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A VARIATIONAL APPROACH TO HETEROCLINIC ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS

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Abstract

Consider the Hamiltonian system of forced pendulum type:

$$(*) \qquad \ddot{q} + V'(q) = 0$$

where V is periodic in the components $q_1, ..., q_n$ of q. Assuming that

$$\mathcal{M} = \{x \in \mathbf{R}^n | V(x) = \max_{\mathbf{R}^n} V\}$$

consists of isolated points, we show for each pair of points ξ, $\eta \in \mathcal{M}$, there is a chain of heteroclinic orbits of $(*)$ joining ξ and η. A lower bound for the number of distinct primitive chains is also given.

AMS (MOS) Subject Classifications: 34C25, 35J60, 58E05, 58F05, 58F22

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A Variational Approach to Heteroclinic
Orbits for a Class of Hamiltonian Systems

To J. L. Lions on the occasion of
his 60^{th} birthday

Introduction.

A large literature has developed in the last decade in which methods from the calculus of variations have been used to prove the existence of periodic solutions of Hamiltonian systems of ordinary differential equations. The recent monograph of Mawhin and Willem [1] provides a sizable bibliography of such works. Aside from equilibria, periodic solutions are the simplest global in time solutions of differential equations. It is only within the past one - two years that attempts have begun to extend the variational approach to such systems to find other kinds of global solutions of Hamiltonian systems [2-10]. Thus far mainly homoclinic orbits have been treated [2, 4-7, 9-10]. However heteroclinic orbits were studied in [3] for the class of second order Hamiltonian systems:

(HS) $\qquad \ddot{q} + V'(q) = 0$

where $q = (q_i, \dots, q_n) \in \mathbf{R}^n$ and V satisfies

$(V_1) \qquad V \in C^1(\mathbf{R}^n, \mathbf{R})$

and

(V_2) There are numbers $T_i > 0$ such that V is T_i periodic in q_i, $1 \leq i \leq n$.

Under (V_1)-(V_2), (HS) models a system of compound pendulum type. Our goal in this paper is to extend one of the main results in [3]. First we will recall what was done in [3]. Let

$$\mathcal{M} = \{x \in \mathbf{R}^n \mid V(x) = \max_{\mathbf{R}^n} V\}.$$

For $q \in C(\mathbf{R}, \mathbf{R}^n)$, if $q(t)$ has a limit as $t \to \infty$ (resp. $t \to -\infty$) we denote it by $q(\infty)$ (resp. $q(-\infty)$). In [3] it was shown that:

Theorem 0.1. If V satisfies (V_1)-(V_2) and $\mathcal{M}$ consists of isolated points, then for each $\beta \in \mathcal{M}$, there is a solution q of (HS) with $q(-\infty) = \beta$, $\dot{q}(-\infty) = 0$, $q(\infty) \in \mathcal{M}\backslash\{\beta\}$, and $\dot{q}(\infty) = 0$.

Note that the condition on $\mathcal{M}$ is a generic one. Theorem 0.1 provides a heteroclinic orbit of (HS) emanating from each $\beta \in \mathcal{M}$ and joining β to $\mathcal{M}\backslash\{\beta\}$. Since (HS) is time reversible, there exists a second such solution which starts in $\mathcal{M}\backslash\{\beta\}$ and terminates at β.

Abusing notation slightly, we denote the basic periodic region for V in $\mathbf{R}^n$ by T^n. It was also proved in [3] that:

Theorem 0.2. If V satisfies (V_1)-(V_2) and $\mathcal{M}|_{T^n}$ consists of a single point, then for each $\beta \in \mathcal{M}$, there are at least $2n$ distinct solutions Q_j of (HS) such that $Q_j(-\infty) = \beta$, $\dot{Q}_j(-\infty) = 0$, $Q_j(\infty) \in \mathcal{M}\backslash\{\beta\}$, $\dot{Q}_j(\infty) = 0$, $1 \le j \le 2n$.

In the setting of Theorem 0.2, two solutions Q and q are distinct if $Q(\cdot) - q(\cdot) \notin \mathcal{M}$.

Note that the simple pendulum corresponds to $n = 1$ with (HS) given by

$$\ddot{q} + \sin q = 0. \tag{0.3}$$

For this case, Theorem 0.2 and the time reversibility of (0.3) provide the same count of heteroclinic solutions as given by the usual phase portrait of (HS).

Simple minimization arguments are used to obtain Theorem 0.1 and 0.2. Recently P. Felmer [8] extended the above results to a more general class of Hamiltonian systems of the form

$$\begin{aligned} \dot{p} &= -H_q(p,q) \\ \dot{q} &= H_p(p,q) \end{aligned} \tag{0.4}$$

where $p, q \in \mathbf{R}^n$ and $H(p,q) = K(p,q) + V(q)$. Felmer assumed the potential energy and associated set $\mathcal{M}$ are as in Theorem 0.1, the kinetic energy K satisfies (V_2) as a function of q, and some other technical conditions hold. The special case $K = \frac{1}{2}|p|^2$ corresponds to (HS). The class of problems treated in [8] includes the classical compound pendulum system. The variational structure associated with (0.4) is more complicated than its analogue for (HS) and it is not known whether the minimization arguments of [3] can be applied in any direct way to treat (0.4). Instead Felmer uses minimax arguments based on the so-called Saddle Point Theorem.

