, or "mass" operator is modified. In the latter, the PTSS is truncated: a finite number of interaction orders is retained, while the higher-orders are discarded. For example, let us consider the former case (I) first:

A. Series Modification: First Order Dyson Equation

As applied to the various forms of the Dyson Equation [cf. (2.35), (3.2)], the form of the equivalent deterministic scattering operators (EDSO), $\hat{Q}_1^{(d)}$, $\hat{Q}_{12}^{(d)}$, . . ., $\hat{Q}_{12 \ldots m}^{(d)}$, cf. (2.34), is modified in some fashion, suggested by the physics of the problem. This EDSO is altered, usually by truncation, while the infinite operator series implied by $(1 - \hat{\eta}_{1,\infty}^{(d)-1})$, etc., cf. (2.35), etc.,

remains. Let us examine, for example, the first-order Dyson equation (2.35) for the mean field $<\alpha>$ and postulate a first-order independent structure for $\hat{Q}$:

I. First-Order Independence:

$$\therefore \left< \prod_{j=1}^{n} \hat{Q}_j \right> = \prod_{j=1}^{n} <\hat{Q}_j> \qquad := \bullet \; \bullet \; \ldots \; \bullet_n \; , \qquad (3.5)$$

and from (3.5) in (2.37a,b) and then in (2.35) we see at once that the ED_1SO, $\hat{Q}_1^{(d)}$, is directly

$$\hat{Q}_1^{(d)} \big|_{indep.} = \hat{A}_o^{(1)} = <\hat{Q}> = \bullet; \quad \hat{A}_m^{(1)} = 0, \quad m \geq 1; \quad \therefore \hat{\eta}_{1,\infty}^{(d)} \equiv \hat{M}_\infty <\hat{Q}> , \qquad (3.6)$$

where $\hat{Q}_1^{(d)}$ reduces to a single term, so that (2.35) becomes at once

$$<\alpha> = \alpha_H + \hat{M}_\infty <\hat{Q}> <\alpha> = (\hat{1} - \hat{M}_\infty <\hat{Q}>)^{-1} \alpha_H \qquad (3.7)$$

with the following equivalent (deterministic) diagrams [cf. (3.2), (3.3)]:

$$(3.7a)$$

Thus, in the case of purely volume scatter ($<\hat{Q}> = 0$), $\hat{Q}_1^{(d)} = 0$, and $\therefore <\alpha> = \alpha_H$, here, which clearly shows that the postulate (3.5) is much too restrictive to provide meaningful results in most cases.

In a similar way, we may employ a more structured, second-order assumption on the statistical character of $\hat{Q}$. This is the second-order "independent" or (generalized) *Bourret* or *Bi-local approximation*, defined by

II. Second-Order Independence

$$
\left\langle \prod_j^n \hat{Q}_j \right\rangle = \prod_{j=1}^{n/2} \langle \hat{Q}_i^{(2)} \rangle \quad ; n = \text{even} \quad := \frown \frown \ldots \frown_{(n)} . \quad (3.8a)
$$

$$
= \prod_i^{(n-1)/2} \langle \hat{Q}_i^{(2)} \rangle \langle \hat{Q} \rangle \quad ; n = \text{odd} \quad := \frown \frown \ldots \frown_{(n)} \bullet \quad (3.8b)
$$

$$
(\hat{Q}_{i=0} \equiv \hat{1}) .
$$

For the odd-order case we can also set, alternatively,

$$
\left\langle \prod_j^n \hat{Q}_j \right\rangle = \langle \hat{Q} \rangle \prod_1^{(n-1)/2} \langle \hat{Q}_i^{(2)} \rangle ; \; n = \text{odd} := \bullet \; \frown \frown \ldots \frown_{(n)} . \quad (3.8c)
$$

