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A NOTE ON THE GAUSSIAN INTEGRATION FORMULA

by
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A Note On The Gaussian Integration Formula

by

C. Richardson

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A Note On The Gaussian Integration Formula

ABSTRACT

The Gaussian integration method is described together with the 2-point implementation for numerical integration. A FORTRAN program is given and the result of a test example.

BACKGROUND

1. This work was originally completed some years ago and the software, which is incorporated in a Maths Library on the UDT computer facility, has been used extensively. Gauss' formula provides a method for calculating the definite integral of a given function.

THEORY

2. If $P_m(x)$, $P_n(x)$, $x \in R$, are polynomials of degree m,n respectively, with $m \geq n$, then polynomials q(x), r(x) exist such that:

$$P_m(x) = P_n(x)q(x) + r(x) \qquad 1.$$

where the degree of q(x) is m-n and the degree of r(x) is n-1 at most. The polynomials q(x), r(x) are unique and called the quotient and remainder of $P_m(x)$ with reference to $P_n(x)$.

3. Consider:

$$\int_{-1}^{+1} P_{2n+1}(x)dx$$

and let $P_{n+1}(x)$ be a Legendre polynomial of degree n+1. Then by equation 1:

$$\int_{-1}^{+1} P_{2n+1}(x)dx = \int_{-1}^{+1} \left\{P_{n+1}(x)q_n(x) + r_n(x)\right\}dx \qquad 2.$$

But any polynomial of degree n can be expressed as a linear combination of Legendre polynomials of degree $k \leq n$. Thus

$$q_n(x) = \sum_{k=0}^{n} a_k P_k(x) \qquad 3.$$

and therefore, using the orthogonal properties of Legendre functions:

$$\int_{-1}^{+1} P_{n+1}(x)q_n(x)dx = 0 \qquad 4.$$

gives:

$$\int_{-1}^{+1} P_{2n+1}(x)dx = \int_{-1}^{+1} r_n(x)dx \qquad 5.$$

4. We seek solutions of the form:

$$\int_{-1}^{+1} P_{2n+1}(x)dx = \sum_{k=0}^{n} a_k P_{2n+1}(x_k)$$

$$= \sum_{k=0}^{n} a_k \left\{ P_{n+1}(x_k) q_n(x_k) + r_n(x_k) \right\} \qquad 6.$$

but equation 5 shows the integral is independant of the choice of $q_n(x)$ and therefore equation 6 is true iff x_k are the n+1 roots of $P_{n+1}(x)$. Let:

$$\omega_{n+1}(x) = \prod_{k=0}^{n} (x-x_k) \qquad 7.$$

then expanding in partial fractions gives:

$$\frac{r_n(x)}{\omega_{n+1}(x)} = \sum_{k=0}^{n} \frac{r_n(x_k)}{\omega'_{n+1}(x_k)(x-x_k)} \qquad 8.$$

Integration of equation 8 gives:

$$\int_{-1}^{+1} r_n(x)dx = \int_{-1}^{+1} \sum_{k=0}^{n} \frac{\omega_{n+1}(x) r_n(x_k) dx}{\omega'_{n+1}(x_k)(x-x_k)} \qquad 9.$$

Therefore:

$$\int_{-1}^{+1} P_{2n+1}(x)dx = \sum_{k=0}^{n} a_k P_{2n+1}(x_k) \qquad 10.$$

where:

$$a_k = \int_{-1}^{+1} \frac{\omega_{n+1}(x)dx}{\omega'_{n+1}(x_k)(x-x_k)} \qquad 11.$$

5. If we use the Hermite interpolation formula to represent a function f(x) by a polynomial of degree 2n+1, the error in the integral formula is given by:

$$e_n = \frac{f^{(2n+2)}(\hat{x})}{2n+2)!} \int_{-1}^{+1} \prod_{k=0}^{n} (x-x_k)^2 dx \qquad 12.$$

where $-1 \leq \hat{x} \leq +1$.

6. There are n+1 values of x_k, and hence this method of integration is called the n+1 point method. For the Gaussian 2-point integral formula put n=1 and then x_k k=0,1 are the zeros of $P_2(x)$, ie x_k are given by:

$$(3x^2-1)/2 = 0$$

therefore:

$$x_k = \pm 1/\sqrt{3}$$

and:

$$\omega_{n+1}(x) = (x^2 - 1/3)$$

Hence the weighting factors are:

$$a_k = \int_{-1}^{+1} \frac{x \pm 1/\sqrt{3}}{2x} dx$$

$$= 1$$

Finally the formula is:

$$\int_{-1}^{+1} f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right) + \frac{f^{(4)}(\hat{x})}{135} \qquad 13.$$