In this paper, Theorem 0.1 will be generalized in another direction. Namely we will show for any $\beta, \gamma \in \mathcal{M}$, there exists a "chain" of heteroclinic orbits of (HS) joining β and γ. A more precise statement together with a proof will be given in the following section.

§1. A generalization of Theorem 0.1.

To formulate our generalization of Theorem 0.1, the functional setting of [3] as well as some of the technical tools developed there are needed. Let

$$E = \{q \in W^{1,2}_{\text{loc}}(\mathbf{R}, \mathbf{R}^n) \mid \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt < \infty\}.$$

Then E is a Hilbert space under the norm

$$\|q\| = \left(\int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt + |q(0)|^2 \right)^{1/2}. \tag{1.1}$$

Note that $q \in E$ implies q is continuous on $\mathbf{R}$.

To treat (HS), it can be assumed without loss of generality that

$$0 \in \mathcal{M} \text{ and } V(0) = 0. \tag{1.2}$$

For $q \in E$, set

$$I(q) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\dot{q}(t)|^2 - V(q(t)) \right) dt. \tag{1.3}$$

The heteroclinic solutions of (HS) were obtained in [3] as minima of I restricted to an appropriate subset of E. The following inequality (Lemma 3.6 of [3]) played a useful role in that process. For $A \subset \mathbf{R}^n$, let

$$B_\rho(A) = \{x \in \mathbf{R}^n \mid |x - A| < \rho\}.$$

Lemma 1.4: Let

$$0 < \rho < \frac{1}{2} \min_{\xi \in \mathcal{M} \setminus \{0\}} |\xi|.$$

Set

$$\alpha(\rho) = \min_{x \notin B_\rho(\mathcal{M})} -V(x).$$

If $w \in E$ and $w(t) \notin B_\rho(\mathcal{M})$ for all $t \in [r, s]$, then

$$I(w) \geq \sqrt{2\alpha(\rho)} |w(r) - w(s)|.$$

The next result is a small extension of Proposition 3.11 of [3]. It provides some regularity at $\pm\infty$ for prospective initial points of I.

Proposition 1.5: If $w \in E$ and

$$\int_a^{\infty} \left(\frac{1}{2} |\dot{w}|^2 - V(w) \right) dt < \infty$$

$$\left(\text{resp. } \int_{-\infty}^{a} \left(\frac{1}{2} |\dot{w}|^2 - V(w) \right) dt < \infty\right)$$

for some $a \in \mathbf{R}$, then $w(\infty)$ (resp. $w(-\infty)) \in \mathcal{M}$.

Now the class of sets on which I will be studied can be introduced. Let ϵ satisfy:

$$0 < \epsilon < \frac{1}{3} \min_{\xi \in \mathcal{M}\setminus\{0\}} |\xi| \equiv \frac{\gamma}{3} \tag{1.6}$$

and $\xi \in \mathcal{M}$. Set

$$\begin{aligned}\Gamma_\epsilon(\xi) = \{w \in E \,|w(-\infty) = 0, w(\infty) = \xi, \text{ and} \\ w(t) \notin B_\epsilon(\mathcal{M}\setminus\{0,\xi\}) \text{ for all } t \in \mathbf{R}\}.\end{aligned} \tag{1.7}$$

Define

$$c_\epsilon(\xi) = \inf_{w \in \Gamma_\epsilon(\xi)} I(w) \tag{1.8}$$

and

$$c_\epsilon = \inf_{\xi \in \mathcal{M}\setminus\{0\}} c_\epsilon(\xi). \tag{1.9}$$

It was shown in [3] (Proposition 3.12) that

Proposition 1.10. For each ϵ and ξ as above, there is a $q = q_{\epsilon,\xi} \in \Gamma_\epsilon(\xi)$ such that $I(q) = c_\epsilon(\xi)$, i.e. q minimizes $I\big|_{\Gamma_\epsilon(\xi)}$.

Lemma 1.4 shows $I(q_{\epsilon,\xi}) \to \infty$ as $|\xi| \to \infty$. Hence in (1.9) only finitely many numbers $c_\epsilon(\xi)$ are involved and the infimum is in fact a minimum. If $c_\epsilon = c_\epsilon(\varsigma)$ and $q_\epsilon = q_{\epsilon,\varsigma}$, so $I(q_\epsilon) = c_\epsilon$, simple comparison and regularity arguments given in [3] show

$$q_\epsilon(t) \cap \partial B_\epsilon(\mathcal{M}\setminus\{0,\varsigma\}) = \emptyset \text{ for all } t \in \mathbf{R}$$

provided that ϵ is sufficiently small. Hence q_ϵ is the desired heteroclinic solution of (HS) as claimed in Theorem 0.1.