Consequently, the ED_1SO, $\hat{Q}_i^{(d)}$), (2.37), (2.37a,b), becomes here

$$
A_0^{(1)} + \qquad \leftarrow A_1^{(1)} \rightarrow \qquad - \qquad \leftarrow A_2^{(1)} \rightarrow \qquad + \ldots:
$$

$$
\hat{Q}_i^{(d)} |_B = \langle \hat{Q} \rangle + [\langle \hat{Q} \hat{M}_\infty \hat{Q} \rangle - \langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle] - [\langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \hat{M}_\infty \hat{Q} \rangle - \langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle] + \ldots
$$

$$
(3.9a)
$$

or

$$
\hat{Q}_i^{(d)} |_B = \langle \hat{Q} \rangle + [\langle \hat{Q} \hat{M}_\infty \hat{Q} \rangle - \langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle] - [\langle \hat{Q} \hat{M}_\infty \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle - \langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle \hat{M}_\infty \langle \hat{Q} \rangle] + \ldots
$$

$$
(3.9b)
$$

In diagram form these are:

$$
(3.9c)
$$

Now, unlike the purely independent case (3.5), (3.6), ... the series for $\hat{Q}_i^{(d)}$ does not terminate, so that in this second-order situation we are generally forced to truncate the series to obtain manageable approximations.

Note from (2.37b) *that when* $\langle \hat{Q} \rangle = 0$, which represents purely volume scatter effects

without boundaries or interfaces, *the bi-local* or *Bourret approximation* (3.8) *for* $\hat{Q}_1^{(d)}$ *reduces to a single term*:

$$\langle \hat{Q} \rangle = \bullet = 0:$$

$$\hat{Q}_1^{(d)}\big|_{\text{Bourret:B}} = \langle \hat{Q}\hat{M}_\infty\hat{Q}\rangle \ (= \hat{A}_1^{(1)}); \quad \hat{A}_m^{(1)} = 0, m \geq 2: \quad \text{(diagram)}, \tag{3.10}$$

for this first-order equivalent deterministic medium. [As we have just seen (cf. (3.9a,b)), with interfaces the Bourret approximation (3.8) yields a non-terminating series.] This compact result suggests the utility of the bi-local approximation in applications, wherever it appears reasonable to describe the principal scattering effects of the medium through second-order spatial (and temporal) correlations at most.

The various mean fields $\langle \alpha \rangle$ are diagrammed here in (3.3) when $\hat{Q}_1^{(d)})\big|_B$ is given either by (3.9) or by (3.10) in the unbounded (unlayered) cases. For the latter we have explicitly

$$\langle \hat{Q} \bullet 0: \quad \Longleftrightarrow \big|_{\text{Bourret}} \bullet \frac{i}{1-\text{(diagram)}} \text{(diagram)} \bullet (1+(\text{diagram})^{(n)>1}) \text{(diagram)}; \tag{3.11}$$

for the associated FOS and PTSS. Unlike the strictly independent case I above, here $\langle \alpha \rangle \neq \alpha_H$ and $\langle \alpha \rangle$ now represents a meaningful expression for the mean scattered field when there are no interfaces, cf. remarks following Eq. (3.7a). Equation (3.11) is the generalization of Tatarskii's Eq. (26), ¶61, [17], based on the "classical" Helmholtz equation (2.6), with time-independent $\varepsilon = \varepsilon(R,t)$, to arbitrary (linear) media and input signals. The corresponding *analytic* solution, similarly generalizing Eq. (26a) of Tatarskii (ibid.) is given here by (2.42), modified for random media, Sec. 2.7.

B. Series Truncation

In the cases above the "mass operator" for EDSO, $\hat{Q}_1^{(d)}$, etc., is approximated either as an infinite or terminated series [cf. (3.9), (3.6), (3.10)], thus altering the form but not the fact that the resulting approximate solution is still an infinite, though sub-series of the original PTSS. On the other hand, *truncation* replaces the exact (or previously approximated) infinite operator series with a finite operator series. Of these latter there are two principal types: I: *Taylor's series approximations*, which yield analytic forms directly, and II: *Born approximations*, which constitute a hierarchy of truncated operator series.