7. In general $\int_a^b f(x)dx$ can be transformed to the form of equation 13. Furthermore the accuracy can be improved by subdividing the interval a,b. Let each subinterval have half width h, where:

$$h = (b-a)/2n$$

Then:

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \int_{2kh+a}^{2(k+1)h+a} f(x)dx$$

$$= \sum_{k=0}^{n-1} h\int_{-1}^{+1} f(x)dx'$$

where:

$$x = x'h + h(2k+1) + a$$

and so by equation 13:

$$\int_a^b f(x)dx = h\sum_{k=0}^{n-1}\left\{f\left(a+h(2k+1+1/\sqrt{3})\right)+f\left(a+h(2k+1-1/\sqrt{3})\right)\right\}+\frac{h.n.h^4f^{(4)}(\hat{x})}{135} \qquad 14.$$

8. The error term in equation 14 can be written:

$$e_n = \frac{(b-a)^5 f^{(4)}(\hat{x})}{2^5 n^4 135}$$

Taking 2n intervals the error will be:

$$e_{2n} = \frac{(b-a)^5 f^{(4)}(\hat{x})}{2^5 2^4 n^4 135}$$

$$= \frac{e_n}{16}$$

Put:

$$\int_a^b f(x)dx = I_n + e_n$$

$$= I_{2n} + e_{2n}$$

then a closer approximation is given by:

$$\int_a^b f(x)dx = I_{2n} + (I_{2n} - I_n)/15 \qquad 15.$$

9. Gauss' integral formula is not suitable to evaluate an integral of a function given in tabular form, but when the function is given as an analytic expression the method is very useful. A FORTRAN procedure is described to compute:

$$I = \int_a^b f(x)dx + \varepsilon$$

where:

$$|\varepsilon| \leq tol$$

```
      REAL FUNCTION GAUSS_INTEGRATION(F,A,B,TOL)

C     Description:
C             Integrates the function  F(x) over the range (A,B), where F is a
C             real function  (to be provided).  The repeated  two-point  Gauss
C             method is used  until successive approximations  differ  by less
C             than 15*TOL.

C     Author:
C             C Richardson, ARE (Portland)

C     History:
C             Issue   1.0      22 July      1982
C             Mod     1.1       6 August    1985

C     Modifications:
C             1.1     To adopt coding standard

C     Subroutine Arguments:

      REAL            F                                   !(R) Function
     *                A,                                  !(R) Limits
     *                Tol                                 !(R) Required accuracy

C     Local Variables:

      REAL            P1,P2,                              !Integration estimates
     *                H,                                  !Subinterval half width
     *                Cd,G

      INTEGER         Nc                                  !Number of subintervals

C     Main Entry Point:

      Nc=8                                                !Number of subintervals
      DO WHILE ((Nc .LT. 31).OR.(ABS(P2-P1) .GT. 15.*tol))
        P1=P2                                             !Save last estimate
        H=(B-A)/(2*Nc)                                    !Subinterval half-width
        Cd=A-H
        G=H/1.73205081
        P2=0.                                             !Partial sum
        DO M=1,Nc
          Cd=Cd+H+H
          P2=P2+F(Cd-G)
          P2=P2+F(Cd+G)
        ENDDO
        P2=P2*H                                           !Estimated result
        Nc=2*Nc
      ENDDO
      GAUSS_INTEGRATION=P2+(P2-P1)/15.
      RETURN
      END
```

```
        PROGRAM GAUSS_INTEGRATION_TEST

C       Description:
C               Test  Gaussian integration  by computing the integral of  Sin(x)
C               over the range (A,B).

C       Author:
C               C Richardson, ARE (Portland)

C       History:
C               Issue   1.0     22 July     1982

C       Local Variables:

        REAL            A,B,                            !Integration interval
     *                  P                               !Workspace

        EXTERNAL F

C       Main Entry Point:

100     FORMAT(A)

        TYPE *,'The integration interval A,B must be specified'
        A=ASKR('Enter value of A  (Degrees)')
        B=ASKR('Enter value of B  (Degrees)')

        CALL PLOT_INIT_('WINDOW=A4-V')                  !Initialise graphics
        CALL CHAR_SIZE(1.5,2.0)
        CALL GRID_LIN_LIN(30.,50.,180.,200.,A,-1.,B,1.,1)
        CALL CURVE(F,A,B,1.)                            !Draw function over A,B
        CALL MOVE((A+B)/2.,-1.)                         !Annotate axes
        CALL CHAR_POSN(-7.5,-2.)
        CALL PLOT_TEXT('Angle (Degrees)',15)
        CALL MOVE(A,0.)
        CALL CHAR_POSN(-4.,0.)
        CALL CHAR_ANGLE(90.)
        CALL CHAR_POSN(-3.,0.)
        CALL PLOT_TEXT('Sin(X)',6)
        CALL CHAR_ANGLE(0.)
        P=GAUSS_INTEGRATION(F,A,B,.00001)               !Calculate numerical integral
        P=P/57.29578
        CALL MOVE(A,1.)                                 !Annotate graph
        CALL CHAR_POSN(0.,1.5)
        CALL PLOT_TEXT('Numeric  Integral=',18)
        CALL PLOT_NUM(P,3,4)
        P=COSD(A)-COSD(B)                               !Calculate analytic integral
        CALL MOVE(A,1.)                                 !Annotate graph
        CALL CHAR_POSN(0.,0.5)
        CALL PLOT_TEXT('Analytic Integral=',18)
        CALL PLOT_NUM(P,3,4)
        CALL ORIGIN(0.,0.)                              !Plot title
        CALL SCALE(1.,1.)
        CALL MOVE(30.,20.)
        CALL CHAR_SIZE(1.8,2.4)
        CALL PLOT_TEXT(
     *    'Fig  1. Gaussian Integration Of The Function Sin(X)',51)
        ACCEPT 100,P
        CALL PLOT_FIN
        END

        FUNCTION F(X)                                   !A typical function
        F=SIND(X)
        RETURN
        END
```