Now our extension of Theorem 0.1 can be stated. For $\eta, \xi \in \mathcal{M}$, let

$$\Gamma(\eta,\xi) = \{w \in E \mid w(-\infty) = \eta, w(\infty) = \xi\} \tag{1.11}$$

and

$$c(\eta,\xi) = \inf_{w \in \Gamma(\eta,\xi)} I(w). \tag{1.12}$$

If $\eta = 0$, we write $\Gamma(\xi)$, $c(\xi)$ respectively for (1.11), (1.12). For simplicity we take $\eta = 0$ for what follows and show there is a "chain" of heteroclinic solutions of (HS) joining 0 and ξ corresponding to $c(\xi)$. More precisely

Theorem 1.13. Assume V satisfies (V_1)-(V_2) and $\mathcal{M}$ consists of isolated points. Then for each $\xi \in \mathcal{M}\backslash\{0\}$, there is a $j = j(\xi) \in \mathbf{N} \cup \{0\}$, distinct points $\xi_1, \ldots, \xi_j \in \mathcal{M}\backslash\{0, \xi\}$ and solutions $Q_1, \ldots, Q_{j+1}$ of (HS) such that if $\xi_0 = 0$ and $\xi_{j+1} = \xi$, $Q_i \in \Gamma(\xi_{i-1}, \xi_i)$ and $\dot{Q}_i(\pm\infty) = 0$, $1 \le i \le j+1$. Moreover

$$I(Q_i) = c(\xi_{i-1}, \xi_i), \quad 1 \le i \le j+1 \tag{1.14}$$

and

$$\sum_{i=1}^{j+1} I(Q_i) = c(\xi). \tag{1.15}$$

Proof. The proof involves ideas from [3] and the recent paper [8]. For each ϵ satisfying (1.6), let $q_{\epsilon,\xi}$ be the function given by Proposition 1.10. Set

$$\mathcal{T}_{\epsilon,\xi} = \mathcal{T}_{\epsilon,\xi}(q_{\epsilon,\xi}) = \{\tau \in \mathbf{R} \mid q_{\epsilon,\xi}(\tau) \in \partial B_\epsilon(\mathcal{M}\backslash\{0,\xi\}).$$

Since $q_{\epsilon,\xi} \in \Gamma_\epsilon(\xi)$, $\mathcal{T}_{\epsilon,\xi}$ is compact. In [3] it was shown in Proposition 3.18 and Corollary 3.24 that $q_{\epsilon,\xi}$ is a solution of (HS) on $\mathbf{R}\backslash\mathcal{T}_{\epsilon,\xi}$ with $\dot{q}_{\epsilon,\xi}(\pm\infty) = 0$. The chain of heteroclinic solutions of (HS) will be obtained by analyzing what happends to $q_{\epsilon,\xi}$ as $\epsilon \to 0$. Two cases arise:

Case 1: There exists a $\delta > 0$ and a decreasing sequence $\epsilon_m \to 0$ as $m \to \infty$ with $\epsilon_1 < \delta$ and such that

$$q_{\epsilon_m,\xi}(t) \notin B_\delta(\mathcal{M}\backslash\{0,\xi\}). \tag{1.16}$$

Let $q_m \equiv q_{\epsilon_m,\xi}$. By (1.16), $\mathcal{T}(q_m) = \emptyset$. Moreover $(q_m) \subset \Gamma_\delta(\xi)$. Since for $\epsilon < \bar{\epsilon}$, $\Gamma_\epsilon(\xi) \supset \Gamma_{\bar{\epsilon}}(\xi)$, (1.7)-(1.8) show $c_\epsilon(\xi)$ is a nonincreasing function of ϵ as $\epsilon \to 0$. By (1.16), $q_m \in \Gamma_{\epsilon_j}(\xi)$ for all $m, j \in \mathbf{N}$. Hence $c_\epsilon(\xi)$ must be constant for $\epsilon \le \epsilon_1$ and it can be assumed that q_m is independent of m. Consequently the first statement of Theorem 1.13 holds with $j = 1$ and $Q_1 = q$. That (1.14) is also valid follows from the argument given below for the analogous result for the more complicated Case 2.

Case 2: For any sequence $\epsilon_m \to 0$ as $m \to \infty$,

$$\min |q_{\epsilon_m}(t) - \partial B_{\epsilon_m}(\mathcal{M}\backslash\{0,\xi\})| \to 0 \tag{1.17}$$

To prove Theorem 1.13 for this case, note first that q_m is only determined up to a phase shift. Indeed, for all $t \in \mathbf{R}$, $q_m(t+\theta) \in \Gamma_{\epsilon_m}(\xi)$ and $I(q_m) = I(q_m(t+\theta))$. Since $q_m \in \Gamma_{\epsilon_m}(\xi)$, there is a unique $s_m \in \mathbf{R}$ such that $q_m(s_m) \in \partial B_{\frac{1}{3}}(0)$ and $q_m(t) \in B_{\frac{1}{3}}(0)$ for $t < s_m$. Thus choosing $\theta = s_m$, it can be assumed that $q_m(0) \in \partial B_{\frac{1}{3}}(0)$ and $q_m(t) \in B_{\frac{1}{3}}(0)$ for $t < 0$.