Let us consider I first:

I. Taylor Series Approximations:

Although the exact solution $<\alpha>$ of, say, the first-order Dyson equation (2.35), (3.2), requires an infinite PTSS, cf. (2.25), we can reduce the evaluation of $<\alpha>$ to a purely algebraic solution by means of a Taylor's series expansion with truncation. Thus, if we expand $<\alpha>$:

$$<\alpha(R',t')> = <\alpha(R,t)> + (R'-R) \cdot \nabla_R <\alpha(R,t)> + (t'-t) \frac{\partial}{\partial t} <\alpha(R,t)> + \dots , \qquad (3.12)$$

and keep only the leading term, we have directly the first-order approximate form

$$\left[\hat{M}_\infty \hat{Q}_1^{(d)} <\alpha> \right] \doteq <\alpha(R,t)> \hat{M}_\infty \hat{Q}_1^{(d)} 1 \qquad (3.13a)$$

where $\hat{M}_\infty \hat{Q}_1^{(d)} 1$ is simply an (analytic) function of (R,t), viz.:

$$\hat{n}_1^{(d)} 1 = \hat{M}_\infty \hat{Q}_1^{(d)} 1 = \int M_\infty(R,t \mid R',t') \, dR'dt' \int \hat{Q}_1^{(d)}(R',t' \mid R'',t'') 1 \, dR''dt'' \equiv n_1^{(d)}(R,t). \quad (3.13b)$$

Applying (3.13a,b) to (2.35) gives the approximate *analytic* result

$$\boxed{\; <\alpha^{(1)}(R,t)> \doteq \frac{\alpha_H(R,t)}{1 - n_1^{(d)}(R,t)} \; , \;} \qquad (3.14)$$

where the exact or various approximate expressions for the equivalent deterministic medium operator $\hat{Q}_1^{(d)}$ may be employed, cf. ((2.37a,b), (3.6), (3.10)). Thus, for the fully independent (I) and bi-local approximations (II), $<\hat{Q}> = 0$, we have specifically (and exactly) here:

$$n_{1\text{-ind}}^{(d)} = \hat{M}_\infty <\hat{Q}> 1 \; ; \quad n_{1\text{-ind}}^{(d)} = <\hat{Q} \hat{M}_\infty \hat{Q}> 1, \quad <\hat{Q}> = 0. \qquad (3.15)$$

Clearly, there may be some points (foci) or lines (caustics) in space where $<\alpha^{(1)}> \rightarrow \infty$, when the denominator of (3.14) vanishes. The reality of these singularities depends on how well (3.14) approximates $<\alpha>$ itself: the true foci, etc. may not be infinite, but may be located close to the singularities of the approximation. Thus, (3.14) may serve as a guide as to potentially real foci, etc.

Finally, the error in using (3.14), say, is generally difficult to assess, because the true

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of $<\alpha>$ is not usually available. However, in the cases where the conditions for (2.41) are obeyed we can use the exact result (2.42) to determine the error vis-à-vis using (3.14) for $<\alpha>$.

II. Born Approximations:

Unlike the above procedures A, B, which essentially approximate the "mass operator," $Q^{(d)}$, with an infinite or truncated series, cf. (3.9) or (3.10), we may truncate the PTSS [(2.13), (3.3d), etc.] directly. The resulting finite series is called a *Born approximation* series, whose order depends on the stage at which truncation is applied. Thus, for the first-order Dyson equation (3.3c) for the mean field $<\alpha>$ we can construct the following hierarchy of approximations and associated diagrams:

$n = 0$:

 (3.16a)

$n = 1$:
first Born approximation

 (3.16b)

$n = 2$:
second Born approximation

$$<\alpha> \doteq \sum_{n=0}^{} <(M_\infty \hat{Q})^{(n)}>\alpha_H ;$$

etc.   etc.   etc. $(n \geq 3)$.

The nth order FOR is simply

$(FOR)_n$:
n^{th}-order Born approximation

 (3.16d)

Unlike the approaches above which use modifications of the "mass operator," $Q^{(d)}$, etc., but where the resulting PTSS remain infinite, or where equivalently, the corresponding FOR and FOS [cf. (3.11)] contain closed feedback loops, all Born approximations contain only feed-forward loops, as indicated in (3.16d). This is the immediate consequence, of course, of the truncation of the direct PTSS (3.3d).