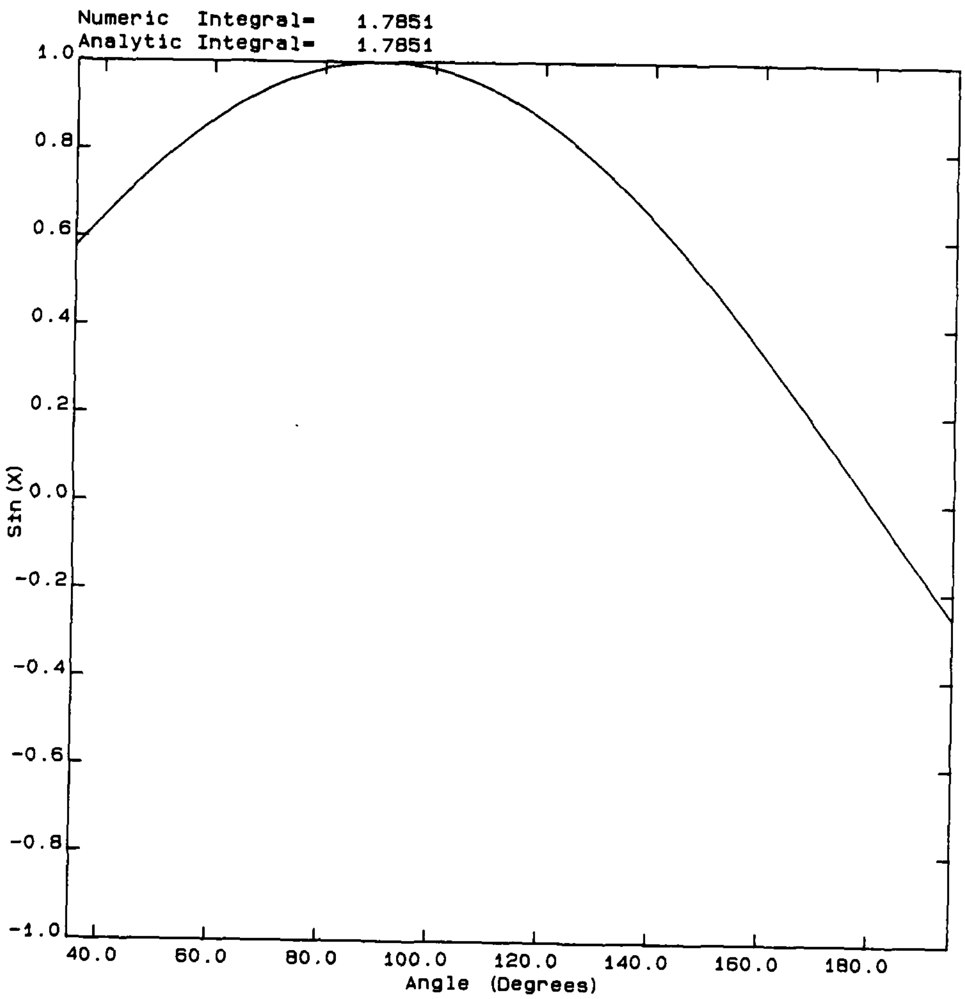

Fig 1. Gaussian Integration Of The Function Sin (X)