Next observe that by Lemma 1.4, whenever a segment of the curve q_m joins $\partial B_{\epsilon_m}(\eta)$ to $\partial B_{\epsilon_m}(\varsigma)$ for $\eta \neq \varsigma \in \mathcal{M}$, this segment makes a positive contribution to $I(q_m)$ whose magnitude $\to \infty$ as $|\eta - \varsigma| \to \infty$ and which is bounded from below by a positive number independently of m and of $\eta, \varsigma \in \mathcal{M}$. By this observation and (1.17), it follows that there is a $j \in \mathbf{N}$, precisely j points $\xi_i, \ldots, \xi_j \in \mathcal{M}\backslash\{0, \xi\}$, and a subsequence of m's such that

$$(1.18) \qquad \min_{t\in\mathbf{R}} |q_m(t) - \xi_k| \to 0$$

as $m \to \infty$ along this subsequence, $1 \le k \le j$. Moreover there is a $\beta > 0$ such that for $\eta \in \mathcal{M}\backslash\{\xi_i \mid 0 \le i \le j+1\}$,

$$(1.19) \qquad \min_{t\in\mathbf{R}} |q_m(t) - \eta| \ge \beta > 0$$

along the subsequence. By (1.18), there is a t_{m_k} such that

$$(1.20) \qquad q_m(t_{m_k}) \to \xi_k, \qquad 1 \le k \le j$$

as $m \to \infty$ along the subsequence. In fact t_{m_k} can be chosen so that it is the smallest value of t satisfying

$$(1.21) \qquad |q_m(t) - \xi_k| = \min_{s\in\mathbf{R}} |q_m(s) - \xi_k|$$

along the subsequence. Passing to further subsequences if necessary, it can be assumed that

$$(1.22) \qquad t_{m_k} > t_{m_i} \qquad \text{for} \quad 1 \le i < k$$

for all m along this subsequence.

Remark: See what follows repeated use of further subsequences will be necessary. This will be evident from the context and so as not to belabor the exposition, we will make no explicit mention of the fact other than this caveat.

Now the solutions $Q_1, \ldots, Q_{j+1}$ can be obtained. The construction will be divided into two steps. First we get Q_1. Then, assuming Q_{i-1} has been found, Q_i will also be constructed.

Set $Q_{1m}(t) = q_m(t), \ -\infty < t < t_{m_1}$. Then Q_{1m} is a solution of (HS) for this range of t and even for a large interval if $|q_m(t_{m_1}) - \xi_1| > \epsilon_m$. Also

$$(1.23) \qquad \begin{cases} \text{(i)} & Q_{1m}(0) \in \partial B_{\frac{\tau}{3}}(0) \\ \text{(ii)} & Q_{1m}(t) \in B_{\frac{\tau}{3}}(0), t < 0, \\ \text{(iii)} & \frac{1}{2}|\dot{Q}_{1m}(t)|^2 + V(Q_{1m}(t)) = 0, t < 0. \end{cases}$$

By Lemma 1.4, the functions q_m are bounded in $L^\infty(\mathbf{R}, \mathbf{R}^n)$. Hence by (HS), (Q_{1m}) are bounded in $C^2((-\infty, t_{m1}), \mathbf{R}^n)$. Consequently by these bounds, (HS), and (1.23), Q_{1m} converges in C^2_{loc} to a solution Q_1 of (HS) satisfying

$$(1.24) \qquad \begin{cases} \text{(i)} & Q_1(0) \in \partial B_{\frac{\tau}{3}}(0). \\ \text{(ii)} & Q_1(t) \in B_{\frac{\tau}{3}}(0), t < 0. \\ \text{(iii)} & \frac{1}{2}|\dot{Q}_1(t)|^2 + V(Q_1(t)) = 0, t < 0. \end{cases}$$

Moreover by the C^2_{loc} convergence, for any $r < 0$,

$$(1.25) \qquad \int_r^0 \left(\frac{1}{2}|\dot{Q}_1(t)|^2 - V(Q_1(t))\right) dt \le c_{\epsilon_1}(\xi).$$

Hence

$$(1.26) \qquad \int_{-\infty}^0 \left(\frac{1}{2}|\dot{Q}_1|^2 - V(Q_1)\right) dt \le c_{\epsilon_1}(\xi).$$

Therefore by Proposition 1.5, $Q_1(-\infty) \in \mathcal{M}$. Now (1.24) (ii) shows $Q_1(-\infty) = 0$ and (1.24) (iii) implies $\dot{Q}_1(-\infty) = 0$.

It remains to find the range of values of $t \ge 0$ for which Q_1 is defined. Two cases will be analyzed: (a) $t_{m_1} \to \infty$ as $m \to \infty$ and (b) t_{m_1} has a bounded subsequence.

Case (a): $t_{m_1} \to \infty$ as $m \to \infty$.

Then the above argument shows Q_{1m} converges to Q_1 satisfying (HS) for all $t \in \mathbf{R}$, (1.24) (iii) holds for all $t \in \mathbf{R}$, and

$$(1.27) \qquad \int_{-\infty}^{\infty} \left[\frac{1}{2}|\dot{Q}_1|^2 - V(Q_1)\right] dt \le c_{\epsilon_1}(\xi).$$

Hence by Proposition 1.5, $Q_1(\infty) \in \mathcal{M}$. We claim $Q_1(\infty) = \xi_1$. By (1.19), $Q_1(\xi) \in \{\xi_i \mid 0 \le i \le j+1\}$. Suppose $Q_1(\infty) = \xi_i, \ i \ne 1$. Then $Q_1(s) \to \xi_i$ as $s \to \infty$ for e.g. $s \in \mathbf{N}$. Since $Q_{1m}(s) \to Q_1(s)$ as $m \to \infty$, there is an $m = m(s)$ such that $Q_{1m(s)}(s) = q_{m(s)}(s) \to \xi_i$ as $s \to \infty$. Now separate but related arguments are required to exclude the possibilities of $i = 0$ and $i \ge 2$.