Examples: Ocean Volume and Surface Channels: The Received Field X

As examples, we noted that both the important cases of weak volume scatter in the ocean and ocean surface scatter are well approximated by at most the scattering element interacting with the homogeneous field α_H, the former because multiple scatter effects are usually quite ignorable, the latter from the physical geometry of the wave surface, which likewise discourages multiple scatter except within a typical wave crest-to-crest domain (except possibly at small grazing angles),

cf. Fig. 2.1, (n = 1). Similar considerations usually apply for ocean bottoms, as well, cf. Fig. 2.1. We shall, however, consider only the modes M = S, V here, observing also that O(30–40 db) they are effectively independent (see [4], Ref. 21, thesis]): there is negligible coupling in the ocean between surface (bottom) and volume scatter.)

Accordingly, a first-Born approximation suffices, and we write for the received field, X, cf. (2.43),

$$X = \hat{R}\alpha = \hat{R}\ (\rightarrow \mathbb{O} + \rightarrow \leftarrow o \rightarrow \mathbb{O}) = \hat{R}(\alpha_H + \hat{M}\hat{Q}_M\alpha_H) \rightarrow : \hat{M}; \leftarrow o: \hat{Q}_M$$

$$\hat{M} \qquad S \text{ or } V \qquad : \hat{M}$$

$$M = S, V. \quad \mathbb{O} = -G_T \tag{3.17a}$$

The mean wave (X) is, with $< \hat{Q}_M > = \bullet$,

$$\therefore <X> = \hat{R}<\alpha> \doteq \hat{R}(\rightarrow \mathbb{O} + \rightarrow \bullet \rightarrow \mathbb{O}) = \left\{ \hat{R}(\alpha_H + \hat{R}\hat{M}<\hat{Q}_M>\alpha_H) \right\} . \tag{3.17b}$$

$$X_H + \qquad X_I$$

The second moment of X, (3.17a), is found to be, on averaging over the phases (ϕ) of the input signal,

$$<<X_1 X_2>>_\phi = <\hat{R}_1 \hat{R}_2 <\alpha_1 \alpha_2>>_\phi$$

$$\doteq \left\langle \hat{R}_1 \hat{R}_2 [\hat{1} + \hat{M}_1 \hat{M}_2 \cdot \left\{ <\hat{Q}_1^{(0)}\hat{Q}_2^{(0)}>_S + <q_1^{(1)}q_2^{(1)}> + \sum_{k=2}^{\infty} <q_1^{(k)}q_2^{(k)}> \right\} \alpha_{H_1}\alpha_{H_2}] \right\rangle_\phi \tag{3.18a}$$

$$\langle\langle X_1 X_2 \rangle\rangle_\bullet = \langle \hat{R}_1 \hat{R}_2 (\hat{1}$$

	$k = 0$	$k = 1$	$(k \geqslant 2)$
	$+ \rightarrow o \rightarrow o$	$+ \rightarrow \bullet \rightarrow \bullet$	$+ \rightarrow \bullet \rightarrow \bullet)$
	11 22	11 22	11 22 H_{12}
= "homog." unscat. {	"classical": coherent FOM: coh.	+ incoherent + incoherent	multiple-scatter: incoher. (0)

where

$$O \equiv \hat{q} = \hat{Q} - <\hat{Q}>_R = \leftarrow O - O = L\left\{ v^{(0)} = \sum_k (<v^{(k)}>_R + \Delta v^{(k)}) \right\}$$

$$\hat{Q}^{(0)} \equiv <\hat{Q}>_R = 0, \quad \therefore <\hat{Q}^{(0)}>_S = <\hat{Q}> = \bullet = <O>_R$$

$$\overset{\frown}{} \equiv < \;>_{S:space\ av.\ x\ \theta}$$

$$\underset{1 \quad 2}{\overset{\frown}{\bullet \quad \bullet}} = <\hat{q}_1\ \hat{q}_2> \text{etc.} \quad \overset{\frown}{} = < \;> = < \;>_{R,S} \tag{3.19}$$

Note that the (total) average, $< \;>$, as in [30], Appendix A.1 et. seq. involves both an average over radiation events and an average over random spatial (S) and parameter values (θ), embodied in $< \;>_S$, cf. (3.14) and, particularly, [30], Appendix A.2.