If $i = 0$, for $m = m(s)$, define

$$(1.28) \qquad \begin{cases} w_m(t) = 0 & t \le s-1 \\ \quad = (t-(s-1))q_m(s) & s-1 \le t \le s \\ \quad = q_m(t) & t > s. \end{cases}$$

Then $w_m \in \Gamma_{\epsilon_m}(\xi)$ and

$$(1.29) \qquad I(q_m) - I(w_m) = \int_{-\infty}^{s} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right) dt - \int_{s-1}^{s} \left(\frac{1}{2}|q_m(s)|^2 - V(w_m)\right) dt.$$

As s and therefore $m \to \infty$, the second term in (1.29) approaches 0 since $q_m(s) \to 0$ while by Lemma 1.4,

$$(1.30) \qquad \int_{-\infty}^{s} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right) dt \ge \frac{\gamma}{2}\sqrt{2\alpha\left(\frac{\gamma}{6}\right)}$$

since for large m, the curve q_m must join 0 to $\partial B_{\frac{\gamma}{3}}(0)$ to $\partial B_{\frac{\gamma}{6}}(0)$ to a neighborhood of ξ_1. Consequently for large s,

$$(1.31) \qquad I(q_m) > I(w_m),$$

contrary to Proposition 1.10. Hence $\xi_i = 0$ is not possible.

Next if $Q_1(\infty) = \xi_i$, $i \ge 2$, let s and $m(s)$ be as above. Let $\varphi_s \in C^1([0,1], \mathbb{R}^n \backslash B_\epsilon(\mathcal{M}))$ be a curve joining $q_m(s)$ to $q_m(t_{mi})$. Since $q_m(s)$ and $q_m(t_{mi})$ approaches ξ_i as $s \to \infty$, we can assume $\varphi_s(t) \to \xi_i$ and $\dot{\varphi}_s(t) \to 0$ uniformly in t as $s \to \infty$. Define

$$(1.32) \qquad \begin{cases} w_m(t) = q_m(t) & t < s \\ \quad = \varphi_s(t-s) & s \le t \le s+1 \\ \quad = q_m(t-s-1+t_{mi}) & t \ge s+1. \end{cases}$$

Then $w_m \in \Gamma_{\epsilon_m}(\xi)$ and

$$(1.33) \qquad I(q_m) - I(w_m) = \int_{s}^{t_{mi}} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right) dt - \int_{s}^{s+1} \left(\frac{1}{2}|\dot{\varphi}_s|^2 - V(\varphi_s)\right) dt.$$

As $s \to \infty$, the second term in (1.33) approaches 0 while

$$(1.34) \qquad \int_{s}^{t_{m_i}} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right) dt \ge 2\sqrt{2\alpha\left(\frac{\gamma}{3}\right)} \cdot \frac{\gamma}{3}.$$

Thus for large s, (1.31) holds, again a contradiction.

Now for both cases we have shown $Q_1(\infty) = \xi_1$. Letting $t \to \infty$ in (1.24) (iii) yields $\dot{Q}_1(\infty) = 0$ and the first statement of Theorem 1.13 holds for Case (a).

Case (b). t_{m1} has a bounded subsequence. Then there is a $\tau > 0$ such that $t_{m1} \to \tau$. Moreover $Q_{1m}(t)$ converges to $Q_1(t)$ in C^2_{loc} for $t \in (-\infty, \tau)$ with $Q_1 \in C((-\infty, \tau], \mathbf{R}^n)$, Q_1 is a solution of (HS) for $t < \tau$, (1.24) (iii) holds for $t < \tau$, and $Q_1(\tau) = \xi_1$. Letting $t \to \tau$, (1.24) (iii) then shows that $\dot{Q}_1(t) \to 0$ as $t \to \tau$ and (HS) implies $\ddot{Q}_1(t) \to 0$ as $t \to \tau$. These facts imply Q_1 can be extended in C^2 to $[\tau, \infty)$ as a solution of (HS) via $Q_1(t) \equiv \xi_1$ for $t \geq \tau$.

Thus we have determined Q_1 for both Cases (a) and (b).

Remark 1.35: If $V \in C^2(\mathbf{R}^n, \mathbf{R})$, by the reasoning of Case (b) and the (then) uniqueness of solutions of the initial value problem for (HS), $Q_1(t) = \xi_1$ for $t \geq \tau$ implies $Q_1(t) \equiv \xi_1$. Hence Case (b) is not possible.

Remark 1.36: The argument given in Case (a) also shows $Q_1(t) \neq \xi_i$, $0 \leq i \leq j+1$ except possibly for $i = 0$ in a t interval containing $-\infty$ and $i = 1$ in a t interval containing ∞.

It remains to show that (1.14) holds for Q_1, i.e.

$$I(Q_1) = c(\xi_1). \tag{1.37}$$

Since $Q_1 \in \Gamma(\xi_1)$,

$$I(Q_1) \geq c(\xi_1). \tag{1.38}$$

If equality does not hold in (1.38), there is a $w \in \Gamma(\xi_1)$ such that

$$I(w) < I(Q_1). \tag{1.39}$$

Define δ by

$$I(Q_1) - I(w) = 5\delta. \tag{1.40}$$

By a weak lower semicontinuity argument,

$$\int_{-\infty}^{0} \left(\frac{1}{2}|\dot{Q}_1|^2 - V(Q_1) \right) dt \leq \underline{\lim}_{m \to \infty} \int_{-\infty}^{0} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m) \right) dt. \tag{1.41}$$

Hence for large m,

$$(1.42)\qquad \int_{-\infty}^{0}\left(\frac{1}{2}|\dot{Q}_1|^2 - V(Q_1)\right)dt - \delta \le \int_{-\infty}^{0}\left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right)dt.$$

Choose τ so that

$$(1.43)\qquad \int_{\tau}^{\infty}\left(\frac{1}{2}|\dot{Q}_1|^2 - V(Q_1)\right)dt < \delta.$$

Note that $\tau < t_{m1}$ for m large. For $t \in [0,\tau]$, $q_m(t) \to Q_1(t)$ uniformly in C^2. Hence for m large,

$$(1.44)\qquad \int_{0}^{\tau}\left(\frac{1}{2}|\dot{Q}_1|^2 - V(Q_1)\right)dt - \delta \le \int_{0}^{\tau}\left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right)dt.$$

Combining (1.40) and (1.42)-(1.44) yields

$$(1.45)\qquad I(w) + 2\delta = I(Q_1) - 3\delta \le \int_{-\infty}^{\tau}\left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right)dt$$

for m large. Choose σ such that

$$(1.46)\qquad I(w) = \int_{-\infty}^{\sigma}\left(\frac{1}{2}|\dot{w}|^2 - V(w)\right)dt + \delta.$$

For m satisfying (1.45), choose σ_m so that $w(\sigma_m) \in \partial B_{\epsilon_m}(\xi_1)$ and $w(t) \notin \overline{B}_{\epsilon_m}(\xi_1)$ for $t < \sigma_m$.

We can assume for m sufficiently large, $\sigma_m > \sigma$. Otherwise $\sigma_m \le \sigma$ for all large m. Since $\sigma_{m+1} > \sigma_m$, $\sigma_m \to \overline{\sigma} \le \sigma$ and $w(\overline{\sigma}) = \xi_1$. But then w can be replace above by a new function $\overline{w}(t) = w(t)$, $t \le \overline{\sigma}$ and $\overline{w}(t) = \xi_1$, $t > \overline{\sigma}$. For this new choice of w, $\sigma_m > \sigma$ for large m.

Recall that

$$(1.47)\qquad r_m \equiv |q_m(t_{m1}) - \xi_1| \ge \epsilon_m.$$

Let $\varphi_m \in C^1([\sigma_m, \sigma_m + 1], \overline{B}_{i_m}(\xi_1) \backslash B_{\epsilon_m}(\xi_1))$ such that $\varphi_m(\sigma_m) = w(\sigma_m)$ and $\varphi_m(\sigma_m + 1) = q_m(t_{m1})$. For m sufficiently large, φ_m can be chosen so that

$$(1.48)\qquad \int_{\sigma_m}^{\sigma_m+1}\left(\frac{1}{2}|\dot{\varphi}_m|^2 - V(\varphi_m)\right)dt < \delta.$$

Set

$$W_m(t) = w(t) \qquad t \le \sigma_m \tag{1.49}$$
$$= \varphi_m(t) \qquad \sigma_m < t < \sigma_m + 1$$
$$= q_m(t - \sigma_m - 1 + t_{m1}) \qquad t \ge \sigma_m + 1.$$

Then by construction $W_m \in \Gamma_{\epsilon_m}(\xi)$. Moreover by (1.48)-(1.49),

$$I(w) \ge \int_{-\infty}^{\sigma_m} \left(\frac{1}{2}|\dot{W}_m|^2 - V(W_m)\right) dt \tag{1.50}$$
$$\ge \int_{-\infty}^{\sigma_m+1} \left(\frac{1}{2}|\dot{W}_m|^2 - V(W_m)\right) dt - \delta.$$

Hence by (1.45) and (1.50),

$$\int_{-\infty}^{\sigma_m+1} \left(\frac{1}{2}|\dot{W}_m|^2 - V(W_m)\right) dt + \delta < \int_{-\infty}^{\tau} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right) dt \tag{1.51}$$

for m large. Finally observe from (1.49) that

$$\int_{\sigma_m+1}^{\infty} \left(\frac{1}{2}|\dot{W}_m|^2 - V(w_m)\right) dt = \int_{t_{m1}}^{\infty} \left(\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right) dt. \tag{1.52}$$