It is the decomposition principle, embodied in $\hat{Q}_M$, (M = S, V, B), ([30], (A.2-5) et seq.), which allows us to resolve the inhomogeneity operator into sets of distinct and (statistically) independent entities as specifically exhibited in (3.18a) and (3.18b). Moreover, and most important, the entities (k = 0, k = 1, k $\geq$ 2) in (3.18a) and (3.18b) above and in (3.17), with the help of (3.19) have an explicit physical interpretation. Thus, the terms k = 0 contain both the *coherent* and incoherent radiation contributions, from all orders (k $\geq$ 1) of radiation interactions, e.g., single and multiple scattering (k $\geq$ 2), for example. Consequently, if $v^{(k)}$ represents the density of (illuminated) k-coupled scattering elements (k > 1), then $<v^{(k)}>_R$ is the coherent radiative contribution, where $< \;>_R$ is the average over the radiation events associated with the ensemble of potential (re-)radiating sources. In a similar way

$$\Delta v^{(k)} \equiv v^{(k)} - <v^{(k)}>_R \tag{3.20a}$$

is the *fluctuation* in the density of k-coupled radiating elements, and is always associated with incoherent radiation. Accordingly, we have

$$v^{(0)} \equiv \sum_{k=1}^{\infty}(<v^{(k)}>_R + \Delta v^{(k)}) = \sum_{k=1}^{\infty} v^{(k)} \equiv v \equiv <v^{(0)}>_R + \Delta v^{(0)} \tag{3.20b}$$

showing the basically coherent and incoherent elements of the scattering or source region containing these various k $\geq$ 1 types of elements. The (average) density of coherent radiators is $<v^{(0)}>_R = \Sigma <v^{(k)}>_R$, since $<\Delta v^{(k)}>_R = 0$. The number of "radiation events" occurring in a small region $d\Lambda$ of the illuminated or emitting domain Λ is dN, so that $v = dN/d\Lambda = v^{(0)}$. A taxonomy and interpretation for the dN, similar to (3.20a), (3.20b), can be constructed: one now has

$$dN^{(0)} \equiv \sum_{k=1}^{\infty}(<dN^{(k)}>_R + dn^{(k)}) = \sum_{k=1}^{\infty} dN^{(k)}. \tag{3.20c}$$

The relations (3.21) et seq. obviously apply if one replaces v etc. by dN.

Since the medium in question is linear, the scattering or inhomogeneity operator $\hat{Q}$ is a linear functional of the $v^{(k)}$, i.e., of $v^{(0)}$, e.g., $\hat{Q} = \mathcal{L}(v^{(0)})$, thus containing all types of radiation interaction effects. Moments of $v^{(0)}$, or more generally, of $\hat{Q}_M$, which appear in (3.17) and (3.18), above, are readily indicated: we have

$$<\hat{Q}> = \mathcal{L}\{<v^{(0)}>\}$$

where

$$<v^{(0)}> = <<v^{(0)}>_R>_S. \qquad (3.21)$$

This is different from zero, in scattering, if specular reflection geometry (both on a surface an din a volume) is available. The second-order moments are instructive:

$$<\hat{Q}_1\hat{Q}_2> = \mathcal{L}\{<v_1^{(0)}v_2^{(0)}>\} \qquad (3.21a)$$

where now

$$<v_1^{(r)}v_2^{(0)}> = <<\Delta v_1^{(0)}\Delta v_2^{(0)}>_R>_S + <<v_1^{(0)}>_R<v_2^{(0)}>_R>_S. \qquad (3.21b)$$

with

$$\Delta v^{(0)} \equiv \sum_{k=1} \Delta v^{(k)}.$$

Equation (3.21b) contains both incoherent ($\sim \Delta v^{(0)}$) and coherent terms ($\sim <\Delta v^{(0)}>_R$), of all orders ($k \geq 1$) of coupled scattering or radiating elements. The associated covariance of $\hat{Q}$ is

$$K_{\hat{Q}} = \mathcal{L}\{<v_1^{(0)}v_2^{(0)}> - <v_1^{(0)}> <v_2^{(0)}>\} \qquad (3.21c)$$

which embodies wholly incoherent radiation.