Adding (1.52) to (1.51) and recalling that $t_{m1} > \tau$ for m large gives

$$I(W_m) + \delta \le I(q_m). \tag{1.53}$$

But (1.53) violates the choice of q_m in Proposition 1.10. Hence (1.37) holds

It remains to do the analogue of the above for $Q_2, \ldots, Q_{j+1}$. Suppose that $Q_1, \ldots, Q_{i-1}$ have been constructed and (1.14) holds for these functions. We will show how to construct Q_i. Let $\bar{t}_{m,i-1}$ be the largest value of t such that

$$|q_m(t) - \xi_i| = |q_m(t_{m,i-1}) - \xi_i|. \tag{1.54}$$

By the comparison argument of Case (a) following (1.26), it can be assumed that $t_{mi} > \bar{t}_{m,i-1}$ for large m. Hence there is a $\sigma_{mi} \in (\bar{t}_{m,i-1}, t_{mi})$ such that

$$\begin{cases} q_m(\sigma_{mi}) \in \partial B_{\epsilon_1}(\xi_{i-1}) \\ q_m(t) \notin \overline{B}_{\epsilon_1}(\xi_{i-1}), t > \sigma_{mi}. \end{cases} \tag{1.55}$$

Define

$$Q_{im}(t) = q_m(t + \sigma_{m,i}) \qquad t \in (\bar{t}_{m,i-1} - \sigma_{mi}, t_{mi} - \sigma_{mi}). \tag{1.56}$$

Then Q_{im} satisfies (HS) for this range of t's and $Q_{im}(0) \in \partial B_{\epsilon_1}(\xi_{i-1})$. As was the case with Q_{1m}, these functions are bounded in $C^2(\bar{t}_{m,i-1}-\sigma_{mi}, t_{mi}-\sigma_{mi}, \mathbf{R}^n)$ and converge in C^2_{loc} (for t in some maximal interval) to a solution Q_i of (HS) such that $Q_i(0) \in \partial B_{\epsilon_1}(\xi_{i-1})$ and $Q_i(t) \notin B_{\epsilon_1}(\xi_{i-1})$ for $t > 0$.

There are now 3 cases to consider

(i) Both sequences $(\bar{t}_{m,i-1} - \sigma_{mi})$, $(t_{mi} - \sigma_{mi})$ are unbounded.

(ii) Only one sequence is unbounded.

(iii) Both sequences are bounded.

Case (i). Both sequences are unbounded.

Consequently, as with Q_1, Q_i satisfies (HS) on $\mathbf{R}$ and

$$\int_S \left(\frac{1}{2}|\dot{Q}_i|^2 - V(Q_i)\right) dt \le c_{\epsilon_1}(\xi) \tag{1.57}$$

for $S = (-\infty, 0]$ and $[0,\infty)$. Proposition 1.5 then shows $Q_i(\pm\infty)$ exists and belongs to $\mathcal{M}$. Moreover the arguments given for Q_1 show $Q_i(-\infty) = \xi_{i-1}$ and $Q_i(\infty) = \xi_i$. Since Q_i satisfies (HS) on $\mathbf{R}$,

$$\frac{1}{2}|\dot{Q}_i(t)|^2 + V(Q_i(t)) \equiv \text{ constant } = A, \quad t \in \mathbf{R}. \tag{1.58}$$

Letting $t \to -\infty$ in (1.58) yields $A \ge 0$. By (1.57)-(1.58),

$$c_{\epsilon_1}(\xi) \ge \int_{-\infty}^0 (A - 2V(Q_i))dt \ge \int_{-\infty}^0 A dt. \tag{1.59}$$

Hence $A = 0$. Therefore letting $t \to \pm\infty$ in (1.58) yields $\dot{Q}_i(\pm\infty) = 0$.

To verify (1.14) for this case, we argue essentially as with Q_1, the only difference being that if $i \ne j+1$, $w(t)$ must be joined to q_m both for large positive and negative values of t. We omit the details.

Case (ii). Only one sequence is unbounded.

Suppose e.g. $(t_{mi} - \sigma_{mi})$ is bounded. The arguments of Case (i) still show $Q_i(-\infty) = \xi_{i-1}$ and (1.58) holds for all t for which the solution is defined with $A = 0$. Since $(t_{mi} - \sigma_{mi})$ is bounded and $Q_{im}(t_{mi} - \sigma_{mi}) \to \xi_i$ as $m \to \infty$, there is a $\tau > 0$ such that $t_{mi} - \sigma_{mi} \to \tau$ as $m \to \infty$ and $Q_i(\tau) = \xi_i$. Then by the arguments given for Case (b) of Q_1, Q_i extends in a C^2 fashion as a solution of (HS) to $[\tau, \infty)$ via $Q_i(t) = \xi_i$ for $t \ge \tau$. We also get (1.14) as in Case (i).

Case (iii). Both sequences are bounded.

As in Case (ii), there are constants $\sigma < 0$ and $\tau > 0$ such that $\bar{t}_{m,i-1} - \sigma_{mi} \to \sigma$, $t_{mi} - \sigma_{mi} \to \tau$, $Q_i(\sigma) = \xi_{i-1}$, $Q_i(\tau) = \xi_i$, and Q_i is a solution of (HS) on (σ, τ). We can continuously extend Q_i to $\mathbf{R}$ via $Q_i(t) = \xi_{i-1}$ for $t < \sigma$ and $Q_i(t) = \xi_i$ for $t > \tau$. We claim this extension belongs to $C^2(\mathbf{R}, \mathbf{R}^n)$ (and therefore is a C^2 solution of (HS) with $\dot{Q}_i(\pm\infty) = 0$). Equation (1.58) cannot be used as earlier to get a C^2 extension since a prior A may be positive. Hence a more indirect approach will be employed. The argument of Case (i) again implies (1.24) holds here. Now a slight variant of an argument from Proposition 3.18 of [3] can be employed.