As in all these approximate situations, the evaluation of error remains a major, very difficult problem, because the exact results are generally unknown. We must rely, instead, on our physical intuition, aided by the (essentially) local physics governing the inhomogeneity operator $\hat{Q}$ in question. This is one reason why the physical "anatomization" of the so-far canonical $\hat{Q}$ is critically important in specific cases.

3.3 An Estimation Procedure for the Mean Operator

When the elements ($-G_T$, $\hat{M}_\infty$) of the "input-output" structure of Fig. 2.2 are available, estimates of the "mass operator" $\hat{Q}_1^{(d)}$ may be made to any degree of accuracy from the empirical or a priori mean field $<\alpha>$, in a controlled way, in principle, with determinable error. This is a form of "system identification" problem, typical of control theory [24]. Here, in essence, one
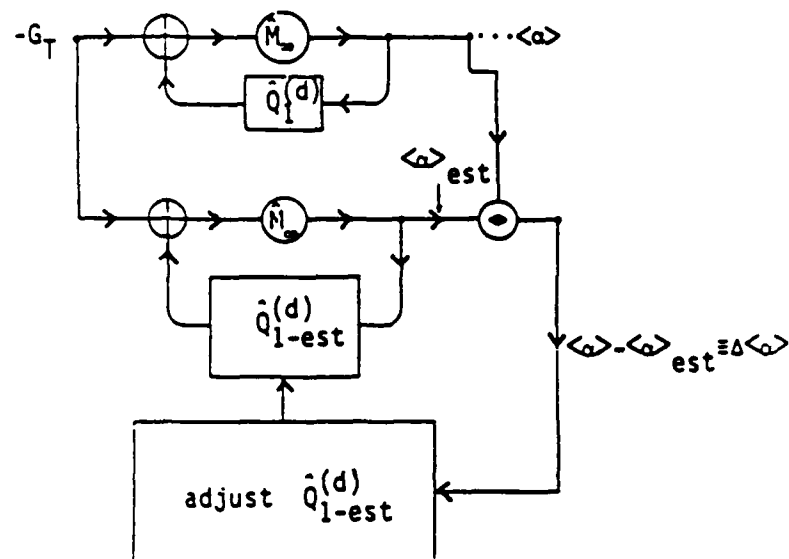
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Fig. 3.1   Feedback-operational method for obtaining the (first-order) equivalent medium (integral) operator $\hat{Q}_1^{(d)}$.

introduces a "black-box" or system with a controllable input, observes the resulting output(s), and attempts to infer the operations of the system which relates the two. A key feature here is that the system in question is known to be linear, so that unique relations are established.