Let $\underline{\sigma} < \sigma$ and $\bar{\tau} > \tau$. For $\psi \in C^1(\mathbf{R}, \mathbf{R}^n)$ with the support of ψ in $[\underline{\sigma}, \bar{\tau}]$ and $\delta \in \mathbf{R}$, the function $Q_i + \delta\psi \in \Gamma(\xi_{i-1}, \xi_i)$. By (1.14),

$$I(Q_i + \delta\psi) \geq I(Q_i). \tag{1.60}$$

Since (1.60) holds for all such δ and ψ,

$$I'(Q_i)\psi \equiv \int_{\underline{\sigma}}^{\bar{\tau}} (\dot{Q}_i \cdot \dot{\psi} - V'(Q_i) \cdot \psi) dt = 0 \tag{1.61}$$

for all $\psi \in W_0^{1,2}([\underline{\sigma}, \bar{\tau}], \mathbf{R}^n)$. Thus Q_i is a weak solution of the equation

$$\begin{cases} \ddot{Q} + V'(Q_r) = 0, & \underline{\sigma} < t < \bar{\tau} \\ Q(\underline{\sigma}) = \xi_{i-1}, & Q(\bar{\tau}) = \xi_i. \end{cases} \tag{1.62}$$

But the inhomogeneous linear system (1.62) has a unique C^2 solution u which can be written down explicitly. Taking the inner product with ψ and integrating by parts implies

$$\int_{\underline{\sigma}}^{\bar{\tau}} (\dot{u} \cdot \dot{\psi} - V'(Q_i) \cdot \psi) dt = 0 \tag{1.63}$$

for all $\psi \in W_0^{1,2}([\underline{\sigma}, \bar{\tau}], \mathbf{R}^n)$. Comparing (1.61) and (1.63) shows

$$\int_{\underline{\sigma}}^{\bar{\tau}} (\dot{Q}_i - \dot{u}) \cdot \dot{\psi} dt = 0 \tag{1.64}$$

for all $\psi \in W_0^{1,2}([\underline{\sigma}, \bar{\tau}], \mathbf{R}^n)$. Thus Q_i is a weak solution of the equation

$$\begin{cases} \ddot{Q} + V'(Q_r) = 0 & \underline{\sigma} < t < \bar{\tau} \\ Q(\underline{\sigma}) = \xi_{i-1}, & Q(\bar{\tau}) = \xi_i. \end{cases} \tag{1.62}$$

But the inhomogeneous linear system (1.62) has a unique C^2 solution u which can be written down explicitly. Taking the inner product with ψ and integrating by parts implies

$$\int_{\underline{\sigma}}^{\bar{\tau}} (\dot{u} \cdot \dot{\psi} - V'(Q_i) \cdot \psi) dt = 0 \tag{1.63}$$

for all $\psi \in W_0^{1,2}([\underline{\sigma},\overline{\tau}],\mathbf{R}^n)$. Comparing (1.61) and (1.63) shows

$$\int_{\underline{\sigma}}^{\overline{\tau}} (\dot{Q}_i - \dot{u}) \cdot \dot{\psi} dt = 0 \tag{1.64}$$

for all $\psi \in W_0^{1,2}([\underline{\sigma},\overline{\tau}],\mathbf{R}^n)$. But $Q_i - u$ belongs to this space. Consequently $Q_i = u$ on $[\underline{\sigma},\overline{\tau}]$, $Q_i \in C^2(\mathbf{R},\mathbf{R}^n)$, and the construction of Q_i is complete.

Finally to complete the proof of Theorem 1.13, (1.15) must be verified. By modifying the functions Q_1 near ξ_1, Q_{j+i} near ξ_j, and Q_i near ξ_{i-1} and ξ_i, $i \neq 1, j+1$, a function $Q \in \Gamma(\xi)$ can be constructed such that $I(Q)$ differs from $\sum_1^{j+1} I(Q_i)$ by an arbitrary small amount. It follows that

$$c(\xi) \leq \sum_{i=1}^{j+1} I(Q_i). \tag{1.65}$$

If strict inequality held in (1.64), there is a $w \in \Gamma(\xi)$ such that

$$I(w) < \sum_{i=1}^{j+1} I(Q_i). \tag{1.66}$$

But then, by earlier arguments, for large m a function $\overline{w}_m \in \Gamma_{\epsilon_m}(\xi)$ can be constructed such that

$$I(\overline{w}_m) < \sum_{i=1}^{j+1} I(Q_i). \tag{1.67}$$

Again, by earlier arguments, this leads to

$$I(w) < I(q_m) = c_{\epsilon_m}(\xi) \tag{1.68}$$

for large m, a constraaiction.

The proof of Theorem 1.13 is complete.

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