For (2.34) to be a *physically* (vis-à-vis mathematically) useful device, we must be able to indicate a physical procedure for determining $\hat{Q}_1^{(d)}$. This may be done according to the scheme shown here in Fig. 3.1. This scheme produces a converging series of estimates, $\hat{Q}_{1\text{-est}}^{(d)}$, for $\hat{Q}_1^{(d)}$, by iteration, *when [<α>, –G_T] are provided experimentally*, and when $\hat{M}_\infty$ is either empirically or analytically given. The basic approach is conceptually quite simple: with a starting estimate $\hat{Q}_{\text{est-1}}^{(d)}$ of $\hat{Q}_1^{(d)}$, one runs through the second loop, using the (experimentally) observed <α>, the given source, –G_T, and the known global descriptions of the homogeneous component of the medium ($\hat{M}_\infty$ here, cf. Sec. 2.2), to get an <α>$_\text{est}$. This is then subtracted from the observed <α> from the first-loop, which in turn leads to an adjustment of $\hat{Q}_{1\text{-est}}^{(d)}$, a second comparison, with a further reduction in the error Δ<α> = <α> – <α>$_\text{est}$ and $\Delta\hat{Q}_1^{(d)} = \hat{Q}_1^{(d)} - \hat{Q}_{1\text{-est}}^{(d)}$. Various optimum and suboptimum schemes from control theory (e.g., Wiener filtering, gradient-climbing, etc., now, however, extended to *four* dimensions) may be used to drive the errors Δ<α>, $\Delta\hat{Q}_1^{(d)}$ essentially to zero, so that $\hat{Q}_{1\text{-est}}^{(d)} \rightarrow \hat{Q}_1^{(d)}$. This technique may also suggest acceptable approximations in experimental situations. In fact, approximating $\hat{Q}_1^{(d)}$ here is analogous to choosing some approximation to $\hat{Q}_1^{(d)}$, (2.37), like $\hat{A}_0$, or $\hat{A}_1$, cf. (2.37 a,b,c) et seq.

In all cases the particular physics of the problem will dictate the procedural details, the approximations (usually required), and the class of "hardware" to be employed, e.g., minicomputer, minisupercomputer, or supercomputer. Some form of the latter appears necessary

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in most cases, cf. Sec. 4 ff., since these problems are generally on the scale of long-range weather prediction. In fact, these problems may be considered as a form of "underwater weather" prediction, especially when the medium (and interface) behavior is properly included. It is hoped that the preceding material may provide at least a stimulus to the much-needed and as yet nonexistent computational solutions to this important class of problems.

4. Concluding Remarks: Analytic Steps to the "Interface Problem"

As we have noted at the beginning of this paper, a principal purpose of this study is to bring the formal analytic-operational treatment to the point where the computational problems of implementing these formal results numerically can be addressed. We call this the "Interface Problem," where the interface lies between the general mathematical formulation, outlined here, and software design and hardware performance.

The various steps which can bring the analytical treatment to this interface (the dotted line) are summarized in Table 4.1.

Our philosophy here has been to start with as general a "top-down" approach as possible, including those physical features which can ultimately influence practical solutions. Among such features are beam patterns, signal waveform, and, of course, geometry, as well as the relevant structures of the medium inhomogeneities. Proceeding with particular applications, we usually find that approximations and simplifications can be made which can significantly reduce the very heavy computational burdens implied by the operational forms of the desired "solutions." The latter, of course, are both deterministic and stochastic, and a full treatment will need to include both these components. The present problems are on the scale of weather prediction problems. In fact, they are themselves models of "underwater weather." With the recent advances in computer capabilities (and reduced costs) we may expect these problems to be within range of numerical solution: The present effort has been presented, in part, to stimulate their numerical investigation, as well as the achievement of further analytic solutions.

Table 4.1

Step	Deterministic Inhomogeneities	Step	Random Inhomogeneities
1.	*Obtain the PDE and b.c.'s + i.c.'s*; Eq. (2.1) et seq.; = "Dynamical Equation"; all representations are equivalent here	1.	Obtain the PDE, b.c.'s + i.c.'s; Eq. (2.1) et seq., for a typical representation
		1a.	Construct the Langevin Equation, Eq. (2.4) et seq.
2.	Obtain the infinite-medium Green's function g_∞, cf. (2.24), (2.26), (2.27)	2.	Same
3.	*Operational Solutions*: These are complete: PTSS, FOS from the FOR, cf. (2.13), Fig. 2.2,, Sec. 2.3, 2.4 for $\hat{Q}$, the inhomogeneity operator, cf. (2.5b), (2.6a). (2.9)	3.	Solutions are the *statistics* of the field α, e.g., $<\alpha>$, $<\alpha_1\alpha_2>$, etc., and *pdf*'s $w_1(\alpha)$, $w_2(\alpha_1,\alpha_2)$, ...
		3a.	Construct *Dyson's equation* (first-order), Sec. 2.5 for Equivalent Deterministic Media (EDM), sol. $<\alpha>$
		3b.	Higher order: $\hat{Q}_{12}^{(d)}$, for $<\alpha_1\alpha_2>$, etc. PTSS; FOS forms (2.35); Some analytic solutions, Sec. 2.7
4.	Approximate $\hat{Q}$, local and distributed. Use *diagram methods*, Sec. 3.2 [$\hat{Q}$ is usually local, cf. (2.5b), (2.6a), (2.9)]	4.	Approximate $\hat{Q}_1^{(d)}$, etc., Sec. 2.5; *Diagram methods* assist; Sec. 3, Sec. 3.2; $\hat{Q}_1^{(d)}$ is *non-local*; all higher-order $\hat{Q}^{(d)}$'s are also non-local
5.	*Analytic solutions* (for volumes only), cf. Sec. 2.7, Eq. (2.42)	5.	*Analytic solutions* (volumes only), cf. Sec. 2.7, Eq. (2.42) modified, 2nd ¶; 1st order. Higher-order by similar way, but very complicated.
6.	*Other Approximations*: Ray tracing, normal modes (time-independent) [21]; $\hat{Q}(R\ \vert...)$—time-harmonic solutions; $\hat{Q}(t\ \vert...)$—space harmonic sol.; $\hat{Q}(R,t\ \vert...)$—sol. not harmonic. Medium non-reciprocal, so Generalized Huygens Principal *not* available. Path Integrals, Super-Eikonal Methods [22], [23]	6.	*Same*, for (EDM), (EDM)$_2$, via $\hat{Q}_1^{(d)}$, etc. (See [12] for details.) Realization of diagram approximations
7.	*Computational Solutions*: special cases only (no boundaries) [7], [10]. General software: to be estimated, approx. and exact.	7.	*Computational Solutions*: generally unknown. To be constructed, approx. and exact = Major Problem.

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Glossary of Principal Symbols

$\alpha, \alpha(R,t)$	=	total field, at $P(R,t)$
Q_T, Q_R	=	beam patterns
α_H	=	homogeneous field
Br_1	=	Bromwich contour; (two-sided Laplace transform)
c_0	=	mean sound speed
∇c	=	sound speed gradient
∇^2	=	Laplacian (in rectangular coördinates)
$\varepsilon, \varepsilon(R,t)$	=	(function of) index of refraction
$\hat{\eta}, \hat{\eta}^{(d)}$	=	field renormalization operators (FRO), (EDFRO)
FD	=	Feynman diagram
FOR	=	feedback operational representation
FOS	=	feedback operational solution
FRO	=	Field Renormalization Operator
$_1F_1$	=	confluent hypergeometric function
$\mathcal{F}, \mathcal{F}^{-1}$	=	Fourier transforms
$\hat{G}, G$	=	imput source operators and functions
g_∞	=	Green's function of an unbounded medium
$\hat{i}_x, \hat{i}_y, \hat{i}_z$	=	unit vectors
$k, \mathbf{k}$	=	wave numbers
$\hat{L}^{(0)}$	=	differential operator for homogeneous part of medium
$\mathcal{L}$	=	linear operator, cf. Eq. (2.22)
$\hat{M}, \hat{M}_\infty$	=	integral Green's function operator
$N_0, N_{01}^{(d)}$	=	field renormalization functions
PTSS	=	perturbation theoretical series solution
$\hat{Q}, \hat{Q}_1^{(d)}, \hat{Q}_{S,V,B,...}$	=	inhomogeneity operators
$\mathbf{R}$	=	position vector in rectangular coördinates
$\hat{R}$	=	receiving array operator
S_{in}	=	input signal
s	=	(complex) angular frequency
$T_M^{(N)}$	=	medium operator
$\mathbf{X}, \mathbf{X}(t)$	=	receiver array outputs to signal processor
$\mathcal{Y}_{0,\infty}, \hat{\mathcal{Y}}_{0,\infty}, \hat{\Upsilon}_{0,\infty}$	=	Fourier and Laplace transform of g_∞
$\hat{1}$	=	unit operator

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